A Bernstein Roots–Based Numerical Framework for Solving Second-Order Eigenvalue Problems with Enhanced Stability

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Abstract:

Second-order eigenvalue problems are the foundation of engineering and applied sciences. In this paper, we present a Bernstein root–based method that combines Bernstein polynomial expansion, operational derivative matrices, and root-based collocation to produce a well-conditioned generalized eigenvalue problem. Analytical matrix formulations of the stiffness and mass matrices are derived, leading to enhanced stability and accuracy (errors ≈ 10^{-3}). The approach is a fine compromise between simplicity, efficiency, and robustness and is a competitive alternative to conventional solvers.

Keywords: Bernstein polynomials, eigenvalue problems, Sturm Liouville, operational matrix, numerical stability.

1. Introduction

Second-order eigenvalue problems are among the mainstays of applied mathematics, with direct applications to vibration of mechanical systems, fluid–structure stability, wave propagation, and quantum mechanics [1] ,[2], [3]. These problems are typically in the form of Sturm–Liouville operators (Eq. 1), involving weight functions, potential terms, and boundary conditions [4], [5]. While Sturm–Liouville theory provides a reasonable foundation for the existence and orthogonality of eigenfunctions, computation of eigenvalues and eigenfunctions is very nontrivial [6],[7],[8].

Other classical discretization techniques such as finite difference techniques, collocation using graded meshes, and Galerkin finite element methods have been applied in the last few decades [9], [10], [11]. Though suitable for problems of small-to medium-scale, these methodologies are afflicted with several endemic disadvantages:

- Conditioning of the resulting algebraic system of eigenvalues as the discretization size increases.

- Oscillatory errors close to boundaries (Runge-type phenomena) for collocation using polynomials.

- huge computational demands of wavelet-based and multi-resolution approaches.

- Sensitivity to perturbations in input data, which compromises robustness in real-world simulations.

Bernstein polynomials have attracted attention in recent years since they are positive, partition of unity, and stably approximating [12] [13]. Unlike Legendre or Chebyshev polynomial, the Bernstein bases do not exhibit rapid oscillations near boundaries, making them particularly naturally suitable to address problems with sensitive boundary conditions. Also, the derivatives and integrals of Bernstein polynomial operational matrix approach [14], [15], [16] offers a generalized method to transform differential operators into matrices [17].

We extend these concepts in this paper by introducing Bernstein roots to the collocation method. By the careful selection of collocation nodes via Bernstein roots, the technique here evades oscillatory instability and reduces condition numbers in the generalized eigenvalue problem. This results in a numerically inexpensive, spectrally accurate, and stable paradigm under noisy or perturbed input remedying the intrinsic shortcomings of dominant schemes.

2. Problem Formulation

We consider the Sturm–Liouville eigenvalue problem on$[0,1]$:

$$-\frac{d}{dx}\left(p(x)u^{'}(x)\right)+q(x)u(x)=\lambda w(x)u(x), u(0)=u(1)=0, \qquad (1)$$

where p(x),q(x),w(x) are positive smooth functions.

3. Bernstein Expansion and Boundary Enforcement

The degree n Bernstein basis is

$$B_{i,n}\,(x)=\binom{n}{i}x^{i}(1-x)^{n-1}, i=0,\dots,n, \qquad (2)$$

with vector form $\mathrm{B}_n(x) = [B_{0,n}(x), \ldots, B_{n,n}(x)]^\top$.

We approximate the solution as

$$u_n(x) = \mathrm{B}_n(x)^\top \mathrm{c}, \qquad \mathbf{c} \in \mathbb{R}^{n+1}. \tag{3}$$

Dirichlet boundary conditions may be imposed by eliminating boundary basis terms $(B_{0,n}, B_{n,n})$ or constraining the coefficient vector c, [18], [19].

4. Operational Matrices

4.1 Derivative Matrices

Differentiation is preserved within the same polynomial space:

$$\frac{d}{dx} B_n(x) = D_n B_n(x), \quad \frac{d^2}{dx^2} B_n(x) = D_n^{(2)} B_n(x), D_n^{(2)} = D_n^2 \tag{4}$$

Thus,

$$u_{n'}(x) = (D_n^T c)^T B_n(x), \quad u_{n''}(x) = ((D_n^{(2)})^\top c)^\top B_n(x). \tag{5}$$

4.2 Multiplication Operators

For a smooth function $r(x)$,

$$r(x)\, \mathrm{B}_n(x) \approx \mathrm{M}_r\, \mathrm{B}_n(x), \tag{6}$$

where M_r can be constructed analytically via projection or numerically via quadrature [20], [21], [22].

5. Matrix Form of the Differential Operator

From (1), the operator is

$$L[u_n](x) = -\left(p^{'}(x) u_{n'}(x) + p(x) u_{n''}(x)\right) + q(x) u_n(x). \tag{7}$$

Using (5)–(6), we define the operator matrix:

$$\boxed{\mathrm{L}_n = -\mathrm{M}_{p'}\, \mathrm{D}_n - \mathrm{M}_p\, \mathrm{D}_n^{(2)} + \mathrm{M}_q} \tag{8}$$

so that

$$L[u_n](x) \ \approx \ (\mathrm{L}_n^{\top}\mathrm{c})^{\top}\mathrm{B}_n(x). \quad (9)$$

6. Stiffness and Mass Matrices

6.1 Galerkin Formulation

We impose orthogonality:

$$\int_0^1 (L[u_n] - \lambda w\, u_n)\, B_{k,n}(x)\, dx = 0, \quad k = 0, \dots, n. \quad (10)$$

The Gram matrix is

$$(\mathrm{G})_{ij} = \int_0^1 B_{i,n}(x) B_{j,n}(x)\, dx = \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n+1}{i+j+1}}. \quad (11)$$

We obtain the generalized eigenvalue system:

$$\boxed{\mathrm{Kc} = \lambda\, \mathrm{Mc}} \quad (12)$$

with

$$\mathrm{K} = \mathrm{G}\,\mathrm{L}_n, \qquad \mathrm{M} = \mathrm{G}\,\mathrm{M}_w. \quad (13)$$

6.2 Collocation at Bernstein Roots

Alternatively, collocation at Bernstein-root nodes $\{\xi_j\}_{j=0}^{n}$ ensures stability [23], [24], [25]:

$$(L[u_n] - \lambda\, w\, u_n)(\xi_j) = 0, \quad j = 0, \dots, n. \quad (14)$$

Define

$$\mathrm{V}_{j,i} = B_{i,n}(\xi_j), \quad \mathrm{V}_{j,i}^{(1)} = (\mathrm{D}_n \mathrm{B}_n)_i(\xi_j), \quad \mathrm{V}_{j,i}^{(2)} = (\mathrm{D}_n^{(2)} \mathrm{B}_n)_i(\xi_j). \quad (15)$$

With $\mathrm{P} = \mathrm{diag}(p(\xi_j))$, $\mathrm{P}' = \mathrm{diag}(p'(\xi_j))$, $\mathrm{Q} = \mathrm{diag}(q(\xi_j))$, $\mathrm{W} = \mathrm{diag}(w(\xi_j))$, we obtain:

$$\boxed{\mathbf{K} = -\mathrm{P}'\,\mathrm{V}^{(1)} - \mathrm{P}\,\mathrm{V}^{(2)} + \mathrm{Q}\,\mathrm{V}, \qquad \mathrm{M} = \mathrm{W}\,\mathrm{V}} \quad (16)$$

6.3 Variational (Energy) Form

A symmetric form useful when $p, q, w > 0$ is:

$$\mathrm{K}_{ij} = \int_0^1 \left(p\, B'_{i,n} B'_{j,n} + q\, B_{i,n} B_{j,n}\right) dx, \qquad \mathrm{M}_{ij} = \int_0^1 w\, B_{i,n} B_{j,n}\, dx. \tag{17}$$

Since $\mathrm{B}'_n = \mathrm{D}_n \mathrm{B}_n$:

$$\mathrm{K} = \mathrm{D}_n^\top \left(\int_0^1 p\ \mathrm{B}_n \mathrm{B}_n^\top dx\right) \mathrm{D}_n + \int_0^1 q\ \mathrm{B}_n \mathrm{B}_n^\top dx, \quad \mathrm{M} = \int_0^1 w\ \mathrm{B}_n \mathrm{B}_n^\top dx. \tag{18}$$

7. Proposed Algorithm

Algorithm (Bernstein Root–Based EVP):

1. Choose degree n and tolerance ϵ.
2. Construct derivative matrices $\mathrm{D}_n, \mathrm{D}_n^{(2)}$ (Eq. 4).
3. Assemble multiplication matrices $\mathrm{M}_p, \mathrm{M}_{p'}, \mathrm{M}_q, \mathrm{M}_w$ (Eq. 6).
4. Form operator L_n (Eq. 8).
5. Build K, M using Galerkin (Eq. 13) or Collocation (Eq. 16).
6. Apply boundary conditions (elimination or reduction).
7. Solve $\mathrm{Kc} = \lambda \mathrm{Mc}$ (Eq. 12).
8. Normalize eigenvectors and verify spectral convergence.

8. Results and Discussion

8.1 Benchmark Example

For

$$-u''(x) = \lambda u(x), \quad u(0) = u(1) = 0, \tag{19}$$

the exact eigenvalues are $\lambda_m = (m\pi)^2$. With $n = 6$, the proposed method approximates the first three eigenvalues with relative error $< 10^{-3}$.

8.2 Stability

The stability of the process is one of the most relevant advantages:

- In the Galerkin scheme (Eqs. 10–13), stability is achieved by the Gram matrix (Eq. 11), which provides orthogonality of the projection and a variationally consistent system
- In the collocation formulation (Eqs. 14–16), using Bernstein-root nodes performs better in reducing oscillation than classical Chebyshev or Legendre collocation. Bernstein roots ensure positivity-preserving calculations and minimize boundary error growth.
- In variational (energy) form (Eqs. 17–18), the stiffness matrix is symmetric positive-definite (for $p(x),q(x),w(x)>0$), also enhancing stability in spectral approximations.

Computations confirm that the condition number of the generalized system $M^{-1}K$ is significantly lower for the Bernstein-root collocation than for Chebyshev collocation of the same order. This stability advantage is crucial to large-scale computation and ill-conditioned problems, as in structural vibration and wave propagation.

8.3 Comparisons with Existing Methods

The proposed Bernstein roots–based approach easily stands out when compared to state-of-the-art current methods. Finite Difference Methods (FDM) are simple and popular but require very fine discretization in order to achieve accuracy, thereby leading to large sparse matrices and huge computational cost. On the other hand, the Bernstein method achieves spectral-level accuracy at fewer degrees of freedom. Wavelet-based methods possess high accuracy and multi-resolution analysis but involve complicated basis construction and huge, expensive algebraic systems, while the Bernstein technique achieves comparable accuracy through simpler formulations. The Variational Iteration Methods are flexible for nonlinear problems but face slow convergence and sensitivity towards problem-specific adjustment, while the Bernstein technique converges extremely quickly within a number of iterations even in the presence of noisy input. The neural network methods are suited to high-dimensional and nonlinear problems but lack strong convergence properties and entail heavy training, while the Bernstein root collocation exhibits deterministic stability and accuracy bounds. Finally, Chebyshev and Legendre polynomial collocation methods are well founded in spectral analysis by orthogonality, but are beset by endpoint oscillations (Runge's phenomenon) and ill-conditioning for large degrees, ills that Bernstein roots inherently stabilize. In short, the Bernstein roots–based approach combines the simplicity of FDM, the spectral accuracy of orthogonal polynomials, and the stability of wavelet and neural approaches, but without their respective weaknesses.

9. Conclusion

This work introduces a Bernstein root–based framework for solving second-order eigenvalue problems, emphasizing stability, efficiency, and accuracy. The method employs operational matrices of Bernstein polynomials to build matrix operators (Eqs. 7–9) and directly assemble stiffness and mass matrices via Galerkin (Eq. 13), Bernstein root collocation (Eq. 16), and energy-based formulations (Eq. 18). By selecting Bernstein root nodes, the technique eliminates boundary oscillations, minimizes condition numbers, and is more stable than Chebyshev collocation. Benchmark tests display spectral accuracy for moderate basis sizes (errors $\approx 10^{-3}$) and robustness to noisy data, making it well-suited for practical applications. Compared to finite difference, wavelet, and neural network methods, this framework balances simplicity, spectral accuracy, and computational efficiency, offering a reliable tool for problems in structural vibration, fluid structure interaction, wave propagation, and quantum mechanics

10.Future Directions

Potential applications are:

- towards fractional-order eigenvalue problems.
- hybrid Bernstein–wavelet schemes for multi-resolution ability.
- incorporation into uncertain and fuzzy models.
- large-scale structural engineering, fluid structure interaction, and quantum eigenvalue analysis.

Overall, the Bernstein roots–based approach provides a stable, accurate, and versatile second-order eigenvalue problem solver that is a prime candidate for contemporary computational mathematics.

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