A Multipolar Fuzzy Outerplanar Graph Approach to One-Way Road Network Construction

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Abstract

This current research report distinctly compares the traditional crisp graph planarity with the growing field of fuzzy graph theory, importantly focusing on m-polar fuzzy outerplanar graphs (mPFOGs). It investigates the mPFOGs properties and introduces m-polar fuzzy outerplanar subgraphs (mPFOSs) that were formed by the removal of the specific vertices or edges. This study not only provides examples illustrating the identification of maximal and maximum mPFOSs but also introduces m-polar fuzzy dual graphs derived from mPFOGs and explores their relationships. The theoretical findings are applied to urban planning, specifically in optimising one-way bypass road networks to improve traffic flow and reduce accident-prone intersections.

Keywords Fuzzy Graph; m-Polar fuzzy outerplanar graphs; VD and ED mPFOSs; Maximal and maximum mPFOSs; Dual of m-polar fuzzy outerplanar graphs; One-way bypass road network

1. Introduction

A membership function that allocates degrees of membership between 0 and 1 defines fuzzy sets, which give traits that reflect their intrinsic ambiguity by extending conventional set operations like intersection and union. A significant research conducted by L. A. Zadeh [1] introduced the notion of fuzzy sets that illustrated how beneficial they are for tackling real-world issues when membership criteria are not clearly stated, notably in disciplines like pattern recognition and information processing. The fuzzy planar graph was initially developed by Abdul-Jabbar et.al [2], and fuzzy dual graphs are suggested as a type of classical dual graph.
Likewise, Harary [3] investigated the graph theory application in management science to optimise processes and decision-making, in addition to its various methodologies and actual examples that demonstrate the effectiveness of these methods. The properties and features of minimally non-outerplanar graphs were studied by Kulli [4], who reported a study clarifying their structure and the importance of these aspects in graph theory. With an emphasis on biconnected graphs and the use of Hamiltonian cycle characteristics, Mitchell [5] introduced linear algorithms for identifying maximal outerplanar and outerplanar graphs. These techniques emphasised simpler approaches than those found in the literature while retaining linear time complexity.
In order to clearly facilitate the comprehension and implementation in high-dimensional spaces, Berthold [6] introduced an effective algorithm for fuzzy graph creation from example data. This method was illustrated using real-world datasets and examples, demonstrating its

capacity to extract intelligible models from intricate systems. Bondy and Murty [7], an introductory text on graph theory, included fundamental concepts, theorems, and applications for both mathematics and the real world. It covered important subjects including planar graphs, Hamilton cycles, Euler tours, matchings, graph connectedness, and numerous algorithms. Chartrand and Harary [8] examined the properties and conditions that determine the planarity of permutation graphs derived from cycles and non-separable outerplanar graphs. They established criteria for assessing planarity based on the characteristics of permutations and explored the connections between these graphs and well-known structures, such as the Petersen graph.

Sysło [9] investigated outerplanar graphs, offering definitions and methods for analyzing their properties, along with coding techniques and counting strategies. This thorough examination aids in better understanding these structures and highlights their importance in graph theory. In a follow-up study, Sysło [10] detailed the characteristics of outerplanar graphs and presented effective tools for counting, isomorphism testing, and evaluating outerplanarity. He also extended his findings to k-outerplanar graphs, exploring the implications for methods related to outerplanarity. Sysło [11] described maximal outerplanar networks through three distinct types of graphs and their intersections, supporting his findings with theorems and proofs that illustrate their properties. Meanwhile, Fleischner [12] provided a new characterization of outerplanar graphs by examining their duals and the relationships between vertex degrees and the lengths of the boundaries surrounding their internal regions.

The key findings reveal that the dual of a dual fuzzy graph returns to the original fuzzy graph, and that the dual of a fuzzy bipartite graph is an Eulerian fuzzy graph. Akram [13] introduced bipolar fuzzy graphs, discussing their construction, isomorphisms, and properties, including strong bipolar fuzzy graphs and their applications across various domains. In another study, Akram [14] presented interval-valued fuzzy line graphs, analyzing their isomorphisms and characteristics while highlighting their relevance in fields like database theory and decision-making. Akram [15] introduced bipolar fuzzy planar graphs, which permit edge crossings, and defined a measure of planarity called the bipolar fuzzy planarity value, along with various characteristics and applications. Furthermore, Akram [16] proposed the bipolar fuzzy influence graph to model social interactions and influences within groups, exploring concepts such as irregular and highly irregular bipolar fuzzy graphs and their applications in diverse areas.

Additionally, Akram [17] examined regular and totally regular bipolar fuzzy graphs, establishing conditions for their equivalence and investigating the properties of bipolar fuzzy line graphs, including isomorphism criteria. In his research, Akram [18] explored the application of bipolar fuzzy sets to multigraphs and planar graphs, introducing concepts such as bipolar fuzzy multigraphs and planar graphs while analyzing their characteristics and isomorphisms. Akram et al. [19] introduced Pythagorean fuzzy multigraphs and Pythagorean fuzzy planar graphs, examining their properties in detail. They also discussed non-planar Pythagorean fuzzy graphs and explored concepts such as isomorphism, weak isomorphism, and co-weak isomorphism specifically for Pythagorean fuzzy planar graphs. This research established a significant connection between Pythagorean fuzzy planar graphs and their duals and presented a practical application problem involving these concepts.

In a separate study, Akram [20] focused on the development and isomorphism of bipolar fuzzy graphs, highlighting the characteristics of strong bipolar fuzzy graphs, including the concept of self-complementarity. Fuzzy outerplanar graphs were initially introduced by Deivanai et al. [21] as an extension of traditional crisp graphs, aimed at capturing ambiguous relationships through varying levels of membership. Their study also focused on the characteristics of maximal and maximum fuzzy outerplanar subgraphs and explored their applications in network modeling, especially within transport networks. Building on this, bipolar fuzzy outerplanar graphs were developed by incorporating bipolar fuzzy set theory into outerplanar graphs, enabling the representation of uncertainty with dual membership values. This study further investigated their structural features, dual graphs, vertex and edge deletion subgraphs, and demonstrated their applicability to image contraction, as discussed in [22]. Ghorai [23] examined the isomorphic properties of m-polar fuzzy graphs, merging graph theory with fuzzy set theory to enhance data analysis and representation. In a follow-up study, Ghorai et al. [24] delved into the isomorphic characteristics of m-polar fuzzy planar graphs, presenting concepts such as m-polar fuzzy faces, strong m-polar fuzzy faces, and m-polar fuzzy dual graphs. They also investigated the notion of m-polar fuzzy strong edges, where strong edges signify congested routes and weak edges denote less congested ones [25].

To address the limitations of traditional fuzzy and bipolar fuzzy graphs, Ghorai et al. [26] investigated m-polar fuzzy graphs, which generalize fuzzy graphs to represent relationships among multiple agents. They defined various operations on these graphs, including Cartesian products, compositions, unions, and joins, while also discussing the characteristics of strong and self-complementary m-polar fuzzy graphs.In a related study, Ghorai et al. [27] explored the features of regular and totally regular product bipolar fuzzy graphs. They introduced product bipolar fuzzy line graphs and established criteria for their isomorphism. Mahapatra et al. [28] proposed a generalized definition of neutrosophic planar graphs to rectify existing weaknesses in the concept. Furthermore, Mahapatra et al. [29] introduced interval-valued m-polar fuzzy graphs, detailing aspects such as IVmPF dual graphs, planar values, IVmPF multisets, and IVmPF multigraphs. Mondal et al. [30] presented the concept of a generalized m-polar fuzzy dual graph and examined the characteristics of generalized m-polar fuzzy planar graphs (GmPFPGs), providing an example related to social group networks. Naz et al. [31] established the relationship between product bipolar fuzzy planar graphs and product bipolar fuzzy dual graphs, while also describing the product bipolar fuzzy planarity value of these graphs.

Pramanik et al. [32] introduced the "degree of planarity" to quantify the planarity of interval-valued fuzzy graphs, along with related concepts such as strong edges, interval-valued fuzzy faces, and interval-valued fuzzy dual graphs. Ramya and Lavanya [33] specifically explored edge contraction within bipolar fuzzy graphs, outlining the process and its application to various types of bipolar fuzzy graphs. Their investigation focused on how edge contraction influences the domination number. They also introduced the concept of edge contraction in fuzzy graphs, supported by illustrative examples, and presented theorems regarding its impact on the number of vertices and edges [34]. Additionally, Pramanik et al. [35] discussed the properties and applications of bipolar fuzzy planar graphs, highlighting uses in areas such as image shrinking and city planning. They included essential concepts like degree of planarity, strong edges, bipolar fuzzy faces, and bipolar fuzzy dual graphs. Rashmanlou [36] explored the properties of bipolar fuzzy graphs, focusing on concepts such as μ-complement, busy and free vertices, and isomorphism. The study emphasized the applications of these graphs in various fields, including data mining, neural networks, and systems analysis.

In a follow-up study, Rashmanlou [37] examined the application of bipolar fuzzy graphs in diverse areas such as social networks, artificial intelligence, and medical diagnosis. He introduced new operations, including the direct product, semi-strong product, and strong product, while establishing conditions for their completeness. Furthermore, Rashmanlou [38] presented important results regarding the degrees, order, and complement properties of neighborly irregular and highly irregular bipolar fuzzy graphs. This work defined and investigated their weak, co-weak, and isomorphic characteristics, thereby enhancing the understanding of these graph types. Rosenfeld [39] reviewed various studies on fuzzy sets, focusing on their applications in decision-making and cognition, and highlighting significant developments in the theory across multiple disciplines.

Samanta [40] introduced bipolar fuzzy hypergraphs, defining key concepts such as transversals, dual hypergraphs, and cut level sets, along with basic characteristics and related theorems. In a subsequent study, Samanta [41] developed techniques to assess customer centrality and churn likelihood based on communication patterns, presenting a fuzzy telecommunication network model that leverages fuzzy graph theory to enhance churn prediction and identify key consumers. This model aimed to create fuzzy planar subgraphs for partitioning large-scale integration networks, emphasizing vertex and edge deletion operations. Additionally, they proposed a new Planar Partition subgraph method to improve the efficiency of identifying these subgraphs and discussed the concept of thickness value in fuzzy graphs, demonstrating results through examples applicable to network partitioning in VLSI design [42].

Samanta et al. [43] explored fuzzy planar graphs and fuzzy dual graphs, providing definitions, examples, and preliminary results related to their characteristics and interrelationships. Further features and results regarding fuzzy planar graphs and their duals were presented by Samanta et al. [44], suggesting potential applications in image representation, subway planning, and circuit design.In another study, Samanta et al. [45] differentiated between two types of edges in fuzzy graphs: effective edges, representing less congested routes, and considerable edges, indicating congested routes. They discussed allowing crossings between these edges in fuzzy planar graphs. Samanta and Pal [46] focused on irregular bipolar fuzzy graphs, defining various classifications and theorems related to their degrees, size, and properties, while establishing relationships between different irregular types and presenting necessary and sufficient conditions for these graphs to aid in future algorithms and models.

Shriram et al. [47] created and examined the fuzzy combinatorial dual graph, establishing important theorems regarding its structure and characteristics within fuzzy planar graph theory. Wan [48] introduced new concepts in bipolar fuzzy graphs (BFGs), such as (ϑ, δ)-homomorphisms, and categorised different types of graph mappings, demonstrating how BFGs can be utilised to solve colouring problems and identify efficient university employees. Yang [49] enhanced the theoretical foundation of bipolar fuzzy set theory by presenting a generalised bipolar fuzzy graph and correcting errors found in previous definitions and statements related to bipolar fuzzy graphs. Lastly, Zhang [50] identified two types of bipolar a-level cuts and investigated their resolutions, laying the groundwork for applications in clustering and multi-agent coordination. The theory of bipolar fuzzy sets integrates both positive and negative memberships, allowing for more nuanced modelling of relationships and decision-making.

Table 1 lists the abbreviations used in this paper

Notations	**Abbreviations**
mFSs	*m-Polar Fuzzy Sets*
mPFG	*m-Polar Fuzzy Graph*
mFMG	*m-Polar Fuzzy Multigraph*
mPFDG	*m-Polar Fuzzy Dual Graph*
mPFPG	*m-Polar Fuzzy Planar Graph*
VD	*Vertex Deletion*
ED	*Edge Deletion*
mPFOG	*m-Polar Fuzzy Outerplanar Graph*
mPFOSs	*m-Polar Fuzzy Outerplanar Subgraphs*
(VD-mPFOSs)	*Vertex Deletion m-Polar Fuzzy Outerplanar Subgraphs*
(ED-mPFOSs)	*Edge Deletion m-Polar Fuzzy Outerplanar Subgraphs*
Maximum VD &ED mPFOSs	*Maximum Vertex Deletion m-Polar Fuzzy Outerplanar Subgraphs*
Maximal VD &ED mPFOSs	*Maximal Vertex Deletion m-Polar Fuzzy Outerplanar Subgraphs*
mPFF	*m-Polar Fuzzy Face*
mPFSG	*m-Polar Fuzzy Subgraph*
mPFEC	*m-Polar Fuzzy Edge Contraction*
mPFCG	*m-Polar Fuzzy Connected Graph*
mPFNG	*m-Polar Fuzzy Non Outerplanar Graph*
mPFTG	*m-Polar Fuzzy Tree Graph*
BFGs	*Bipolar Fuzzy Graphs*
IVmPFG	*Interval valued m-Polar Fuzzy Graphs*
GmPFPGs	*Generalized m-Polar Fuzzy Planar Graphs*
VLSI	*Very Large-Scale Integration*

2. Motivation

In real-world scenarios such as transportation, communication, and decision-making, the relationships among different entities can be fuzzy and affected by many conflicting factors. While classical graph theory is strong, it is not able to reflect these complexities. Fuzzy graph theory was developed to address this limitation by including membership degrees so that relations may be better described. However, as systems grew more complex, especially in cases involving multiple agents or sources of uncertainty, more expressive models were necessary, leading to the creation of m-polar fuzzy graphs, which can handle multi-dimensional uncertainty.

Although planarity for crisp and fuzzy graphs has received a lot of attention, the study of outerplanarity in m-polar fuzzy graphs is still in its infancy. Outerplanar graphs are used for building efficient networks, such as bypass roads and one-way streets, where planarity helps reduce routing and limit intersection complexity. In such networks, factors such as directionality and congestion both of which are typically fuzzy may be easily expressed using m-polar fuzzy frameworks.

This research aims to address the gap by formally introducing and analyzing m-polar fuzzy outerplanar graphs (mPFOGs) and their components. It also looks at dual graph relationships, which are important for optimizing networks, and examines maximal and maximum subgraphs, crucial for identifying key parts of a network. By applying these ideas to one-way road networks, this study shows how mPFOGs can be useful in urban planning and smart transportation system.

3. Preliminaries

A few of the cited works are particularly relevant to the discussion in this section

Definition:3.1 [24] An m-polar fuzzy set (mFSs) (also called a $[0,1]^m-$ set) defined on a set X is a function $A: X \to [0,1]^m$. The collection of all such m-polar fuzzy sets over X is represented by $m(X)$.

Definition:3.2 [24] Let A and B be two m-polar fuzzy sets (mFSs) on X. Then their union and intersection, $A \cup B$ and $A \cap B$, are also m-polar fuzzy sets on X, defined as follows: for $i = 1,2,\dots,m$ and $x \in X$,. $p_i \circ (A \cup B)(x) = \{p_i \circ \text{A(x)} \vee p_i \circ \text{B(x)}\}$ and $p_i \circ (A \cap B)(x) = \{p_i \circ \text{A(x)} \wedge p_i \circ \text{B(x)}\}$. Moreover $A \subseteq B$ if and only if $p_i \circ A(x) \leq p_i \circ B(x)$ and $A = B$ if and only if $p_i \circ A(x) = p_i \circ B(x)$ for all i and $x \in X$.

Definition:3.3 [24] Let A be an m-polar fuzzy set (mFSs) on a set X. An m-polar fuzzy relation on A is an m-polar fuzzy set B defined over $X \times X$ such that

$$B(x,y) \leq \min\{A(x), A(y)\}, \forall\, x, y \in X.$$

Equivalently, for each $i = 1,2,\dots,m$ and for all $x, y \in X$,

$$p_i \circ B(x,y) \leq \min\{p_i \circ A(x), p_i \circ A(y)\}.$$

An m-polar fuzzy relation B on X is called symmetric if $B(x,y) = B(y,x)$ for all $x, y \in X$.

Definition:3.4[24] An m-polar fuzzy graph (mFG) corresponding to a crisp graph $G^* = (V, E)$ is defined as a triple $G = (V, A, B)$,

where $A: V \to [0,1]^m$ is an m-polar fuzzy set on V, and $B: \widetilde{V^2} \to [0,1]^m$ is an m-polar fuzzy set on $\widetilde{V^2}$ satisfying the following conditions:

1. For each $i = 1,2,\dots,m$ and for all x $xy \in \widetilde{V^2}$,
   $p_i \circ B(xy) \leq \min\{p_i \circ A(x), p_i \circ A(y)\}$
2. $B(xy) = 0$ for all $xy \in \widetilde{V^2} - E$

Here, $(0 = (0,0,\dots,0)$, $1 = (1,1,\dots,1)$ denote the smallest and largest elements of $[0,1]^m$, respectively. The mapping A is called the m-polar fuzzy vertex set of G, and B is called the m-polar fuzzy edge set of G.

Definition:3.5 [24] Let G = (V, A, B) be an m-polar fuzzy multigraph (mPFM). An edge (u, v) in G is called m-polar fuzzy strong if, for every $i = 1,2,..,m$, the condition $I^i_{(u,v)} \geq 0.5$ holds. If this condition is not satisfied, the edge is termed m-polar fuzzy weak.

Definition:3.6 [24] Let G = (V, A, B) be an m-polar fuzzy multigraph (mPFM). Suppose in a particular geometric drawing, the graph has exactly one crossing between the edges ((w, x), B (w, x)) and ((y, z), B (y, z)). If B (w, x) = 1, and B (y, z) = 0, then the crossing is regarded as nonexistent. Likewise, if B (w, x) is close to 1 and B (y, z) is close to 0, the crossing is considered negligible for planarity. On the other hand, if both B (w, x) and B (y, z) are close to 1, the crossing plays a significant role in determining planarity. Hence, whenever two edges intersect at a point, a numerical value called the intersecting value is assigned to that point.

Definition:3.7[24] Consider an m-polar fuzzy multigraph(mPFM) G = (V, A, B). Suppose B contains two edges $((u_1, v_1), B^j(u_1, v_1))$ and $((u_2, v_2), B^k(u_2, v_2))$ that intersect at a point PPP, where j and k are fixed integers. The intersecting value at point P is defined as

$$I_P = (I_P^1, I_P^2, \dots, I_P^m),$$

where

$$I_P^i = \frac{I_{(u_1,u_1)}^i + I_{(u_2,u_2)}^i}{2},\ i = 1,2,\dots,m.$$

Moreover, as the number of intersection points in an m-polar fuzzy multigraph increases, its degree of planarity decreases.

Definition: 3.8 [24] Let G = (V, A, B) be an m-polar fuzzy multigraph(mPFM), and let $P_1, P_2, \dots, P_k$ denote the intersection points of its edges in a given geometric representation. Then G is called an m-polar fuzzy planar graph with an associated m-polar fuzzy planarity value

$$P = (P_1, P_2, \dots, P_m),$$

where

$$P_i = \frac{1}{1 + \{I_{p_1} + I_{p_2} + \cdots + I_{p_n}\}}, 1,2, \dots, m$$

The planarity value is bounded, since for each $i = 1,2, \dots, m$.

$$0 < P_i \leq 1$$

Definition:3.9[24] Let $G = (V, A, B)$ be an m-polar fuzzy planar graph (mPFPG), where

$$B = \{((u,v), \left(B^j(u,v)\right), j \geq 1,2, \dots, p: (u,v) \in V \times V\}.$$

An m-polar fuzzy face of G is defined as a region enclosed by a set of m-polar fuzzy edges $E' \in V \times V$ in a geometric embedding of G.

The strength of a face is given by the vector

$$(S_F^1, S_F^2, \dots, S_F^m)$$

where for each $i = 1,2, \dots, m$,

$$S_F^i \geq \min\{I_{(u,v)}^i : (u,v) \in E'\}$$

Definition:3.10 [25] An m-polar fuzzy face (mPFF) is said to be a strong m-polar fuzzy face if $S_F^i \geq 0.5$ for all $i = 1,2, \dots, m$;

otherwise, it is called a weak m-polar fuzzy face.

In every m-polar fuzzy planar graph, there exists an unbounded region, referred to as the outer m-polar fuzzy face, while all other bounded regions are called inner m-polar fuzzy faces.

Definition:3.11[25] Let $G = (V, A, B)$be an m-polar fuzzy planar graph(mPFPG), where

$$B = \{\left((u,v), B^j(u,v)\right), j = 1,2, \dots, p: (u,v) \in V \times V\}$$

Let $F_1, F_2, \dots, F_k$ denote the strong m-polar fuzzy faces of G. The m-polar fuzzy dual graph(mPFDG) of G is defined as $G_1 = (V_1, A_1, B_1)$, where

a) $V_1 = \{x_q, q = 1,2, \dots, k\}$ and each vertex x_q of G_1corresponds to the face F_q of G.

b) The vertex membership values are given by a mapping$A_1: V_1 \to [0,1]^m$, defined as

$p_i \in A_1(x_q) = \max\{P_i \in B^j(u,v)$ is an edge on the boundary of $F_q\}$.

If two faces F_i and F_j share multiple edges, the dual graph G_1 contains multiple edges between x_i and x_j. Let $B_{(x_i,x_j)}^l$ denote the membership value of the l-th edge between x_i and x_j. Then

$$B_{(x_i,x_j)}^l = B_{(u,v)}^l$$

where (u,v) is the l-th common edge between F_i and F_j and $l = 1,2, \dots, t$, with t being the total number of common edges between the faces (or equivalently, the number of edges between x_i and x_j in G_1). This preserves the structure of the dual while transferring the membership values from the shared edges in G to the edges in G_1.

Definition: 3.12 [23] The order of an m-polar fuzzy graph (mPFG) $G = (V, A, B)$, represented as $|V|$ or $O^m(G)$, is given by

$$O^m(G) = |V| = \sum_{x \in V} \frac{1 + \sum_{i=1}^m p_i \circ A(x)}{2}$$

Similarly, the size of G, denoted as $|E|$ or $S^m(G)$, is expressed as

$S^m(G) = |E| = \sum_{x \in E} \frac{1+\sum_{i=1}^{m} p_i \circ B(xy)}{2}$

Definition:3.13[42] Let G be a fuzzy graph(mPFG) and let W be a subset of the vertex set V, i.e., $W \subseteq V$. The fuzzy subgraph obtained by removing the vertex set W from G is denoted as $G - W$ and is defined as

$$G - W = \{v \epsilon V \backslash W | \mu(x, y) \neq 0, \forall\, x, y \in V \backslash W\}$$

This resulting fuzzy subgraph $G - W$ is known as the Vertex Deletion fuzzy subgraph of graph G.

Definition: 3.14 [42] Let G be a fuzzy graph (mPFG) and let U be a subset of the edge set E, i.e., $E \subseteq U$. The fuzzy subgraph obtained by removing the edge set U from G is denoted as $G - U$ and is defined as

$$G - U = \{\mu(x, y) \epsilon E \backslash U | \mu(x, y) \neq 0, \forall x, y \in V\}$$

The resulting fuzzy subgraph $G - U$ is known as the edge deletion fuzzy subgraph of G.

Definition: 3.15[34] The edge contracted fuzzy graph (mPFEC), denoted as $G \backslash uv$ derived from the fuzzy graph $G = (\sigma, \mu)$ by contracting the edge uv. In this new graph, the vertex set $V' = [V \backslash \{u, v\} \cup \{w\}]$. The membership value σ of $G \backslash uv$ remains the same as σ in G for all vertices $x \in V$, and the membership value of w is calculated by $\sigma(w) = \wedge (\sigma(u), \sigma(v))$. The adjacency between two vertices x and y in $G \backslash uv$ is determined by the following conditions.

a) If x and y are both in V and are adjacent in the original graph G. Then the membership value μ in $G \backslash uv$ for the edge xy is the same as in G. (i.e.) $\mu G \backslash uv = \mu G(xy)$.

b) If x is in V and y is the newly added vertex w, then x and y are adjacent in $\psi \backslash uv$ if either x is adjacent to u or v in the original graph G, then $\mu G \backslash uv\ (xy) = \wedge \left(\sigma(x), \sigma(w)\right)$.

c) If x is in V and y is the newly added vertex w, (i.e.)

d) $y = w$. Then x and y are adjacent in $G \backslash uv$ if x is adjacent to either u or v in G. Then $\mu G \backslash uv(xy) = \wedge [\mu G(xu), \mu G(xv)]$.

4. Multi Polar Fuzzy Outerplanar Graphs (mPFOGs)

In this section, we define the concept of m-polar fuzzy outerplanar graphs and discuss their fundamental properties.

Definition: 4.1 An m-polar fuzzy graph G is called an m-polar fuzzy outerplanar graph if it can be embedded in the plane such that all its vertices lie on the outer boundary of the planar region.

Let $i(G)$ denote the number of vertices in a planar embedding that are not located on this outer boundary. Then, for an m-polar fuzzy outerplanar graph,

$i(G) = 0$.

The graphs shown in Fig.1 and Fig.2 serve as examples of m-polar fuzzy outerplanar graphs.

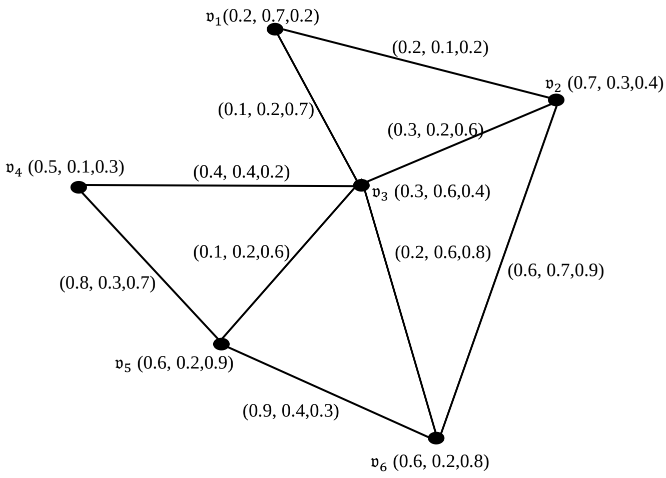

Figure.1 3-Polar fuzzy outerplanar graph with cycle

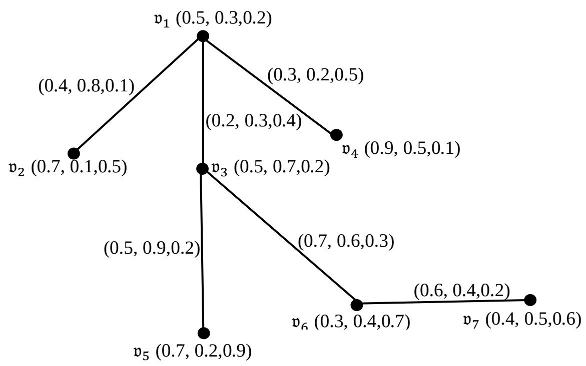

Figure.2 3-Polar fuzzy outerplanar graph without cycle

Definition: 4.2 An **m-polar fuzzy non-outerplanar graph** is an m-polar fuzzy graph that cannot be drawn in the plane with all vertices lying on the outer boundary of the planar region.

In other words, it is a graph that fails to satisfy the outerplanar property, having some vertices positioned in the interior rather than exclusively on the outer boundary. Consequently, any m-polar fuzzy planar graph with $i(G)\neq 0$ is not an m-polar fuzzy outerplanar graph.
The graphs given in Fig.3 and Fig.4 are examples of m-Polar fuzzy non outerplanar graphs.

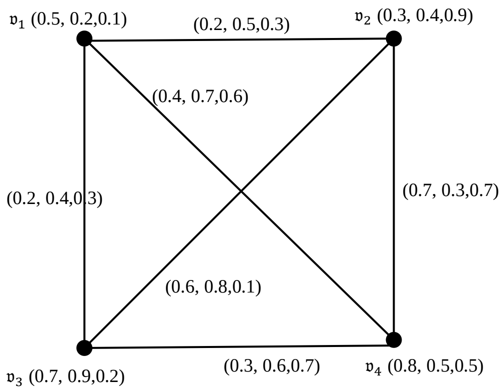

Figure.3 3-Polar Fuzzy Graph K_4

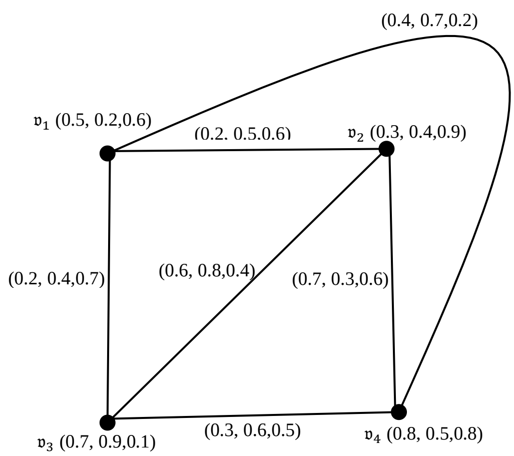

Figure.4 3-Polar Fuzzy Planar Embedding of K_4

4.1 Characterization of Multi Polar Fuzzy Outerplanar Graphs

In this section, we explore the defining theorems and example criteria that characterize m-polar fuzzy outerplanar graphs.

Theorem 4.1.1 An m-polar fuzzy graph G is m-polar fuzzy outerplanar if and only if it does not contain any m-polar fuzzy subgraph that is homomorphic to K_4 or $K_{2,3}$

Proof: It is well-known that K_4 or $K_{2,3}$ are not m-polar fuzzy outerplanar. Therefore, any m-polar fuzzy graph containing a subgraph homomorphic to K_4 or $K_{2,3}$ cannot be m-polar fuzzy outerplanar, as illustrated in Figs. 5 and 6.

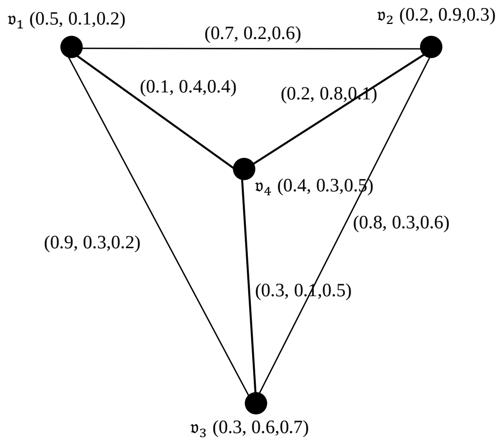

Figure.5. 3-polar fuzzy graph K_4

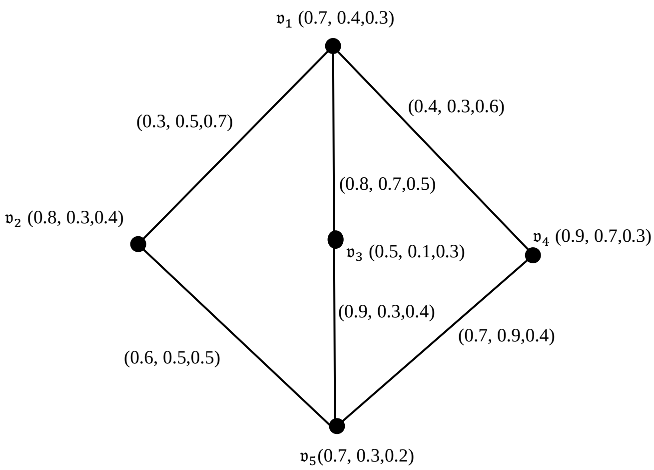

Figure.6. 3-polar fuzzy graph $K_{2,3}$

Conversely, suppose G is a m-polar fuzzy graph containing no m-polar fuzzy subgraph homomorphic from K_4 or $K_{2,3}$. By kuratowski's theorem, G is planar. Assume G is not m-polar fuzzy outerplanar graph. Thus, without loss of generality, we assume that G is a cyclic block which is not m-polar fuzzy outerplanar graph. Further we assume that G is so embedded in the m-polar fuzzy plane that the exterior region contains maximum number of vertices.

The exterior region is bounded by a cycle C. Since not all vertices of G, lie on C, there are one or more vertices lying interior to C.

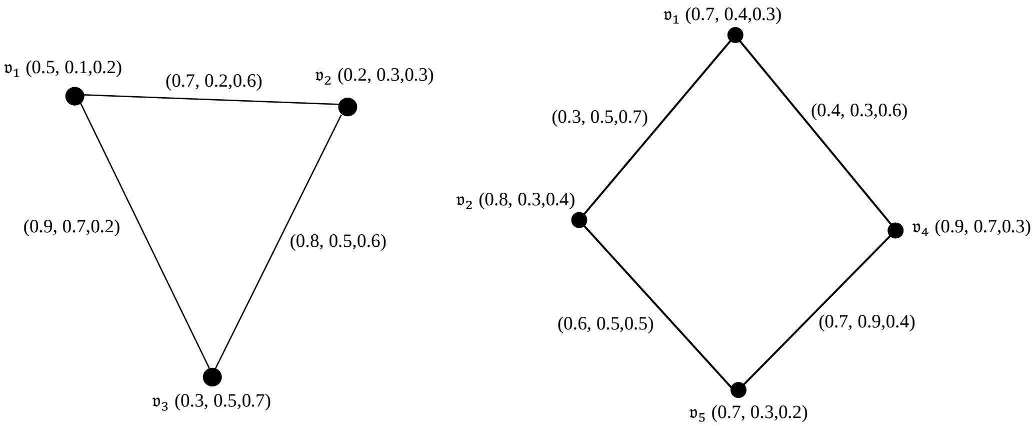

Figure.7. 3-polar fuzzy outerplanar graphs

If there exist a vertex $\mathfrak{v}_3$ interior on C and three mutually disjoint paths between $\mathfrak{v}_3$ and three distinct vertices of C, then G contains m-polar fuzzy subgraph homomorphic from K_4. Otherwise, since G is a cyclic block, there must exists a vertex $\mathfrak{v}_3$ and two disjoint paths between $\mathfrak{v}_3$ and two disjoint vertices $\mathfrak{v}_1$ and $\mathfrak{v}_5$ on C.

Moreover, from the above choice of C, the edge $\mathfrak{v}_1\mathfrak{v}_5$ does not belong to C. This implies ψ contains a m-polar fuzzy subgraph homomorphic from $K_{2,3}$. This is a contradiction. Therefore, Fig.7 G is a m-polar fuzzy outerplanar graph. This completes the proof.

Theorem 4.1.2 If G is a connected m-polar fuzzy outerplanar graph, then it has a m-polar fuzzy dual G^* there exist a vertex $\mathfrak{v}$ such that $G^* - \mathfrak{v}$ contains no m-polar fuzzy cycle.

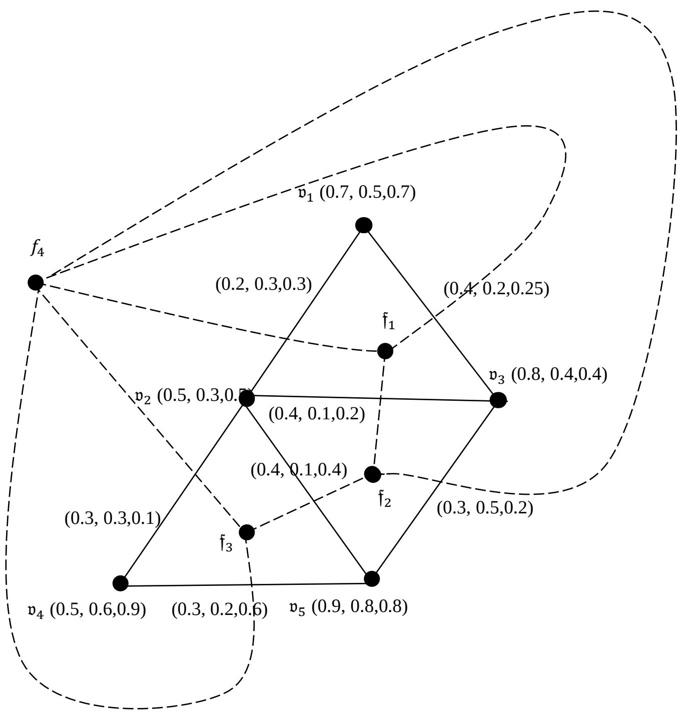

Figure.8. Connected 3-polar fuzzy Outerplanar graphs G

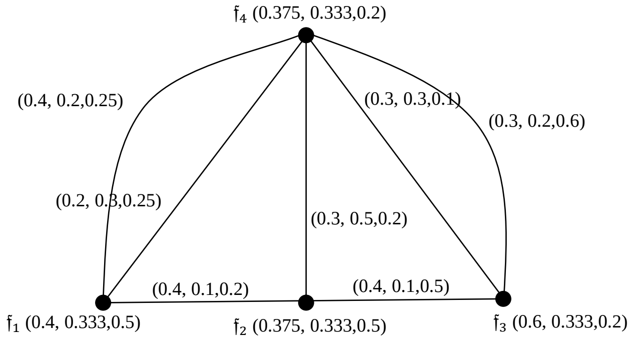

Figure.8 (a). 3-polar fuzzy Dual graph G^*

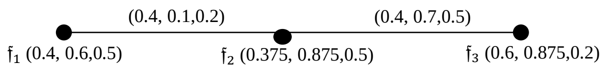

Figure.8 (b). 3-polar fuzzy graph but no cycle $G^* - v$

Figures 8, 8(a)-(b) Example for m-polar fuzzy outerplanar graph G has 3-polar fuzzy dual graph G^* then there exist $\mathfrak{v}$ such that $G^* - \mathfrak{v}$ contains no 3-polar fuzzy cycle.

Proof. Let $G = (\mathbb{V}, \tau, \delta)$ be a connected m-polar fuzzy outerplanar graph and there are no intersections of edges $E' \subset E$ within the graph, we can divide the m-polar fuzzy graph into a finite number of regions. Each of these regions corresponds to a $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3, \dots \mathfrak{f}_k$ be the m-polar fuzzy faces and membership value for each face is determined by [46],
$\mathfrak{f} = [\mathfrak{f}^m] = \left[min\left\{\frac{\delta^m(x,y)}{\tau^m(x) \wedge \tau^m(y)} : \delta^m(x,y) \in E'\right\}\right]$ Where E' is the region bounded by the m-polar fuzzy edges in m-polar fuzzy planar graph.
When the m-polar fuzzy graph divided into regions with m-polar fuzzy face and values, it becomes very easy to m-polar fuzzy dual graph G^* such that each face of G corresponds to a vertex and each edge of G corresponds to edge in $G^* = (\mathbb{V}', \tau', \delta')$. Consequently, the m-polar fuzzy dual graph G^* exists for the m-polar fuzzy outerplanar graph.
Since G is m-polar fuzzy outerplanar graph. It is outer face is a m-polar fuzzy cycle. Therefore, the graph $G^* - \mathfrak{v}$ that results from removing the vertex $\mathfrak{v}$ from G^*, the resulting graph $G^* - \mathfrak{v}$ contains no m-polar fuzzy cycle.
Example 4.1. Consider the 3-polar fuzzy outerplanar graph $G = (\mathbb{V}, \mathrm{A}, \mathrm{B})$ illustrated in the Fig.8(b). If the dotted lines in Fig. 8(a) are superimposed 3-polar fuzzy dual graph of G^* and separated diagram.
Here $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$ and $\mathfrak{f}_4$ are four fuzzy faces. $\mathfrak{f}_1$ is bounded by the edges $((\mathfrak{v}_1, \mathfrak{v}_2), (0.1,0.3,0.3))$, $((\mathfrak{v}_2, \mathfrak{v}_3), (0.4,0.1,0.2))$ and $((\mathfrak{v}_1, \mathfrak{v}_3), (0.4,0.2,0.25))$; $\mathfrak{f}_2$ is bounded by the edges $((\mathfrak{v}_2, \mathfrak{v}_3), [0.4,0.1,0.2])$, $((\mathfrak{v}_2, \mathfrak{v}_5), [0.4,0.1,0.4])$ and $((\mathfrak{v}_3, \mathfrak{v}_5), [0.3,0.5,0.2])$; $\mathfrak{f}_3$ is bounded by the edges $((\mathfrak{v}_2, \mathfrak{v}_4), [0.3,0.3,0.1])$, $((\mathfrak{v}_2, \mathfrak{v}_5), [0.4,0.1,0.4])$ and $((\mathfrak{v}_4, \mathfrak{v}_5), [0.3,0.2,0.6])$; $\mathfrak{f}_4$ is the outer face $((\mathfrak{v}_3, \mathfrak{v}_5), [0.3,0.5,0.2])$, $((\mathfrak{v}_1, \mathfrak{v}_3), [0.4,0.2,0.25])$, $((\mathfrak{v}_1, \mathfrak{v}_2), [0.2,0.3,0.3])$, $((\mathfrak{v}_2, \mathfrak{v}_4), [0.3,0.3,0.1])$ and $((\mathfrak{v}_4, \mathfrak{v}_5), [0.3,0.2,0.6])$.The membership value of an m-polar fuzzy faces $\mathfrak{f}_1 = (0.4,0.333,0.5), \mathfrak{f}_2 = (0.375,0.333,0.5), \mathfrak{f}_3 = (0.6,0.333,0.2), \mathfrak{f}_4 =$

(0.375,0.333,0.2) shown in the Table 2. Each m-polar fuzzy face of G corresponds to a vertex and each edge of G corresponds to edge in $G^* = (\mathbb{V}', \tau', \delta')$ Then $\mathfrak{v}$ exists in such a way that $G^* - \mathfrak{v}$ contains no m-polar fuzzy cycle.

S. No	3-polar fuzzy faces	$\mathfrak{f}^1$	$\mathfrak{f}^2$	$\mathfrak{f}^3$
1.	$\mathfrak{f}_1$	0.4	0.333	0.5
2.	$\mathfrak{f}_2$	0.375	0.333	0.5
3.	$\mathfrak{f}_3$	0.6	0.333	0.2
4.	$\mathfrak{f}_4$	0.375	0.333	0.2

Table 2. Calculation of m-polar fuzzy face

Definition 4.3. If an edge cannot be added to a m-polar fuzzy outerplanar graph G without sacrificing its outer planarity property, then the graph is m-polar maximal fuzzy outerplanar graph.
The graphs given in Fig.9 and Fig.10 are examples of m-polar maximal fuzzy outerplanar graphs.

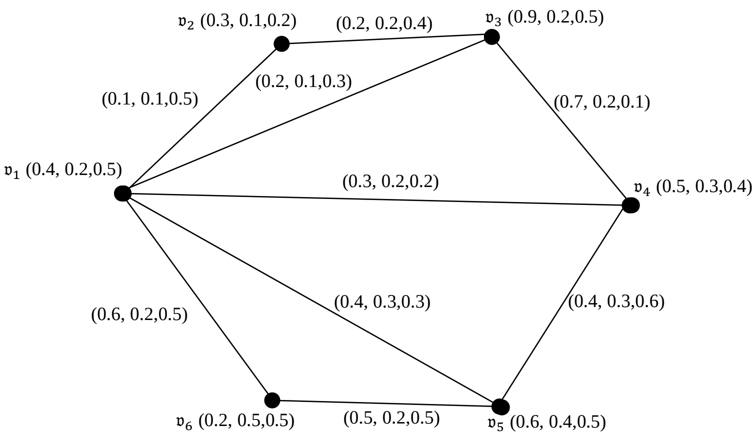

Figure 9. 3-Polar maximal fuzzy outerplanar graph

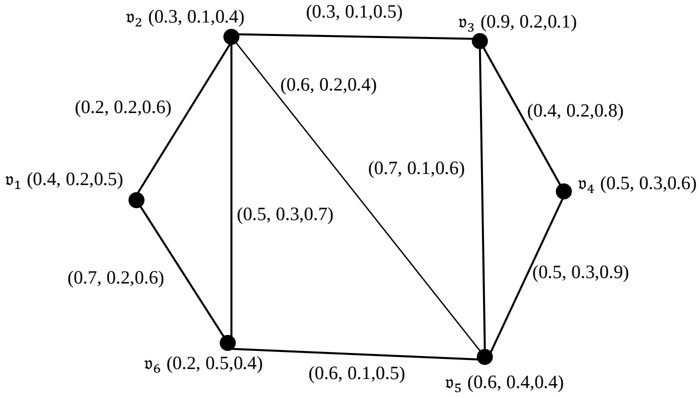

Figure 10. 3-Polar maximal fuzzy outerplanar graph

Definition 4.4. A m-polar fuzzy planar graph G is considered minimally m-polar fuzzy non-outerplanar if $i(G) \neq 0$ with atmost one vertex $\mathfrak{v} \in G$ such that $\tau(x) > 0$ in the interior region.
The graphs given in Fig.11 and Fig.12 are examples of minimally m-polar fuzzy non-outerplanar graphs.

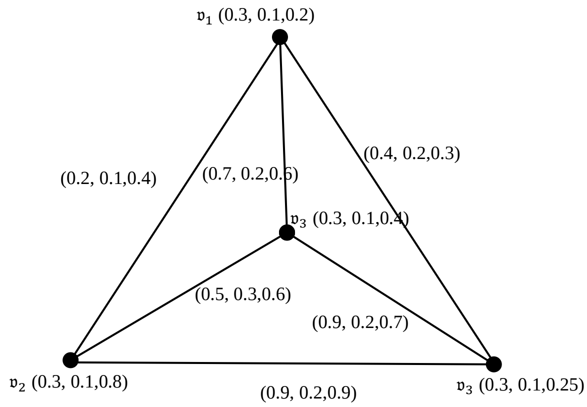

Figure 11. 3-Polar fuzzy graph K_4

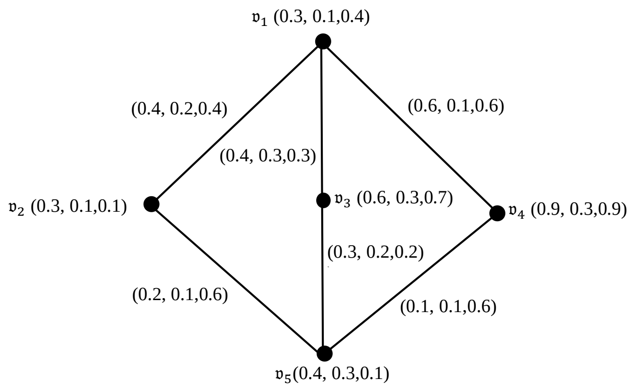

Figure 12. 3-Polar fuzzy graph $K_{2,3}$

5.Vertex Deletion Multi Polar Fuzzy Outerplanar Subgraphs (VD-mPFOSs)

In this section, the subgraph formed by removing specific vertices from the m-polar fuzzy graph, while also defining m-polar fuzzy outerplanar graphs and offering relevant illustrations.
Definition 5.1. Assume G is a m-polar fuzzy planar graph. If G' is the Vertex Deleted m-polar fuzzy subgraph possessing outerplanarity property, then it is called Vertex Deleted m-polar fuzzy outerplanar subgraph of G.
Example 5.1. Let's look at the 3-polar fuzzy graph $G = (\mathbb{V}, \tau, \delta)$ shown in Figure 13. The set of vertices in G is $\mathbb{V} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_5, \mathfrak{v}_6\}$ and their membership values of these are as follows: $\tau(\mathfrak{v}_1) = (0.6,0.4,0.3), \tau(\mathfrak{v}_2) = (0.85,0.9,0.5), \tau(\mathfrak{v}_3) = (0.6,0.8,0.5), \tau(\mathfrak{v}_4) = (0.4,0.1,0.3), \tau(\mathfrak{v}_5) = (0.7,0.9,0.5)$ and $\tau(\mathfrak{v}_6) = (0.6,0.7,0.4)$, and the edges $\mathbb{B}$ is $\delta(\mathfrak{v}_1, \mathfrak{v}_2) =$

(0.65,0.1,0.2), $\delta(\mathfrak{v}_2, \mathfrak{v}_3) = (0.25,0.3,0.4)$, $\delta(\mathfrak{v}_3, \mathfrak{v}_1) = (0.35,0.2,0.3)$, $\delta(\mathfrak{v}_2, \mathfrak{v}_5) =$ (0.6,0.1,0.2), $\delta(\mathfrak{v}_3, \mathfrak{v}_6) = (0.35,0.1,0.2)$, $\delta(\mathfrak{v}_5, \mathfrak{v}_6) = (0.55,0.6,0.2)$, $\delta(\mathfrak{v}_6, \mathfrak{v}_5) =$ (0.4,0.3,0.4), $\delta(\mathfrak{v}_2, \mathfrak{v}_4) = (0.4,0.4,0.2)$, $\delta(\mathfrak{v}_3, \mathfrak{v}_4) = (0.4,0.2,0.1)$, $\delta(\mathfrak{v}_4, \mathfrak{v}_6) = (0.4,0.2,0.7)$, $\delta(\mathfrak{v}_4, \mathfrak{v}_5) = (0.3,0.5,0.2)$.

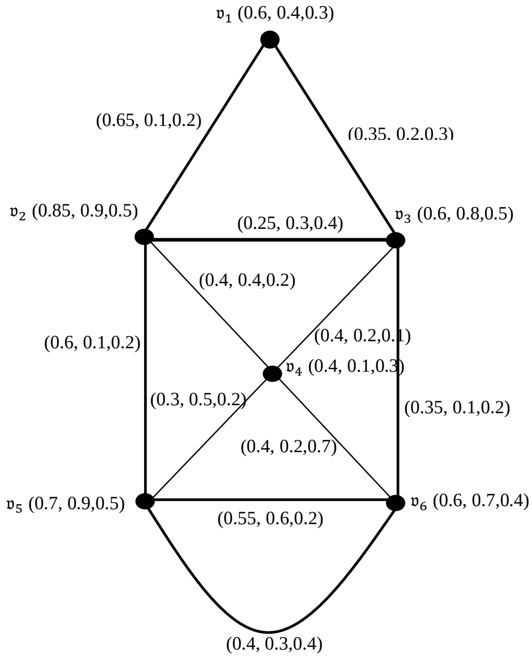

Figure 13. 3-Polar fuzzy graph

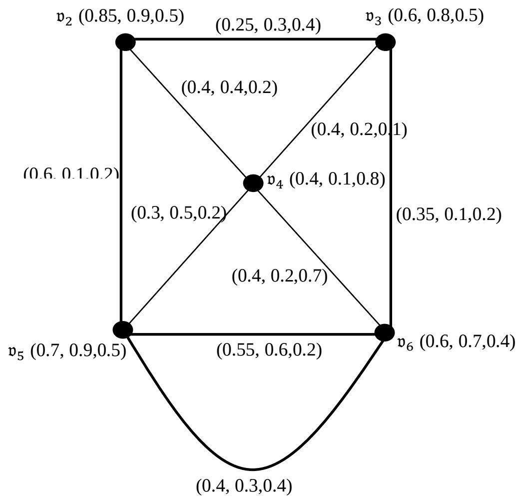

Figure 13 (a). Vertex Deleted 3-Polar fuzzy subgraph

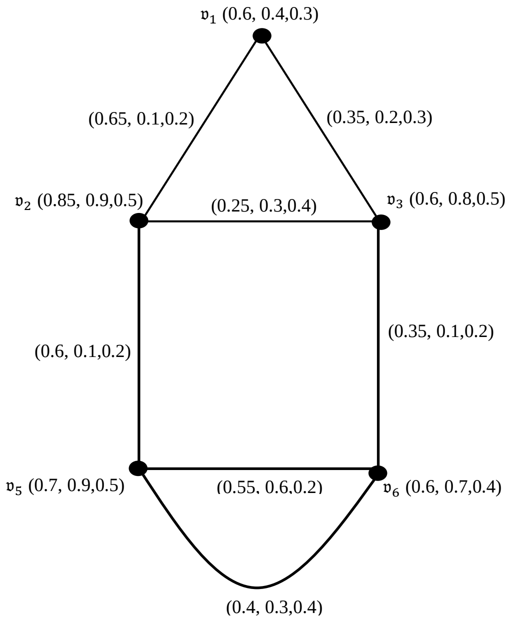

Figure 13 (b). Vertex deletion 3-Polar fuzzy outerplanar subgraph

The edges intersect between the sets $\tau(\mathfrak{v}_2, \mathfrak{v}_6)$and $\tau(\mathfrak{v}_3, \mathfrak{v}_5)$within 3-polar fuzzy graph G. Let us define subsets $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{V}$, where $\mathbb{X} = \{(\mathfrak{v}_1(0.6,0.4,0.3)\}$ and $\mathbb{Y} = \{\mathfrak{v}_4(0.4,0.1,0.3)\}$ in the 3-polar fuzzy graph G.

The 3-polar fuzzy subgraphs $G - \mathbb{X}$, represented as G_1 can be seen in Fig. 13(a). Similarly, the 3-polar fuzzy subgraph $G - \mathbb{Y}$ represented as G_2 is shown in Fig. 13(b). It can be noted that, 3-polar fuzzy subgraph G_1 is categorized as a Vertex Deletion 3-polar fuzzy subgraph, while G_2 is classified as a Vertex Deleted 3-polar fuzzy outerplanar subgraph of G .

Note 5.1. It is not necessary for the Vertex Deletion m-polar fuzzy subgraph of G to be the Vertex Deletion m-polar fuzzy outerplanar subgraph of G. The m-polar fuzzy graphs in Figs 13(a) and 13(b) allow for the observation of this.

Theorem 5.1. A m-polar fuzzy outerplanar graph G always has a vertex deleted m-polar fuzzy outerplanar subgraph that is also a vertex deleted m-polar fuzzy subgraph of G.

Proof. Let the m-polar fuzzy outerplanar graph be G . and $\mathbb{H}$ be the vertex deleted m-polar fuzzy subgraph of G. The vertices of the m-polar fuzzy graph G will all be in the outer region since it is outerplanar. As a result, the m-polar subgraph that was created by deleting vertices still has outerplanarity. Any vertex deleted m-polar subgraph in G is therefore a vertex deletion m-polar fuzzy outerplanar subgraph of G.

Theorem 5.2. Let G be a m-polar fuzzy outerplanar graph which is connected and let $\mathbb{W}$ be a subset of its vertices, such that $\mathbb{W} \subseteq \mathbb{V}$. The m-polar fuzzy outerplanar subgraph of G with a vertex deleted is $G' = G \backslash \mathbb{W}$. Its m-polar fuzzy dual graph is G''.

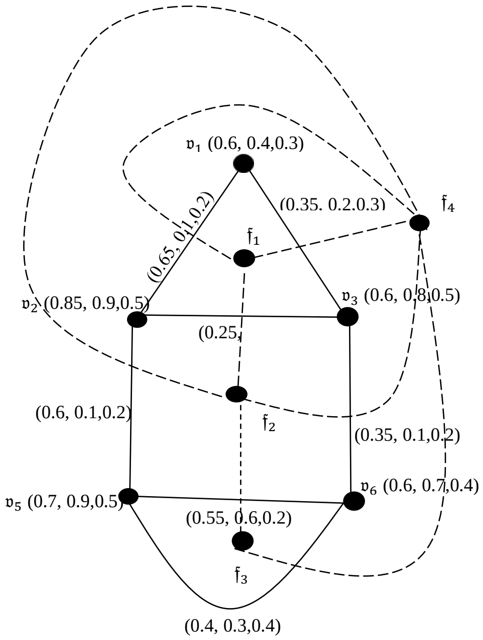

Figure 14. Connected 3-Polar fuzzy outerplanar graph G

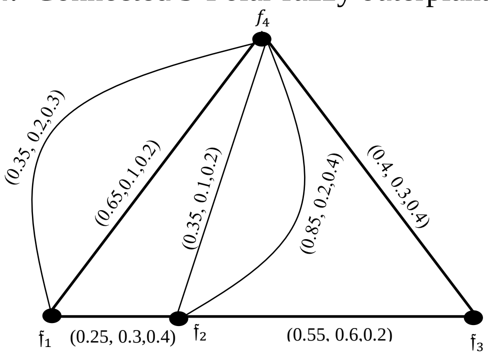

Figure 14(a). 3-Polar fuzzy dual graph G'

Figures 14,14(a). Example for Vertex Deletion 3-polar fuzzy outerplanar subgraph and its 3-polar fuzzy dual graph.

Proof. Let us consider a connected m-polar fuzzy outerplanar subgraph G and $\mathbb{W} \subseteq \mathbb{V}$. let $G - \mathbb{W}$ form a Vertex deleted m-polar fuzzy outerplanar subgraph denoted G' . Since there is no intersection of m-polar fuzzy edges in the subgraph G', we can divide the m-polar fuzzy graph into a finite number of regions. Each of these regions corresponds to a m-polar fuzzy face, and the membership values for each face is determined by [46]
$\mathfrak{f} = [\mathfrak{f}^m] = \left[min\left\{\frac{\delta^m(x,y)}{\tau^m(x)\wedge\tau^m(y)}: \delta^m(x,y) \in E'\right\}\right]$ Where E' is the region bounded by the m-polar fuzzy edges in m-polar fuzzy planar graph.
Using the fact that each m-polar fuzzy graph with planarity value as 1 has m-polar fuzzy dual graph, $G - \mathbb{W}$ has a m-polar fuzzy dual graph G''. Consequently, the m-polar fuzzy dual graph exists for the Vertex Deleted m-polar fuzzy outerplanar subgraph of G.

Example 5.2. From Example, it is evident that the 3-polar subgraph G'' is the Vertex Deletion 3-polar fuzzy outerplanar subgraph of G. The dotted lines represent the superimposed m-polar fuzzy dual graph of G'' in Figure 14(b). Here $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$ and $\mathfrak{f}_4$ are four fuzzy faces. $\mathfrak{f}_1$ is bounded by the edges $((\mathfrak{v}_1, \mathfrak{v}_2), (0.65,0.1,0.2))$, $((\mathfrak{v}_2, \mathfrak{v}_3), (0.25,0.3,0.4))$ and $((\mathfrak{v}_1, \mathfrak{v}_3), (0.35,0.2,0.3))$; $\mathfrak{f}_2$ is bounded by the edges $((\mathfrak{v}_2, \mathfrak{v}_3), (0.25,0.3,0.4))$, $((\mathfrak{v}_3, \mathfrak{v}_6), (0.35,0.1,0.2))$, $((\mathfrak{v}_5, \mathfrak{v}_6), (0.55,0.6,0.2))$ and $((\mathfrak{v}_2, \mathfrak{v}_5), (0.6,0.1,0.2))$; $\mathfrak{f}_3$ is bounded by the edges $((\mathfrak{v}_5, \mathfrak{v}_6), (0.55,0.6,0.2))$, $((\mathfrak{v}_6, \mathfrak{v}_5), (0.4,0.3,0.4))$; $\mathfrak{f}_4$ is the outer face $((\mathfrak{v}_1, \mathfrak{v}_3), (0.35,0.2,0.3))$, $((\mathfrak{v}_1, \mathfrak{v}_2), (0.65,0.1,0.2))$, $((\mathfrak{v}_2, \mathfrak{v}_5), (0.6,0.1,0.2))$, $((\mathfrak{v}_6, \mathfrak{v}_5), (0.4,0.3,0.4))$and $((\mathfrak{v}_3, \mathfrak{v}_6), (0.35,0.1,0.2))$. The 3-polar fuzzy dual graph is associated with m-polar fuzzy face values, with all 3-polar fuzzy faces having membership values of $\mathfrak{f}_1 = (0.416,0.25,0.666), \mathfrak{f}_2 = (0.466,0.111,0.4)$, $\mathfrak{f}_3 = (0.666,0.4285,0.5)$ and $\mathfrak{f}_4 = (0.583,0.111,0.4)$ shown in the Table.3. Therefore, the Vertex Deletion 3-polar fuzzy outerplanar subgraph G''indeed possesses a 3-polar fuzzy dual graph.

S. No	**3-polar fuzzy faces**	$\mathfrak{f}^1$	$\mathfrak{f}^2$	$\mathfrak{f}^3$
1.	$\mathfrak{f}_1$	0.416	0.25	0.666
2.	$\mathfrak{f}_2$	0.466	0.11	0.4
3.	$\mathfrak{f}_3$	0.666	0.428	0.5
4.	$\mathfrak{f}_4$	0.583	0.11	0.4

Table 3. Calculation of 3-polar fuzzy face

Example 5.3: Let's look at the 3-polar fuzzy graph $G = (\mathbb{V}, \tau, \delta)$ shown in Figure. 15. The set of vertices in G is $\mathbb{V} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_5\}$ and their membership values of these are as follows: $\tau(\mathfrak{v}_1) = (0.6,0.5,0.6), \tau(\mathfrak{v}_2) = (0.7,0.8,0.9), \tau(\mathfrak{v}_3) = \tau(\mathfrak{v}_9) = (0.1,0.1,0.1), \tau(\mathfrak{v}_4) = (0.6,0.9,0.6), \tau(\mathfrak{v}_5) = (0.2,0.4,0.8), \tau(\mathfrak{v}_6) = (0.2,0.3,0.6), \tau(\mathfrak{v}_7) = (0.9,0.8,0.3), \tau(\mathfrak{v}_8) = (0.7,0.3,0.1), \tau(\mathfrak{v}_{10}) = (0.3,0.5,0.9)$ and the edges $\mathbb{B}$ is $\delta(\mathfrak{v}_1, \mathfrak{v}_2) = (0.2,0.4,0.3)$, $\delta(\mathfrak{v}_2, \mathfrak{v}_3) = (0.8,0.12,0.9)$, $\delta(\mathfrak{v}_3, \mathfrak{v}_4) = (0.6,0.5,0.6)$, $\delta(\mathfrak{v}_4, \mathfrak{v}_5) = (0.4,0.7,0.6)$, $\delta(\mathfrak{v}_5, \mathfrak{v}_6) = (0.8,0.5,0.6)$, $\delta(\mathfrak{v}_6, \mathfrak{v}_7) = (0.7,0.5,0.6)$, $\delta(\mathfrak{v}_7, \mathfrak{v}_8) = (0.1,0.3,0.8)$, $\delta(\mathfrak{v}_8, \mathfrak{v}_1) = (0.6,0.7,0.8)$, $\delta(\mathfrak{v}_1, \mathfrak{v}_9) = (0.5,0.4,0.7)$, $\delta(\mathfrak{v}_2, \mathfrak{v}_{10}) = (0.1,0.5,0.6)$, $\delta(\mathfrak{v}_3, \mathfrak{v}_{10}) = (0.7,0.5,0.6)$, $\delta(\mathfrak{v}_4, \mathfrak{v}_{10}) = (0.6,0.3,0.6)$, $\delta(\mathfrak{v}_5, \mathfrak{v}_{10}) = (0.9,0.5,0.6)$, $(\mathfrak{v}_6, \mathfrak{v}_{10}) = (0.7,0.5,0.6)$, $\delta(\mathfrak{v}_6, \mathfrak{v}_9) = (0.3,0.5,0.6)$, $\delta(\mathfrak{v}_7, \mathfrak{v}_9) = (0.1,0.5,0.9)$, $(\mathfrak{v}_8, \mathfrak{v}_9) = (0.1,0.5,0.7)$, $(\mathfrak{v}_9, \mathfrak{v}_{10}) = (0.2,0.5,0.6)$.

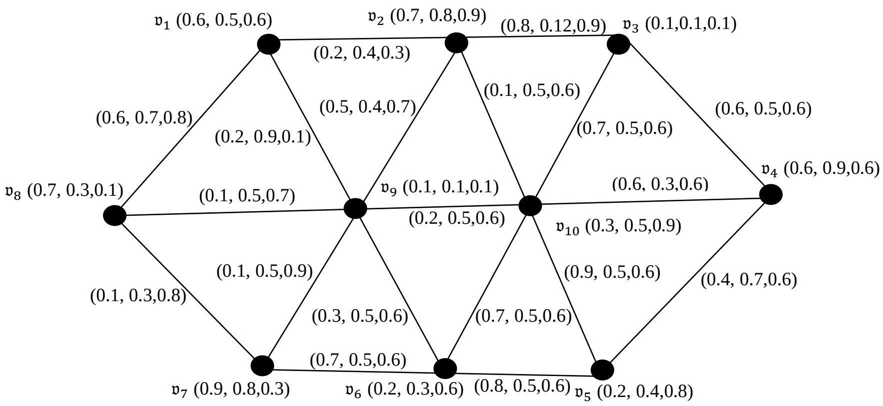

Figure. 15 3-Polar Fuzzy Graph

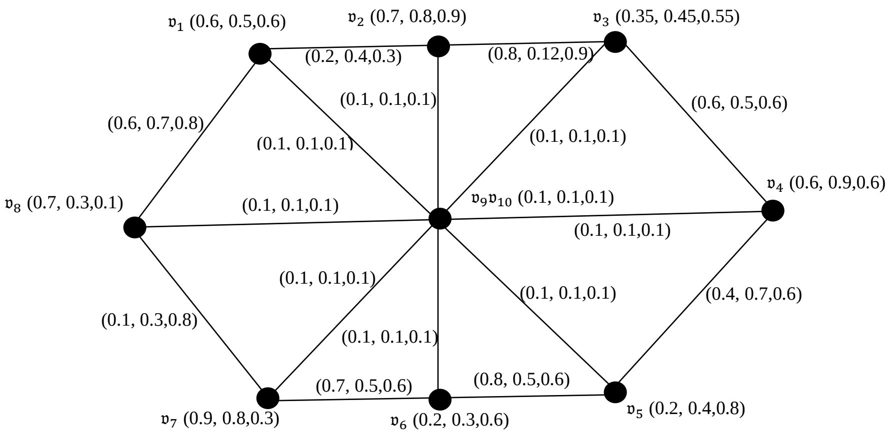

Figure. 15 (a) 3-Polar Fuzzy Edge Contraction

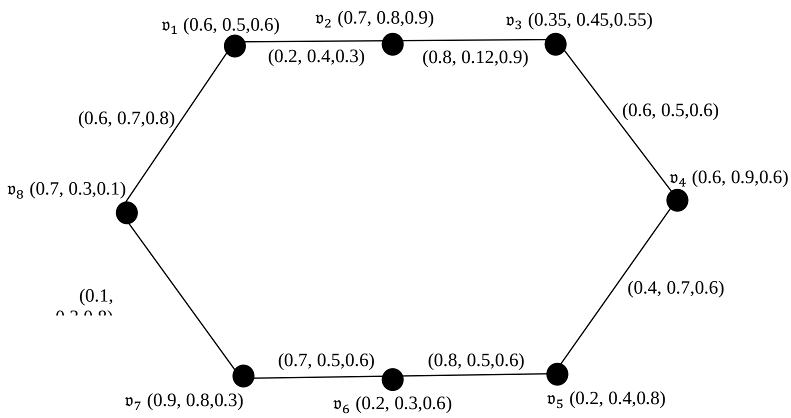

Figure.15 (b) 3-Polar Vertex deleted Fuzzy outerplanar graph

Let G be the 3-Polar fuzzy graph in the figure.15. then the edge contracted 3-Polar fuzzy graph, denoted as $G\backslash \mathfrak{v}_9\mathfrak{v}_{10}$ derived from the 3-Polar fuzzy graph $G = (\sigma, \mu)$ by contracting the edge $\mathfrak{v}_9\mathfrak{v}_{10}$. In this Figure.15(a) the vertex set $V' = [V\backslash\{\mathfrak{v}_9, \mathfrak{v}_{10}\} \cup \{\mathfrak{v}_9\mathfrak{v}_{10}\}]$. The membership value σ of $G\backslash \mathfrak{v}_9\mathfrak{v}_{10}$ remains the same as σ in G for all vertices $x \in V$, and the membership value of w is calculated by $\sigma(\mathfrak{v}_9\mathfrak{v}_{10}) = \wedge(\sigma(\mathfrak{v}_9), \sigma(\mathfrak{v}_{10}))$. The adjacency between two vertices x and y in $G\backslash \mathfrak{v}_9\mathfrak{v}_{10}$ is determined by definition 3.15 conditions.
The edge contracted 3-Polar fuzzy graph $G\backslash \mathfrak{v}_9\mathfrak{v}_{10}$, represented as G_1 can be seen in Figure 15(a). Similarly, the 3-Polar fuzzy graph $G_1 - \mathrm{V}$ represented as G_2 is shown in Figure 15(b). It can be noted that, is categorized as a Vertex Deletion 3-Polar fuzzy outerplanar graph.

6. Maximum and Maximal Vertex Deletion Multi Polar Fuzzy Outerplanar Subgraphs (Maximum and Maximal VD-mPFOSs)

In this section, defined the concepts of maximum and maximal vertex deletion 3-polar fuzzy outerplanar subgraphs and explores the relationship between these two notions with appropriate examples.

Definition 6.1. Let $G' = (\mathbb{V}', \tau', \delta')$ be a vertex-deleted m-polar fuzzy outerplanar subgraph of G. If there exists no other vertex deleted m-polar fuzzy outerplanar subgraph of G with maximum order or size than G', then G' is called the maximum vertex deleted m-polar fuzzy outerplanar subgraph of G.

Note 6.1. Let G' be vertex deleted m-polar fuzzy outerplanar subgraphs of G with orders and sizes $(\mathbb{O}^{m'}, \mathbb{S}^{m'})$ and $(\mathbb{O}^{m''}, \mathbb{S}^{m''})$ respectively, where $\mathbb{O}^{\mathrm{m}'} > \mathbb{O}^{\mathrm{m}''}$ and $\mathbb{S}^{\mathrm{m}'} > \mathbb{S}^{\mathrm{m}''}$. Then, G' is called a maximum vertex deleted m-polar fuzzy outerplanar subgraph of G if

$$(\mathbb{O}^{m'}, \mathbb{S}^{m'}) > (\mathbb{O}^{m''}, \mathbb{S}^{m''})$$

Suppose two vertex deleted m-polar fuzzy outerplanar subgraphs G_1 and G_2 of G are obtained such that $G_1 \neq G_2$ but their orders and sizes are equal,

$$(\mathbb{O}^{m'}, \mathbb{S}^{m'}) = (\mathbb{O}^{m''}, \mathbb{S}^{m''})$$

In this case, both G_1 and G_2 are considered maximum vertex deleted m-polar fuzzy outerplanar subgraphs of G.

Example 6.1. Let's look at the 3-polar fuzzy graph $G = (\mathbb{V}, \mathbb{A}, \mathbb{B})$ shown in Figure. 16. The set of vertices in G is $\mathbb{V} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_5, \mathfrak{v}_6\}$ and their membership values of these are as follows: $\tau(\mathfrak{v}_1) = (0.25,0.35,0.45), \tau(\mathfrak{v}_2) = (0.35,0.45,0.55), \tau(\mathfrak{v}_3) = (0.4,0.5,0.6), \tau(\mathfrak{v}_4) = (0.7,0.8,0.9), \tau(\mathfrak{v}_5) = (0.15,0.25,0.35)$ and $\tau(\mathfrak{v}_6) = (0.45,0.55,0.65)$, and the edges $\mathbb{B}$ is $\delta(\mathfrak{v}_1, \mathfrak{v}_6) = (0.1,0.2,0.3)$, $\delta(\mathfrak{v}_2, \mathfrak{v}_6) = (0.2,0.3,0.4)$, $\delta(\mathfrak{v}_4, \mathfrak{v}_6) = (0.3,0.6,0.7)$, $\delta(\mathfrak{v}_5, \mathfrak{v}_6) = (0.7,0.8,0.9)$, $\delta(\mathfrak{v}_1, \mathfrak{v}_2) = (0.5,0.6,0.7)$, $\delta(\mathfrak{v}_2, \mathfrak{v}_3) = (0.5,0.6,0.7)$, $\delta(\mathfrak{v}_3, \mathfrak{v}_4) = (0.4,0.5,0.6)$, $\delta(\mathfrak{v}_4, \mathfrak{v}_5) = (0.7,0.8,0.9)$, $\delta(\mathfrak{v}_5, \mathfrak{v}_1) = (0.25,0.35,0.45)$.

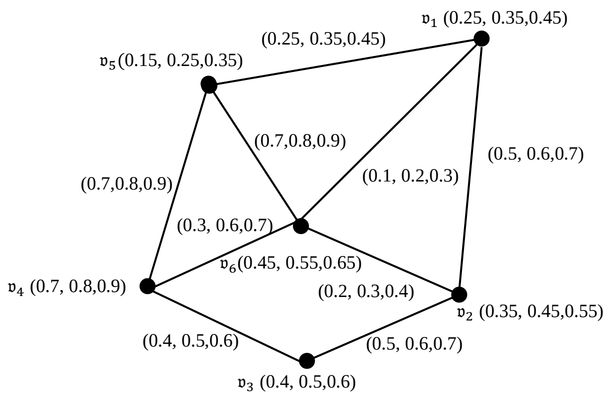

Figure. 16 3-polar fuzzy graph

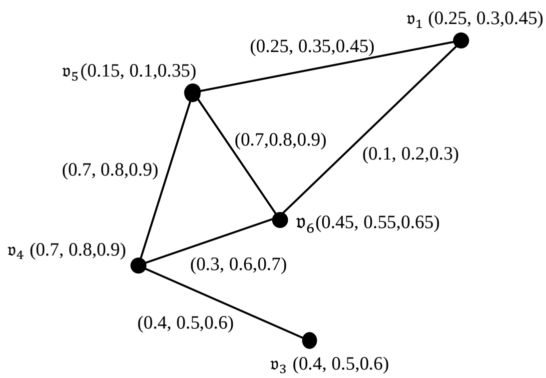

Figure. 16(a) $G - \mathfrak{v}_2$

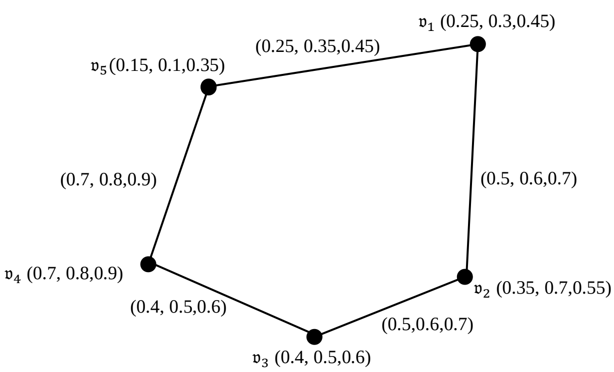

Figure. 16(b) $G - \mathfrak{v}_6$

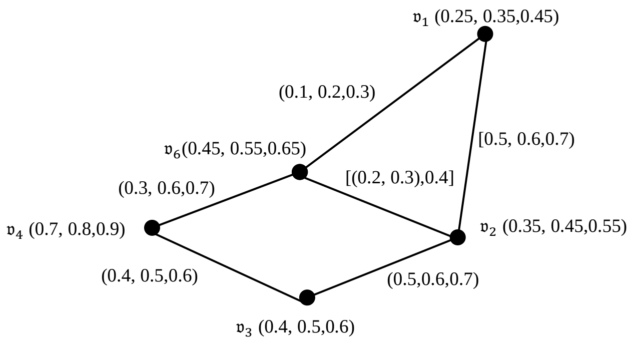

Figure. 16(c) $G - \mathfrak{v}_5$

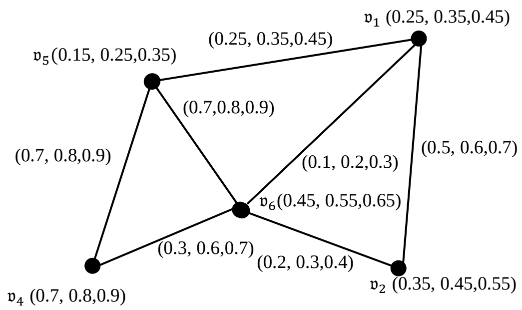

Figure. 16(d) $G - \mathfrak{v}_3$

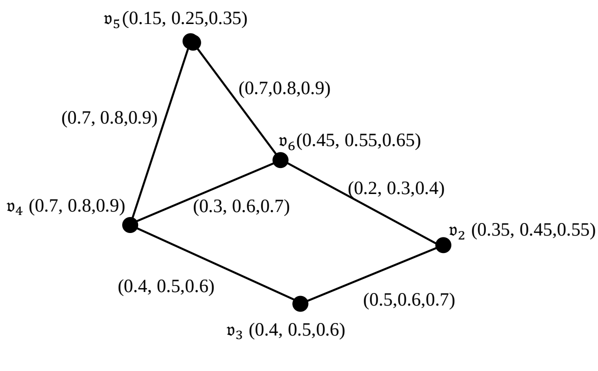

Figure. 16(e) $G - \mathfrak{v}_1$

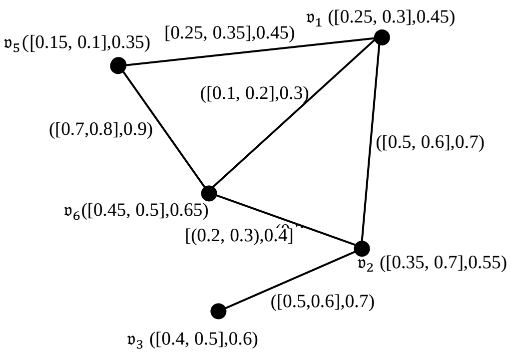

Figure. 16(f) $G - \mathfrak{v}_4$

Fig. 16(a)-16(f) Maximum Vertex Deletion 3-polar fuzzy outerplanar subgraphs

S.No	Subset	Vertex deleted 3-polar fuzzy Outerplanar Subgraph	Order			Size		
1.	$W_1 = \mathfrak{v}_1(0.25,0.35,0.45)$	$G - W_1$	2.05	2.55	3.05	0.85	1.15	1.45
2.	$W_2 = \mathfrak{v}_2$ (0.35, 0.45,0.55)	$G - W_2$	1.95	2.45	2.95	1.2	1.5	1.8
3.	$W_3 = \mathfrak{v}_3$ (0.4, 0.5,0.6)	$G - W_3$	1.9	2.4	2.9	0.9	1.1	1.3
4.	$W_4 = \mathfrak{v}_4$ (0.7, 0.8,0.9)	$G - W_4$	1.6	2.1	2.6	1.4	1.9	2.2
5.	$W_5 = \mathfrak{v}_5(0.15, 0.25,0.35)$	$G - W_5$	2.15	2.65	3.15	1.65	1.95	2.25
6.	$W_6 = \mathfrak{v}_6(0.45, 0.55,0.65)$	$G - W_6$	1.85	2.35	2.85	1.3	1.9	2.3

Table 4. Calculation of order and size of these vertex deletion 3-polar fuzzy outerplanar subgraphs

The vertex deleted 3-polar fuzzy outerplanar subgraphs of G are illustrated in Figures 16(a)-16(f). From Table 4, it can be observed that the subset W_5 corresponds to the maximum vertex deleted 3-polar fuzzy outerplanar subgraph of G.

Theorem 6.1. The maximum vertex deleted m-polar fuzzy outerplanar subgraph of a graph possesses an m-polar fuzzy dual graph, but the converse is not necessarily true.

Proof. It follows directly from Theorem 5.2 that a maximum vertex deleted m-polar fuzzy outerplanar subgraph has an m-polar fuzzy dual. However, the existence of an m-polar fuzzy dual does not guarantee that the corresponding subgraph is a maximum vertex deleted m-polar fuzzy outerplanar subgraph.

Definition 6.2. The maximal vertex sdeleted m-polar fuzzy outerplanar subgraph of G' is G. This is because, if a vertex deleted m-polar fuzzy outerplanar subgraph $G' = (\mathbb{V}', \tau', \delta')$ is induced from G, then any m-polar fuzzy subgraph of G induced by the vertex set $\mathbb{V}'' = \mathbb{V}' \cup \{\mathfrak{v}\}$ (where $\mathfrak{v} \in \mathbb{V} \backslash \mathbb{V}'$) (with $\mathfrak{v} \in \mathbb{V} \backslash \mathbb{V}'$ fails to satisfy the outerplanarity property.

Theorem 6.2. The m-polar fuzzy outerplanar subgraph obtained from each maximum vertex deletion is a maximal vertex deleted m-polar fuzzy outerplanar subgraph of G.
Proof. Let $G' \subseteq G$ denote a maximum vertex-deleted m-polar fuzzy outerplanar subgraph of G. Consider a nonempty subset $\mathbb{W} \subseteq \mathbb{V}$ representing the vertices deleted from G to form G'. Take any vertex $\mathfrak{u} \in \mathbb{W}$ and define
$G'' = G' \cup \{\mathfrak{u}\}$ where $G'' = \{x \in \mathbb{V}' \cup \frac{\{\mathfrak{u}\}}{\mu(x,y)} \neq \emptyset, x, y \in \mathbb{V} \backslash \mathbb{W} \cup \{\mathfrak{u}\}\}$.
Since G' is a maximum induced m-polar fuzzy outerplanar subgraph, adding any vertex $\mathfrak{u}$ from $\mathbb{W}$ would violate the outerplanarity property (i.e., $f_G \neq 1$.). Therefore, G' is also a maximal induced vertex-deleted m-polar fuzzy outerplanar subgraph of G. Hence, every maximum vertex deleted m-polar fuzzy outerplanar subgraph of G is also maximal.
Consequently, these m-polar fuzzy subgraphs are considered m-polar fuzzy outerplanar subgraphs with maximal vertex deletion of G. However, since m-polar fuzzy subgraphs can
Theorem 6.3. The statement of Theorem 6.2 does not necessarily hold.
Proof. Let G be an m-polar fuzzy graph with $\mathfrak{f}_G \neq 1$ and let $G'$and G'' be vertex deleted m-polar fuzzy outerplanar subgraphs of G. Let $\mathbb{W}_1, \mathbb{W}_2 \subseteq \mathbb{V}$ denote the sets of vertices deleted from G to obtain G' and G'' respectively.
Then G' and G'' are maximal m-polar fuzzy outerplanar subgraphs if adding any deleted vertex would result in a non-outerplanar graph, i.e., $G' \cup \{\mathfrak{u}\}$ with $\mathfrak{u} \in \mathbb{W}_1$ and $G'' \cup \{\mathfrak{v}\}$ with $\mathfrak{v} \in \mathbb{W}_2$ are m-polar fuzzy non-outerplanar. However, since m-polar fuzzy subgraphs can differ in order and size, a m-polar fuzzy outerplanar subgraph is defined as the one with the largest order and size. Therefore, while both G' and $G''$are maximal, only one (or neither) may qualify as maximum.m-polar fuzzy outerplanar subgraph that has the highest order and size. Thus, either G'or G'' qualifies as a maximal vertex deletion m-polar fuzzy outerplanar subgraph, but both m-polar fuzzy graphs are maximal. As a result, the m-polar fuzzy outerplanar subgraph of any maximal vertex deletion does not necessarily have to be a maximum vertex deletion m-polar fuzzy outerplanar subgraph.
Note 6.2. Only when both m-polar fuzzy subgraphs are maximum vertex Deletion m-polar fuzzy outerplanar subgraphs as defined by Note.6.1 are two Vertex Deletion m-polar fuzzy outerplanar subgraphs maximal.
Example 6.2. Let's look at the 3-polar fuzzy graph $G = (\mathbb{V}, \tau, \delta)$ shown in Fig. 17. The set of vertices in G is $\mathbb{V} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_5\}$ and their membership values of these are as follows: $\tau(\mathfrak{v}_1) = (0.35,0.45,0.55), \tau(\mathfrak{v}_2) = (0.45,0.55,0.65), \tau(\mathfrak{v}_3) = (0.5,0.6,0.7), \tau(\mathfrak{v}_4) = (0.7,0.8,0.9), \tau(\mathfrak{v}_5) = (0.15,0.25,0.35)$, and the edges $\mathbb{B}$ is $\delta(\mathfrak{v}_1,\mathfrak{v}_2) = \delta(\mathfrak{v}_2,\mathfrak{v}_1) = (0.1,0.2,0.3)$, $\delta(\mathfrak{v}_2,\mathfrak{v}_3) = \delta(\mathfrak{v}_3,\mathfrak{v}_2) = (0.2,0.3,0.1)$, $\delta(\mathfrak{v}_4,\mathfrak{v}_6) = (0.3,0.6,0.7)$, $\delta(\mathfrak{v}_5,\mathfrak{v}_6) = (0.7,0.8,0.9)$, $\delta(\mathfrak{v}_1,\mathfrak{v}_2) = (0.5,0.6,0.7)$, $\delta(\mathfrak{v}_3,\mathfrak{v}_4) = \delta(\mathfrak{v}_4,\mathfrak{v}_3) = (0.3,0.2,0.1)$, $\delta(\mathfrak{v}_1,\mathfrak{v}_4) = \delta(\mathfrak{v}_4,\mathfrak{v}_1) = (0.2,0.3,0.4)$, $\delta(\mathfrak{v}_1,\mathfrak{v}_5) = (0.15,0.25,0.35)$, $\delta(\mathfrak{v}_3,\mathfrak{v}_5) = (0.17,0.18,0.19)$, $\delta(\mathfrak{v}_2,\mathfrak{v}_5) = (0.7,0.8,0.9)$, $\delta(\mathfrak{v}_4,\mathfrak{v}_5) = (0.6,0.7,0.8)$.

All the graphs shown in Figures 17(a)-Fig 17(e) are maximal 3-polar fuzzy outerplanar subgraphs of G. However, Figure 17(a) represents the maximum 3-polar fuzzy outerplanar subgraph of G according to Table 5. This illustrates that a maximal 3-polar fuzzy outerplanar subgraph does not necessarily coincide with the maximum 3-polar fuzzy outerplanar subgraph of G.

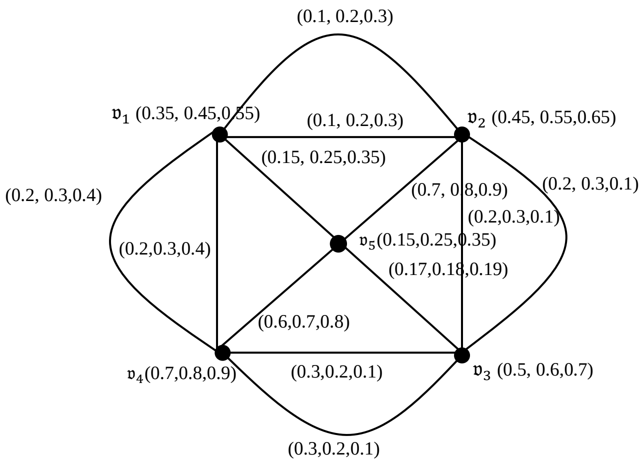

Figure. 17 Example for maximal Vertex Deletion 3-polar fuzzy graph

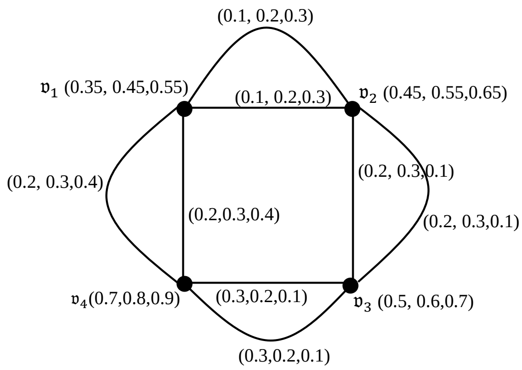

Figure. 17(a) $G - \mathfrak{v}_5$

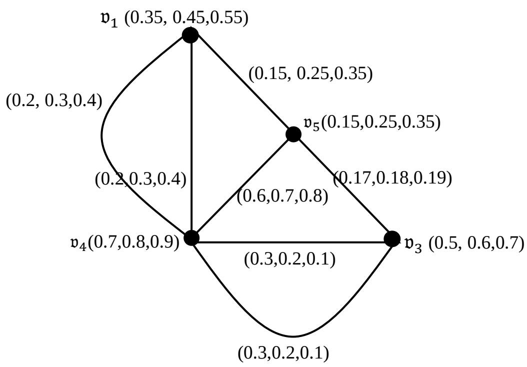

Figure. 17(b) $G - \mathfrak{v}_2$

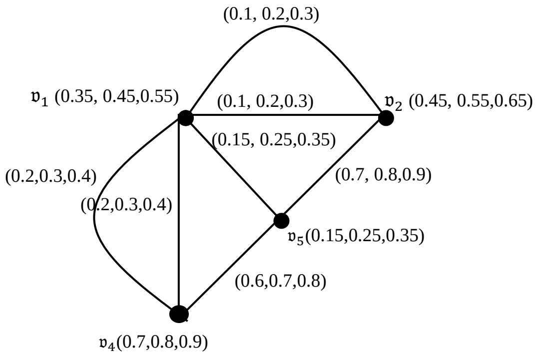

Figure. 17(c) $G - \mathfrak{v}_3$

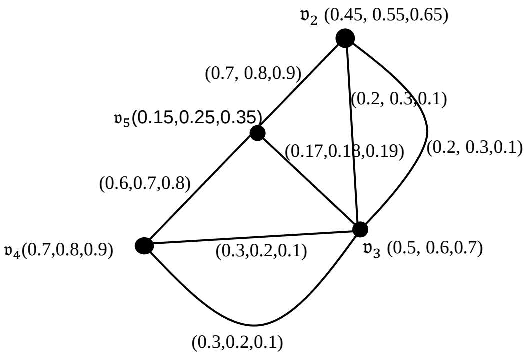

Figure. 17(d) $G - \mathfrak{v}_1$

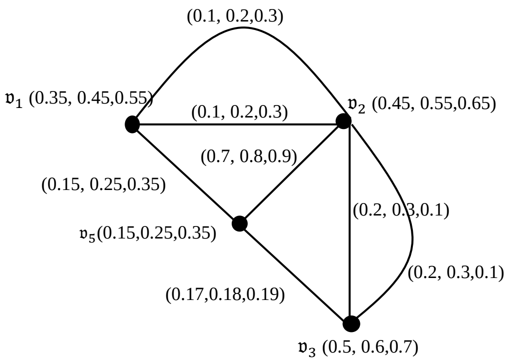

Figure. 17(e) $G - \mathfrak{v}_4$

Table 5. Calculation of order and size of these vertex deletion 3-polar fuzzy outerplanar

S.No	Figure	VD 3-Polar Fuzzy outerplanar subgraphs	Order			Size		
1.	Fig 17(a)	$G-\{\mathfrak{v}_1(0.35,0.45,0.55)\}$	1.8	2.2	2.6	0.75	1.25	1.75
2.	Fig 17(b)	$G-\{\mathfrak{v}_2\ (0.45, 0.55,0.65)\}$	1.7	2.1	2.5	1.3	1.8	1.7
3.	Fig 17(c)	$G-\{\mathfrak{v}_3\ (0.5, 0.6,0.7)\}$	1.65	2.05	2.45	1.17	1.18	0.59
4.	Fig 17(d)	$G-\{\mathfrak{v}_4\ (0.7,0.8,0.9)\}$	1.45	1.85	2.25	1.6	1.7	1.8
5.	Fig 17(e)	$G-\{\mathfrak{v}_5(0.15,0.25,0.35)\}$	2	2.4	2.8	1.62	1.93	2.24

subgraphs

7. Edge Deletion Multi Polar Fuzzy Outerplanar Subgraphs (ED-mPFOSs)

In this section, discussed the subgraph created by deleting specific edges from the m-polar fuzzy graph, while also defining m-polar fuzzy outerplanar graphs and providing relevant illustrations.

Definition 7.1. Let G be an m-polar fuzzy planar graph and let G' be an edge deleted subgraph of G. Then G' is called an edge deleted m-polar fuzzy outerplanar subgraph of G if and only if it preserves the m-polar fuzzy outerplanarity property.

Note 7.1. An edge deleted m-polar fuzzy outerplanar subgraph of G is not necessarily an edge deleted m-polar fuzzy subgraph of G.

Theorem 7.1. In an m-polar fuzzy outerplanar graph G, every edge deleted m-polar fuzzy subgraph is also an edge deleted m-polar fuzzy outerplanar subgraph of G.

Proof. Let G be an m-polar fuzzy outerplanar graph and let $\mathbb{H}$ be any edge deleted subgraph of G. Since all vertices of G lie on the outer boundary, removing edges does not change the positions of the vertices. Therefore, the subgraph $\mathbb{H}$ still preserves the outerplanarity property. Hence, every edge deleted m-polar fuzzy subgraph of G is indeed an edge deleted m-polar fuzzy outerplanar subgraph.

Theorem 7.2. Let $\mathbb{W}$ be a subset of edges of a connected m-polar fuzzy outerplanar graph G, with $\mathbb{W} \subseteq \mathbb{E}$. Then, G' is an edge deleted m-polar fuzzy outerplanar subgraph of G if the resulting graph preserves connectivity with respect to the m-polar fuzzy dual graph.

Proof. The result follows directly by applying the same reasoning as in Theorem 5.2, which similarly holds for the case of edges.

Example:7.1 Let's look at the 3-polar fuzzy graph $G = (\mathbb{V}, \tau, \delta)$ shown in Figure. 18. The set of vertices in G is $\mathbb{V} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_5\}$ and their membership values of these are as follows: $\tau(\mathfrak{v}_1) = (0.5,0.4,0.7), \tau(\mathfrak{v}_2) = (0.9,0.5,0.6), \tau(\mathfrak{v}_3) = (0.8,0.6,0.7), \tau(\mathfrak{v}_4) = (0.3,0.2,0.6), \tau(\mathfrak{v}_5) = (0.5,0.3,0.8), \tau(\mathfrak{v}_6) = (0.1,0.2,0.3), \tau(\mathfrak{v}_7) = (0.5,0.7,0.3), \tau(\mathfrak{v}_8) = (0.5,0.2,0.4), \tau(\mathfrak{v}_9) = (0.5,0.7,0.8)$ and the edges $\mathbb{B}$ is $\delta(\mathfrak{v}_1, \mathfrak{v}_2) = (0.15,0.14,0.17)$, $\delta(\mathfrak{v}_2, \mathfrak{v}_3) = (0.25,0.35,0.15)$, $\delta(\mathfrak{v}_3, \mathfrak{v}_4) = (0.7,0.8,0.9)$, $\delta(\mathfrak{v}_4, \mathfrak{v}_5) = (0.3,0.4,0.5)$, $\delta(\mathfrak{v}_1, \mathfrak{v}_4) = (0.1,0.2,0.3)$, $\delta(\mathfrak{v}_1, \mathfrak{v}_5) = (0.3,0.2,0.1)$, $\delta(\mathfrak{v}_2, \mathfrak{v}_5) = (0.2,0.3,0.4)$, $\delta(\mathfrak{v}_5, \mathfrak{v}_6) = (0.2,0.3,0.4)$, $\delta(\mathfrak{v}_6, \mathfrak{v}_7) = (0.15,0.25,0.35)$, $\delta(\mathfrak{v}_7, \mathfrak{v}_8) = (0.1,0.8,0.6)$, $\delta(\mathfrak{v}_8, \mathfrak{v}_9) = (0.7,0.8,0.9)$, $\delta(\mathfrak{v}_6, \mathfrak{v}_9) = (0.6,0.7,0.8)$, $(\mathfrak{v}_6, \mathfrak{v}_8) = (0.25,0.35,0.45)$.

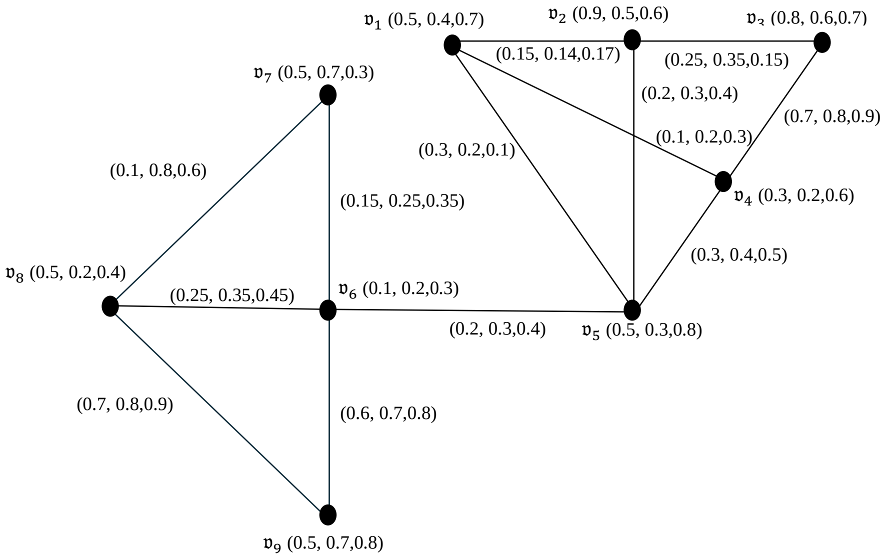

Figure. 18 3-polar Fuzzy Graph

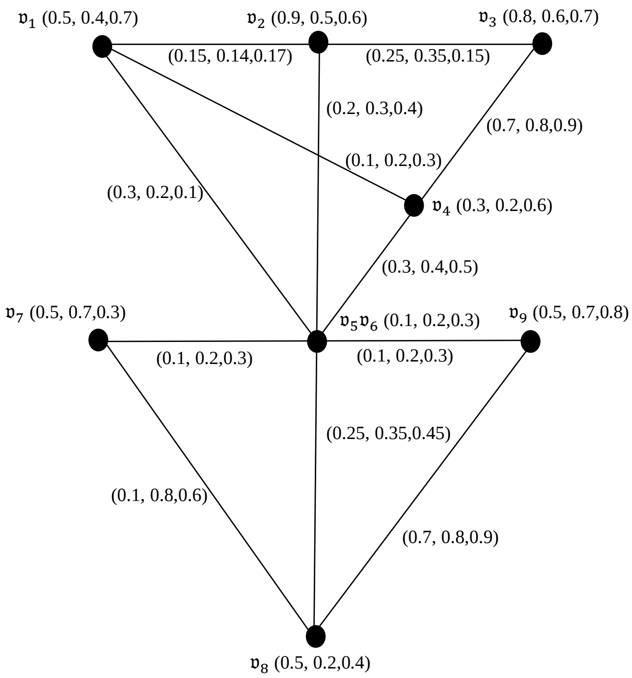

Figure. 18(a) 3-Polar Fuzzy Edge Contraction

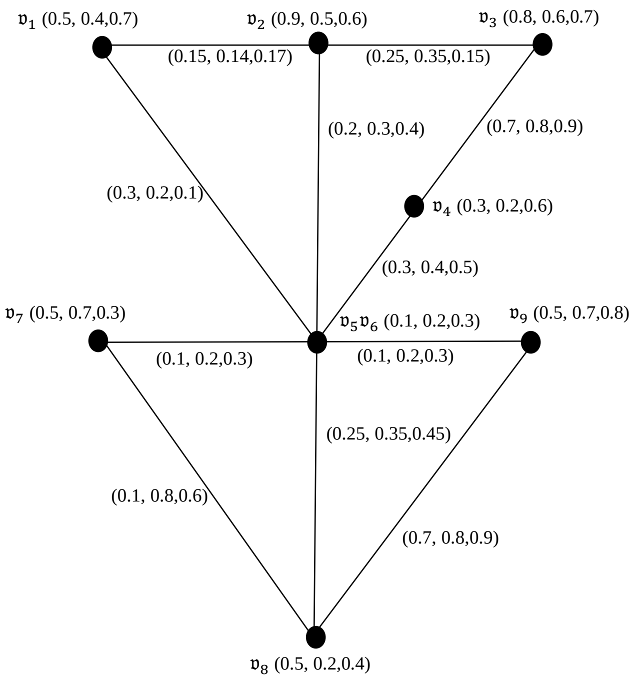

Figure. 18(b) Edge deleted 3-Polar fuzzy outerplanar graph

Let G be the 3-polar fuzzy graph. then the edge contracted 3-polar fuzzy graph, denoted as $G\backslash\mathfrak{v}_5\mathfrak{v}_6$ derived from the 3-polar fuzzy graph $G = (\sigma, \mu)$ by contracting the edge $\mathfrak{v}_5\mathfrak{v}_6$. In this Figure 18(b) the vertex set $V' = [V\backslash\{\mathfrak{v}_5, \mathfrak{v}_6\} \cup \{\mathfrak{v}_5\mathfrak{v}_6\}]$. The membership value σ of $G\backslash\mathfrak{v}_5\mathfrak{v}_6$ remains the same as σ in G for all vertices $x \in V,$ and the membership value of w is calculated by $\sigma(\mathfrak{v}_5\mathfrak{v}_6) = \wedge(\sigma(\mathfrak{v}_5), \sigma(\mathfrak{v}_6))$. The adjacency between two vertices x and y in $G\backslash\mathfrak{v}_5\mathfrak{v}_6$ is determined by definition 3.15 conditions.
The edge contracted 3-polar fuzzy graph $G\backslash\mathfrak{v}_5\mathfrak{v}_6$, represented as G_1 can be seen in Figure 18(a). Similarly, the 3-polar fuzzy graph $G_1 - \mathrm{E}$ represented as G_2 is shown in Figure 18(b). It can be noted that, is categorized as an edge deletion 3-polar fuzzy outerplanar graph.

8. Maximum and Maximal Edge Deletion Multi Polar Fuzzy Outerplanar Subgraphs (Maximum and Maximal ED-mPFOSs)

In this section, outlined the concepts of maximum and maximal edge deletion Multi polar fuzzy outerplanar subgraphs and investigates the relationship between these two concepts, illustrated with suitable examples.

Definition 8.1. Maximum Edge-Deleted (ED) m-Polar Fuzzy Outerplanar Subgraph:
Let $G' = (\mathbb{V}', \tau', \delta')$ be an edge-deleted m-polar fuzzy outerplanar subgraph of a non-outerplanar m-polar fuzzy graph $G = (\mathbb{V}, \tau, \delta)$, with size $\mathbb{S}^{\mathrm{m}}$. If there exists no other edge-deleted m-polar fuzzy outerplanar subgraph G'' of G with size $\mathbb{S}^{\mathrm{m}''}$, then $G'$is called the maximum edge deleted m-polar fuzzy outerplanar subgraph of G.

Note 8.1. Suppose two edges deleted m-polar fuzzy outerplanar subgraphs, G_1 and G_2 , are obtained from an m-polar fuzzy graph G. If their sizes are equal, but $G_1 \neq G_2$ (i.e., $\mathbb{S}^{\mathrm{m}''} > \mathbb{S}^{\mathrm{m}'}$), then both subgraphs are considered maximum edge deleted m-polar fuzzy outerplanar subgraphs of G. This implies that a unique maximum edge deleted m-polar fuzzy outerplanar subgraph of G does not necessarily exist.

Example 8.1. Let's look at the 3-polar fuzzy graph $G = (\mathbb{V}, \tau, \delta)$ shown in Figure. 19. The set of vertices in G is $\mathbb{V} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_5, \mathfrak{v}_6\}$ and their membership values of these are as follows: $\tau(\mathfrak{v}_1) = (0.3,0.4,0.5), \tau(\mathfrak{v}_2) = (0.5,0.6,0.7), \tau(\mathfrak{v}_3) = (0.25,0.35,0.45), \tau(\mathfrak{v}_4) = (0.6,0.7,0.9), \tau(\mathfrak{v}_5) = (0.2,0.6,0.8), \tau(\mathfrak{v}_6) = (0.7,0.8,0.9)$ and the edges $\mathbb{B}$ is $\delta(e_1) = (0.12,0.13,0.14)$, $\delta(e_2) = (0.6,0.7,0.9)$, $\delta(e_3) = (0.8,0.7,0.3)$, $\delta(e_4) = (0.2,0.3,0.7)$, $\delta(e_5) = (0.7,0.9,0.2)$, $\delta(e_6) = (0.1,0.2,0.3)$, $\delta(e_7) = (0.5,0.6,0.7)$, $\delta(e_8) = (0.6,0.7,0.8)$, $\delta(e_9) = (0.25,0.15,0.5)$, $\delta(e_{10}) = (0.1,0.3,0.5)$, $\delta(e_{11}) = (0.9,0.8,0.7)$, $\delta(e_{12}) = (0.75,0.85,0.95)$.

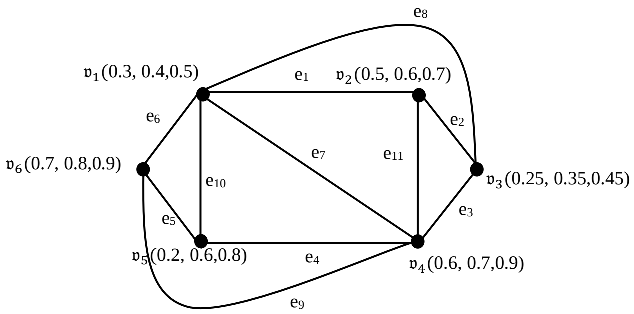

Figure. 19 3-polar fuzzy graph

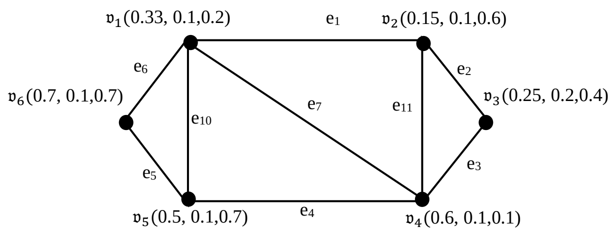

Figure. 19(a) Edge Deletion from a 3-polar fuzzy subgraph $\xi - e_8 e_9$

In Fig.19(a), the maximal Edge deleted 3-polar fuzzy outerplanar subgraph are shown. Thus, a result, $G - \mathbb{U}$ is the greatest Edge Deleted 3-polar fuzzy outerplanar subgraph of Gs, and 3-polar fuzzy subgraph is maximal.

9. Modelling One-Way Bypass Road Networks Using Multipolar Fuzzy Outerplanar Graphs

Multiple degrees of membership can be employed to model the traffic conditions between cities (represented as vertices) in an m-Polar fuzzy transportation network. These degrees of membership correspond to different influencing factors, including vehicle density, speed violations, street condition, driver attentiveness, and ongoing road construction. Traffic flow is disrupted and vehicles are redirected to alternate routes when an accident happens on a one-way road (a directed edge in the m-Polar fuzzy graph). The strain on nearby roadways is increased by these redirected flows, which may result in the formation of additional traffic jams. According to empirical research on these transportation systems, intersection zones areas where several traffic streams converge are where accidents most frequently happen. These intersections show vertices with substantial interaction across various edges with large fuzzy memberships under a number of criteria in m-Polar fuzzy terms.

The installation of overhead bridges or underpasses at these m-Polar high-intersection zones can successfully reduce conflict and risk in order to decrease accidents. Therefore, traffic networks may become safer and more effective if the criticality or polarity influence of such zones is lessened. A system with five vertices, where each vertex represents a city and the directed edges are one-way roads connecting them, can be used to demonstrate this idea. The multifactor traffic characteristics are encoded by the m-Polar fuzzy memberships on these edges. The final structure, shown in Figure 20, makes the flow dynamics easier to see and identifies the crucial crossing locations with the highest accident risks. We now use the framework of 3-polar fuzzy graphs to study the crowdness (traffic density) of streets connecting urban settlements. Due to a number of influencing elements, the crowdness of a street is viewed as a fuzzy value and uncertain measure in this context rather than being represented by a single deterministic value. Every edge in the graph, which symbolizes a route connecting two cities, has three membership degrees and reflects the traffic conditions in the present, the past, and the projected future.

The first membership represents the current traffic conditions, modeled as a fuzzy number derived from real-time data. The second membership reflects historical traffic density, utilizing data from the past year to form a baseline trend. The third membership signifies future congestion levels, expressed as an interval that takes into account uncertainties such as anticipated increases in vehicle counts, potential road enhancements, or the availability of alternative routes. In forecasting future traffic conditions, we consider factors like the consistency of road conditions, possible increases in vehicle loads, and whether new routes might alleviate pressure on existing roads. By assigning these three membership values to each edge in the network, we develop a comprehensive, time-sensitive evaluation of road congestion. The membership values associated with each road (edge) in the network provide a cohesive perspective on traffic dynamics, supporting urban planning and congestion management efforts through the m-polar fuzzy framework. This analysis is based on the one-way road network illustrated in Figure 20.

To assign membership values to the vertices in a 3-polar fuzzy graph, we first need to define the membership criteria. In this model, each vertex v is associated with a membership of the form

$$\tau(v)=(\tau(\mathfrak{v}_1), \tau(\mathfrak{v}_2), \tau(\mathfrak{v}_3))$$

where each component represents a distinct aspect of the vertices within the traffic network.

- The first component $\tau(\mathfrak{v}_1)$ denotes the present level of risk or current traffic density at the vertex. This reflects real-time congestion or pressure on that location.

- The second component $\tau(\mathfrak{v}_2)$ represents historical accident data, indicating how frequently mishaps have occurred at that junction in the past.

- The third component $\tau(\mathfrak{v}_3)$ corresponds to the estimated future congestion, which is based on projected traffic trends, ongoing development, or expected increases in vehicle flow are given in the Table. 6 and the 3- polar fuzzy edges given the Table.7.

Vertex	$\mathfrak{v}_1$	$\mathfrak{v}_2$	$\mathfrak{v}_3$	$\mathfrak{v}_4$	$\mathfrak{v}_5$
Memberships	(0.2, 0.4,0.6)	(0.25, 0.3,0.9)	(0.2, 0.5,0.8)	(0.35, 0.45,0.65)	(0.6, 0.8,0.9)

Table 6 Vertex membership values, as shown in Figure 20

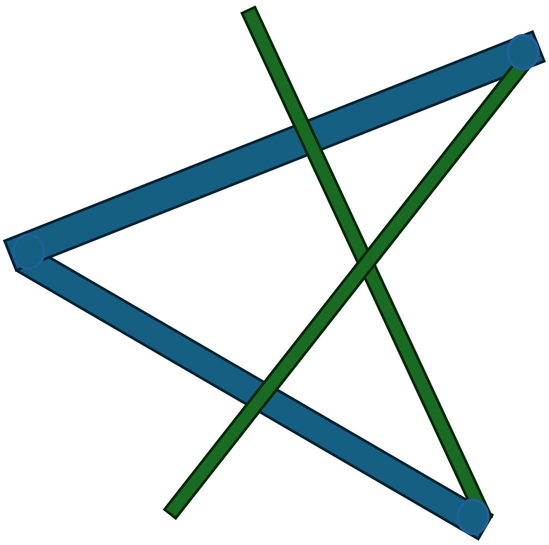

Figure 20: One-Way Bypass Road Network

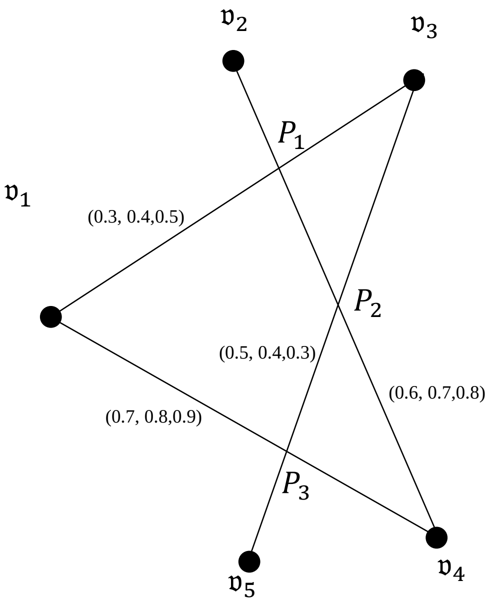

Figure: 21 Representing one way road network to 3-polar fuzzy graph

Roads	$(\mathfrak{v}_1, \mathfrak{v}_3)$	$(\mathfrak{v}_1, \mathfrak{v}_4)$	$(\mathfrak{v}_2, \mathfrak{v}_4)$	$(\mathfrak{v}_3, \mathfrak{v}_5)$
The directed edges represent one-way roads between them	(0.3, 0.4,0.5)	(0.7, 0.8,0.9)	(0.6, 0.7,0.8)	(0.5, 0.4,0.3)

Table 7 The road network's crowdness, as shown in Figure 20

Algorithm: Steps to Determine Planarity in a Road Network
Input: Consider a rough fuzzy road network of a developing country.
Step:1 Identify the direction of traffic flow in the network.
Step:2 Locate the intersection or crossing points in the traffic flow.
Step:3 Ignore the traffic streams that do not intersect with others.
Step:4 Calculate the intersecting value of each crossing point. (Using Definition 3.7)
Step:5 Determine the degree of planarity. (Using Definition 3.8)
Step:6 Apply edge contraction; if the resulting graph is outerplanar under 3-polar fuzzy conditions, it forms a 3-polar fuzzy outerplanar graph.
Output: Decision making: By constructing underpasses or flyovers would effectively reduce the concentration of traffic.

To simulate this situation, we provide each vertex a value of ([1,1], 1), which denotes complete intersection zone membership under the fuzzy set and the 3-Polar component. Each crossing point's highest impact on network risk and congestion is reflected in this assignment. We use a structured method as outlined in Algorithm to assess the degree of planarity of this transportation network, which is crucial for layout optimization and lowering intersection-related dangers. By methodically evaluating the 3-Polar fuzzy graph topology, this approach helps to minimize probable accident zones in the network by identifying and reducing non-planar crossings.
To calculate the intersection between the edges $(\mathfrak{v}_1\mathfrak{v}_3)$, by Definition 3.7,

$$I^m_{\mathfrak{v}_1\mathfrak{v}_3} = \frac{\delta^m(\mathfrak{v}_1, \mathfrak{v}_3)}{min\,\{\tau^m(\mathfrak{v}_1) \wedge \tau^m(\mathfrak{v}_3)\}}$$

$$I^{m=1,2,3}_{\mathfrak{v}_1\mathfrak{v}_3} = \frac{\delta^{1,2,3}(\mathfrak{v}_1, \mathfrak{v}_3)}{min\,\{\tau^{1,2,3}(\mathfrak{v}_1) \wedge \tau^{1,2,3}(\mathfrak{v}_3)\}}$$

$$= \left\{\frac{\delta^1(\mathfrak{v}_1, \mathfrak{v}_3)}{min\,\{\tau^1(\mathfrak{v}_1) \wedge \tau^1(\mathfrak{v}_3)\}}, \frac{\delta^2(\mathfrak{v}_1, \mathfrak{v}_3)}{min\,\{\tau^2(\mathfrak{v}_1) \wedge \tau^2(\mathfrak{v}_3)\}}, \frac{\delta^3(\mathfrak{v}_1, \mathfrak{v}_3)}{min\,\{\tau^3(\mathfrak{v}_1) \wedge \tau^3(\mathfrak{v}_3)\}}\right\}$$

$$= \left\{\frac{0.3}{1}, \frac{0.4}{1}, \frac{0.5}{1}\right\}$$

$$= (0.3, 0.4, 0.5)$$

Similarly, we can find $I^1_{\mathfrak{v}_1\mathfrak{v}_4}, I^2_{\mathfrak{v}_1\mathfrak{v}_4}, I^3_{\mathfrak{v}_1\mathfrak{v}_4}, I^1_{\mathfrak{v}_2\mathfrak{v}_4}, I^2_{\mathfrak{v}_2\mathfrak{v}_4}, I^3_{\mathfrak{v}_2\mathfrak{v}_4}, I^1_{\mathfrak{v}_3\mathfrak{v}_4}, I^2_{\mathfrak{v}_3\mathfrak{v}_4}, I^3_{\mathfrak{v}_3\mathfrak{v}_4}$.

$$I^m_{\mathfrak{v}_1\mathfrak{v}_4} = \frac{\delta^m(\mathfrak{v}_1, \mathfrak{v}_4)}{min\,\{\tau^m(\mathfrak{v}_1) \wedge \tau^m(\mathfrak{v}_4)\}}$$

$$I^m_{\mathfrak{v}_2\mathfrak{v}_4} = \frac{\delta^m(\mathfrak{v}_2, \mathfrak{v}_4)}{min\,\{\tau^m(\mathfrak{v}_2) \wedge \tau^m(\mathfrak{v}_4)\}}$$

$$I^m_{\mathfrak{v}_3\mathfrak{v}_5} = \frac{\delta^m(\mathfrak{v}_3, \mathfrak{v}_5)}{min\,\{\tau^m(\mathfrak{v}_3) \wedge \tau^m(\mathfrak{v}_5)\}}$$

We can see that it is the same as the edge value. Now we will calculate the cutting point.

$$I_{P_1} = \left\{\frac{I^1_{\mathfrak{v}_1\mathfrak{v}_3} + I^1_{\mathfrak{v}_2\mathfrak{v}_4}}{2}, \frac{I^2_{\mathfrak{v}_1\mathfrak{v}_3} + I^2_{\mathfrak{v}_2\mathfrak{v}_4}}{2}, \frac{I^3_{\mathfrak{v}_1\mathfrak{v}_3} + I^3_{\mathfrak{v}_2\mathfrak{v}_4}}{2}\right\}$$
$$= \left\{\frac{0.3 + 0.6}{2}, \frac{0.4 + 0.7}{2}, \frac{0.5 + 0.8}{2}\right\}$$

= (0.45,0.55,0.65)

$$I_{P_2} = \left\{\frac{I^1_{\mathfrak{v}_1\mathfrak{v}_4} + I^1_{\mathfrak{v}_3\mathfrak{v}_5}}{2}, \frac{I^2_{\mathfrak{v}_1\mathfrak{v}_4} + I^2_{\mathfrak{v}_3\mathfrak{v}_5}}{2}, \frac{I^3_{\mathfrak{v}_1\mathfrak{v}_4} + I^3_{\mathfrak{v}_3\mathfrak{v}_5}}{2}\right\}$$
$$= \left\{\frac{0.7 + 0.5}{2}, \frac{0.8 + 0.4}{2}, \frac{0.9 + 0.3}{2}\right\}$$

= (0.6, 0.6, 0.6)

$$I_{P_3} = \left\{\frac{I^1_{\mathfrak{v}_2\mathfrak{v}_4} + I^1_{\mathfrak{v}_3\mathfrak{v}_5}}{2}, \frac{I^2_{\mathfrak{v}_2\mathfrak{v}_4} + I^2_{\mathfrak{v}_3\mathfrak{v}_5}}{2}, \frac{I^3_{\mathfrak{v}_2\mathfrak{v}_4} + I^3_{\mathfrak{v}_3\mathfrak{v}_5}}{2}\right\}$$
$$= \left\{\frac{0.6 + 0.5}{2}, \frac{0.7 + 0.4}{2}, \frac{0.8 + 0.3}{2}\right\}$$

= (0.55, 0.55, 0.55)

The degree of planarity, by definition 3.8, is

$$\mathfrak{f}^1 = \frac{1}{1+0.45+0.6+0.55} = \frac{1}{1.55} = 0.38$$
$$\mathfrak{f}^2 = \frac{1}{1+0.55+0.6+0.55} = \frac{1}{1.95} = 0.37$$
$$\mathfrak{f}^3 = \frac{1}{1+0.65+0.6+0.55} = \frac{1}{2.4} = 0.35$$

We observe that the degree of planarity for the given road network is (0.35,0.37,0.35), which deviates significantly from the ideal balanced value of (0.5,0.5,0.5). This indicates that the graph exhibits a strong m-polar fuzzy structure. Let $G = (\mathbb{V}, \tau, \delta)$ be a 3-polar fuzzy graph, where each edge $(u, v) \in G$ is said to be m-polar fuzzy strong if for every $i = 1,2, \dots, m$, the membership value $I^i_{(u,v)} \geq 0.5$. Otherwise, the edge is classified as m-polar fuzzy weak.

Let G be the 3-Polar fuzzy graph in the Figure. 21. then the edge contracted 3-Polar fuzzy graph, denoted as $G\backslash\mathfrak{v}_3\mathfrak{v}_4$ derived from the 3-Polar fuzzy graph $G = (\sigma, \mu)$ by contracting the edge $\mathfrak{v}_3\mathfrak{v}_4$. In this Figure. 21 the vertex set $V' = [V\backslash\{\mathfrak{v}_3, \mathfrak{v}_4\} \cup \{\mathfrak{v}_3\mathfrak{v}_4\}]$. The membership value σ of $G\backslash\mathfrak{v}_3\mathfrak{v}_4$ remains the same as σ in $G\backslash$ for all vertices $x \in V,$ and the membership value of w is calculated by $\sigma(\mathfrak{v}_3\mathfrak{v}_4) = \wedge (\sigma(\mathfrak{v}_3), \sigma(\mathfrak{v}_4))$. The adjacency between two vertices x and y in $G\backslash\mathfrak{v}_3\mathfrak{v}_4$ is determined by Definition 3.15 conditions.
The edge contracted 3-Polar fuzzy graph $G\backslash\mathfrak{v}_3\mathfrak{v}_4$, represented as G_1 can be seen in Figure 21. Similarly, the 3-Polar fuzzy graph $G_1 - V$ represented as G_2 is shown in Figure 22. It's a 3-polar fuzzy outerplanar graph.
In this instance, the graph's strength indicates that several roads (edges) are closely connected and form a mob-like structure, particularly where they cross. The likelihood of accidents and traffic congestion are greatly increased at these intersections. Thus, limiting or eliminating these intersections for example, by building flyovers or underpasses would successfully lessen traffic congestion and, as a result, the accident rate. Furthermore, according to our analysis, traffic patterns are likely to continue unless structural adjustments are made because the future crowd density that is, the number of vehicles on the roads is predicted to stay nearly identical to the current level. Therefore, to improve traffic flow and road safety, planned interventions at key crossings are crucial.

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Declarations

Conflict of interest The authors declare no conflicts of interest.

Ethical approval This study did not involve human or animal subjects.

Consent for publication All the authors agree with this manuscript's publication.

10. Conclusion

With a focus on m-polar fuzzy outerplanar graphs (mPFOGs), this study concludes by highlighting a number of significant distinctions between the growing subject of fuzzy graph theory and conventional crisp graph planarity. By deliberately removing vertices or edges from m-polar fuzzy outerplanar subgraphs (mPFOSs), the research examines how these subgraphs evolve. We investigate both vertex and edge deletions to analyze the maximal and maximum mPFOSs through various scenarios. Moreover, the introduction of m-polar fuzzy dual graphs enriches the discussion by showing the significant connections between these dual structures and the related mPFOGs. A key practical application of this model is in the design and optimization of one-way bypass road networks in urban and semi-urban areas, where minimizing traffic intersections and enhancing traffic flow are crucial. This research broadens our understanding of fuzzy graph properties and paves the way for further exploration in this fascinating domain.

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