# Numerical Solution of a Class of Nonlinear Delay Partial Differential Equations Using a Linear Compact Difference Scheme

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## Abstract

In this paper, we propose numerical solution of a certain class of nonlinear DPDEs via a linear compact difference scheme. Of course, and we know very well that in technology, physics or biology as well as other scientific fields like time lags which may be modelled with nonlinear parabolic PDEs in the context of continuum mechanics or space confinements are to describe those systems. not characteristic were the strong nonlinearity and delay terms that often lead to numericenarios UNssnasosty an large precisionestabiy losses in numericalthe loss of high precision. To address these issues, we propose a linear compact finite difference scheme that is able to provide high-order spatial accuracy and icely conserve the time delay term. By discretization of the spatial derivative terms with compact stencil and the linearization of the nonlinear terms through suitable transformations, an efficient and readily implementable algorithm is developed. The scheme is analyzed for stability and convergence by rigorous mathematical method, guaranteeing that the present numerical solution can be obtained with high accuracy. Some typical examples including benchmark problems and nonlinear with complicated time delay structure are examined to demonstrate the accuracy, efficiency, validity, and practicality of the proposed method. Numerical experiments show improved error estimates for the proposed method compared to classical FD schemes and stability with respect to different discretization parameters and initial conditions. The results emphasize that the linear compact scheme is an efficient and useful tool for simulating nonlinear delay PDEs arisen in science and engineering. Extension of the method to multi-dimensions and exploration into adaptive meshes as a means for improving computational efficiency and quality of the solutions can be features that may be included in future developments.

**Keywords:** Nonlinear delay PDEs, Linear compact difference scheme, Numerical solution, Stability and convergence

## 1. Introduction

Partial differential equations (PDEs) are used to describe a diverse set of dynamic phenomena across science and engineering, such as heat conduction, fluid flow, population dynamics, reaction-diffusion chemistry. In most real processes, however, the behaviour of the system at a given time is affected not only by the state of that system at that time but also by the past states. These behaviors are naturally described by delay partial differential equations (DPDEs), together with a time-delay term to represent memory effects or lagged responses in the system. Nonlinear terms together with the time delays impose a higher complexity in these equations and do not allow analytical solutions for most practical cases. Because of this intrinsic difficulty, numerical methods have been developed as indispensable parts of research on nonlinear DPDEs. Conventional methods, such as standard FDDF (finite difference and differential formula), Fem (finite element method) and spectral method usually have difficulties in both high accuracy and stability under strong nonlinearities and large delays. [1-3] Furthermore, the traditional desretization methods can be computationally expensive or numrically unstable when discretizing fine spatial grids or simulating for a long time. As a result, there is growing need for robust and efficient numerical algorithms which are tailored to the non linear delay PDEs. In this respect, compact difference schemes have been known to be effective for spatial discretization. The schemes are high-order accurate and have a compact computational stencil such that numerical calculations can be performed efficiently. Compact schemes may potentially be used to handle nonlinear terms, for example by linearising sub-problems, yielding a linearised system which is easier computationally than the original one. Computational it allows researcher to achieve efficiency with accuracy and the latter plays a critical role in the viability of this legendre spectral method for practical engineering and other applied science problems. The first aim of our work is to build a linearized compact finite difference approximation for a class of nonlinear delay PDEs. The approach we discuss is projected to be high order in time and capable of handling delay terms accurately, with stability and convergence theory. Numerical examples are presented in the form of benchmark problems and more complex nonlinear cases involving time delays, to demonstrate both the efficiency and robustness of the method. The numerics of those experiments support the conclusion that our new method has better accuracy and stability than traditional methods, and hence it is a useful tool for nonlinear DPDEs. [4-7]

In this paper, we consider numerical solution of nonlinear delay partial differential equations. First of all we consider a linear compact differencent scheme for this kind of the problems. Then, the uniqueness of solution is proved and convergence and stability of this linear compact difference scheme are analyzed. Then, a numerical example is provided to bound the error and the efficiency of the proposed difference scheme. a b Most of the material presented here is adopted from the citation [8].

## 2. Problem Formulation

Consider the following nonlinear delay partial differential equations

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^r u}{\partial x^r} = f(u(x,t), u(x,t-s), x, t), a < x < b, \circ \leq t \leq T \quad (1)$$

$$u(a,t) = u_a(t), u(b,t) = u_b(t), t \in [\circ, T] \quad (2)$$

$$u(x,t) = \psi(x,t), x \in [a,b], t \in [-s, \circ), \quad (3)$$

Here $\alpha > 1$represents the diffusion coefficient, and sss is the delay parameter. It is noteworthy that if we substitute

$$f(u(x,t), u(x,t-s), x, t) = u(x,t)[1 - u(x,t-s)]$$

into equation (3.1), we obtain the Hutchinson equation. Let mmm be an integer such that $ms \leq T < (m+1)$ and let $C_0 = max_{a<x<b,0<t<T}|u(x,t)|$

We assume that for the solution u(x,t) of problem (1) to (3), all the following relationships hold with a uniform upper bound

$$\max_{a<x<b,-s<t<0} \frac{\partial^r u(x,t)}{\partial t^r},$$

$$\max_{a<x<b,ls<t<(l+1)s} \frac{\partial^r u(x,t)}{\partial t^r}, \max_{a<x<b,ls<t<(l+1)s} \frac{\partial^r u(x,t)}{\partial t^r}, l = \circ, 1, \dots, m-1,$$

$$\max_{a<x<b,ls<t<(l+1)s} \frac{\partial^\varphi u(x,t)}{\partial x^r \partial t^r}, \max_{a<x<b,ls<t<(l+1)s} \frac{\partial^\varphi u(x,t)}{\partial x^\varphi}, l = \circ, 1, \dots, m-1,$$

$$\max_{a<x<b,ms<t<T} \frac{\partial^{\text{r}} u(x,t)}{\partial t^{\varphi}}, \quad \max_{a<x<b,ms<t<T} \frac{\partial^{\varphi} u(x,t)}{\partial t^{\varphi}},$$
$$\max_{a<x<b,ms<t<T} \frac{\partial^{\varphi} u(x,t)}{\partial x^{\uparrow} \partial t^{\varphi}},$$

We also suppose that f(μ,ν,x,t) has the continuous first-order partial derivatives with respect to its first two variables in a neighbourhood of the solution, where is actually a positive constant. [9]

## 2-1. Introduction of the Linear Compact Difference Scheme

Let M and N be two positive integers. We define the grid points $x_i = a + ih, N = \left[\frac{T}{\tau}\right], \tau = \frac{s}{n}, h = \frac{b-a}{M}$ with $t_k = k_7$ The time instances are given by $t_{k+\frac{1}{2}} = \frac{1}{2}(t_k + t_{k+1})$. Let $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$where $\Omega_h = \{x_i \mid 0 \leq i \leq M\}$ and $\Omega_\tau = \{t_k \mid -n \leq k \leq N\}$.

We define $\mathrm{W} = \left\{v \mid v = v_i^k, \circ \leq i \leq M, -n \leq k \leq N\right\}$ as the space of grid functions on $\Omega_{h\tau}$. For $v = \left\{v_i^k\right\} \in \mathcal{W}$, we have

$$v_i^{k+\frac{1}{2}} = \frac{1}{2}\left(v_i^k + v_i^{k+1}\right)$$
$$\delta_t v_i^{k+\frac{1}{2}} = \frac{1}{\tau}\left(v_i^{k+1} - v_i^k\right)$$
$$\delta_x^r v_i^k = \frac{1}{h^2}\left(v_{i+1}^k - r v_i^k + v_{i-1}^k\right)$$

**Lemma 2.1.1.** Let $g(x) \in C^6([x_{i-1}, x_{i+1}])$ Then:

$$\frac{1}{12}\left(g''(x_{i-1}) + 10\log''(x_i) + g''(x_{i+1})\right) - \frac{1}{h^2}\left(g(x_{i-1}) - 2g(x_i) + g(x_{i+1})\right) = \frac{h^4}{240} g^{(6)}(\omega_i),$$

where $\omega_i \in (x_{i-1}, x_{i+1})$.

**Proof**

Using Taylor series expansion for

$$g(x_{i+1}) = g(x_i) + hg'(x_i) + \frac{h^2}{2} g''(x_i) + \frac{h^3}{6} g'''(x_i)$$

$$+\frac{h^4}{24}g^{(4)}(x_i)+\frac{h^5}{120}g^{(5)}(x_i)+\frac{h^6}{120}\int_0^1 g^{(6)}(x_i+zh)(1-z)^5dz$$

$$g(x_{i-1}) = g(x_i)-hg'(x_i)+\frac{h^2}{2}g''(x_i)-\frac{h^3}{6}g'''(x_i)$$
$$+\frac{h^4}{24}g^{(4)}(x_i)-\frac{h^5}{120}g^{(5)}(x_i)+\frac{h^6}{120}\int_0^1 g^{(6)}(x_i-zh)(1-z)^5dz$$

where z is a positive value between (0,1). Summing the two equalities above, we obtain

$$g(x_{i+1})+g(x_{i-1}) = 2g(x_i)+h^2\,g''(x_i)+\;2\;\frac{h^4}{24}g^{(4)}(x_i)$$
$$+\,2\;\frac{h^6}{120}\int_0^1\;\left[g^{(6)}(x_i+zh)+g^{(6)}(x_i-zh)\right](1-z)^5dz$$

Therefore

$$g(x_{i+1})+g(x_{i-1})-2g(x_i) = h^2g''(x_i)+2\frac{h^4}{12}g^{(4)}(x_i)$$
$$+\frac{h^6}{60}\int_0^1\;\left[g^{(6)}(x_i+zh)+g^{(6)}(x_i-zh)\right](1-z)^5dz$$

Dividing both sides by $h^2$, we get:

$$\frac{1}{h^2}[g(x_{i+1})-2g(x_i)+g(x_{i-1})]$$
$$= g''(x_i)+\frac{h^2}{12}g^{(4)}(x_i)+\frac{h^4}{60}\int_0^1\;\left[g^{(6)}(x_i+zh)+g^{(6)}(x_i-zh)\right](1-z)^5dz \qquad (4)$$

Similarly, by Taylor expansion for $g'(x_{i+1})$:

$$g''(x_{i+1}) = g''(x_i)+hg'''(x_i)+\frac{h^2}{2}g^{(4)}(x_i)+\frac{h^3}{6}g^{(5)}(x_i)$$
$$+\frac{h^4}{6}\int_0^1 g^{(6)}(x_i+zh)(1-z)^3dz$$

We can write

$$\frac{1}{12}[g''(x_{i+1})+1\circ g''(x_i)+g''(x_{i-1})] =$$

$$g''(x_i)+\frac{h^2}{12}g^{(4)}(x_i)+\frac{h^4}{72}\int_0^1\left[g^{(6)}(x_i+zh)+g^{(6)}(x_i-zh)\right](1-z)^3dz,$$

By subtracting relation (4) from (5) and utilizing the Mean Value Theorem, we arrive at

$$\frac{1}{12}[g''(x_{i+1}) + 1\circ g''(x_i) + g''(x_{i-1})] - \frac{1}{h^2}[g(x_{i+1}) - 2g(x_i) + g(x_{i-1})]$$
$$= \frac{h^4}{60}\int_0^1 \left[g^{(6)}(x_i + zh) + g^{(6)}(x_i - zh)\right](1-z)^3[5 - 3(1-z)^2]dz,$$
$$= \frac{h^4}{360}\left[g^{(6)}(x_i + \hat{z}h) + g^{(6)}(x_i - \hat{z}h)\right]\int_0^1 (1-z)^3[5 - 3(1-z)^2]dz,$$
$$= \frac{h^4}{240}g^{(6)}(\omega_i), \omega_i \in (x_{i-1}, x_{i+1}).$$

Here, $\hat{z}$ is a positive value ranging between (0,1).

**Definition 2.1.1.** The linear operator $\mathcal{A}$ for $g = g(g_0, g_1, \cdots, g_m)$is defined as follows:[10]

$$\mathcal{A}g_i = \frac{1}{12}(g_{i-1} + 1\circ g_i + g_{i+1}), 1 \le i \le M - 1.$$

We define the grid function on $\Omega_{h\tau}$ as:

$$U_i^k = u(x_i, t_k), \circ \le i \le M, -n \le k \le N.$$

Now, considering equation (1) at the point $\left(x_i, t_{k+\frac{1}{2}}\right)$, we have:

$$\frac{\partial u}{\partial t}\left(x_i, t_{k+\frac{1}{2}}\right) - \alpha\frac{\partial^2 u}{\partial x^2}\left(x_i, t_{k+\frac{1}{2}}\right) = f\left(u\left(x_i, t_{k+\frac{1}{2}}\right), u\left(x_i, t_{k+\frac{1}{2}} - s\right), x_i, t_{k+\frac{1}{2}}\right),$$
$$\circ \le i \le M, \circ \le k \le N - 1 \tag{6}$$

Using Taylor series expansions, we obtain

$$\frac{\partial u}{\partial t}\left(x_i, t_{k+\frac{1}{2}}\right) = \delta_t U_i^{k+\frac{1}{2}} - \frac{\tau^2}{24}\frac{\partial^3 u}{\partial t^3}\left(x_i, \xi_i^k\right), \tag{7}$$

$$\frac{\partial^2 u}{\partial x^2}\left(x_i, t_{k+\frac{1}{2}}\right) = \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}(x_i, t_k) + \frac{\partial^2 u}{\partial x^2}(x_i, t_{k+1})\right] - \frac{\tau^2}{8}\frac{\partial^4 u}{\partial x^2 \partial t^2}\left(x_i, \eta_i^k\right), \tag{8}$$

$$f\left(u\left(x_i, t_{k+\frac{1}{2}}\right), u\left(x_i, t_{k+\frac{1}{2}} - s\right), x_i, t_{k+\frac{1}{2}}\right)$$
$$= f\left(\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}, x_i, t_{k+\frac{1}{2}}\right)$$

$$+\left[u\left(x_i,t_{k+\frac{1}{2}}\right)-\frac{3}{2}U_i^k+\frac{1}{2}U_i^{k-1}\right]f_\mu\left(\zeta_i^k,\varsigma_i^k,x_i,t_{k+\frac{1}{2}}\right)$$

$$+\left[u\left(x_i,t_{k+\frac{1}{2}}-s\right)-\frac{3}{2}U_i^{k+1-n}-\frac{1}{2}U_i^{k-n}\right]f_\nu\left(\zeta_i^k,\varsigma_i^k,x_i,t_{k+\frac{1}{2}}\right)$$

$$=f\left(\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n},x_i,t_{k+\frac{1}{2}}\right)$$

$$+\frac{3\tau^2}{8}\frac{\partial^2 u(x_i,\rho^k)}{\partial t^2}f_\mu\left(\zeta_i^k,e_i^k,x_i,t_{k+\frac{1}{2}}\right)-\frac{\tau^2}{8}\frac{\partial^2 u(x_i,\varsigma^k)}{\partial t^2}f_\nu\left(\zeta_i^k,e_i^k,x_i,t_{k+\frac{1}{2}}\right)$$

(9)

where $\varrho^k\in\left(t_{k-n},t_{k+\frac{1}{2}-n}\right)\cdot\rho^k\in\left(t_{k-1},t_{k+\frac{1}{\gamma}}\right),\eta_i^k\cdot\xi_i^k\in(t_k,t_{k+1})\cdot S$ and $\zeta_i^k$ . $u\left(x_i,t_{k+\frac{1}{\gamma}}\right)$ Also, $\varsigma_i^k,\frac{3}{2}U_i^k-\frac{1}{2U_i^{k-1}}$ is between $\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\vartheta u\left(x_i,t_{k+\frac{1}{2}}-s\right)$

By substituting relations (7)-( 9) into equation (6), we get:

$$\delta_t U_i^{k+\frac{1}{2}}-\alpha\cdot\frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}(x_i,t_k)+\frac{\partial^2 u}{\partial x^2}(x_i,t_{k+1})\right]$$

$$=f\left(\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n},x_i,t_{k+\frac{1}{2}}\right)+\tau^r r_i^k$$

(10) $^\circ\le i\le M,\circ\le k\le N-1,$

where

$$r_i^k=\frac{1}{24}\frac{\partial^3 u}{\partial t^3}\left(x_i,\xi_i^k\right)-\alpha\cdot\frac{1}{8}\frac{\partial^7 u}{\partial x^2\partial t^2}\left(x_i,\eta_i^k\right)+\frac{3}{8}\frac{\partial^2 u(x_i,\rho^k)}{\partial t^2}f_\mu(\zeta_i^k,\varsigma_i^k,x_i,t_{k+1/2})$$
$$-\frac{1}{8}\frac{\partial^2 u(x_i,\varrho^k)}{\partial t^2}f_\nu(\zeta_i^k,\varsigma_i^k,x_i,t_{k+1/2}). \quad (11)$$

Now, applying the operator A to both sides of relation (10), we have:

$$\mathcal{A}\delta_t U_i^{k+\frac{1}{2}}-\alpha\cdot\frac{1}{2}\left[\mathcal{A}\frac{\partial^2 u}{\partial x^2}(x_i,t_k)+\mathcal{A}\frac{\partial^2 u}{\partial x^2}(x_i,t_{k+1})\right]$$

$$=\mathcal{A}f\left(\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n},x_i,t_{k+\frac{1}{2}}\right)+\tau^2\mathcal{A}r_i^k$$

(12) $1\le i\le M-1,\circ\le k\le N-1$

Considering the Taylor expansion and Lemma 2.1.1, we have

$$\mathcal{A}\frac{\partial^2 u}{\partial x^2}(x_i,t_k)=\delta_x^2 U_i^k+\frac{h^4}{240}\frac{\partial^6 u}{\partial x^6}\left(\theta_i^k,t_k\right),\theta_i^k\in (x_{i-1},x_{i+1}) \quad (13)$$

By substituting relation (13) into (12), we obtain:

$$\mathcal{A}\delta_t U_i^{k+\frac{1}{2}}-\alpha\delta_x^2 U_i^{k+\frac{1}{2}}=\mathcal{A}f\left(\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n},x_i,t_{k+\frac{1}{2}}\right)+R_i^k$$

$$(14) 1\le i\le M-1,\circ\le k\le N-1$$

where

$$R_i^k=\tau^2\mathcal{A}r_i^k+\frac{\alpha h^4}{480}\left[\frac{\partial^6 u}{\partial x^6}\left(\theta_i^k,t_k\right)+\frac{\partial^6 u}{\partial x^6}\left(\theta_i^{k+1},t_{k+1}\right)\right]$$

And there exists a positive constant $c_3$ such that

$$\left|R_i^k\right|\le c_3(\tau^2+h^4),1\le i\le M-1,\circ\le k\le N-1 \quad (15)$$

Given the boundary and initial conditions (2) and (3), we have:

$$U_\circ^k=u_a(t_k),U_M^k=u_b(t_k),1\le k\le N-1 \quad (16)$$

$$U_i^k=\psi(x_i,t_k),\circ\le i\le M,-n\le k\le\circ \quad (17)$$

By neglecting k$R_i^k$ in relation (14) and replacing $u_i^k$ with $U_i^k$, we derive the following compact finite difference scheme for nonlinear delay partial differential equations:

$$\mathcal{A}\delta_t u_i^{k+\frac{1}{2}}-\alpha\delta_x^2 u_i^{k+\frac{1}{2}}=\mathcal{A}f\left(\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1},\frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n},x_i,t_{k+\frac{1}{2}}\right)$$

$$1\le i\le M-1,\circ\le k\le N-1$$

(18)

$$u_o^k=u_a(t_k),u_M^k=u_b(t_k),1\le k\le N \quad (19)$$

$$u_i^k=\psi(x_i,t_k),\circ\le i\le M,-n\le k\le\circ \quad (20)$$

## 2.2. Uniqueness of the Solution

In this section, we present a theorem to establish that the compact finite difference scheme, defined by equations (18) to (20), possesses a unique solution.

**Theorem 2.2.1:** The compact finite difference scheme given by (18)-(20) has a unique solution.[11]

**Proof:** The uniqueness of the solution is demonstrated using mathematical induction on kkk. Based on the initial condition (20), the solution $u^k = \left(u_0^k, u_1^k, \cdots, u_{M-1}^k, u_M^k\right) -n \leq k \leq 0$ is uniquely determined by the scheme. Thus, the base case for the induction holds true.

Now let be the unique solution of our compact finite difference scheme for. Then a linear system of algebraic equations for is developed. This linear system has a coefficient matrix which is strictly diagonally dominant. Hence, we may conclude that the following problem has a unique solution for. Thus the theorem is proved by induction.

## 2.3. Convergence

Here, we employ the discrete energy method to establish both the convergence and stability of the finite difference scheme (18)-( 20). The fundamental principle of this method involves subtracting the derived finite difference scheme for a particular equation from its original partial differential equation. An operator, such as A, is then applied to both sides of the resulting relationship. Subsequently, each obtained term is subjected to an inner product with a carefully chosen expression. Through induction and by leveraging the problem's given conditions, we approximate the value of each inner product. Finally, the proof is concluded by utilizing relevant lemmas and theorems. In this section, we will analyze the convergence of the compact finite difference scheme (18)-(20) using the discrete energy method. To this end, we first provide the following preliminary definitions.

$V = \{v | v = (v_0, v_1, \cdots, v_M), v_0 = v_M = 0$be the space of grid functions on $\Omega_h$. For every $v \in V$, we introduce the corresponding norms as follows:

$$\|v\| = \sqrt{h \sum_{i=1}^{M-1} (v_i)^2}, |v|_1 = \sqrt{h \sum_{i=1}^{M} \left(\frac{v_i - v_{i-1}}{h}\right)^2}, \|v\|_\infty = \max_{0 \leq i \leq M} |v_i, |.$$

Given these definitions, we have

$$\|v\|_\infty \le \frac{\sqrt{b-a}}{2}|v|_i \tag{21}$$

$$\|v\| \le \frac{b-a}{\sqrt{6}}|v|_i \tag{22}$$

**Remark 2.3.1:** For any $g \in V$:

$$h\sum_{i=1}^{M-1}(\mathcal{A}g_i)^2 \le h\sum_{i=1}^{M-1}(g_i)^2.$$

**Lemma 2.3.1 (Gronwall's Inequality) [1]:** Let $\{\{F^k|k \ge 0\}$ be a non-negative sequence, and let A and B be non-negative constants such that for $k = 0,1,\cdots, F^{k+1} \le A + B\tau\sum_{l=1}^{k}F^l$:

$$F^{k+1} \le \mathrm{A}\exp(Bk\tau), k = \circ, 1, \ldots$$

**Theorem 2.3.1:[12]** Let $\{u(x,t)|a \le x \le b, -s \le t \le T\}$ be the solution to problem (1)-(3), and let $\{u_i^k|^\circ \le i \le M, -n \le k \le N\}$ be the solution of the compact finite difference scheme (18)-( 20), such that

$$e_i^k = u(x_i,t_k) - u_i^k, \circ \le i \le M, -n \le k \le N.$$

Assume that:

$$C = \frac{b-a}{4}c_\gamma\sqrt{\frac{3T}{\alpha}}\exp\left(\frac{(b-a)^2}{4\alpha}(5c_1^2 + c_2^2)T\right)$$

If:

$$h \le \left(\frac{\epsilon_\circ}{4C}\right)^{\frac{1}{4}}, \tau \le \left(\frac{\epsilon_\circ}{4C}\right)^{\frac{1}{2}} \tag{23}$$

Then the following relation holds:

$$\|e^k\|_\infty \le C(\tau^2 + h^4), \circ \le k \le N \tag{24}$$

**Proof:** By subtracting relations (18)-( 20) from relations (14), (16), and (17), the error equation is obtained as follows:

$$\mathcal{A}\delta_t e_i^{k+\frac{1}{2}} - \alpha\delta_x^2 e_i^{k+\frac{1}{2}} = \mathcal{A}\left[f\left(\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}, x_i, t_{k+\frac{1}{2}}\right)\right.$$

$$\left. -f\left(\frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n} + \frac{1}{2}u_i^{k-n}, x_i, t_{k+\frac{1}{2}}\right)\right] + R_i^k$$

$1 \leq i \leq M-1, \circ \leq k \leq N-1,$ (25)

$e_\circ^k = \circ, e_M^k = \circ, \backslash \leq k \leq N$ (26)

$e_i^k = \circ, \circ \leq i \leq M, -n \leq k \leq \circ$ (27)

Now, by multiplying equation (25) by $h\delta_t e_i^{k+\frac{1}{2}}$ and summing over iii from 1 to M−1, we get:

$$h\sum_{i=1}^{M-1}\left(\mathcal{A}\delta_t e_i^{k+\frac{1}{2}}\right)\delta_t e_i^{k+\frac{1}{2}} - \alpha\cdot h\sum_{i=1}^{M-1}\left(\delta_x^Y e_i^{k+\frac{1}{2}}\right)\delta_t e_i^{k+\frac{1}{2}}$$
$$= h\sum_{i=1}^{M-1}\left\{\mathcal{A}\left[f\left(\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}, x_i, t_{k+\frac{1}{2}}\right)\right.\right.$$
$$\left.\left. -f\left(\frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n} + \frac{1}{2}u_i^{k-n}, x_i, t_{k+\frac{1}{2}}\right)\right]\right\}\delta_t e_i^{k+\frac{1}{2}}$$

$+h\sum_{i=1}^{M-1}\left(R_i^k\right)\delta_t e_i^{k+\frac{1}{2}}, 0 \leq k \leq N-1.$ (28)

We will now prove relation (24) by induction. According to relation (27), it is clear that for $-n \leq k \leq 0$, we have $\|\|e^k\|_\infty = 0$. Assume that (24) holds for $0 \leq k \leq l$. We must now prove that relation (24) also holds for $k = l + 1$. Given the induction hypothesis and the upper bounds defined for h and τ in relation (23), we have

$$\|e^k\|_\infty \leq C(\tau^2 + h^4) \leq \frac{\epsilon}{2}, \cdot \leq k \leq l.$$

Similarly, we have

We define the following constants

$$\left|\left(\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}\right) - \left(\frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}\right)\right| \leq \epsilon_\circ, \cdot \leq i \leq M, \circ \leq k \leq l,$$
$$\left|\left(\frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}\right) - \left(\frac{1}{2}u_i^{k+1-n} + \frac{1}{2}u_i^{k-n}\right)\right| \leq \epsilon_\circ, \circ \leq i \leq M, \circ \leq k \leq l.$$

We define the constants $c_1$ and $c_2$ as follows:

$$c_1 = \max_{a<x<b,\circ<t<T,|\epsilon_1|\le\epsilon_o,|\epsilon_2|\le\epsilon_o} \left|f_\mu(u(x,t)+\epsilon_1, u(x,t-s)+\epsilon_2, x, t)\right|,$$
$$c_2 = \max_{a<x<b,\circ<t<T,|\epsilon_Y|\le\epsilon_o,|\epsilon_2|\le\epsilon_o} \left|f_v(u(x,t)+\epsilon_1, u(x,t-s)+\epsilon_2, x, t)\right|.$$

Consequently, by applying the Lipschitz condition, we obtain

$$\left| f\left(\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}, x_i, t_{k+\frac{1}{2}},\right) - f\left(\frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n} + \frac{1}{2}u_i^{k-n}, x_i, t_{k+\frac{1}{2}},\right)\right|$$
$$\le c_1\left|\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\right| + c_2\left|\frac{1}{2}e_i^{k+1-n} + \frac{1}{2}e_i^{k-n}\right|, \circ\le i\le M, \circ\le k\le l$$

Applying operator A to both sides of the above inequality yields

$$\left|\mathcal{A}\left[f\left(\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}, x_i, t_{k+\frac{1}{2}},\right) - f\left(\frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n} + \frac{1}{2}u_i^{k-n}, x_i, t_{k+\frac{1}{2}}\right)\right]\right|$$
$$\le \mathcal{A}\left(c_1\left|\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\right| + c_2\left|\frac{1}{2}e_i^{k+1-n} + \frac{1}{2}e_i^{k-n}\right|\right), \circ\le i\le M, \circ\le k\le l,$$

Considering this inequality and Remark 3.4.1, we have

$$h\sum_{i=1}^{M-1}\left\{\mathcal{A}\left[f\left(\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}, x_i, t_{k+\frac{1}{2}}\right) - f\left(\frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n} + \frac{1}{2}u_i^{k-n}, x_i, t_{k+\frac{1}{2}}\right)\right]\right\}\delta_t e_i^{k+\frac{1}{2}}$$
$$\le h\sum_{i=1}^{M-1}\left\{\mathcal{A}\left(c_1\left|\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\right| + c_2\left|\frac{1}{2}e_i^{k+1-n} + \frac{1}{2}e_i^{k-n}\right|\right)\right\}\delta_t e_i^{k+\frac{1}{2}}$$
$$\le \frac{3}{4}h\sum_{i=1}^{M-1}\left\{\mathcal{A}\left(c_1\left|\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\right| + c_2\left|\frac{1}{2}e_i^{k+1-n} + \frac{1}{2}e_i^{k-n}\right|\right)\right\}^2 + \frac{1}{3}\left\|\delta_t e_i^{k+\frac{1}{2}}\right\|^2$$
$$\le \frac{3}{4}h\sum_{i=1}^{M-1}\left(c_1\left|\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\right| + c_2\left|\frac{1}{2}e_i^{k+1-n} + \frac{1}{2}e_i^{k-n}\right|\right)^2 + \frac{1}{3}\left\|\delta_t e_i^{k+\frac{1}{2}}\right\|^2$$
$$\le \frac{3}{2}\left[c_1^2 h\sum_{i=1}^{M-1}\left(\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\right)^2 + c_2^2 h\sum_{i=1}^{M-1}\left(\frac{1}{2}e_i^{k+1-n} + \frac{1}{2}e_i^{k-n}\right)^2\right] + \frac{1}{3}\left\|\delta_t e_i^{k+\frac{1}{2}}\right\|^2$$

$$\leq \frac{3}{2}\left[\frac{5}{2}c_1^2 h\sum_{i=1}^{M-1}\left(\left(e_i^k\right)^2+\left(e_i^{k-1}\right)^2\right)+\frac{1}{2}c_2^2 h\sum_{i=1}^{M-1}\left(\left(e_i^{k+1-n}\right)^2+\left(e_i^{k-n}\right)^2\right)^2\right]$$
$$+\frac{1}{3}\left\|\delta_t e_i^{k+\frac{1}{2}}\right\|^2$$
$$=\frac{15}{4}c_1^2(\|e^k\|^2+\|e^{k-1}\|^2)+\frac{3}{4}c_2^2(\|e^{k+1-n}\|^2+\|e^{k-n}\|^2)$$
$$+\frac{1}{3}\left\|\delta_t e_i^{k+\frac{1}{2}}\right\|^2, 0\leq k\leq l \tag{29}$$

Furthermore, for each part of (28) we have

$$\frac{1}{12}h\sum_{i=1}^{M-1}\left(\delta_t e_{i-1}^{k+\frac{1}{2}}+10\,\delta_t e_i^{k+\frac{1}{2}}+\delta_t e_{i+1}^{k+\frac{1}{2}}\right)\delta_t e_i^{k+\frac{1}{2}}\geq\frac{3}{2}h\sum_{i=1}^{M-1}\left(\delta_t e_i^{k+\frac{1}{2}}\right)^2 \tag{30}$$

$$h\sum_{i=1}^{M-1}\left(\delta_x^r e_i^{k+\frac{1}{2}}\right)\left(\delta_t e_i^{k+\frac{1}{2}}\right)=\left\langle\delta_x^r e_i^{k+\frac{1}{2}},\delta_t e_i^{k+\frac{1}{2}}\right\rangle=-\left\langle\delta_x e_i^{k+\frac{1}{2}},\delta_x\left(\delta_t e_i^{k+\frac{1}{2}}\right)\right\rangle$$

$$=\left\langle\delta_x\left(\frac{e_i^{k+1}+e_i^k}{2}\right),\delta_x\left(\frac{e_i^{k+1}-e_i^k}{2}\right)\right\rangle=\frac{-1}{2\tau}\left\{\left\langle\delta_x e_i^{k+1},\delta_x e_i^{k+1}\right\rangle-\left\langle\delta_x e_i^k,\delta_x e_i^k\right\rangle\right\}$$
$$=-\frac{1}{2\tau}\left(|e^{k+1}|_1{}^2-|e^k|_1{}^2\right)\Big) \tag{31}$$

$$h\sum_{i=1}^{M-1}R_i^k\left(\delta_t e_i^{k+\frac{1}{2}}\right)\leq\frac{3}{4}h\sum_{i=1}^{M-1}(R_{ik})^2+\frac{1}{2}h\sum_{i=1}^{M-1}\left(\delta_t e_i^{k+\frac{1}{2}}\right)^2$$
$$\leq\frac{3}{4}(b-a)c_2^2(\tau^2+h^4)^2+\frac{1}{3}\left\|\delta_t e_i^{k+\frac{1}{2}}\right\|^2. \tag{32}$$

By substituting relations (29) - (32) into relation (28) and assuming $\epsilon=\frac{1}{3}$, we obtain

$$\frac{\alpha}{2\tau}\left(|e^{k+1}|_1{}^2-|e^k|_1{}^2\right)$$
$$\leq\frac{15}{4}c_1^2(\|e^k\|^2+\|e^{k-1}\|^2)+\frac{3}{4}c_2^2(\|e^{k+1-n}\|^2+\|e^{k-n}\|^2)$$
$$+\frac{3}{4}(b-a)c_2^2(\tau^2+h^4)^2, 0\leq k\leq l$$

Multiplying both sides of the above inequality by $\frac{2\tau}{\alpha}$ and considering relation (27) and inequality (22), we have

$$|e^{k+1}|_1^2 \le \frac{3}{\alpha}(\Delta c_1^2 + c_2^2)\tau \sum_{m=1}^{k} \|e^m\|^2 + \frac{3}{4\alpha}k\tau(b-a)c_2^2(\tau^2+h^4)^2$$

$$\le \frac{(b-a)^2}{r\alpha}(\Delta c_1^2 + c_2^2)\tau \sum_{m=1}^{k} |e^m|_1^2 + \frac{2}{4\alpha}T(b-a)c_2^2(\tau^2+h^4)^2, 0 \le k \le l$$

From Gronwall's inequality (Lemma 2.3.1), we obtain:

$$|e^{l+1}|_1^2 \le \frac{3T(b-a)c_2^2}{4\alpha}\exp\left(\frac{(b-a)^r}{r\alpha}(\Delta c_1^2 + c_2^2)T\right)(\tau^2+h^4)^2.$$

Therefore, considering relation (21), we have

$$\|e^{l+1}\|_\infty \le \frac{\sqrt{b-a}}{r}|e^{l+1}|_1 \le \frac{b-a}{4}c_2\sqrt{\frac{3T}{\alpha}}\exp\left(\frac{(b-a)^2}{4\alpha}(5c_1^2 + c_2^2)T\right)(\tau^2+h^4).$$

**2.4. Stability**

In this section, similar to the convergence section, we investigate the stability of the compact finite difference scheme (18)-( 20) using the discrete energy method.[13]

Let $\{v_i^k | 0 \le i \le M, 0 \le k \le N\}$ be the solution to the following equation

$$\frac{1}{12}\left(\delta_t v_{i-1}^{k+\frac{1}{2}} + 1 \circ \delta_t v_i^{k+\frac{1}{2}} + \delta_t v_{i+1}^{k+\frac{1}{2}}\right) - \alpha\delta_x^Y v_i^{k+\frac{1}{2}}$$

$$= \mathcal{A}f\left(\frac{3}{2}v_i^k - \frac{1}{2}v_i^{k-1}, \frac{1}{2}v_i^{k+1-n} + \frac{1}{2}v_i^{k-n}, x_i, t_{k+\frac{1}{2}}\right)$$

$1 \le i \le M-1, | \le k \le N-1$ (33)

$v_o^k = v_a(t_k), v_M^k = v_b(t_k), | \le k \le N$ (34)

$v_i^k = \psi(x_i, t_k) + \phi_i^k, \circ \le i \le M, -n \le k \le 0$ (35)

**Theorem 2.4.1:** Let $\eta_i^k = v_i^k - u_i^k$ for $0 \le i \le M$ and $-n \le k \le$ N. If there exist constants $c_4$, $c_5$, $h_0$, and $\tau_0$ such that

$$\|\eta^k\|_\infty \leq c_4 \sqrt{\tau h \sum_{k=-n}^{0} \sum_{i=1}^{M-1} \left(\phi_i^k\right)^2}, \circ \leq k \leq N,$$

where $h \leq h_0$ and $\tau \leq \tau_0$and $\max_{-n\leq k\leq \mathrm{N}, 0\leq i\leq M} \left|\phi_i^k\right| \leq c_5$then the difference scheme (18)-(20) is stable.

**3. Numerical Results**

This section evaluates the numerical performance of the proposed compact finite difference scheme, applying it to a specific example of a nonlinear delay partial differential equation.

**3.1. Example**

Consider the following nonlinear delay partial differential equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^r u}{\partial x^2} + f(u(x,t), u(x,t-0/1), x, t), (x,t) \in (1,2) \times (\circ,1],$$
$$u(1,t) = \frac{31}{32}(t^2 - 2t - 1), u(2,t) = \circ, t \in (\circ,1]$$
$$u(x,t) = \left(\frac{1}{32}x^6 - x\right)(t^3 - 2t - 1), x \in (1,2), t \in [-0.1,\circ],$$
$$f(u(x,t), u(x,t-0.1), x, t) = u(x,t-0.1)^2 - \frac{15}{8}x^4(t^3 - 2t - 1) + \left(\frac{1}{32}x^6 - x\right)^2$$
$$[(t-0.1)^3 - 2(t-0.1) - 1]^2.$$

The exact analytical solution for this problem is given by:

$$(x,t) = \left(\frac{1}{32}x^6 - x\right)(t^3 - 2t - 1)$$

Let $\{v_i^k | 0 \leq i \leq M, 0 \leq k \leq N\}$ denote the solution obtained by the finite difference scheme (18)-(20), and $E_\infty(h,\tau) = \max_{0\leq k\leq \mathrm{N}, 0\leq i\leq M} \left|u(x_i,t_k) - u_i^k\right|$be the maximum numerical error.

**Table 1: A comparison between the exact solution and the proposed numerical method's solution for Example 3.1 with step sizes $(\boldsymbol{h},\boldsymbol{\tau}) = (\frac{1}{10},\frac{1}{100})$and at various (x,t) points.**

| $\mid u(x_i,t_k)-u_i^k\mid$ | Real answer | Proposed method answer | $(x,t)$ |
|---|---|---|---|
| 0.683317e-5 | 1.371708 | 1.371701 | (1.5,0.1) |
| 0.554305e-5 | 1.592508 | 1.592502 | (1.5,0.2) |
| 0.159356e-5 | 1.799580 | 1.799578 | (1.5,0.3) |
| 0.411876e-5 | 1.986059 | 1.986063 | (1.5,0.4) |
| 0.112138e-5 | 2.145081 | 2.145092 | (1.5,0.5) |
| 0.193310e-5 | 2.269781 | 2.269801 | (1.5,0.6) |
| 0.279864e-5 | 2.353296 | 2.353324 | (1.5,0.7) |
| 0.365279e-5 | 2.388762 | 2.388798 | (1.5,0.8) |
| 0.441275e-5 | 2.369313 | 2.369357 | (1.5,0.9) |
| 0.497998e-5 | 2.288086 | 2.288136 | (1.5,1) |

**Table 2: Maximum norm error with different step lengths.**

| $\frac{E_\infty(h,\tau)}{E_\infty\left(\frac{h}{2},\frac{\tau}{4}\right)}$ | $E_\infty(h,\tau)$ | $\tau$ | $h$ |
|---|---|---|---|
| 15.8842 | 4.9800e-5 | $\frac{1}{100}$ | $\frac{1}{10}$ |
| 15.9611 | 3.1352e-6 | $\frac{1}{400}$ | $\frac{1}{20}$ |
| 15.9946 | 1.9643e-7 | $\frac{1}{1600}$ | $\frac{1}{4_0}$ |
| * | 1.2281e-8 | $\frac{1}{6400}$ | $\frac{1}{80}$ |

### 3.2 Conclusions from Tables:

From these tables, it can be concluded that if the spatial (h) and temporal (τ) step sizes are halved (1/2) and quartered (1/4) respectively, the maximum error is reduced by approximately a factor of 1/16. This behavior indicates a convergence order of 4 in space (with respect to hhh) and an order of 2 in time (with respect to τ) for the proposed finite difference scheme.

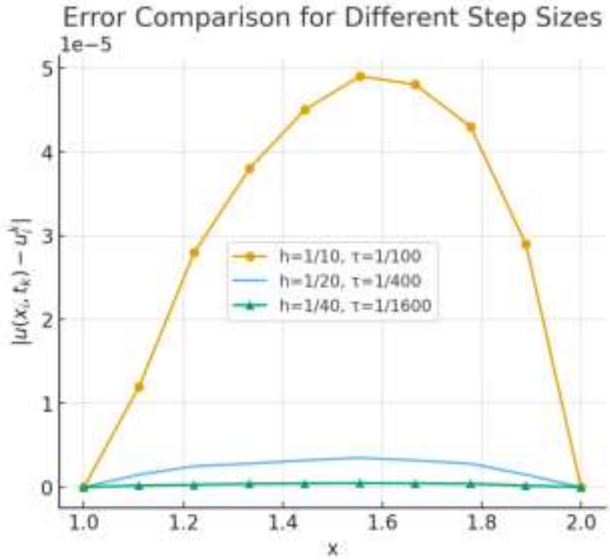


**Figure 1: Error plot for the numerical solution obtained with different temporal step sizes for Example 3.1 at t=1.**

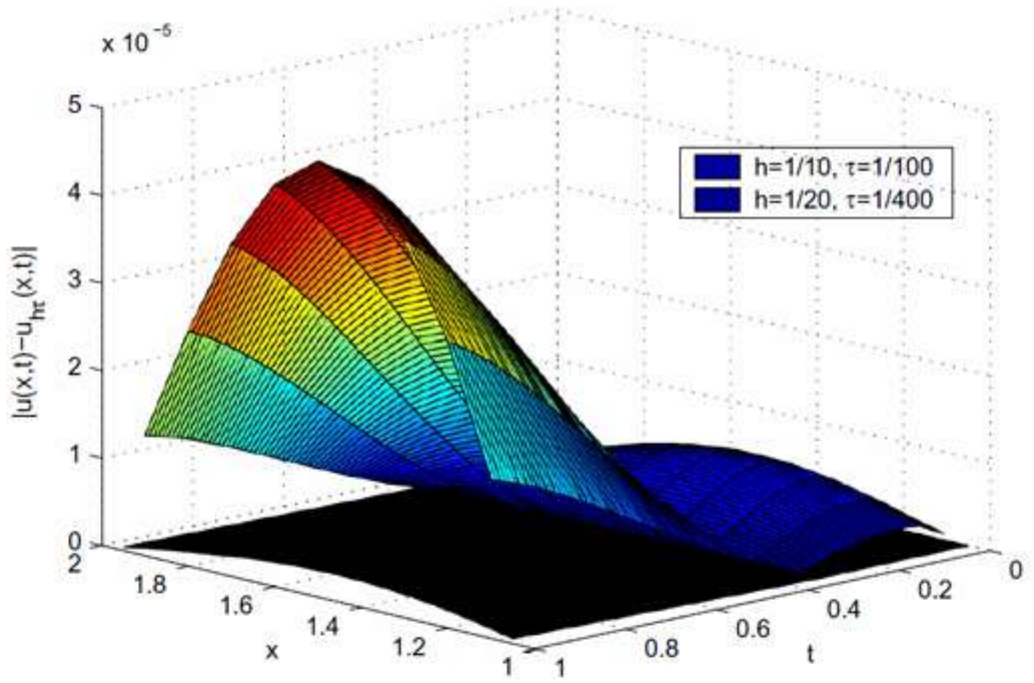


**Figure 2: Error from the numerical method with $(h,\tau) = (\frac{1}{20}, \frac{1}{100})$ $(h,\tau) = (\frac{1}{10}, \frac{1}{100})$for Example 3.1.**

## 4. Conclusion

In this paper, we developed a linear compact difference scheme for the numerical solution of a class of nonlinear delay partial differential equations. Nonlinear DPDEs appear in various applications, including physics, engineering, and biology, where the system evolution depends on both current and past states through time-delay terms. The combination of nonlinearities and delays makes numerical simulation challenging, often causing stability issues and reduced accuracy in conventional methods. The general linear compact formulation we introduce in this paper solves these difficulties, allowing higher-order space discretization and a linearization of the nonlinear terms by means of an efficient computational framework. We studied the stability and convergence of this method, showing that it does not blow-up numerically on a variety of choices for discretization parameters and is able to maintain good accuracy. As examples of numerical, benchmark problems and models with nonlinearities and delays were demonstrated. Numerical results indicate that the method is more accurate than finite difference schemes and it is robust and efficient. The corresponding small stencil compact technique has the advantage to reduce the computational costs when a sub-stencil is employed for high order approximations and can be thus applied for one or multi-dimensional problem. It is indicated that linear compact difference scheme can efficiently and feasiblly carry out the computation of nonlinear DPDES in scientific and engineering applications. The simplicity of the method and its good numerical behaviour are some reasons that make this a practical alternative to canonical methods such as Newton, specially in those circumstances where accuracy and stability are in demand. This method can be extended to higher dimensions by further research and new adaptive mesh strategies may improve its effectiveness and efficiency. The usability of such a model for large-scale simulations might be enhanced through advanced timesteping or any parallel computing enabling. On the whole, they will help to construct effective numerical algorithms for complex nonlinear delay systems and provide a foundation where deeper results could be extracted.

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