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The Detour Chromatic Number of a Graph

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Abstract

In this paper, we introduce the concept detour chromatic number of a graph. Also detour chromatic number of several classes of graph are determined and some of its general properties are studied. Let G be a connected graph. A subset $S \subseteq V(G)$ is said to be a detour chromatic set of G if S is both a detour set and a chromatic set of G. A detour chromatic set S is said to be minimum detour chromatic set of G if there exists no detour chromatic set S' of G with $|S'| < |S|$. The cardinality of a minimum detour chromatic set of G is the detour chromatic number of G. It is denoted by $\chi_{d_n}(G)$.

Keywords: Detour number, Chromatic number, Detour Chromatic number.

AMS: 05C12

1. Introduction

By a graph $G=(V,E)$ we mean a finite, connected, undirected graph with neither loops nor multiple edges. The *order* $|V|$ and *size* $|E|$ of G are denoted by p and q respectively. For graph theoretic terminology we refer to west[8]. A vertex v of G is said to be an *extreme vertex* if the subgraph induced by its neighborhood is complete. The set of all extreme vertices is denoted by $Ext(G)$. For vertices x and y in a connected graph G, the *detour distance* $D(x,y)$ is the length of a longest $x-y$ path in G. An $x-y$ path of length $D(x,y)$ is called an $x-y$ detour. The closed interval $I_D[x,y]$ consists of all vertices lying on some $x-y$ detour of G. For $S \subseteq V, I_D[S] = \cup_{x,y \in S} I_D[x,y]$. A set S of vertices is a detour set if $I_D[S] = V(G)$, and the minimum cardinality of a detour set is the detour number $dn(G)$. A detour set of cardinality $dn(G)$ is called a *minimum detour set* [2, 3,].

A k −coloring of G is a function $c: V(G) \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for every adjacent vertices $u, v \in V(G)$. The *chromatic number* of G denoted by $\chi(G)$, is the smallest k for which G has a k −coloring. A graph having chromatic number k is called a $k-$ chromatic graph. Let G be a k-chromatic graph. A set $S \subseteq V(G)$ is called chromatic set if S contains all k vertices of different colors in G. The chromatic number of a graph was studied in [6, 7].

A vertex of degree 0 is called an *isolated vertex* and a vertex of degree 1 is called an *end vertex* or a *pendant vertex*. A vertex that is adjacent to a pendant vertex is called a

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support vertex. A vertex of degree $p - 1$ is called a *full vertex*. The set of all full vertices is denoted by $Fx(G)$. A cycle of length three is also a *triangle* and a graph G containing no triangles is called *triangle-free*.

Definition 1.1

Let G be a connected graph. A subset $S \subseteq V(G)$ is said to be a detour chromatic set of G if S is both a detour set and a chromatic set of G. A detour chromatic set S is said to be minimum detour chromatic set of G if there exists no detour chromatic set S' of G with $|S'| < |S|$. The cardinality of a minimum detour chromatic set of G is the detour chromatic number of G. It is denoted by $\chi_{d_n}(G)$. Any detour chromatic set S of cardinality $\chi_{d_n}(G)$ is called a χ_{d_n} – set of G or $\chi_{d_n}(G) -$ set.

Example 1.2

Consider the graph G in Figure 1.1. $S = \{v_6, v_7, v_8\}$ is a unique minimum detour set of G and $S_1 = \{v_1, v_4\}$ is a chromatic set of G but S is not a chromatic set of G and S_1 is not a detour set of G. Thus, $\chi_{d_n}(G) \geq 4$. It is clear that $S_2 = \{v_1, v_6, v_7, v_8\}$ is a detour set and a chromatic set of G. Therefore, S_2 is a detour chromatic set of G and hence $\chi_{d_n}(G) = 4$.

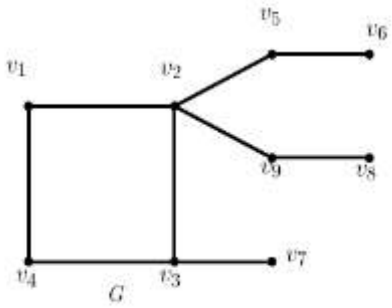

Figure 1.1

Here $d(G) = 3$ and $\chi(G) = 2$. Thus, detour chromatic number is different from both detour number as well as chromatic number.

2 Some Basic Results

Theorem 2.1. *[3] Every end vertex of G belongs to every detour set of G.*

Theorem 2.2. *[3] For a non-trivial tree, dn(G) = k, where k is the number of end vertices of G.*

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3 The Detour Chromatic Number of a Graph

Remark 3.1

For any connected graph G, there can be more than one detour chromatic set.

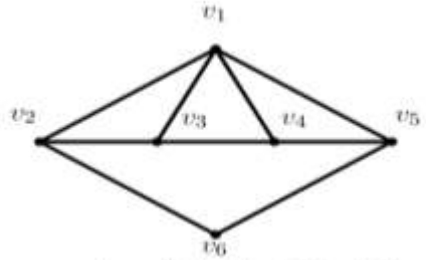
For the graph G in Figure 1.1, $S_1 = \{v_3, v_6, v_7, v_8\}, S_2 = \{v_5, v_6, v_7, v_8\}, S_3 = \{v_4, v_6, v_7, v_8\}$ and $S_4 = \{v_6, v_7, v_8, v_9\}$ are four detour chromatic sets of cardinality four.

Thus, the minimum detour chromatic set of G is not be unique.

Remark 3.2

There is no relation connecting $d(G)$ and $\chi(G)$.

Consider the connected graph G in Figure 3.2.

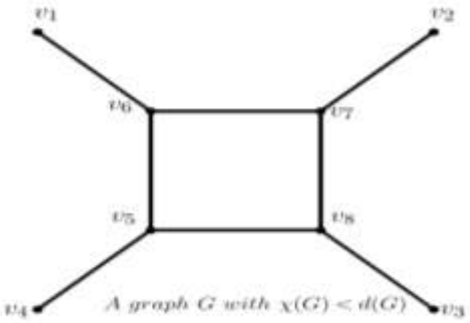


A graph G with $\chi(G) > d(G)$

Figure 3.2

Here $S = \{v_1, v_2, v_5\}$ is a minimum chromatic set of G and so $\chi(G) = 3$. But it can be easily verified that $S_1 = \{v_1, v_6\}$ is a minimum detour set of G and so $d(G) = 2$. Therefore, $\chi(G) > d(G)$.

Now, consider the connected graph G in Figure 3.3.



A graph G with $\chi(G) < d(G)$

Figure 3.3

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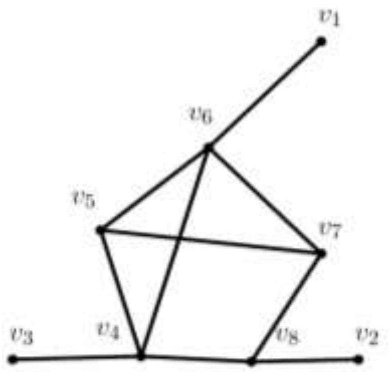
Here $S = \{v_1, v_2, v_3, v_4\}$ is a minimum detour set of G and so $d(G) = 4$. But it is clear that $S_1 = \{v_1, v_2\}$ is a minimum chromatic set of G and so $\chi(G) = 2$. Therefore, $\chi(G) < d(G)$.

Now, consider the connected graph G in Figure 3.4.



A graph with $d(G) = \chi(G)$

Figure 3.4

Here, $S = \{v_1, v_2, v_3\}$ is a minimum detour set as well as minimum chromatic set of G. Hence $d(G) = \chi(G) = 3$.

Theorem 3.3

If G is a connected graph of order $n \geq 2$, then $2 \leq d(G) \leq \chi_{d_n}(G) \leq n$.

Proof.

A detour set needs minimum two vertices and so $d(G) \geq 2$. Since every detour chromatic set of G is a detour set, $d(G) \leq \chi_{d_n}(G)$. Moreover, the set of all vertices in G form a detour chromatic set of G and so $\chi_{d_n}(G) \leq n$. Hence $2 \leq d(G) \leq \chi_{d_n}(G) \leq n$.

Remark 3.4

The bounds in Theorem 3.3 are strict. For the connected graph G given in Figure 3.5, $d(G) = 3, \chi_{d_n}(G) = 4$ and $n = 7$.

Therefore, $2 < d(G) < \chi_{d_n}(G) < n$.

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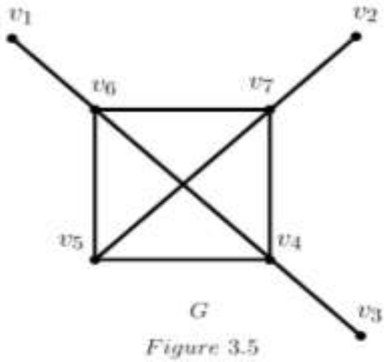

Figure 3.5

Corollary 3.5

Let G be any connected graph of order n. If $d(G) = n$, then $\chi_{d_n}(G) = n$.

Proof.

The proof is straight forward from Theorem 3.3.

Remark 3.6

The converse of corollary 3.5 need not be true. For the graph G in Figure 3.6, $\chi_{d_n}(G) = 5 = n$. But $d(G) = 2$.

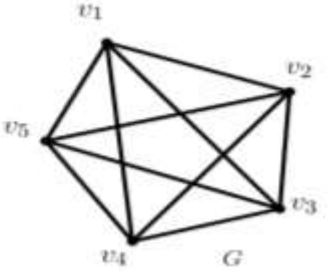

Figure 3.6

Corollary 3.7

Let G be any connected graph of order $n \geq 2$. If $\chi_{d_n}(G) = 2$, then $d(G) = 2$.

Proof.

The proof is straight forward from Theorem 3.3.

Remark 3.8

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The converse of corollary 3.7 need not be true always. For the graph G in Figure 3.2, $d(G) = 2$, but $\chi_{d_n}(G) = 3$.

Observation 3.9

The detour chromatic number of some standard graphs can be easily found and are given below:

(i) For the complete graph $K_n (n \geq 2), \chi_{d_n}(K_n) = n$.

(ii) For the star graph $K_{1,n-1} (n \geq 2), \chi_{d_n}(K_{1,n-1}) = n$.

(iii) For the path graph $P_n (n \geq 2), \chi_{d_n}(P_n) = \begin{cases} 2 \text{ } if \text{ } n \text{ } is \text{ } even \\ 3 \text{ } if \text{ } n \text{ } is \text{ } odd \end{cases}$.

(iv) For the cycle graph $C_n (n \geq 3), \chi_{d_n}(C_n) = \begin{cases} 2 \text{ } if \text{ } n \text{ } is \text{ } even \\ 3 \text{ } if \text{ } n \text{ } is \text{ } odd \end{cases}$.

(v) For the complete bipartite graph $K_{m,n}, (1 \leq m \leq n), \chi_{d_n}(K_{m,n}) = 2$.

(vi) For the fan graph $G = K_1 + P_{n-1} (n \geq 3), \chi_{d_n}(G) = 3$.

(vii) For the wheel graph $G = K_1 + C_{n-1} (n \geq 4), \chi_{d_n}(G) = \begin{cases} 3 \text{ } if \text{ } n-1 \text{ } is \text{ } even \\ 4 \text{ } if \text{ } n-1 \text{ } is \text{ } odd \end{cases}$.

Theorem 3.10

Let G be a connected graph of order $n \geq 2$. Then $2 \leq \chi(G) \leq \chi_{d_n}(G) \leq n$.

Proof.

Since G be a connected graph of order $n \geq 2$, we need minimum two colours for a proper colouring G. Therefore, $\chi(G) \geq 2$. Also, every detour chromatic set of G is a chromatic set, $\chi(G) \leq \chi_{d_n}(G)$. Moreover, $V(G)$ for a detour chromatic set of G. Thus, $\chi_{d_n}(G) \leq n$. Hence, $2 \leq \chi(G) \leq \chi_{d_n}(G) \leq n$.

Remark 3.11

The bounds in Theorem 3.10 are sharp. For the path graph $P_n, \chi(P_n) = 2$, for the cycle graph $C_{2n+1}, (n \geq 1), \chi(C_{2n+1}) = 3$. For the star graph $K_{1,n-1}, \chi_{d_n}(K_{1,n-1}) = n$.

Also, all the inequalities in Theorem 3.10, are strict. For the connected graph G, given in Figure 3.7, $\chi(G) = 3, \chi_{d_n}(G) = 4$ and $n = 7$. Therefore, $2 < \chi(G) < \chi_{d_n}(G) < n$.

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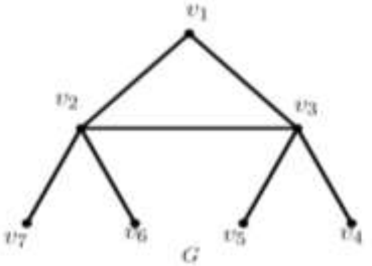

Figure 3.7

Corollary 3.12

Let G be any connected graph of order n. If $\chi(G) = n$, then $\chi_{d_n}(G) = n$.

Proof.

The proof is straight forward from Theorem 3.10.

Remark 3.13

The converse of corollary 3.12 need not be true. For the star graph $K_{1,n-1}$, $\chi_{d_n}(K_{1,n-1}) = n$ but $\chi(K_{1,n-1}) = 2$.

Corollary 3.14

Let G be any connected graph of order n. If $\chi_{d_n}(G) = 2$, then $\chi(G) = 2$.

Proof.

The proof is straight forward from Theorem 3.10.

Remark 3.15

The converse of corollary 3.14 need not be true. For the graph G in Figure 1.1, $\chi(G) = 2$, but $\chi_{d_n}(G) = 4$.

Theorem 3.16

Every end vertices of G belong to every detour chromatic of G.

Proof.

Since every detour chromatic set of G is a detour set, the result follows from Theorem 2.1

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Observation 3.17

If G is a connected graph, we have the following.

(i) A minimum detour set S in G which is also a chromatic set of G, then S is $\chi_{d_n}-$ set of G.

(ii) A minimum chromatic set S in G, which is also a detour set of G, then S is a $\chi_{d_n}-$ set of G.

Remark 3.18

If S is a χ_{d_n}- set of G, then S need not be a minimum detour set as well as minimum chromatic set of G

For example, consider the connected graph G in Figure 1.1. Here $S = \{v_1, v_6, v_7, v_8\}$ is a χ_{d_n}- set of G and so $\chi_{d_n}(G) = 4$. Also, $S_1 = \{v_6, v_7, v_8\}$ is a detour set of G and $S_2 = \{v_1, v_4\}$ is a chromatic set of G.

Theorem 3.19

Let G be any connected graph and if the set of all end vertices S of G is a detour chromatic set of G, then S is the unique minimum detour chromatic set of G.

Proof.

Let S be the set of all end vertices of G, which is a detour chromatic set of G. Then, $\chi_{d_n}(G) \leq |S|$. But, by Theorem 3.16, $\chi_{d_n}(G) \geq |S|$. Hence, $\chi_{d_n}(G) = |S|$. Therefore, that S is a minimum detour chromatic set of G. Moreover, by Theorem 3.16, every minimum detour chromatic set must contain S and so that S is the unique minimum detour chromatic set of G.

Theorem 3.20

Every full degree vertex of a connected graph G of order n belong to every detour chromatic set of G.

Proof

Let S be a detour chromatic set of G and let u be a full degree vertex of G. So $\deg(u) = n - 1$. Let $u_1, u_2, \ldots, u_{n-1}$ be the vertices of G adjacent with u. To prove $u \in S$. Suppose $u \notin S$. Let $c_1, c_2, \ldots, c_{n-1}$ be the colours assigns to $u_1, u_2, \ldots, u_{n-1}$, respectively

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which may be distinct. Since $u \notin S$, by proper colouring, u must assign any one of the colour from the set $\{c_1, c_2, \dots, c_{n-1}\}$. Without loss of generality, we assign u as colour c_1. This shows that u and u_1 receive same colour. Since $\deg(u) = n - 1$, that u and u_1 are adjacent. It follows that S is not a detour chromatic set of G, which is a contradiction. Hence $u \in S$.

4 Conclusion

In this paper, we get a deep knowledge about detour chromatic number of a graph. It has many applications in the field of social networking and modern technologies. For our future work we can extend it for large families of graphs.

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