New subclasses of bi-univalent functions related with Hurwitz–Lerch Zeta function of complex order

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Abstract

In this article, the authors use the Hurwitz–Lerch zeta function associated with the subclasses of bi-univalent functions. Three new subclasses of bi-univalent functions are introduced. For these new classes, the authors obtain first two initial coefficient bounds. Furthermore, the famous Fekete-Szegö inequality is also drive for this new subclass of functions. Some findings improved results already available in the literature.

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1. Introduction

Indicate $\mathcal{A}$ be the class of all functions $f : \mathbb{E} \to \mathbb{C}$ defined by

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n \tag{1.1}$$

which are analytic in open unit disk $\mathbb{E} := \{\xi \in \mathbb{C} : |\xi| < 1\}$. Indicate $\mathcal{S}$ be the subclass of $\mathcal{A}$ which are univalent in $\mathbb{E}$. Fix $0 \leq \delta < 1$. Let $\mathcal{S}^*(\delta)$, $\mathcal{C}(\delta)$ and $\mathcal{R}(\delta)$ be the subclasses of class $\mathcal{S}$ which is well known the class of starlike, convex and the class of functions whose derivatives have positive real part of order δ respectively. The analytic descriptions of the above classes are given by

$$\mathcal{S}^*(\delta)=\left\{f\in\mathcal{S}:\Re\left(\frac{\xi f'(\xi)}{f(\xi)}\right)>\delta,0\leq\delta<1\right\},$$

$$\mathrm{C}(\delta)=\left\{f\in\mathcal{S}:\Re\left(1+\frac{\xi f''(\xi)}{f'(\xi)}\right)>\delta,0\leq\delta<1\right\}$$

and

$$\mathcal{R}(\delta)=\{\mathrm{f}\in\mathcal{S}:\ \Re(\mathrm{f}'(\xi))>\delta,0\leq\delta<1\}.$$

The Koebe one-quarter theorem [11] ensures the range of every function of the class $\mathcal{S}$ contains the disc of radius 1/ 4. Hence, every univalent function $f\ \in\ \mathcal{S}$ has an inverse f^{-1} which is introduced by

$$f^{-1}\big(f(\xi)\big)=\xi,\qquad \xi\in\mathbb{E}$$

and

$$f\big(f^{-1}(\zeta)\big)=\zeta,\ \ \zeta\in\left(|\zeta|<\frac{1}{4}\right).$$

The inverse function $h\equiv f^{-1}$ is given by the power series

$$h(\zeta)\equiv\ f^{-1}(\zeta)=\zeta-a_2\zeta^2+(2a_2^2-a_3)\zeta^2+(5a_2^3-5a_2a_3+a_4)\zeta^4+\cdots. \tag{1.2}$$

A function $f\in\mathcal{A}$ is said to be bi-univalent in $\mathbb{E}$ if f and it's inverse $h\equiv f^{-1}$ are univalent in $\mathbb{E}$. Let Σ be the class of all bi-univalent functions in $\mathbb{E}$. In 1985 Louis de Branges [7] show that the Bieberbach Conjecture which states that, for each $f\in\mathcal{S}$ given by (1.1) then following inequality holds

$$|a_n|\leq\ n,n\in\mathbb{N}\backslash\{1\}.$$

Brannan and Clunie [4] conjectured that $|a_2\ |\leq\sqrt{2}$. Further in 1981, Styer and Wright [26] observed that there exist functions in Σ for which $|a_2|\ >\ 4/3$ and in 1984 Tan [27] proved that $|a_2|\leq\ 1.485$. For more details, see [1, 6, 17, 18, 28].

For two holomorphic functions $f_1,f_2\in\ \mathcal{A}$ and their Taylor series are given by

$$f_1(\xi)=\xi+\sum_{n=2}^{\infty}A_n\xi^n\ \ and\ f_2(\xi)=\xi+\sum_{n=2}^{\infty}B_n\xi^n,$$

then the function $f_1*f_2:\ \mathbb{E}\ \rightarrow\ \mathbb{C}$ defined by

$$(f_1 * f_2)(\xi) = f_1(\xi) * f_2(\xi) = \xi + \sum_{n=2}^{\infty} A_n B_n \xi^n$$

is called the convolution of f_1 and f_2. The function $f_1 * f_2$ belongs to $\mathcal{A}$ and it is also called the Hadamard product of f_1 and f_2.

1.1 **Hurwitz–Lerch Zeta function.** Hurwitz–Lerch Zeta function [22,16] plays an important role in various fields of mathematical sciences, especially in geometric function theory. A general Hurwitz–Lerch Zeta function $\Psi(\xi,\lambda,b)$ defined in [24] by

$$\Psi(\xi,\lambda,b) := \sum_{n=2}^{\infty} \frac{\xi^n}{(n+b)^{\lambda}}$$

where $b \in C \setminus \{B_0\}, \lambda \in \mathbb{C}, \Re(\lambda) > 1$ and $B_0 := \mathbb{Z} \setminus \{\mathbb{N}\}$.

Now we consider the normalized function

$$£(\xi,\lambda,\vartheta) = (1+n)^{\vartheta}\left[\Psi(\xi,\lambda,b) - \frac{1}{\lambda^n}\right] = \xi + \sum_{n=2}^{\infty}\left(\frac{1+\lambda}{n+\lambda}\right)^{\vartheta}\xi^n.$$

Motivated by the investigation of operators we introduced an operator $\Lambda_{\lambda}^{\vartheta}: \mathcal{A} \to \mathcal{A}$ defined by

$$\Lambda_{\lambda}^{\vartheta} f(\xi) = £(\xi,\lambda,\vartheta) * f(\xi) = \xi + \sum_{n=2}^{\infty}\left(\frac{1+\lambda}{n+\lambda}\right)^{\vartheta} a_n \xi^n.$$

Hence, we have

$$\Lambda_{\lambda}^{\vartheta} f(\xi) = \xi + \sum_{n=2}^{\infty} \Phi_{\mathrm{n}}(\lambda,\vartheta) a_n \xi^n,$$

where

$$\Phi_{\mathrm{n}}(\lambda,\vartheta) = \left(\frac{1+\lambda}{n+\lambda}\right)^{\vartheta}.$$

Remark 1. For the choice of parameters λ and ϑ we have:

(i) If $\vartheta = 0$ then

$$\Lambda_{\vartheta}^{0} f(\xi) \equiv f(\xi).$$

(ii) If $\vartheta = 1$ and $\lambda = 0$ then

$$\Lambda_0^1 f(\xi) \equiv \int_0^{\xi} \frac{f(u)}{u} du = \mathcal{L}f(\xi) = \xi + \sum_{n=2}^{\infty} \frac{a_n}{n} \xi^n$$

is well known as Alexander operator [2].

(iii) If $\vartheta = 1$ and $\lambda = b$, $(b > -1)$ then

$$\Lambda_b^1 f(\xi) \equiv \mathcal{F}_b f(\xi) = \frac{1+b}{\xi^b} \int_0^{\xi} u^{b-1} f(u) du = \xi + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right) a_n \xi^n$$

is well known as Bernardi operator [3].

(iv) If $\vartheta = \eta, (\eta > 0)$ and $\lambda = 1$ then

$$\Lambda_1^{\eta} f(\xi) \equiv \mathcal{J}^{\eta} f(\xi) = \xi + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^{\eta} a_n \xi^n$$

is well known as Jung–Kim–Srivastava integral operator [10].

Let $\mathcal{P}_{\mu}(c, \delta)$ be the class of analytic functions $x(\xi)$ analytic in $\mathbb{E}$ normalized by the conditions $x(0) = 1$ and $x(0) > 1$ and

$$\int_0^{2\pi} \left| \Re \left\{ \frac{1 + \frac{1}{c}[x(\xi) - 1] - \delta}{1 - \delta} \right\} \right| d\theta \leq 2\pi,$$

where $\mu \geq 2, 0 \leq \delta < 1$ and $c \in \mathbb{C} \backslash \{0\}$. $\mathcal{P}_{\mu}(c, \delta)$ was introduced in [13]. A function $x \in \mathcal{P}_{\mu}(c, \delta)$ then there exists two analytic functions $x_1(\xi)$ and $x_2(\xi)$ belongs to the class $\mathcal{P}(c, \delta)$ such that

$$x(\xi) = \frac{\mu}{4}[x_1(\xi) - x_2(\xi)] + \frac{1}{2}[x_1(\xi) + x_2(\xi)].$$

Remark 2. For the choice of parameters μ, δ and ϑ we have:

(i) If $c = 1$ then, $\mathcal{P}_{\mu}(1, \delta) \equiv \mathcal{P}_{\mu}(\delta)$ was introduced in [14].

(ii) If $c = 1$ and $\delta = 0$ then, $\mathcal{P}_{\mu}(1, 0) \equiv \mathcal{P}_{\mu}$ was introduced in [15].

(iii) If $c = 1, \delta = 0$ and $\mu = 2$ then, $\mathcal{P}_2(1, 0) \equiv \mathcal{P}$ is the well – known class of Caratheodory functions of positive real part.

Lemma 1. [13] A function $x \in \mathcal{P}_{\mu}(c, \delta)$ given in the form

$$x(\xi) = 1 + \sum_{n=1}^{\infty} x_n \xi^n = 1 + x_1\xi + x_2\xi^2 + x_3\xi^3 + \cdots, \qquad \xi \in \mathbb{E},$$

then

$$|x_n| \leq \mu|c|(1-\delta), \qquad \forall\ n \in \mathbb{N}.$$

This result is sharp.

In this article, the authors use the hurwitz–lerch zeta function associated with the subclasses of bi-univalent functions. We introduce three new subclasses of bi-univalent functions. For these new classes, the authors obtain first two initial coefficient bounds. Furthermore, the famous Fekete-Szegö inequality are also drive for this new subclass of functions. Some findings improved results already available in the literature.

2. Main Definition

Definition 1. For fixed $2 \leq \mu \leq 4$, $c \in \mathbb{C} \setminus \{0\}, \lambda \in \mathbb{C}, \alpha, \upsilon \geq 0$ and $0 \leq \delta < 1$. We say that the function $f \in \Sigma$ is in the class $\mathcal{R}_{\Sigma,\lambda,\alpha}^{\vartheta,\upsilon}(c,\mu,\delta)$ if the following conditions hold:

$$\alpha\xi\Lambda_\lambda^\vartheta f''(\xi) + (\upsilon - 2\alpha)\Lambda_\lambda^\vartheta f'(\xi) + (1 - \upsilon + 2\alpha)\frac{\Lambda_\lambda^\vartheta f(\xi)}{\xi} \in \mathcal{P}_\mu(c,\delta)$$

and

$$\alpha\zeta\Lambda_\lambda^\vartheta h''(\zeta) + (\upsilon - 2\alpha)\Lambda_\lambda^\vartheta h'(\zeta) + (1 - \upsilon + 2\alpha)\frac{\Lambda_\lambda^\vartheta h(\zeta)}{\zeta} \in \mathcal{P}_\mu(c,\delta)$$

where h is the inverse function of f given in (1.2).

Remark 3. For the choice of parameters in Definition 1 we have the following cases:

(i) If $\vartheta = 0, \upsilon = 1 + 2\alpha$ and $c = 1$ then $\mathcal{R}_{\Sigma,\lambda,\alpha}^{0,1+2\alpha}(1,\mu,\delta) \equiv \mathcal{F}_\Sigma^\alpha(\mu,\delta)$ is the class of functions hold the following conditions:

$$f'(\xi) + \alpha\xi f''(\xi) \in \mathcal{P}_\mu(\delta)$$

and

$$h'(\zeta) + \alpha\zeta h''(\zeta) \in \mathcal{P}_\mu(\delta).$$

The family $\mathcal{F}_\Sigma^\alpha(\mu,\delta)$ was initiated by Sharma [20] in 2023.

(ii) If $\vartheta = 0, \upsilon = 1 + 2\alpha, \mu = 2$ and $c = 1$ then $\mathcal{R}_{\Sigma,\lambda,\alpha}^{0,1+2\alpha}(1,2,\delta) \equiv \mathcal{F}_\Sigma^\alpha(\delta)$ is the class of functions hold the following conditions:

$$\Re(f'(\xi) + \alpha\xi f''(\xi)) > \delta$$

and

$$\Re\big(h'(\zeta) + \alpha\zeta h''(\zeta)\big) > \delta.$$

The family $\mathcal{F}_{\Sigma}^{\alpha}(\delta)$ was initiated by Srivastava [23] in 2015.

(iii) If $\vartheta = 0, \alpha = 0$ and $c = 1$ then $\mathcal{R}_{\Sigma,\lambda,0}^{0,\upsilon}(1,\mu,\delta) \equiv \mathcal{G}_{\Sigma}^{\upsilon}(\mu,\delta)$ is the class of functions hold the following conditions:

$$\upsilon f'(\xi) + (1-\upsilon)\frac{f(\xi)}{\xi} \in \mathcal{P}_{\mu}(\delta)$$

and

$$\upsilon h'(\zeta) + (1-\upsilon)\frac{h(\zeta)}{\zeta} \in \mathcal{P}_{\mu}(\delta).$$

The family $\mathcal{G}_{\Sigma}^{\upsilon}(\mu,\delta)$ was initiated by Sharma [20] in 2023.

(iv) If $\vartheta = 0, \alpha = 0, \mu = 2$ and $c = 1$ then $\mathcal{R}_{\Sigma,\lambda,0}^{0,\upsilon}(1,2,\delta) \equiv \mathcal{G}_{\Sigma}^{\upsilon}(\delta)$ is the class of functions hold the following conditions:

$$\Re\left(\upsilon f'(\xi) + (1-\upsilon)\frac{f(\xi)}{\xi}\right) > \delta$$

and

$$\Re\left(\upsilon h'(\zeta) + (1-\upsilon)\frac{h(\zeta)}{\zeta}\right) > \delta.$$

The family $\mathcal{G}_{\Sigma}^{\upsilon}(\delta)$ was initiated by Frasin and Aouf [9] in 2011.

(v) If $\vartheta = 0, \alpha = 0, \upsilon = 1$ and $c = 1$ then then $\mathcal{R}_{\Sigma,\lambda,0}^{0,1}(1,\mu,\delta) \equiv \mathcal{H}_{\Sigma}(\mu,\delta)$ is the class of functions hold the following conditions:

$$f'(\xi) \in \mathcal{P}_{\mu}(\delta)$$

and

$$h'(\zeta) \in \mathcal{P}_{\mu}(\delta).$$

The family $\mathcal{H}_{\Sigma}(\mu,\delta)$ was initiated by Li [12] in 2020.

(vi) If $\vartheta = 0, \alpha = 0, \upsilon = 1, \mu = 2$ and $c = 1$ then then $\mathcal{R}^{0,1}_{\Sigma,\lambda,0}(1,2,\delta) \equiv \mathcal{H}_\Sigma(\delta)$ is the class of functions hold the following conditions:

$$\Re\big(f'(\xi)\big) > \delta$$

and

$$\Re\big(h'(\zeta)\big) > \delta.$$

The family $\mathcal{H}_\Sigma(\delta)$ was initiated by *Srivastava* [25] in 2020.

Definition 2. For fixed $2 \leq \mu \leq 4, c \in \mathbb{C} \setminus \{0\}, \lambda \in \mathbb{C}, \eta \geq 0$ and $0 \leq \delta < 1$. We say that the function $f \in \Sigma$ is in the class $\mathcal{S}^{\vartheta,\eta}_{\Sigma,\lambda}(c,\mu,\delta)$ if the following conditions hold:

$$\frac{\xi \Lambda^{\vartheta}_{\lambda} f'(\xi)}{\Lambda^{\vartheta}_{\lambda} f(\xi)} + \eta \xi \Lambda^{\vartheta}_{\lambda} f''(\xi) \in \mathcal{P}_\mu(c,\delta)$$

and

$$\frac{\zeta \Lambda^{\vartheta}_{\lambda} h'(\zeta)}{\Lambda^{\vartheta}_{\lambda} h(\zeta)} + \eta \zeta \Lambda^{\vartheta}_{\lambda} h''(\zeta) \in \mathcal{P}_\mu(c,\delta)$$

where h is the inverse function of f given in (1.2).

Remark 4. For the choice of parameters in Definition 2 we have the following cases:

(i) If $\vartheta = 0, \eta = 0$ and $c = 1$ then $\mathcal{S}^{0,0}_{\Sigma,\lambda}(1,\mu,\delta) \equiv \mathcal{S}^*_\Sigma(\mu,\delta)$ is the class of functions hold the following conditions:

$$\frac{\xi f'(\xi)}{f(\xi)} \in \mathcal{P}_\mu(\delta)$$

and

$$\frac{\zeta h'(\zeta)}{h(\zeta)} \in \mathcal{P}_\mu(\delta).$$

The family $\mathcal{S}^*_\Sigma(\mu,\delta)$ was initiated by Li [12] in 2020.

(ii) If $\vartheta = 0, \eta = 0, \mu = 2$ and $c = 1$ then $\mathcal{S}^{0,0}_{\Sigma,\lambda}(1,2,\delta) \equiv \mathcal{S}^*_\Sigma(\delta)$ is the class of functions hold the following conditions:

$$\Re\left(\frac{\xi f'(\xi)}{f(\xi)}\right) > \delta$$

and

$$\Re\left(\frac{\zeta h'(\zeta)}{h(\zeta)}\right) > \delta.$$

The family $\mathcal{S}_{\Sigma}^{*}(\delta)$ was initiated by Brannan and Taha [5] in 1985.

Definition 3. For fixed $2 \leq \mu \leq 4, c \in \mathbb{C} \setminus \{0\}, \lambda \in \mathbb{C}, 0 \leq \varkappa \leq 1$ and $0 \leq \delta < 1$. We say that the function $f \in \Sigma$ is in the class $\mathcal{C}_{\Sigma,\lambda}^{\vartheta,\varkappa}(c, \mu, \delta)$ if the following conditions hold:

$$(1-\varkappa)\Lambda_{\lambda}^{\vartheta} f'(\xi) + \varkappa\left(1 + \frac{\xi \Lambda_{\lambda}^{\vartheta} f''(\xi)}{\Lambda_{\lambda}^{\vartheta} f'(\xi)}\right) \in \mathcal{P}_{\mu}(c, \delta)$$

and

$$(1-\varkappa)\Lambda_{\lambda}^{\vartheta} h'(\zeta) + \varkappa\left(1 + \frac{\zeta \Lambda_{\lambda}^{\vartheta} h''(\zeta)}{\Lambda_{\lambda}^{\vartheta} h'(\zeta)}\right) \in \mathcal{P}_{\mu}(c, \delta)$$

where h is the inverse function of f given in (1.2).

Remark 5. For the choice of parameters in Definition 3 we have the following cases:

(i) If $\vartheta = 0,\ \varkappa = 1$ and $c = 1$ then $\mathcal{C}_{\Sigma,\lambda}^{0,1}(1, \mu, \delta) \equiv \mathcal{C}_{\Sigma}(\mu, \delta)$ is the class of functions hold the following conditions:

$$1 + \frac{\xi f''(\xi)}{f'(\xi)} \in \mathcal{P}_{\mu}(\delta)$$

and

$$1 + \frac{\zeta h''(\zeta)}{h'(\zeta)} \in \mathcal{P}_{\mu}(\delta).$$

The family $\mathcal{C}_{\Sigma}(\mu, \delta)$ was initiated by Li [12] in 2020.

(ii) If $\vartheta = 0, \varkappa = 1, \mu = 2$ and $c = 1$ then $\mathcal{C}_{\Sigma,\lambda}^{0,1}(1, 2, \delta) \equiv \mathcal{C}_{\Sigma}(\delta)$ is the class of functions hold the following conditions:

$$\Re\left(1 + \frac{\xi f''(\xi)}{f'(\xi)}\right) > \delta$$

and

$$\Re\left(1+\frac{\zeta h''(\zeta)}{h'(\zeta)}\right)>\delta.$$

The family $\mathcal{C}_{\Sigma}(\delta)$ was initiated by Brannan and Taha [5] in 1985.

(iii) If $\vartheta = 0, \varkappa = 0$ and $c = 1$ then $\mathcal{C}_{\Sigma,\lambda}^{0,0}(1,\mu,\delta) \equiv \mathcal{H}_{\Sigma}(\mu,\delta)$ is the class of functions hold the following conditions:

$$f'(\xi) \in \mathcal{P}_{\mu}(\delta)$$

and

$$h'(\zeta) \in \mathcal{P}_{\mu}(\delta).$$

The family $\mathcal{H}_{\Sigma}(\mu,\delta)$ was initiated by Li [12] in 2020.

(iv) If $\vartheta = 0, \varkappa = 0, \mu = 2$ and $c = 1$ then $\mathcal{C}_{\Sigma,\lambda}^{0,0}(1,2,\delta) \equiv \mathcal{H}_{\Sigma}(\delta)$ is the class of functions hold the following conditions:

$$\Re\big(f'(\xi)\big) > \delta$$

and

$$\Re\big(h'(\zeta)\big) > \delta.$$

The family $\mathcal{H}_{\Sigma}(\delta)$ was initiated by *Srivastava* [25] in 2020.

3. Main Theorems

Theorem 1. For fixed $2 \le \mu \le 4, c \in \mathbb{C}\setminus\{0\}, \lambda \in \mathbb{C}, \alpha, \upsilon \ge 0$ and $0 \le \delta < 1$. A function $f \in \mathcal{R}_{\Sigma,\lambda,\alpha}^{\vartheta,\upsilon}(c,\mu,\delta)$ then

$$|a_2| \le \sqrt{\frac{\mu|c|(1-\delta)}{\Phi_3(\lambda,\vartheta)[2(\alpha+\upsilon)+1]}}, \tag{3.1}$$

$$|a_3| \le \frac{\mu|c|(1-\delta)}{\Phi_3(\lambda,\vartheta)[2(\alpha+\upsilon)+1]} \tag{3.2}$$

and

$$|a_3 - \aleph\, a_2^2| \le \begin{cases} \dfrac{\mu|c|(1-\delta)(1-\aleph)}{\Phi_3(\lambda,\vartheta)[2(\alpha+v)+1]}, & \aleph \le 0 \\ \dfrac{\mu|c|(1-\delta)}{\Phi_3(\lambda,\vartheta)[2(\alpha+v)+1]}, & 0 \le \aleph \le 2 \\ \dfrac{\mu|c|(1-\delta)(\aleph-1)}{\Phi_3(\lambda,\vartheta)[2(\alpha+v)+1]}, & \aleph \ge 2 \end{cases} \tag{3.3}$$

Proof. Consider $f \in \mathcal{R}_{\Sigma,\lambda,\alpha}^{\vartheta,v}(c,\mu,\delta)$. Then, there exist two analytic functions $x(\xi)$ and $y(\zeta)$ belongs to the class $\mathcal{P}_{\mu}(c,\delta)$ such that

$$\alpha\xi\Lambda_\lambda^\vartheta f''(\xi) + (v-2\alpha)\Lambda_\lambda^\vartheta f'(\xi) + (1-v+2\alpha)\frac{\Lambda_\lambda^\vartheta f(\xi)}{\xi} = x(\xi) \tag{3.4}$$

and

$$\alpha\zeta\Lambda_\lambda^\vartheta h''(\zeta) + (v-2\alpha)\Lambda_\lambda^\vartheta h'(\zeta) + (1-v+2\alpha)\frac{\Lambda_\lambda^\vartheta h(\zeta)}{\zeta} = y(\zeta) \tag{3.5}$$

where

$$x(\xi) = 1 + \sum_{n=1}^{\infty} x_n \xi^n = 1 + x_1\xi + x_2\,\xi^2 + x_3\xi^3 + \cdots \tag{3.6}$$

and

$$y(\zeta) = 1 + \sum_{n=1}^{\infty} y_n \zeta^n = 1 + y_1\zeta + y_2\,\zeta^2 + y_3\zeta^3 + \cdots. \tag{3.7}$$

Now using (3.4), (3.5), (3.6) and (3.7), we get

$$(v+1)\Phi_2(\lambda,\vartheta)a_2 = x_1, \tag{3.8}$$

$$[2(\alpha+v)+1]\Phi_3(\lambda,\vartheta)a_3 = x_2, \tag{3.9}$$

$$-(v+1)\Phi_2(\lambda,\vartheta)a_2 = y_1, \tag{3.10}$$

and

$$[4(\alpha+v)+2]\Phi_3(\lambda,\vartheta)a_2^2 - [2(\alpha+v)+1]\Phi_3(\lambda,\vartheta)a_3 = y_2. \tag{3.11}$$

On addition of (3.8) and (3.10) we have $x_1 = -y_1$ and similarly adding (3.9) and (3.11), we get

$$[4(\alpha+v)+2]\Phi_3\,(\lambda,\vartheta)a_2^2 = x_2 + y_2. \tag{3.12}$$

Using Lemma 1 in (3.12), we get

$$|a_2|^2 \leq \frac{\mu|c|(1-\delta)}{[4(\alpha+\upsilon)+2]\Phi_3(\lambda,\vartheta)}. \tag{3.13}$$

Equation (3.13) gives the bound of $|a_2|$ given in (3.1). Similarly Using Lemma 1 in (3.9) gives the bound of $|a_3|$ given in (3.2). Hence, for any fixed real number ℵ and from (3.9) and (3.12), we get

$$a_3 - \aleph\, a_2^2 = \frac{(2-\aleph)x_2 - \aleph\, y_2}{[4(\alpha+\upsilon)+2]\Phi_3(\lambda,\vartheta)}. \tag{3.14}$$

Using Lemma 1 in (3.14), we get

$$|a_3 - \aleph\, a_2^2| \leq \frac{(2-\aleph)x_2 - \aleph\, y_2}{[4(\alpha+\upsilon)+2]\Phi_3(\lambda,\vartheta)}. \tag{3.15}$$

Equation (3.15) gives the bounds given in (3.3). Which completes the proof of Theorem 1.

Theorem 2. For fixed $2 \leq \mu \leq 4, c \in \mathbb{C}\setminus\{0\}, \lambda \in \mathbb{C}, \eta \geq 0$ and $0 \leq \delta < 1$. A function $f \in \mathcal{S}_{\Sigma,\lambda}^{\vartheta,\eta}(c,\mu,\delta)$ then

$$|\mathrm{a}_2| \leq \sqrt{\frac{\mu|c|(1-\delta)}{|2(1+3\eta)\Phi_3(\lambda,\vartheta)-\Phi_2(\lambda,2\vartheta)|}}, \tag{3.16}$$

$$|\mathrm{a}_3| \leq \frac{\mu|c|(1-\delta)}{|2(1+3\eta)\Phi_3(\lambda,\vartheta)-\Phi_2(\lambda,2\vartheta)|} \tag{3.17}$$

and

$$|a_3 - \aleph\, a_2^2| \leq \begin{cases} \dfrac{\mu|c|(1-\delta)(1-\aleph)}{|2(1+3\eta)\Phi_3(\lambda,\vartheta)-\Phi_2(\lambda,2\vartheta)|}, & \aleph \leq \Pi \\ \dfrac{\mu|c|(1-\delta)}{|2(1+3\eta)\Phi_3(\lambda,\vartheta)|}, & \Pi \leq \aleph \leq 2-\Pi \\ \dfrac{\mu|c|(1-\delta)(\aleph-1)}{|2(1+3\eta)\Phi_3(\lambda,\vartheta)-\Phi_2(\lambda,2\vartheta)|}, & \aleph \geq 2-\Pi \end{cases} \tag{3.18}$$

where ℵ is a real number and

$$\Pi = \frac{\Phi_2(\lambda,2\vartheta)}{2(1+3\eta\ )\Phi_3(\lambda,\vartheta)}.$$

Proof. Consider $f \in \mathcal{S}_{\Sigma,\lambda}^{\vartheta,\eta}(c,\mu,\delta)$. Then, there exist two analytic functions $x(\xi)$ and $y(\zeta)$ belongs to the class $\mathcal{P}_{\mu}(c,\delta)$ such that

$$\frac{\xi \Lambda_{\lambda}^{\vartheta} f'(\xi)}{\Lambda_{\lambda}^{\vartheta} f(\xi)} + \eta \xi \Lambda_{\lambda}^{\vartheta} f''(\xi) = x(\xi) \tag{3.19}$$

and

$$\frac{\zeta \Lambda_{\lambda}^{\vartheta} h'(\zeta)}{\Lambda_{\lambda}^{\vartheta} h(\zeta)} + \eta \zeta \Lambda_{\lambda}^{\vartheta} h''(\zeta) = y(\zeta) \tag{3.20}$$

where x(ξ) and y(ζ) are given in (3.6) and (3.7). Now using (3.5), (3.6), (3.19) and (3.20), we have

$$(1+2\eta)\Phi_2(\lambda,\vartheta)a_2 = x_1, \tag{3.21}$$

$$2(1+3\eta)\Phi_3(\lambda,\vartheta)a_3 - \Phi_2(\lambda,2\vartheta)a_2^2 = x_2, \tag{3.22}$$

$$-(1+2\eta)\Phi_2(\lambda,\vartheta)a_2 = y_1, \tag{3.23}$$

and

$$[4(1+3\eta)\Phi_3(\lambda,\vartheta) - \Phi_2(\lambda,2\vartheta)]a_2^2 - 2(1+3\eta)\Phi_3(\lambda,\vartheta)a_3 = y_2. \tag{3.24}$$

On addition of (3.21) and (3.23) we have $x_1 = -y_1$ and similarly adding (3.22) and (3.24), we get

$$2[2(1+3\eta)\Phi_3(\lambda,\vartheta) - \Phi_2(\lambda,2\vartheta)]a_2^2 = x_2 + y_2. \tag{3.25}$$

Using Lemma 1 in (3.25), we get

$$|a_2|^2 \leq \frac{\mu|c|(1-\delta)}{|2(1+3\eta)\Phi_3(\lambda,\vartheta) - \Phi_2(\lambda,2\vartheta)|}. \tag{3.26}$$

Equation (3.26) gives the bound of $|a_2|$ given in (3.16). Now subtracting (3.24) from (3.22) we get

$$4(1+3\eta)\Phi_3(\lambda,\vartheta)a_3 = 4(1+3\eta)\Phi_3(\lambda,\vartheta)a_2^2 + x_2 + y_2. \tag{3.27}$$

Using the value of a_2^2 given in (3.25) in (3.27), we get

$$4(1+3\eta)\Phi_3(\lambda,\vartheta)a_3 = \frac{[4(1+3\eta)\Phi_3(\lambda,\vartheta) - \Phi_2(\lambda,2\vartheta)]x_2 + \Phi_2(\lambda,2\vartheta)y_2}{2(1+3\eta)\Phi_3(\lambda,\vartheta) - \Phi_2(\lambda,2\vartheta)}. \tag{3.38}$$

Using Lemma 1 in (3.28) gives the bound of $|a_3|$ given in (3.17). Hence, for any fixed real number ℵ and from (3.25) and (3.28), we get

$$a_3-\aleph\, a_2^2=\frac{[4(1+3\eta)\Phi_3(\lambda,\vartheta)-\Phi_2(\lambda,2\vartheta)-2(1+3\eta)\Phi_3(\lambda,\vartheta)\aleph]x_2}{4(1+3\eta)\Phi_3(\lambda,\vartheta)[2(1+3\eta)\Phi_3(\lambda,\vartheta)-\Phi_2(\lambda,2\vartheta)]}+\frac{[\Phi_2(\lambda,2\vartheta)-2(1+3\eta)\Phi_3(\lambda,\vartheta)\aleph]y_2}{4(1+3\eta)\Phi_3(\lambda,\vartheta)[2(1+3\eta)\Phi_3(\lambda,\vartheta)-\Phi_2(\lambda,2\vartheta)]}. \tag{3.39}$$

Using Lemma 1 in (3.29), we get

$$a_3-\aleph\, a_2^2=\frac{\mu|c|(1-\delta)|4(1+3\eta)\Phi_3(\lambda,\vartheta)-\Phi_2(\lambda,2\vartheta)-2(1+3\eta)\Phi_3(\lambda,\vartheta)\aleph|}{4(1+3\eta)\Phi_3(\lambda,\vartheta)[2(1+3\eta)\Phi_3(\lambda,\vartheta)-\Phi_2(\lambda,2\vartheta)]}+\frac{\mu|c|(1-\delta)|\Phi_2(\lambda,2\vartheta)-2(1+3\eta)\Phi_3(\lambda,\vartheta)\aleph|}{4(1+3\eta)\Phi_3(\lambda,\vartheta)[2(1+3\eta)\Phi_3(\lambda,\vartheta)-\Phi_2(\lambda,2\vartheta)]}. \tag{3.40}$$

Equation (3.30) gives the bounds given in (3.18). Which completes the proof of Theorem 2.

Theorem 3. For fixed $2\le \mu \le 4, c\in\mathbb{C}\setminus\{0\},\lambda\in\mathbb{C}, 0\le и\le 1$ and $0\le\delta<1$. A function $f\in\mathcal{C}_{\Sigma,\lambda}^{\vartheta,и}(c,\mu,\delta)$ then

$$|a_2|\le\sqrt{\frac{\mu|c|(1-\delta)}{|3(1+и)\Phi_3(\lambda,\vartheta)-4и\Phi_2(\lambda,2\vartheta)|}}, \tag{3.31}$$

$$|a_3|\le\frac{\mu|c|(1-\delta)}{|3(1+и)\Phi_3(\lambda,\vartheta)-4и\Phi_2(\lambda,2\vartheta)|} \tag{3.32}$$

and

$$|a_3-\aleph\, a_2^2|\le\begin{cases}\dfrac{\mu|c|(1-\delta)(1-\aleph)}{|3(1+и)\Phi_3(\lambda,\vartheta)-4и\Phi_2(\lambda,2\vartheta)|}, & \aleph\le\Theta\\ \dfrac{\mu|c|(1-\delta)}{3(1+и)\Phi_3(\lambda,\vartheta)}, & \Theta\le\aleph\le 2-\Theta\\ \dfrac{\mu|c|(1-\delta)(\aleph-1)}{|3(1+и)\Phi_3(\lambda,\vartheta)-4и\Phi_2(\lambda,2\vartheta)|}, & \aleph\ge 2-\Theta\end{cases} \tag{3.33}$$

where ℵ is a real number and

$$\Theta=\frac{4и\Phi_2(\lambda,2\vartheta)}{3(1+и)\Phi_3(\lambda,\vartheta)}.$$

Proof. Consider $f\in\mathcal{C}_{\Sigma,\lambda}^{\vartheta,и}(c,\mu,\delta)$. Then, there exist two analytic functions $x(\xi)$ and $y(\zeta)$ belongs to the class $\mathcal{P}_\mu(c,\delta)$ such that

$$(1-и)\Lambda_\lambda^\vartheta f'(\xi)+и\left(1+\frac{\xi\Lambda_\lambda^\vartheta f''(\xi)}{\Lambda_\lambda^\vartheta f'(\xi)}\right)=x(\xi) \tag{3.34}$$

and

$$(1-и)\Lambda_\lambda^\vartheta f'(\xi)+и\left(1+\frac{\xi\Lambda_\lambda^\vartheta f''(\xi)}{\Lambda_\lambda^\vartheta f'(\xi)}\right)=x(\xi) \tag{3.35}$$

where x(ξ) and y(ζ) are given in (3.6) and (3.7). Now using (3.5), (3.6), (3.34) and (3.35), we have

$$2\Phi_2(\lambda,\vartheta)a_2=x_1, \tag{3.36}$$

$$3(1+и)\Phi_3(\lambda,\vartheta)a_3-4и\Phi_2(\lambda,2\vartheta)a_2^2=x_2, \tag{3.37}$$

$$-2\Phi_2(\lambda,\vartheta)a_2=y_1, \tag{3.38}$$

and

$$[6(1+и)\Phi_3(\lambda,\vartheta)-4и\Phi_2\ (\lambda,2\vartheta)]a_2^2-3(1+и)\Phi_3(\lambda,\vartheta)a_3=y_2. \tag{3.39}$$

On addition of (3.36) and (3.38) we have $x_1=-y_1$ and similarly adding (3.37) and (3.39), we get

$$[6(1+и)\Phi_3(\lambda,\vartheta)-8и\Phi_2(\lambda,2\vartheta)]a_2^2=x_2+y_2. \tag{3.40}$$

Using Lemma 1 in (3.40), we get

$$|a_2|^2\le\frac{\mu|c|(1-\delta)}{|3(1+и)\Phi_3(\lambda,\vartheta)-4и\Phi_2(\lambda,2\vartheta)|}. \tag{3.41}$$

Equation (3.41) gives the bound of $|a_2|$ given in (3.31). Now subtracting (3.39) from (3.37) and using the value of a_2^2 given in (3.40), we get

$$a_3=\frac{[12(1+и)\Phi_3(\lambda\,,\vartheta)-8и\Phi_2(\lambda,2\vartheta)]x_2+\ 8и\Phi_2(\lambda,2\vartheta)y_2}{6(1+и)\Phi_3(\lambda,\vartheta)[6(1+и)\Phi_3(\lambda,\vartheta)-8и\Phi_2(\lambda,2\vartheta)]}. \tag{3.42}$$

Using Lemma 1 in (3.42) gives the bound of $|a_3|$ given in (3.32). Hence, for any fixed real number $\aleph$ and from (3.40) and (3.42), we get

$$\begin{aligned}a_3-\aleph\,a_2^2&=\frac{[12(1+и)\Phi_3(\lambda,\vartheta)-8и\Phi_2(\lambda,2\vartheta)-6(1+и)\Phi_3(\lambda,\vartheta)\aleph]x_2}{6(1+и)\Phi_3(\lambda,\vartheta)[6(1+и)\Phi_3(\lambda,\vartheta)-8и\Phi_2(\lambda,2\vartheta)]}\\&\quad+\frac{[8и\Phi_2(\lambda,2\vartheta)-6(1+и)\Phi_3(\lambda,\vartheta)\aleph]y_2}{6(1+и)\Phi_3(\lambda,\vartheta)[6(1+и)\Phi_3(\lambda,\vartheta)-8и\Phi_2(\lambda,2\vartheta)]}.\end{aligned} \tag{3.43}$$

Using Lemma 1 in (3.43), we get

$$a_3 - \aleph\, a_2^2 = \frac{\mu|c|(1-\delta)|12(1+\text{и})\Phi_3(\lambda,\vartheta) - 8\text{и}\Phi_2(\lambda,2\vartheta) - 6(1+\text{и})\Phi_3(\lambda,\vartheta)\aleph|}{6(1+\text{и})\Phi_3(\lambda,\vartheta)[6(1+\text{и})\Phi_3(\lambda,\vartheta) - 8\text{и}\Phi_2(\lambda,2\vartheta)]} + \frac{\mu|c|(1-\delta)|8\text{и}\Phi_2(\lambda,2\vartheta) - 6(1+\text{и})\Phi_3(\lambda,\vartheta)\aleph|}{6(1+\text{и})\Phi_3(\lambda,\vartheta)[6(1+\text{и})\Phi_3(\lambda,\vartheta) - 8\text{и}\Phi_2(\lambda,2\vartheta)]}. \tag{3.44}$$

Equation (3.44) gives the bounds given in (3.33). Which completes the proof of Theorem 3.

4. Corollaries

With the choice of $\vartheta = 0$, $\upsilon = 1 + 2\alpha$ and $c = 1$ in Theorem 1, we can deduce the following corollary.

Corollary 1. For fixed $2 \le \mu \le 4, \alpha \ge 0$ and $0 \le \delta < 1$. A function $f \in \mathcal{F}_{\Sigma}^{\alpha}(\mu,\delta)$ then

$$|a_2| \le \sqrt{\frac{\mu(1-\delta)}{3(1+2\alpha)}},$$

$$|a_3| \le \frac{\mu(1-\delta)}{3(1+2\alpha)}$$

and

$$|a_3 - \aleph\, a_2^2| \le \begin{cases} \dfrac{\mu(1-\delta)(1-\aleph)}{3(1+2\alpha)}, & \aleph \le 0 \\ \dfrac{\mu(1-\delta)}{3(1+2\alpha)}, & 0 \le \aleph \le 2 \\ \dfrac{\mu(1-\delta)(\aleph-1)}{3(1+2\alpha)}, & \aleph \ge 2 \end{cases}$$

With the choice of $\vartheta = 0, \alpha = 0$ and $c = 1$ in Theorem 1, we can deduce the following corollary.

Corollary 2. For fixed $2 \le \mu \le 4, \upsilon \ge 0$ and $0 \le \delta < 1$. A function $f \in \mathcal{G}_{\Sigma}^{\upsilon}(\mu,\delta)$ then

$$|a_2| \le \sqrt{\frac{\mu(1-\delta)}{1+2\upsilon}},$$

$$|a_3| \leq \frac{\mu(1-\delta)}{1+2\upsilon}$$

and

$$|a_3 - \aleph\, a_2^2| \leq \begin{cases} \dfrac{\mu(1-\delta)(1-\aleph)}{1+2\upsilon}, & \aleph \leq 0 \\ \dfrac{\mu(1-\delta)}{1+2\upsilon}, & 0 \leq \aleph \leq 2 \\ \dfrac{\mu(1-\delta)(\aleph-1)}{1+2\upsilon}, & \aleph \geq 2 \end{cases}$$

With the choice of $\vartheta = 0, \alpha = 0, \upsilon = 1$ and $c = 1$ in Theorem 1, we can deduce the following corollary.

Corollary 3. For fixed $2 \leq \mu \leq 4$ and $0 \leq \delta < 1$. A function $f \in \mathcal{H}_\Sigma(\mu, \delta)$ then

$$|a_2| \leq \sqrt{\frac{\mu(1-\delta)}{3}},$$

$$|a_3| \leq \frac{\mu(1-\delta)}{3}$$

and

$$|a_3 - \aleph\, a_2^2| \leq \begin{cases} \dfrac{\mu(1-\delta)(1-\aleph)}{3}, & \aleph \leq 0 \\ \dfrac{\mu(1-\delta)}{3}, & 0 \leq \aleph \leq 2 \\ \dfrac{\mu(1-\delta)(\aleph-1)}{3}, & \aleph \geq 2 \end{cases}$$

With the choice of $\vartheta = 0$ in Theorem 2, we can deduce the following corollary.

Corollary 4. For fixed $2 \leq \mu \leq 4, c \in \mathbb{C} \setminus \{0\}, \eta \geq 0$ and $0 \leq \delta < 1$. A function $f \in \mathcal{S}_\Sigma^\eta(c, \mu, \delta)$ then

$$|a_2| \leq \sqrt{\frac{\mu|c|(1-\delta)}{1+6\eta}}$$

$$|a_3| \leq \frac{\mu|c|(1-\delta)}{1+6\eta}$$

and

$$|a_3 - \aleph\, a_2^2| \le \begin{cases} \dfrac{\mu|c|(1-\delta)(1-\aleph)}{1+6\eta}, & \aleph \le \dfrac{1}{2(1+3\eta)} \\ \dfrac{\mu|c|(1-\delta)}{1+6\eta}, & \dfrac{1}{2(1+3\eta)} \le \aleph \le \dfrac{3+6\eta}{2(1+3\eta)} \\ \dfrac{\mu|c|(1-\delta)(\aleph-1)}{1+6\eta}, & \aleph \ge \dfrac{3+6\eta}{2(1+3\eta)} \end{cases}$$

With the choice of $\vartheta = 0, \eta = 0$ and $c = 1$ in Theorem 2, we can deduce the following corollary.

Corollary 5. For fixed $2 \le \mu \le 4$ and $0 \le \delta < 1$. A function $f \in \mathcal{S}_{\Sigma}^{*}(\mu, \delta)$ then

$$|a_2| \le \sqrt{\mu(1-\delta)}$$

$$|a_3| \le \mu(1-\delta)$$

and

$$|a_3 - \aleph\, a_2^2| \le \begin{cases} \mu(1-\delta)(1-\aleph), & \aleph \le \dfrac{1}{2} \\ \dfrac{\mu(1-\delta)}{2}, & \dfrac{1}{2} \le \aleph \le \dfrac{3}{2} \\ \mu(1-\delta)(\aleph-1), & \aleph \ge \dfrac{3}{2} \end{cases}$$

With the choice of $\vartheta = 0$ in Theorem 3, we can deduce the following corollary.

Corollary 6. For fixed $2 \le \mu \le 4, c \in \mathcal{C} \setminus \{0\}, 0 \le \varkappa \le 1$ and $0 \le \delta < 1$. A function $f \in \mathcal{C}_{\Sigma}^{\varkappa}(c, \mu, \delta)$ then

$$|a_2| \le \sqrt{\frac{\mu|c|(1-\delta)}{3-\varkappa}},$$

$$|a_3| \le \frac{\mu|c|(1-\delta)}{3-\varkappa}$$

and

$$|a_3 - \aleph\, a_2^2| \leq \begin{cases} \dfrac{\mu|c|(1-\delta)(1-\aleph)}{3-\text{и}}, & \aleph \leq \dfrac{4\text{и}}{3(1+\text{и})} \\ \dfrac{\mu|c|(1-\delta)}{3(1+\text{и})}, & \dfrac{4\text{и}}{3(1+\text{и})} \leq \aleph \leq \dfrac{2(3+\text{и})}{3(1+\text{и})} \\ \dfrac{\mu|c|(1-\delta)(\aleph-1)}{3-\text{и}}, & \aleph \geq \dfrac{2(3+\text{и})}{3(1+\text{и})} \end{cases}$$

With the choice of $\vartheta = 0, \kappa = 1$ and $c = 1$ in Theorem 3, we can deduce the following corollary.

Corollary 6. For fixed $2 \leq \mu \leq 4$ and $0 \leq \delta < 1$. A function $f \in \mathcal{C}(\mu, \delta)$ then

$$|a_2| \leq \sqrt{\frac{\mu(1-\delta)}{2}},$$

$$|a_3| \leq \frac{\mu(1-\delta)}{2}$$

and

$$|a_3 - \aleph\, a_2^2| \leq \begin{cases} \dfrac{\mu(1-\delta)(1-\aleph)}{2}, & \aleph \leq \dfrac{2}{3} \\ \dfrac{\mu(1-\delta)}{6}, & \dfrac{2}{3} \leq \aleph \leq \dfrac{4}{3} \\ \dfrac{\mu(1-\delta)(\aleph-1)}{2}, & \aleph \geq \dfrac{4}{3} \end{cases}$$

Remark 6. (i) Corollary 1 and Corollary 2 verifies the results obtained by Sharma et. al. [20].

(ii) For $\mu = 2$ in Corollary 1 verify the bound of $|a_2|$ and improve the bound of $|a_3|$ obtained by Srivastava [23].

(iii) For $\mu = 2$ in Corollary 2 verify the bound of $|a_2|$ and improve the bound of $|a_3|$ obtained by Frasin and Aouf [9].

(iv) Corollary 3 verifies the results obtained by Sharma et. al. [20] and verify the bound of $|a_2|$ and improve the bound of $|a_3|$ obtained by Li et. al. [12].

(v) Corollary 5 and Corollary 7 verifies the results obtained by Sharma et. al. [20] and verify the bound of $|a_2|$ and improve the bound of $|a_3|$ obtained by Li et. al. [12].

(vi) For $\mu = 2$ in Corollary 5 and Corollary 7 verify the bound of $|a_2|$ and $|a_3|$ obtained by Brannan and Taha [5].

(vii) For $\vartheta = 0, \alpha = 0, \upsilon = 1$ and $\mu = 2$ in Theorem 1 verify the bound of $|a_2|$ and improve the bound of $|a_3|$ obtained by Deniz et. al. [8].

5. Conclusion

In this article, the authors have introduced three new subclasses of Σ, the class of bi-univalent functions related with Hurwitz–Lerch Zeta function in the open unit disk $\mathbb{E}$. The author's established first two initial coefficient bound for the classes $\mathcal{R}^{\vartheta,\upsilon}_{\Sigma,\lambda,\alpha}(c,\mu,\delta)$, $\mathcal{S}^{\vartheta,\eta}_{\Sigma,\lambda}(c,\mu,\delta)$ and $\mathcal{C}^{\vartheta,\varkappa}_{\Sigma,\lambda}(c,\mu,\delta)$. Also, very famous Fekete-Szegö inequality were obtained for these new subclasses. Interesting remarks on the main results including improvements of the earlier bounds were also given. More corollaries and remarks could be reported for the selection of parameters, and those details have been omitted.

Also, the research highlighted in this article can be further developed by analyzing close-to-convex functions [19], ozaki-close-to-convex functions [17], and concave univalent functions [21] with constrained boundary rotation.

Moreover, the research highlighted in this article can be further developed by exploring the q-analogue of a Bessel function, the q-analogue of a Mittag–Leffler-type function, a q-exponential function, and a q-Ruscheweyh derivative with limited boundary rotation and bounded radius rotation. However, these fascinating details and observations are not discussed. Additionally, the same type of findings can be replicated for other significant special functions documented in the literature.

Notation

$\mathcal{A}$ – Class of normalized analytic functions in $\mathbb{E}$.

$\mathcal{S}$ – Class of all normalized univalent functions in $\mathbb{E}$.

$\mathcal{S}^*$ – Class of starlike functions in $\mathbb{E}$.

$\mathcal{C}$ – Class of convex functions in $\mathbb{E}$.

Σ – The class of bi-univalent functions in $\mathbb{E}$.

$\Psi(\xi,\lambda,b)$ – Hurwitz–Lerch Zeta function.

$\mathcal{L}$ – Alexander operator.

$\mathcal{F}_{\mathrm{b}}$ – Bernardi operator.

$\mathcal{J}^{\eta}$ – Jung–Kim–Srivastava integral operator.

Data Availability Statement

No data were used to support this study.

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Conflict of interest

The authors declare no conflicts of interest.

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