

## Some Geometric Properties of Normalized Holomorphic Functions Associated With Ruopoola Derivative Operator

**Jadhav. S.S<sup>1</sup>, Thange. T.G<sup>2</sup>, Mhaske A.S<sup>3</sup>, Khairnar S.K<sup>4</sup>**

<sup>1</sup>Department of Mathematics, Sundarrao More Arts, Commerce, and Science (Sr.) College, Poladpur-Raigad

<sup>2</sup>Department of Mathematics, Yogeshwari Mahavidyalaya, Ambejogai

<sup>3</sup> Department of Mathematics, Dada Patil Mahavidyalaya, Karjat-Ahmednagar

<sup>4</sup>Department of Mathematics, Chandu Kana Thakur Mahavidyalaya, Panvel

### ABSTRACT

In this paper we found new generalized Ruopoola derivative operator. Associated with this operator new generalized class  $\text{Tr}(E_\lambda^m; \gamma, \mu, m, q)$  of normalized holomorphic functions is investigated. Different geometric properties we obtained are Growth theorem, Closure theorem, Integral mean inequality, Extreme point theorem, Coefficient inequality, Convolution and Distortion theorem for given class.

### 1. INTRODUCTION AND PRELIMINARIES.

Let  $N$  denotes subclass of all holomorphic functions in open unit disk  $U = \{z : |z| < 1\}$  normalized with conditions  $f(0) = 0, f'(0) = 1$  given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1.1)$$

**Definition 1.1.** Ruscheweyh in [3] has introduced following differential operator.

$R^n: N \rightarrow N$  defined by

$$\begin{aligned} R^n(f(z)) &= \frac{z}{(1-z)^{n+1}} * f(z) & n \in \mathbb{N} \cup \{0\} \\ &= z + \sum_{k=2}^{\infty} \binom{n+k-1}{n} a_k z^k & (z \in U) \end{aligned} \quad (1.2)$$

Where  $*$  is hadmard product defined in (6.1).

We note that  $R^0 f(z) = f(z), R' f(z) = z f'(z)$

[6] has used following definition 1.1 and 1.2

**Definition 1.2.** A function  $f$  in  $N$  is said to be in  $C(\alpha)$ , if and only if

$$\operatorname{Re} \{f'(z)\} > \alpha \quad (z \in U \text{ and } 0 \leq \alpha < 1) \tag{1.3}$$

**Definition 1.3.** A function  $f$  in  $N$  is said to be in  $CS^*(\alpha)$  if and only if

$$\operatorname{Re} \left\{ \frac{f(z)}{z} \right\} > \alpha \quad (z \in U \text{ and } 0 \leq \alpha < 1) \tag{1.4}$$

We write the classes  $C(0) = C$ ,  $CS^*(0) = CS^*$

**Definition 1.4.** For  $f \in N$ , [1] has introduced following differential operator, known as Al-Oboudi differential operator.

$D^n: N \rightarrow N$  defined by

$$D^0 f(z) = f(z) \tag{1.5}$$

$$D^1 f(z) = (1 - \delta)f(z) + \delta z f'(z) = D_\delta f(z) \quad \delta \geq 0. \tag{1.6}$$

$$D^n f(z) = D_\delta (D^{n-1} f(z)) \tag{1.7}$$

From (1.8) and (1.9) we have

$$D^n (f(z)) = z + \sum_{k=2}^{\infty} [1 + (k - 1)\delta]^n a_k z^k \quad (z \in U) \tag{1.8}$$

We will make use of definition of subordination between analytic functions [2] in our further investigation. Also, [11] has defined the following Rusal differential operator. Expression for this operator is

Let  $n \in N \cup \{0\}$ ,  $0 \leq \lambda \leq 1$ ,  $A_\lambda^n: N \rightarrow N$  defined by

$$A_\lambda^n (f(z)) = z + \sum_{k=2}^{\infty} ([1 + (k - 1)\delta]^n (1 - \lambda) + \lambda^{n+k-1} C) a_k z^k. \tag{1.9}$$

[12] has introduced the Opoola differential operator for  $f(z)$  in  $N$  is denoted by  $D^n(u, \alpha, c)$ . It is defined as below:

$$D^0(u, \alpha, \partial) f(z) = f(z)$$

$$D^1(u, \alpha, \partial) f(z) = z D_c f(z) = z c f'(z) - z(\alpha - u)c + (1 + (u - \alpha - 1)c) f(z)$$

$$D^2(u, \alpha, \partial) f(z) = z D_c (z D_c f(z))$$

$$D^n(u, \alpha, \partial) f(z) = z D_c D^n(u, \alpha, c)$$

After simplification we get

$$D^n(u, \alpha, c)f(z) = z^+ \sum_{k=2}^{\infty} [1 + (k + \alpha - u - 1)c]^n a_k z^k \tag{1.10}$$

Here  $0 \leq c, 0 \leq \mu \leq \alpha$

**Definition 1.5.** For two functions  $f$  and  $g$  analytic in  $U$ , we say that the function  $f(z)$  is subordinate to  $g(z)$  in  $U$  and write

$$f(z) \prec g(z) \quad (z \in U) \tag{1.11}$$

If there exist Schwarz function  $w(z)$ , analytical in  $U$  with  $w(0) = 0$  and  $|w(z)| < 1$  such that  $f(z) = g(w(z))$   $(z \in U)$  (1.12)

J.E. Littlewood has introduced following subordination theorem which we stated as lemma.

We use this lemma to prove integral mean inequality given in theorem 5.1

**Lemma 1.1** Let  $f$  and  $g$  analytic in unit disc and suppose  $g \prec f$ , then for  $0 < t < \infty$

$$\int_0^{2\pi} |g(re^{i\theta})|^t d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^t d\theta \quad (0 \leq r < 1, t > 0) \tag{1.13}$$

Strict equality holds for  $0 \leq r < 1$  unless  $f$  is constant or  $w(z) = \alpha z, |\alpha| = 1$ .

In next section we have defined Ruopoola differential operator. It is generalization of latest Opoola derivative operator given by [12]

**2. Ruopoola Differential Operator, Classes  $Tr(E_\lambda^m; \gamma, \mu, m, q)$  and  $\widetilde{Tr}(E_\lambda^n; \gamma, \mu, m, q)$ .**

We derived the Ruopoola differential operator by making convex combination of Rusal & Opoola differential operators discussed in (1.2) and (1.8) respectively. We also introduced New subclasses  $Tr(E_\lambda^m; \gamma, \mu, m, q)$  and  $\widetilde{Tr}(E_\lambda^n; \gamma, \mu, m, q)$  which are generalization of  $Tr(E_\lambda^m; \gamma, \mu, m, q)$  and  $\widetilde{Tr}(E_\lambda^n; \gamma, \mu, m, q)$  respectively[7].

**Definition 2.1.** Let  $n \in \mathbb{N} \cup \{0\}, 0 \leq \lambda \leq 1, E_\lambda^n: \mathbb{N} \rightarrow \mathbb{N}$  defined by

$$\begin{aligned} E_\lambda^n(f(Z)) &= (1 - \lambda) D^n f(Z) + \lambda R^n f(z). \\ &= (1 - \lambda) z^+ \sum_{k=2}^{\infty} [1 + (k + \alpha - u - 1)\partial]^n a_k z^k + \lambda z^+ \sum_{k=2}^{\infty} \frac{n+k-1}{n} C \lambda a_k z^k \end{aligned} \tag{2.1}$$

$$\begin{aligned} &= (1 - \lambda) z^+ \sum_{k=2}^{\infty} ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda \frac{n+k-1}{n} C) a_k z^k \\ &= z^+ \sum_{k=2}^{\infty} ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda \frac{n+k-1}{n} C) a_k z^k. \end{aligned} \tag{2.2}$$

$$\text{If } n=0, E_\lambda^0 f(z) = f(z). \tag{2.3}$$

$$\lambda=1, E_1^n(f(Z))= R^n f(z). \tag{2.4}$$

$$\lambda=0, E_0^n(f(Z)) = D^n f(Z) \tag{2.5}$$

**Definition 2.2.**  $\text{Tr}(E_\lambda^m; \gamma, \mu, m, q) = \{f \in N : \left| \frac{1}{r\gamma}((r-u)\frac{E_\lambda^m f}{z} + u (E_\lambda^m f)' - r) \right| < q \}$  (2.6)

Where  $z \in U, \gamma \in C \setminus \{0\}, r \in \mathbb{R}^+, 0 < q \leq 1, 0 < \mu \leq r, m \in \mathbb{N} \cup \{0\}, E_\lambda^m f$  is defined in .We illustrate the subclass  $\text{Tr}(E_\lambda^m; \gamma, \mu, m, q)$  with following example.

**Definition 2.3:** If  $f(z) = z + \sum_{k=2}^\infty a_k z^k$   $g(z) = z + \sum_{k=2}^\infty b_k z^k$ , then hadmad product (Convolution) is defined as given bellow

$$f * g = z + \sum_{k=2}^\infty (a_k b_k) z^k \tag{2.7}$$

**EXAMPLE 2.1** If  $f(z) = z$ , then for  $\gamma = 1, \mu = r, m = 0, 0 < q \leq 1$ , show that  $f(z) \in \text{Tr}(E_\lambda^m; \gamma, \mu, m, q)$

for  $\gamma = 1, \mu = r, m = 0, 0 < q \leq 1$ ,

$$\begin{aligned} & \left| \frac{1}{r\gamma}((r-u)\frac{E_\lambda^m f}{z} + u (E_\lambda^m f)' - r) \right| = \left| \frac{1}{r.1}((r-r)\frac{E_\lambda^0 f}{z} + r (E_\lambda^0 f)' - r) \right| \\ & = \left| \frac{1}{r} (| r (E_\lambda^0 f)' - r |) \right| \\ & = | (E_\lambda^0 f)' - 1 | \\ & = | (z)' - 1 | \\ & = | 1 - 1 | \\ & < q. \end{aligned}$$

Hence, by definition (2.2),

$$f(z) \in \text{Tr}(E_\lambda^0; \gamma, \mu, m, q)$$

**Definition 2.4**

Let  $\widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$  be the subclass of  $N$  which satisfies inequality

$$\sum_{k=2}^\infty (r + (k-1)\mu)([1 + (k-1)\partial]^n (1-\lambda) + \lambda^{n+k-1} C) |a_k| < r |\gamma| q \tag{2.8}$$

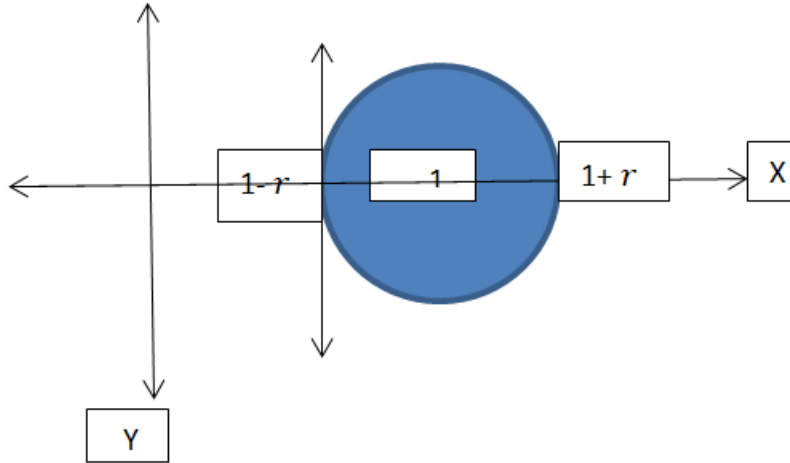
**Example 2.2.**  $\text{Tr}(E_\lambda^0; 1, r, 0, q) \subseteq C(1-q)$

$$\begin{aligned} \text{Tr}(E_\lambda^0; 1, r, 0, q) &= \{f \in N : \left| \frac{1}{r.1}((r-r)\frac{E_\lambda^0 f}{z} + r (E_\lambda^0 f)' - r) \right| < q \} \\ &= \{f \in N : \left| \frac{1}{r} (r (E_\lambda^0 f)' - r) \right| < q \} \\ &= \{f \in N : | ((E_\lambda^0 f)' - 1) | < q \} \end{aligned}$$

$$= \{f \in N: |f'(z) - 1| < q\}$$

Suppose  $g(z) \in Tr(E_\lambda^0; 1, p, 0, q)$

$$\therefore |g'(z) - 1| < q$$



$$\therefore \text{Re}\{g'(z)\} > 1 - r$$

Therefore,  $g(z) \in C(1 - r)$

Hence,  $Tr(E_\lambda^0; 1, r, 0, q) \subseteq C(1 - q)$ .

**Remark 2.1.** For  $\gamma \in C \setminus \{0\}, 0 < q \leq 1, 0 \leq \mu \leq r, m \in N \cup \{0\}$  the following functions are in class  $\widetilde{Tr}(E_\lambda^n; \gamma, \mu, m, q)$

$$f_1(z) = z + \frac{rq|\gamma|}{(r+\mu)((1+\alpha-u)\partial)^{n(1-\lambda)+\lambda^{n+1}C}} z^2 \quad (z \in U)$$

$$f_2(z) = z + \frac{rq|\gamma|}{(r+2\mu)((1+2+\alpha-u)\partial)^{n(1-\lambda)+\lambda^{n+2}C}} z^3 \quad (z \in U)$$

$$f_3(z) = z + \frac{z^2}{(r+\mu)((1+\alpha-u)\partial)^{n(1-\lambda)+\lambda^{n+1}C}} + \frac{(r|\gamma|q-1)}{(r+2\mu)((1+2+\alpha-u)\partial)^{n(1-\lambda)+\lambda^{n+2}C}} z^3 \quad (z \in U)$$

**Example 2.3.**  $Tr(E_\lambda^0; 1, 0, 0, q) \subseteq CS^*(1 - q)$

$$Tr(E_\lambda^0; 1, 0, 0, q) = \{f \in N: \left| \frac{1}{r_1}((r-0)\frac{E_\lambda^0 f}{z} + 0(E_\lambda^0 f)' - r) \right| < q\}$$

$$= \{f \in N: \left| \frac{1}{r}(r\frac{E_\lambda^0 f}{z} - r) \right| < q\}$$

$$= \{f \in N: \left| \frac{f(z)}{z} - 1 \right| < q\}$$

Suppose  $g(z) \in Tr(E_\lambda^0; 1, 0, 0, q)$

$$\therefore \left| \frac{g(z)}{z} - 1 \right| < q$$

$\therefore \frac{g(z)}{z}$  lies in circle around (1,0) of radius  $q$

Therefore,  $\operatorname{Re} \left\{ \frac{g(z)}{z} \right\} > 1 - q$

$$g(z) \in CS^*(1 - q)$$

Hence,  $\operatorname{Tr}(E_\lambda^0; 1, 0, 0, q) \subseteq CS^*(1 - q)$

### 3. Geometric Properties for Classes $\operatorname{Tr}(E_\lambda^m; \gamma, \mu, m, q)$ and $\widetilde{\operatorname{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$

Our first theorem gives sufficient condition for normalized analytic functions to be in

$\operatorname{Tr}(E_\lambda^m; \gamma, \mu, m, \beta)$

**Theorem 3.1** let  $f(z) \in \mathcal{N}$  satisfy

$$\sum_{k=2}^{\infty} (r + (k - 1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} {}_n C) |a_k| \leq r |\gamma| q \quad (3.1)$$

$$\gamma, \mu \in \mathbb{C} \setminus \{0\}, 0 < q \leq 1, 0 < \mu \leq r, m \in \mathbb{N} \cup \{0\}$$

Then  $f \in \operatorname{Tr}(E_\lambda^m; \gamma, \mu, m, q)$

Proof: Assume (3.1) is valid for  $f(z) \in \mathcal{N}$  and  $\gamma (\gamma \in \mathbb{C} \setminus \{0\})$ ,  $q (0 < q \leq 1)$ ,  $\mu (0 < \mu \leq |r|)$ ,  $m \in \mathbb{N} \cup \{0\}$ ,

Using (1.8) we have

$$\begin{aligned} & ((r - u) \frac{E_\lambda^m f}{z} + \mu (E_\lambda^m f)' - r) = \\ & \frac{(r - u)}{z} [Z + \sum_{k=2}^{\infty} ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} {}_n C) a_k z^k] + \\ & \mu [1 + \sum_{k=2}^{\infty} ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} {}_n C) k a_k z^{k-1}] - r \\ & = \frac{(r - u)}{z} Z + \sum_{k=2}^{\infty} \frac{(r - u)}{z} ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} {}_n C) a_k z^k \\ & + u + \sum_{k=2}^{\infty} ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} {}_n C) k u a_k z^{k-1} - r \\ & = (r - u) + \sum_{k=2}^{\infty} (r - u) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} {}_n C) a_k z^k \\ & + u + \sum_{k=2}^{\infty} ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} {}_n C) k u a_k z^{k-1} - p \\ & = \sum_{k=2}^{\infty} (r + (k + \alpha - u - 1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} {}_n C) a_k z^{k-1} \end{aligned}$$

Therefore

$$\begin{aligned} & \left| ((r - \mu) \frac{E_\lambda^m f}{z} + \mu (E_\lambda^m f)' - r) \right| \\ & = \left| \sum_{k=2}^{\infty} (r + (k - 1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} {}_n C) a_k z^{k-1} \right| \\ & = \sum_{k=2}^{\infty} (r + (k - 1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} {}_n C) |a_k| |z^{k-1}| \end{aligned}$$

$$\leq \sum_{k=2}^{\infty} (r + (k - 1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}C}) |a_k| < r|\gamma|q$$

Hence

$$\left| \frac{1}{r\gamma} \left( (r - u) \frac{E_{\lambda}^m f}{z} + u (E_{\lambda}^m f)' - r \right) \right| < q$$

Thus  $f(z) \in \text{Tr}(E_{\lambda}^m; \gamma, \mu, m, q)$

**Corollary 3.1:**  $\widetilde{\text{Tr}}(E_{\lambda}^n; \gamma, \mu, m, q) \subseteq \text{Tr}(E_{\lambda}^m; \gamma, \mu, m, q)$ .

**Proof.** Suppose  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \widetilde{\text{Tr}}(E_{\lambda}^n; \gamma, \mu, m, q)$ .

$$\therefore \sum_{k=2}^{\infty} (r + (k - 1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}C}) |a_k| < r|\gamma|q$$

From theorem 2.1,

$$f(z) \in \text{Tr}(E_{\lambda}^m; \gamma, \mu, m, q)$$

The following example illustrate that converse of the above corollary need not be true.

**Example 3.1 :** If  $f(z) = z + \frac{z^2}{2}$  and  $|r| \geq 1$ . Taking  $\gamma = 1, \mu = r, m = 0, q = 1$ , we will get

$$\begin{aligned} & \left| \frac{1}{r\gamma} \left( (r - u) \frac{E_{\lambda}^m f}{z} + u (E_{\lambda}^m f)' - r \right) \right| \\ &= \left| \frac{1}{r1} \left( (r - r) \frac{E_{\lambda}^0 f}{z} + r (E_{\lambda}^0 f)' - r \right) \right| \\ &= \left| \frac{1}{r} (r (E_{\lambda}^0 f)' - r) \right| \\ &= \left| (E_{\lambda}^0 f)' - 1 \right| \\ &= \left| \left( z + \frac{z^2}{2} \right)' - 1 \right| \\ &= \left| 1 + 2 \frac{z}{2} - 1 \right| \\ &= |z| < 1. \end{aligned}$$

Therefore,

$$f(z) \in \text{Tr}(E_{\lambda}^0; 1, r, 0, 1).$$

But  $\sum_{k=2}^{\infty} (r + (k - 1)\mu) ([1 + (k + \alpha - u - 1)\partial]^0 (1 - \lambda) + \lambda^{0+k-\frac{1}{n}C}) |a_k|$

$$\begin{aligned} &= \sum_{k=2}^{\infty} kr(1 - \lambda + \lambda) |a_k| \\ &= \sum_{k=2}^{\infty} kr |a_k| \\ &= 2r a_2 \\ &= 2r \cdot \frac{1}{2} \\ &= r < r|\gamma|\beta. \end{aligned}$$

Therefore,  $f(z) \notin \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$

Hence,

$$f(z) \in \text{Tr}(E_\lambda^0; 1, p, 0, 1) \text{ but } f(z) \notin \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q).$$

Our next theorem gives coefficient inequalities for  $f(z)$  belonging to class  $\widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$ .

**Theorem 3.2: Coefficient inequality**

$$\text{If } f(z) \in \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q) \text{ then } |a_k| \leq \frac{r|\gamma|q}{(r+(k-1)\mu)([1+(k+\alpha-u-1)\partial]^n(1-\lambda)+\lambda^{n+k-\frac{1}{n}C})} \quad k \geq 2$$

**Proof:** Given that  $f(z) \in \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$

Therefore

$$\sum_{k=2}^{\infty} (r+(k-1)\mu)([1+(k+\alpha-u-1)\partial]^n(1-\lambda)+\lambda^{n+k-\frac{1}{n}C}) |a_k| \leq r|\gamma|q$$

$$(r+(k-1)\mu)([1+(k+\alpha-u-1)\partial]^n(1-\lambda)+\lambda^{n+k-\frac{1}{n}C}) |a_k| \leq r|\gamma|q$$

$$|a_k| \leq \frac{r|\gamma|q}{(r+(k-1)\mu)([1+(k+\alpha-u-1)\partial]^n(1-\lambda)+\lambda^{n+k-\frac{1}{n}C})}$$

Now we prove the following theorem

**Theorem 3.3: Growth theorem**

Let function  $f(z)$  defined by (1.1) be in class  $\widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$ , then

$$|z| - \frac{r|\gamma|q}{[r+\mu]((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+\frac{1}{n}C}} |z|^2 \leq |f(z)| \leq |z| + \frac{r|\gamma|q}{[r+\mu]((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+\frac{1}{n}C}} |z|^2 \tag{3.2}$$

Equality is attained for function  $f(z)$  given by  $f(z) = z + \frac{r|\gamma|q}{[r+\mu]((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+\frac{1}{n}C}} z^2$

**Proof:**  $[r+\mu]((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+\frac{1}{n}C} \sum_{k=2}^{\infty} |a_k|$   
 $\leq \sum_{k=2}^{\infty} (r+(k-1)\mu)([1+(k+\alpha-u-1)\partial]^n(1-\lambda)+\lambda^{n+k-\frac{1}{n}C}) |a_k|$   
 $\leq r|\gamma|q$

Therefore ,

$$\sum_{k=2}^{\infty} |a_k| \leq \frac{r|\gamma|q}{[r+\mu]((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC} \tag{3.3}$$

Also  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$  and using (3.3)

$$\begin{aligned} |f(z)| &= \left| z + \sum_{k=2}^{\infty} a_k z^k \right| \\ &\leq |z| + \left| \sum_{k=2}^{\infty} a_k z^k \right| \\ &\leq |z| + \sum_{k=2}^{\infty} |a_k| |z^k| \\ &\leq |z| + |z|^2 \sum_{k=2}^{\infty} |a_k| \\ &\leq |z| + |z|^2 \frac{r|\gamma|q}{[p+\mu]((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC} \end{aligned} \tag{3.4}$$

similarly,

$$\begin{aligned} |f(z)| &= \left| z + \sum_{k=2}^{\infty} a_k z^k \right| \\ &\geq |z| - \left| \sum_{k=2}^{\infty} a_k z^k \right| \\ &\geq |z| - \sum_{k=2}^{\infty} |a_k| |z^k| \\ &\geq |z| - |z|^2 \sum_{k=2}^{\infty} |a_k| \\ &\geq |z| - |z|^2 \frac{r|\gamma|q}{[r+\mu]((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC} \end{aligned} \tag{3.5}$$

Using (3.4) and (3.5)

$$\begin{aligned} |z| - \frac{r|\gamma|q}{[r+\mu]((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC} |z|^2 \leq |f(z)| \leq |z| + \\ \frac{r|\gamma|q}{[r+\mu]((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC} |z|^2 \end{aligned}$$

Hence

$$\begin{aligned} |z| - \frac{r|\gamma|q}{[r+\mu]((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC} |z|^2 \leq |f(z)| \leq |z| + \\ \frac{r|\gamma|q}{[r+\mu]((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC} |z|^2 \end{aligned}$$

Equality is attained for function  $f(z)$  given by  $f(z) = z + \frac{r|\gamma|q}{[r+\mu]((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC} z^2$

**Theorem 3.4: Distortion theorem**

Let function  $f(z)$  defined by (1.1) be in class  $\widetilde{Tr}(E_\lambda^n; \gamma, \mu, m, q)$  ,then

$$1 - \frac{2r|\gamma|q}{u((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC)} |z| \leq |f^*(z)| \leq 1 + \frac{2r|\gamma|q}{u((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC)} |z| \tag{3.6}$$

Equality attained for function  $f(z)$  given by  $f(z) = z + \frac{r|\gamma|q}{u((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC)} z^2$

**Proof:** Let  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$  and  $f^*(z) = 1 + \sum_{k=2}^{\infty} k a_k z^{k-1}$

$$\begin{aligned} |f^*(z)| &= \left| 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} \right| \\ &\leq 1 + \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} \\ &\leq 1 + |z| \sum_{k=2}^{\infty} k |a_k| \end{aligned} \tag{3.7}$$

$$\text{But } \sum_{k=2}^{\infty} (r + (k - 1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n(1 - \lambda) + \lambda^{n+k-1}nC) |a_k| \leq r|\gamma|q$$

Also

$$2r + (k-2)\mu \geq 0$$

$$2r + k\mu - 2\mu \geq 0$$

$$2r + 2k\mu - 2\mu \geq k\mu$$

$$\frac{k\mu}{2} \leq r + (k-1)\mu$$

Similarly

$$(r + (k - 1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n(1 - \lambda) + \lambda^{n+k-1}nC) \geq \frac{k\mu}{2} ((1 + \alpha - u)\partial)^n(1 - \lambda) + \lambda^{n+1}nC$$

$$\sum_{k=2}^{\infty} \frac{k\mu}{2} ((1 + \alpha - u)\partial)^n(1 - \lambda) + \lambda^{n+1}nC |a_k| \leq$$

$$\sum_{k=2}^{\infty} (r + (k - 1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n(1 - \lambda) + \lambda^{n+k-1}nC) |a_k|$$

$$\leq r|\gamma|q$$

$$\sum_{k=2}^{\infty} k |a_k| \leq \frac{2r|\gamma|q}{u((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC)}$$

From (3.7)

$$|f^*(z)| \leq 1 + \frac{2r|\gamma|q}{u((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC)} |z| \tag{3.8}$$

Similarly

$$\begin{aligned} |f^*(z)| &= \left| 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} \right| \\ &\geq 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} \end{aligned}$$

$$\begin{aligned} &\geq 1 - |z| \sum_{k=2}^{\infty} k |a_k| \\ &\geq 1 - \frac{2r|\gamma|q}{u((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC)} |z| \end{aligned} \tag{3.9}$$

(3.8) and (3.9) we have

$$1 - \frac{2r|\gamma|q}{u((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC)} |z| \leq |f'(z)| \leq 1 + \frac{2r|\gamma|q}{u((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC)} |z|$$

Equality attained for function  $f(z)$  given by  $f(z) = z + \frac{2r|\gamma|q}{u((1+\alpha-u)\partial)^n(1-\lambda)+\lambda^{n+1}nC)} z^2$

In this section we will prove the extreme point theorem. We also find the extreme points for the subclass  $\widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$ .

**Theorem 3.5. Extreme Point Theorem**

Let  $f_1(z) = z$  and

$$f_k(z) = z + \frac{r|\gamma|q}{(r+(k-1)\mu)([1+(k+\alpha-u-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}nC)} z^k \quad (k \geq 2) \tag{3.10}$$

Then  $f(z) \in \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$  if and only if  $f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$

Where  $\lambda_k \geq 0$  and  $\sum_{k=1}^{\infty} \lambda_k = 1$ .

**Proof:** Suppose that  $f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$

$$\begin{aligned} &= \lambda_1 f_1(z) + \sum_{k=2}^{\infty} \lambda_k \left( z + \frac{r|\gamma|q}{(r+(k-1)\mu)([1+(k+\alpha-u-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}nC)} z^k \right) \\ &= (1 - \sum_{k=2}^{\infty} \lambda_k) z + \sum_{k=2}^{\infty} \lambda_k \left( z + \frac{r|\gamma|q}{(r+(k-1)\mu)([1+(k+\alpha-u-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}nC)} z^k \right) \\ &= z + \sum_{k=2}^{\infty} \lambda_k z^k \frac{r|\gamma|q}{(r+(k-1)\mu)([1+(k+\alpha-u-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}nC)} \\ &= z + \sum_{k=2}^{\infty} a_k z^k \quad \text{where } a_k = \frac{r|\gamma|q}{(r+(k-1)\mu)([1+(k+\alpha-u-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}nC)} \lambda_k \end{aligned}$$

Claim:  $f(z) \in \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$

$$\begin{aligned} &\sum_{k=2}^{\infty} [r + \mu(k-1)] ([1+(k+\alpha-u-1)\partial]^n(1-\lambda) + \lambda^{n+k-1}nC) |a_k| \\ &= \sum_{k=2}^{\infty} [r + \mu(k-1)] ([1+(k+\alpha-u-1)\partial]^n(1-\lambda) + \lambda^{n+k-1}nC) \frac{r|\gamma|q}{(r+(k-1)\mu)([1+(k+\alpha-u-1)\partial]^n(1-\lambda)+\lambda^{n+k-1}nC)} \lambda_k \\ &= r|\gamma|q \sum_{k=2}^{\infty} \lambda_k \end{aligned}$$

$$= r |\gamma| q (1 - \lambda_1)$$

$$\leq r |\gamma| q$$

From theorem 2.1  $f(z) \in \widetilde{\text{Tr}} (E_\lambda^n; \gamma, \mu, m, q)$

Conversely suppose that  $f(z) \in \widetilde{\text{Tr}} (E_\lambda^n; \gamma, \mu, m, q)$

Setting  $\lambda_k = \frac{[r + \mu(k-1)] ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}C})}{r |\gamma| q} a_k$  and  $\lambda_1 = 1 - \sum_{k=2}^\infty \lambda_k$

$$\sum_{k=1}^\infty \lambda_k f_k(z) = \lambda_1 f_1(z) + \sum_{k=2}^\infty \lambda_k f_k(z)$$

$$= (1 - \sum_{k=2}^\infty \lambda_k) z + \sum_{k=2}^\infty \lambda_k \left( z + \frac{r |\gamma| q}{(r + (k-1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}C})} z^k \right)$$

$$= z + \sum_{k=2}^\infty \lambda_k \frac{r |\gamma| q}{(r + (k-1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}C})} z^k$$

$$= z + \sum_{k=2}^\infty \frac{[r + \mu(k-1)] ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}C}) a_k}{r |\gamma| q} \frac{r |\gamma| q}{(p + (k-1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}C})} z^k$$

$$= z + \sum_{k=2}^\infty a_k z^k$$

$$= f(z)$$

Hence  $f(z) = \sum_{k=1}^\infty \lambda_k f_k(z)$ .

**Corollary 3.2.** The extreme points of the  $\widetilde{\text{Tr}} (E_\lambda^n; \gamma, \mu, m, q)$  are the functions  $f_1(z) = z$  and

$$f_k(z) = z + \frac{r |\gamma| q}{(r + (k-1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}C})} z^k \quad (k=2, 3, 4, \dots)$$

We now turn to Convolution theorem, which gives that the class  $\widetilde{\text{Tr}} (E_\lambda^n; \gamma, \mu, m, q)$  is closed under convolution.

**Theorem 3.6 Convolution Theorems**

Let  $f, g \in \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad \text{with} \quad a_k \geq 0, \quad b_k \geq 0 \quad \text{and} \quad (a_k b_k)^{\frac{1}{2}} < 1$$

Then  $f * g \in \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$

**Proof:** We have  $f \in \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$  if and only if

$$\sum_{k=2}^{\infty} [r + \mu(k - 1)] ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}} C) |a_k| \leq r |\gamma| q$$

$g \in \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$  if and only if

$$\sum_{k=2}^{\infty} [r + \mu(k - 1)] ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}} C) |b_k| \leq r |\gamma| q$$

By Cauchy Schwarz inequality

$$\sum_{k=2}^{\infty} (t_k |a_k| t_k |b_k|)^{\frac{1}{2}} \leq \left( \sum_{k=2}^{\infty} t_k |a_k| \right)^{\frac{1}{2}} \left( \sum_{k=2}^{\infty} (t_k |b_k|)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$\text{Where } t_k = (r + (k-1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}} C)$$

$$\sum_{k=2}^{\infty} (r + (k - 1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}} C) |a_k b_k|^{\frac{1}{2}} \leq r |\gamma| q \quad (3.11)$$

By assumption  $(a_k b_k)^{\frac{1}{2}} < 1$

Then

$$a_k b_k < (a_k b_k)^{\frac{1}{2}} \quad (3.12)$$

Thus from (3.11) and (3.12)

$$\sum_{k=2}^{\infty} (r + (k - 1)\mu) ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}} C) |a_k b_k| < r |\gamma| q$$

Hence  $f * g \in \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$

In our next example we will show that the class  $\widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$  is convex set.

**Example 3.2.** Let  $f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k, f_j(z) \in \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$ , then

$$\text{for } g(z) = \sum_{k=1}^l c_j f_j(z).$$

$$g(z) \in \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q) \quad , \quad \text{where } \sum_{j=1}^l c_j = 1.$$

**Proof:** Let  $f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k$  with  $f_j(z) \in \widetilde{\text{Tr}}(E_\lambda^n; \gamma, \mu, m, q)$ .

$$\sum_{k=2}^{\infty} [r + \mu(k - 1)] ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-\frac{1}{n}} C) |a_{k,j}| \leq r |\gamma| q$$

$$g(z) = \sum_{k=1}^l c_j f_j(z).$$

$$\begin{aligned}
 &= \sum_{k=1}^l c_j (z + \sum_{k=2}^{\infty} a_{k,j} z^k) \\
 &= z + \sum_{k=1}^l c_j \sum_{k=2}^{\infty} a_{k,j} z^k \\
 &= z + \sum_{k=2}^{\infty} z^k \sum_{j=1}^l c_j a_{k,j} \\
 &= z + \sum_{k=2}^{\infty} e_k z^k \quad \text{where } e_k = \sum_{j=1}^l c_j a_{k,j}
 \end{aligned}$$

Claim:  $g(z) \in \widetilde{\text{Tr}}(E_{\lambda}^n; \gamma, \mu, m, q)$

$$\begin{aligned}
 &\sum_{k=2}^{\infty} [r + \mu(k - 1)] ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} C) |e_k| \\
 &= \sum_{k=2}^{\infty} [r + \mu(k - 1)] ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} C) \left| \sum_{j=1}^l c_j a_{k,j} \right| \\
 &\leq \sum_{j=1}^l (c_j \sum_{k=2}^{\infty} [r + \mu(k - 1)] ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} C) |a_{k,j}|)
 \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{j=1}^l c_j r |\gamma|_q \\
 &\leq r |\gamma|_q.
 \end{aligned}$$

Therefore  $g(z) \in \widetilde{\text{Tr}}(E_{\lambda}^n; \gamma, \mu, m, q)$

In our next example we prove Integral Mean Inequality for the class  $\widetilde{\text{Tr}}(E_{\lambda}^n; \gamma, \mu, m, q)$

**Example 3.3.**  $f(z) \in \widetilde{\text{Tr}}(E_{\lambda}^n; \gamma, \mu, m, q)$  and suppose that

$$\sum_{k=2}^{\infty} ([1 + (k + \alpha - u - 1)\partial]^n (1 - \lambda) + \lambda^{n+k-1} C) |a_k| \leq \frac{r |\gamma|_q}{(r+(j-1)\mu)}$$

Also let the function

$$f_j(z) = z + \frac{r |\gamma|_q}{(r+(j-1)\mu) ([1+(j+\alpha-u-1)\partial]^n (1-\lambda) + \lambda^{n+j-1} C)} z^j \quad (j \geq 2) \tag{3.13}$$

Consider the function  $w(z)$  as given bellow

$$w(z)^{j-1} = \frac{(r+(j-1)\mu)}{r |\gamma|_q} \sum_{k=2}^{\infty} [(1 - \lambda) [1 + (k + \alpha - u - 1)\partial]^n + \lambda^{n+k-1} C] a_k z^{k-1}. \tag{3.14}$$

Then for  $z = re^{i\theta}$  with  $0 < r < 1$

$$\int_0^{2\pi} |E_\lambda^n f(z)|^t d\theta \leq \int_0^{2\pi} |E_\lambda^n f_j(z)|^t d\theta \quad (0 \leq \lambda \leq 1, t > 0)$$

Where  $E_\lambda^n f$  is differential operator defined in (2.2)

**Proof:** We have from definition (2.2)

$$E_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [(1-\lambda)[1+(k+\alpha-u-1)\partial]^n + \lambda {}^{n+k-1}C_n] a_k z^k$$

$$D_\lambda^n f_j(z) = z + \frac{r|\gamma|q}{(r+(j-1)\mu)} z^j$$

For  $z = re^{i\theta}$  with  $0 < r < 1$  we have to show that

$$\int_0^{2\pi} |1 + \sum_{k=2}^{\infty} [(1-\lambda)[1+(k+\alpha-u-1)\partial]^n + \lambda {}^{n+k-1}C_n] a_k z^{k-1}|^t d\theta$$

$$\leq \int_0^{2\pi} |1 + \frac{r|\gamma|q}{(r+(j-1)\mu)} z^{j-1}|^t d\theta \quad (t > 0)$$

By applying Littlewoods subordination theorem, it would sufficient to show that

$$1 + \sum_{k=2}^{\infty} [(1-\lambda)[1+(k+\alpha-u-1)\partial]^n + \lambda {}^{n+k-1}C_n] a_k z^{k-1} \prec 1 + \frac{r|\gamma|q}{(1+(j-1)\mu)} z^{j-1}$$

That is  $t(z) \prec h(z)$

Where  $t(z) = 1 + \sum_{k=2}^{\infty} [(1-\lambda)[1+(k+\alpha-u-1)\partial]^n + \lambda {}^{n+k-1}C_n] a_k z^{k-1}$

$$h(z) = 1 + \frac{r|\gamma|q}{(r+(j-1)\mu)} z^{j-1}$$

That is we want to show that  $t(z) = h(w(z))$ ,  $w(0)=0$  and  $|w(z)| \leq 1$

$$h(w(z)) = 1 + \frac{r|\gamma|q}{(r+(j-1)\mu)} w(z)^{j-1}$$

$$\begin{aligned}
&= 1 + \frac{r|\gamma|_q}{(r+(j-1)\mu)} \frac{(r+(j-1)\mu)}{r|\gamma|_q} \sum_{k=2}^{\infty} [(1-\lambda)[1+(k+\alpha-u-1)\partial]^n + \lambda^{n+k-\frac{1}{n}} C] a_k z^{k-1} \\
&= 1 + \sum_{k=2}^{\infty} [(1-\lambda)[1+(k+\alpha-u-1)\partial]^n + \lambda^{n+k-\frac{1}{n}} C] a_k z^{k-1} \\
&= t(z)
\end{aligned}$$

Therefore  $h(w(z)) = t(z)$  and  $w(0) = 0$

Moreover, we prove that analytic function  $|w(z)| < 1, z \in U$

$$\begin{aligned}
|w(z)^{j-1}| &= \left| \frac{(r+(j-1)\mu)}{r|\gamma|_q} \sum_{k=2}^{\infty} [(1-\lambda)[1+(k+\alpha-u-1)\partial]^n + \lambda^{n+k-\frac{1}{n}} C] a_k z^{k-1} \right| \\
&\leq \frac{(r+(j-1)\mu)}{r|\gamma|_q} \sum_{k=2}^{\infty} [(1-\lambda)[1+(k+\alpha-u-1)\partial]^n + \lambda^{n+k-\frac{1}{n}} C] |a_k| |z^{k-1}| \\
&\leq |z| \frac{(r+(j-1)\mu)}{r|\gamma|_q} \sum_{k=2}^{\infty} [(1-\lambda)[1+(k+\alpha-u-1)\partial]^n + \lambda^{n+k-\frac{1}{n}} C] |a_k| \\
&\leq |z| < 1
\end{aligned}$$

Hence proved.

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