

Hankel and Toeplitz determinant for functions of bounded turning associated with sine hyperbolic function

D. Sophia Ponmalar¹, Prathviraj Sharma² and S. Sivasubramanian^{2,*}

¹Department of Mathematics,
Chennai Institute of technology,
Chennai 600069, Tamilnadu, India

E-Mail: sophiaponmalard@citchennai.net

²Department of Mathematics,
University College of Engineering Tindivanam,
Anna University, Tindivanam 604001, Tamilnadu, India

E-Mails: sirprathvi99@gmail.com, sivasaisastha@rediffmail.com

*Corresponding Author

Abstract

In this article, we obtain upper bounds for the Hankel determinant, Toeplitz determinant and logarithmic coefficients for the functions of bounded turning associated with sine hyperbolic function and in the same time Hankel determinant, Toeplitz determinant and logarithmic coefficients for the inverse map.

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1. Introduction

Let \mathcal{A} represent the family of all functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. Also let \mathcal{S} denote the class of functions in \mathcal{A} which are univalent in \mathbb{E} . If $0 \leq \phi < 1$. In 1936, Robertson [23] introduced the

$\mathcal{S}^*(\phi)$ containing all starlike functions of order ϕ . A function f is said to be in the class $\mathcal{S}^*(\phi)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \phi, \quad z \in \mathbb{E}.$$

Similarly, $\mathcal{C}(\phi)$ containing all convex functions of order ϕ . A function f is said to be in the class $\mathcal{C}(\phi)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \phi, \quad z \in \mathbb{E}.$$

It is also known that a function $f \in \mathcal{C}(\phi)$ if and only if $zf' \in \mathcal{S}^*(\phi)$. Let $\mathcal{R}(\phi)$ represent the family of all functions $f \in \mathcal{A}$ satisfying

$$\Re(f'(z)) > \phi, \quad z \in \mathbb{E}.$$

Functions in $\mathcal{R}(\phi)$ are called of bounded turning of order ϕ . For $\phi = 0, \mathcal{R}(\phi) \equiv \mathcal{R}(0) \equiv \mathcal{R}$, is the family of bounded turning functions or sometimes called functions of bounded rotation introduced by Goodman [10]. An analytic function f is subordinate to an analytic function h , written $f(z) \prec h(z)$, if there is a Schwarz function $\omega(z) : \mathbb{E} \rightarrow \mathbb{E}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, satisfying $f(z) = h(\omega(z))$.

In 1966 Pommerenke [19,20] introduced the concept of hankel determinant. The u th Hankel determinant $\mathcal{H}_u(n)$ of the function f are defined by

$$\mathcal{H}_u(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+u+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+u} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+u+1} & a_{n+u} & \cdots & a_{n+2u-2} \end{vmatrix}.$$

Noor [18] determined the rate of growth of $\mathcal{H}_u(n)$ as $n \rightarrow \infty$ for functions $f(z)$ given by (1.1) with bounded boundary and Ehrenborg [8] studied the Hankel determinant of exponential polynomials.

In particular, by varying the parameters u and n , we get

$$\mathcal{H}_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \text{ as } a_1 = 1$$

and

$$\mathcal{H}_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

In 2017 Thomas and Halim [32] gives the concept of Toeplitz determinant. The u^{th} Toeplitz determinant $\mathcal{T}_u(n)$ of the function f are defined by

$$\mathcal{T}_u(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+u-1} \\ a_{n+1} & a_n & \cdots & a_{n+u-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+u+1} & a_{n+u-2} & \cdots & a_n \end{vmatrix}$$

In particular, by varying the parameters u and n , we get

$$\mathcal{T}_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \text{ as } a_1 = 1,$$

$$\mathcal{T}_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix} = a_2^2 - a_3^2$$

and

$$\mathcal{T}_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_2 \\ a_3 & a_2 & a_1 \end{vmatrix} = 1 - 2a_2^2 + 2a_2^2a_3 - a_3^2, \text{ as } a_1 = 1.$$

The Toeplitz determinant $|\mathcal{T}_u(n)|$ for starlike and convex functions studies by Thomas and Halim [32]. Similarly, the Toeplitz determinant $|\mathcal{T}_u(n)|$ for functions of bounded turning studies by Radhika [21]. For additional resources in this area, refer to [7, 14, 22, 31, 33, 34]. For a function $f \in \mathcal{S}$ the logarithmic coefficients γ_n are defined by

$$f(z) = z \exp \left(2 \sum_{n=2}^{\infty} \gamma_n(f) z^n \right), \quad z \in \mathbb{E}. \tag{1.2}$$

It is well known that the logarithmic coefficients play an important role in geometric function theory. In particular, the logarithmic coefficients of the Koebe function $\frac{z}{(1-z)^2}$ are $\gamma_n = \frac{1}{n}$.

For $f \in \mathcal{S}$, Milin conjecture [17], that

$$\sum_{k=1}^n \sum_{v=1}^n \left(v |\gamma_v|^2 - \frac{1}{4} \right) \leq 0.$$

It is interesting that for the class \mathcal{S} the sharp estimates of logarithmic coefficients are known only for the first two γ_1 and γ_2 :

$$|\gamma_1| \leq 1 \quad \text{and} \quad |\gamma_2| \leq \frac{e+2}{2e} = 0.635 \dots,$$

and it is not known for $n \geq 3$. Equating the coefficients of z, z^2 and z^3 in (1.2), we get

$$\gamma_1 = \frac{a_2}{2}, \quad (1.3)$$

$$\gamma_2 = \frac{1}{2} \left(a_3 - \frac{a_2^2}{2} \right) \quad (1.4)$$

and

$$\gamma_3 = \frac{1}{2} \left(a_4 - a_2 a_3 + \frac{a_2^3}{3} \right). \quad (1.5)$$

For any univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \quad z \in \mathbb{E}$$

and

$$f(f^{-1}(w)) = w, \quad |w| < \frac{1}{4}.$$

A function is considered bi-univalent in \mathbb{E} if both f and its inverse f^{-1} are univalent in \mathbb{E} . In the recent past, several subclasses of analytic univalent functions and bi-univalent functions have been identified and researched; see [4, 5, 25, 26, 27, 28, 29, 30] for further details. Let F be the inverse of a function $f \in \mathcal{S}$ defined by

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n, \quad |w| < \frac{1}{4}. \quad (1.6)$$

Hence, from (1.1) and (1.6), we get

$$A_2 = -a_2,$$

$$A_3 = -a_3 + 2a_2^2$$

and

$$A_4 = -a_4 + 5a_2 a_3 - 5a_2^3.$$

Similarly, we can define logarithmic coefficient for inverse function

$$\Gamma_1 = \frac{A_2}{2}, \quad (1.7)$$

$$\Gamma_2 = \frac{1}{2} \left(A_3 - \frac{A_2^2}{2} \right) \tag{1.8}$$

and

$$\Gamma_3 = \frac{1}{2} \left(A_4 - A_2 A_3 + \frac{A_2^3}{3} \right). \tag{1.9}$$

In the context of univalent functions, Krushkal [13] established and demonstrated the inequality represented as

$$\left| a_n^r - a_2^{r(n-1)} \right| \leq 2^{r(n-1)} - 2^r. \tag{1.10}$$

In 1960, Lawrence Zalcman introduced a significant conjecture within Geometric Function Theory, which posits that the coefficients of the functional class \mathcal{S} fulfill the condition:

$$|a_n^2 - a_{2n-1}| \leq (n - 1)^2.$$

This equality is valid for the famous Koebe function $f(z) = \frac{z}{(1-z)^2}$, including its rotated versions.

Let \mathcal{P} denote the class of analytic functions $q(z)$ in \mathbb{E} satisfying the following conditions

$$q(0) = 1 \text{ and } \Re(q(z)) > 0, \quad z \in \mathbb{E}.$$

Hence, if a function $q \in \mathcal{P}$, then the function $q(z)$ given in the form

$$q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n, \quad z \in \mathbb{E}. \tag{1.11}$$

Lemma 1. [6] If the function $q \in \mathcal{P}$ is given in the form (1.11), then

$$|q_n| \leq 2, \quad \forall \quad n \geq 1.$$

Lemma 2. [16] For any $\lambda \in \mathbb{C}$. If the function $q \in \mathcal{P}$ is given in the form (1.11), then

$$|q_2 - \lambda q_1^2| \leq \begin{cases} 2 - 4\lambda & : \quad \lambda \leq 0, \\ 2 & : \quad 0 \leq \lambda \leq 1, \\ 4\lambda - 2 & : \quad \lambda \geq 1. \end{cases}$$

Lemma 3. [1] If the function $q \in \mathcal{P}$ is given in the form (1.11), with $0 \leq b \leq 1$ and $b(2b - 1) \leq d \leq b$, then

$$|q_3 - 2bq_2q_1 + dq_1^3| \leq 2.$$

Lemma 4. [15] If the function $q \in \mathcal{P}$ is given in the form (1.11), then

$$2q_2 = q_1^2 + y(4 - q_1^2)$$

and

$$4q_3 = q_1^3 + 2yq_1(4 - q_1^2) - y^2q_1(4 - q_1^2) + 2(4 - q_1^2)(1 - |y|^2)x,$$

for some y and x such that $|y| \leq 1$ and $|x| \leq 1$.

Inspired by the previously mentioned studies [2, 3, 9, 11, 12, 24], this article presents the class \mathcal{R}_{sinh}^* which is defined as follows:

Definition 1. A function $f \in \mathcal{S}$ is recognized as part of the \mathcal{R}_{sinh}^* class if it complies with the following subordination condition:

$$f'(z) \prec 1 + sinh(z).$$

In similar way, if the function $F = f^{-1}$ given in the form (1.6) belongs to the class \mathcal{R}_{sinh}^* class if it complies with the following subordination condition:

$$F'(w) \prec 1 + sinh(w).$$

In this article, we obtain upper bounds for the Hankel determinant, Toeplitz determinant and logarithmic coefficients for the functions of bounded turning associated with sine hyperbolic function and in the same time Hankel determinant, Toeplitz determinant and logarithmic coefficients for the inverse map.

2. Coefficient bounds and Fekete-Szegö inequality

In this section, we will examine several concerning coefficient bounds and the Fekete-Szegö inequality relevant to the function $f \in \mathcal{R}_{sinh}^*$ and $f^{-1} = F \in \mathcal{R}_{sinh}^*$.

Theorem 1. If a function $f \in \mathcal{R}_{sinh}^*$ given in the form (1.1), then

$$|a_2| \leq \frac{1}{2}, \tag{2.1}$$

$$|a_3| \leq \frac{1}{3} \tag{2.2}$$

and

$$|a_4| \leq \frac{1}{4}. \tag{2.3}$$

The equalities of these inequalities are derived for functions in the following manner:

$$f_1(z) = \cosh(z) + z - 1,$$

$$f_2(z) = \int_0^z (1 + \sinh(s^2)) ds$$

and

$$f_3(z) = \int_0^z (1 + \sinh(s^3)) ds.$$

Proof. If $f \in \mathcal{R}_{sinh}^*$, then according to Definition 1, there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, we have

$$f'(z) = 1 + \sinh(\omega(z)).$$

Let us consider the function $q \in \mathcal{P}$ and

$$q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n = \frac{1 + \omega(z)}{1 - \omega(z)}.$$

Clearly, we have

$$\omega(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{q_1 z + q_2 z^2 + q_3 z^3 + \dots}{2 + q_1 z + q_2 z^2 + q_3 z^3 + \dots}.$$

Now

$$\begin{aligned} 1 + \sinh(\omega(z)) &= 1 + \sinh\left(\frac{q(z) - 1}{q(z) + 1}\right) \\ &= 1 + \frac{1}{2} q_1 z + \frac{1}{2} \left(q_2 - \frac{1}{2} q_1^2\right) z^2 + \left(\frac{1}{3} q_3 - \frac{1}{2} q_1 q_2 + \frac{7}{48} q_1^3\right) z^3 \\ &\quad + \left(\frac{1}{2} q_4 - \frac{1}{2} q_3 q_1 + \frac{7}{16} q_2 q_1^2 - \frac{1}{4} q_2^2 - \frac{3}{32} q_1^4\right) z^4 + \dots \end{aligned} \tag{2.4}$$

Also,

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \dots \tag{2.5}$$

From (2.4) and (2.5), comparing the coefficient of z, z^2, z^3 and z^4 , we have

$$2a_2 = \frac{1}{2} q_1, \tag{2.6}$$

$$3a_3 = \frac{1}{2} \left(q_2 - \frac{1}{2} q_1^2 \right) \quad (2.7)$$

and

$$3a_4 = \frac{1}{3} q_3 - \frac{1}{2} q_1 q_2 + \frac{7}{48} q_1^3. \quad (2.8)$$

An application of Lemma 1 and Lemma 2 in (2.6) and (2.7), we get results given in (2.1) and (2.2), respectively. An application of Lemma 3 in (2.8), we get results given in (2.3). This completes the proof of Theorem 1.

Theorem 2. If a function $f \in \mathcal{R}_{sinh}^*$ given in the form (1.1), then for any $\lambda \in \mathbb{C}$, we have

$$|a_3 - \lambda a_2^2| \leq \begin{cases} -\frac{\lambda}{4}, & \lambda \leq -\frac{4}{3} \\ \frac{1}{3}, & -\frac{4}{3} \leq \lambda \leq \frac{4}{3} \\ \frac{\lambda}{4}, & \lambda \geq \frac{4}{3}. \end{cases} \quad (2.9)$$

Proof. For any $\lambda \in \mathbb{C}$, and from (2.6) and (2.7), we get

$$a_3 - \lambda a_2^2 = \frac{1}{6} \left[q_2 - \left(\frac{4 + 3\lambda}{8} \right) q_1^2 \right]. \quad (2.10)$$

Now by using triangle inequality in (2.10), we get

$$|a_3 - \lambda a_2^2| \leq \frac{1}{6} \left| q_2 - \left(\frac{4 + 3\lambda}{8} \right) q_1^2 \right|. \quad (2.11)$$

An application of Lemma 2, we get results given in (2.9). This completes the proof of Theorem 2.

When we assign $\lambda = 1$ in Theorem 2, the resulting corollary is as follows:

Corollary 1. If a function $f \in \mathcal{R}_{sinh}^*$ given in the form (1.1), then

$$|a_3 - a_2^2| \leq \frac{1}{3} < 1.$$

Theorem 3. If a function $f \in \mathcal{R}_{sinh}^*$. Let F given by (1.6) be the inverse function of f , then

$$|A_2| \leq \frac{1}{2}, \quad (2.12)$$

$$|A_2| \leq \frac{1}{2} \tag{2.13}$$

and

$$|A_4| \leq 1. \tag{2.14}$$

Proof. Substituting, we get

$$A_2 = -\frac{1}{4}q_1, \tag{2.15}$$

$$A_3 = -\frac{1}{6}\left(q_2 - \frac{5}{4}q_1^2\right) \tag{2.16}$$

and

$$A_4 = -\frac{1}{2}\left(q_3 - \frac{17}{12}q_1q_2 + \frac{21}{32}q_1^3\right). \tag{2.17}$$

Now by using Lemma 1 and Lemma 2, in (2.15) and (2.16) gives (2.12) and (2.13), respectively. Now, from (2.17), we have

$$|A_4| = \frac{1}{2} |q_3 - 2bq_2q_1 + dq_1^3|. \tag{2.18}$$

where

$$b = \frac{17}{24} \quad \text{and} \quad d = \frac{21}{32}.$$

We can see that

$$\frac{85}{288} = b(2b - 1) \leq d \leq b.$$

Therefore, by using Lemma 3, in (2.18) we get (2.14). This completes the proof of Theorem 3.

Theorem 4. If a function $f \in \mathcal{R}_{sinh}^*$. Let F given by (1.6) be the inverse function of f , then for any $\lambda \in \mathbb{C}$, we have

$$|A_3 - \lambda A_2^2| \leq \begin{cases} \frac{2 - \lambda}{4}, & \lambda \leq \frac{2}{3} \\ \frac{1}{3}, & \frac{2}{3} \leq \lambda \leq \frac{11}{3} \\ \frac{\lambda - 2}{4}, & \lambda \geq \frac{11}{3}. \end{cases} \tag{2.19}$$

Proof. For any $\lambda \in \mathbb{C}$, and from (2.15) and (2.16), we get

$$A_3 - \lambda A_2^2 = -\frac{1}{6} \left[q_2 - \left(\frac{10 - 3\lambda}{8} \right) q_1^2 \right]. \tag{2.20}$$

Now by using triangle inequality in (2.10), we get

$$|A_3 - \lambda A_2^2| \leq \frac{1}{6} \left| q_2 - \left(\frac{10 - 3\lambda}{8} \right) q_1^2 \right|. \tag{2.11}$$

An application of Lemma 2, we get results given in (2.9). This completes the proof of Theorem 2.

When we assign $\lambda = 1$ in Theorem 2, the resulting corollary is as follows:

Corollary 2. If a function $f \in \mathcal{R}_{sinh}^*$. Let F given by (1.6) be the inverse function of f , then

$$|A_3 - A_2^2| \leq \frac{1}{3} < 1.$$

Remark 1. Corollary 1 and Corollary 2 are both satisfies the Zalcman conjecture.

3. Hankel determinant

Theorem 5. If a function $f \in \mathcal{R}_{sinh}^*$ given in the form (1.1), then

$$|a_2 a_4 - a_3^2| \leq \frac{1}{9}. \tag{3.1}$$

Proof. From (2.6), (2.7) and (2.8), we get

$$|a_2 a_4 - a_3^2| = \frac{1}{2304} |72q_3 q_1 - 64q_2^2 - 8q_2 q_1^2 + 5q_1^4|.$$

Let us assume that $q_1 = q \in [0, 2]$. Now by using Lemma 4, to express q_2 and q_3 in term of $q_1 = q$, we get

$$|a_2 a_4 - a_3^2| = \frac{1}{2304} |3q^4 + 36q(4 - q^2)(1 - |y|^2)x - 18q^2(4 - q^2)y^2 - 16y^2(4 - q^2)^2|.$$

Let us assume that $|y| = \delta, \delta \in [0, 1]$ and $|x| \leq 1$, we get

$$\begin{aligned} |a_2 a_4 - a_3^2| &\leq \frac{1}{2304} |3q^4 + 36q(4 - q^2)(1 - \delta^2) + 18q^2(4 - q^2)\delta^2 + 16\delta^2(4 - q^2)^2| \\ &= \Phi(\delta, q). \end{aligned}$$

Differentiating $\Phi(\delta, q)$ with respect to δ , we get

$$\frac{\partial \Phi}{\partial \delta} = \frac{1}{576} \delta(4 - q^2)(q - 2)(q - 16) \geq 0.$$

As $\frac{\partial \Phi}{\partial \delta} \geq 0$ on $\delta \in [0, 1]$, therefore $\Phi(\delta, q)$ is increasing on $\delta \in [0, 1]$ and hence $\Phi(\delta, q) \leq \Phi(1, q)$. Putting $\delta = 1$, and assume $\vartheta = q^2, \vartheta \in [0, 4]$, we get

$$\Psi(\vartheta) := \Phi(1, q^2) := \frac{1}{2304} [256 - 56\vartheta + \vartheta^2].$$

Differentiating $\Psi(\vartheta)$ with respect to ϑ , we get

$$\Psi'(\vartheta) = \frac{1}{1152} [\vartheta - 28] < 0.$$

As $\Psi'(\vartheta) < 0$ on $\vartheta \in [0, 4]$, therefore $\Psi(\vartheta)$ is decreasing on $\vartheta \in [0, 4]$. Therefore, the maximum value of Ψ is $\frac{1}{9}$ on $\vartheta = 0$. Hence, we obtain

$$|a_2 a_4 - a_3^2| \leq \frac{1}{9}.$$

This completes the proof of Theorem 5.

Theorem 6. If a function $f \in \mathcal{R}_{sinh}^*$. Let F given by (1.6) be the inverse function of f , then

$$\mathcal{H}_2(2)f^{-1} := |A_2 A_4 - A_3^2| = \frac{1}{9}.$$

Proof. From (2.15), (2.16) and (2.17), we get

$$|A_2 A_4 - A_3^2| = \frac{1}{1152} |36q_3 q_1 - 32q_2^2 + 13q_1^4 - 16q_2 q_1^2|.$$

Let us assume that $q_1 = q \in [0, 2]$. Now by using Lemma 4, to express q_2 and q_3 in term of $q_1 = q$, we get

$$|A_2 A_4 - A_3^2| = \frac{1}{1152} |6q^4 - 6yq^2(4 - q^2) - 9y^2q^2(4 - q^2) - 8y^2(4 - q^2)^2 + 18q(4 - q^2)(1 - |y|^2)x|.$$

Let us assume that $|y| = \delta, \delta \in [0, 1]$ and $|x| \leq 1$, we get

$$|A_2 A_4 - A_3^2| \leq \frac{1}{1152} |6q^4 + 6\delta q^2(4 - q^2) + 9\delta^2 q^2(4 - q^2) + 8\delta^2(4 - q^2)^2 + 18q(4 - q^2)(1 - \delta^2)| := \Phi(\delta, q).$$

Differentiating $\Phi(\delta, q)$ with respect to δ , we get

$$\frac{\partial \Phi}{\partial \delta} = \frac{1}{1152} 2\delta(4 - q^2)(q - 2)(q - 16) + 6q^2(4 - q^2).$$

As $\frac{\partial \Phi}{\partial \delta} \geq 0$ on $\delta \in [0, 1]$, therefore $\Phi(\delta, q)$ is increasing on $\delta \in [0, 1]$ and hence $\Phi(\delta, q) \leq \Phi(1, q)$. Putting $\delta = 1$, and assume $\vartheta = q^2, \vartheta \in [0, 4]$, we get

$$\Psi(\vartheta) := \Phi(1, q^2) := \frac{1}{1152} [128 - 4\vartheta - \vartheta^2].$$

Differentiating $\Psi(\vartheta)$ with respect to ϑ , we get

$$\Psi'(\vartheta) = -\frac{1}{576} [\vartheta + 2] < 0.$$

As $\Psi'(\vartheta) < 0$ on $\vartheta \in [0, 4]$, therefore $\Psi(\vartheta)$ is decreasing on $\vartheta \in [0, 4]$. Therefore, the maximum value of Ψ is $\frac{1}{9}$ on $\vartheta = 0$. Hence, we obtain

$$|A_2 A_4 - A_3^2| \leq \frac{1}{9}.$$

This completes the proof of Theorem 6.

4. Toeplitz determinant

Theorem 7. If a function $f \in \mathcal{R}_{sinh}^*$ given in the form (1.1), the

$$|a_2^2 - a_3^2| \leq \frac{1}{4}. \tag{4.1}$$

Proof. From (2.6) and (2.7), we get

$$|a_2^2 - a_3^2| = \frac{1}{144} |9q_1^2 + 4q_2q_1^2 - 4q_2^2 - q_1^4|.$$

Let us assume that $q_1 = q \in [0, 2]$. Now by using Lemma 4, to express q_2 and q_3 in term of $q_1 = q$, we get

$$|a_2^2 - a_3^2| = \frac{1}{144} |9q^2 - y^2(4 - q^2)^2|.$$

Let us assume that $|y| = \delta, \delta \in [0, 1]$ and $|x| \leq 1$, we get

$$|a_2^2 - a_3^2| = \frac{1}{144} |9q^2 + \delta^2(4 - q^2)^2| := \Phi(\delta, q).$$

Differentiating $\Phi(\delta, q)$ with respect to δ , we get

$$\frac{\partial \Phi}{\partial \delta} = \frac{1}{72} \delta(4 - q^2) \geq 0.$$

As $\frac{\partial \Phi}{\partial \delta} \geq 0$ on $\delta \in [0, 1]$, therefore $\Phi(\delta, q)$ is increasing on $\delta \in [0, 1]$ and hence $\Phi(\delta, q) \leq \Phi(1, q)$. Putting $\delta = 1$, and assume $\vartheta = q^2, \vartheta \in [0, 4]$, we get

$$\Psi(\vartheta) := \Phi(1, q^2) := \frac{1}{144} [\vartheta^2 + \vartheta + 16].$$

Differentiating $\Psi(\vartheta)$ with respect to ϑ , we get

$$\Psi'(\vartheta) = \frac{1}{576} [1 + 2\vartheta] > 0.$$

As $\Psi'(\vartheta) > 0$ on $\vartheta \in [0, 4]$, therefore $\Psi(\vartheta)$ is increasing on $\vartheta \in [0, 4]$. Therefore, the maximum value of Ψ is $\frac{1}{4}$ on $\vartheta = 4$. Hence, we obtain

$$|a_2^2 - a_3^2| \leq \frac{1}{4}.$$

This completes the proof of Theorem 7.

Theorem 8. If a function $f \in \mathcal{R}_{sinh}^*$ given in the form (1.1), the

$$\mathcal{J}_3(1) := |1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2| \leq \frac{5}{3}.$$

Proof. Since $f \in \mathcal{R}_{sinh}^*$ given in the form (1.1), then

$$\mathcal{J}_3(1) := |1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2|.$$

Taking modulus and applying the triangle inequality, we get

$$|\mathcal{J}_3(1)| \leq 1 + 2|a_2|^2 + |a_3| |a_3 - 2a_2^2|.$$

Using (2.1), (2.2) and applying Theorem 2, we get

$$|\mathcal{J}_3(1)| \leq 1 + \frac{1}{2} + \frac{1}{6} = \frac{5}{3}.$$

This completes the proof of Theorem 8.

Theorem 9. If a function $f \in \mathcal{R}_{sinh}^*$. Let F given by (1.6) be the inverse function of f , then

$$|A_2^2 - A_3^2| \leq \frac{1}{2}.$$

Proof. From (2.15) and (2.16), we get

$$|A_2^2 - A_3^2| = \frac{1}{576} |36q_1^2 - 16q_2^2 - 25q_1^4 + 40q_2q_1^2|.$$

Let us assume that $q_1 = q \in [0, 2]$. Now by using Lemma 4, to express q_2 and q_3 in term of $q_1 = q$, we get

$$|A_2^2 - A_3^2| = \frac{1}{576} |36q^2 - 9q^4 - 4y^2(4 - q^2)^2 + 12yq^2(4 - q^2)|.$$

Let us assume that $|y| = \delta, \delta \in [0, 1]$ and $|x| \leq 1$, we get

$$|A_2^2 - A_3^2| \leq \frac{1}{576} [36q^2 + 9q^4 + 4\delta^2(4 - q^2)^2 + 12\delta q^2(4 - q^2)] := \Phi(\delta, q).$$

Differentiating $\Phi(\delta, q)$ with respect to δ , we get

$$\frac{\partial \Phi}{\partial \delta} = \frac{1}{72} \delta(4 - q^2)^2 3q^2(4 - q^2) \geq 0.$$

As $\frac{\partial \Phi}{\partial \delta} \geq 0$ on $\delta \in [0, 1]$, therefore $\Phi(\delta, q)$ is increasing on $\delta \in [0, 1]$ and hence $\Phi(\delta, q) \leq \Phi(1, q)$. Putting $\delta = 1$, and assume $\vartheta = q^2, \vartheta \in [0, 4]$, we get

$$\Psi(\vartheta) := \Phi(1, q^2) := \frac{1}{576} [\vartheta^2 + 52\vartheta + 64].$$

Differentiating $\Psi(\vartheta)$ with respect to ϑ , we get

$$\Psi'(\vartheta) = \frac{1}{288} [1 + 104\vartheta] > 0.$$

As $\Psi'(\vartheta) > 0$ on $\vartheta \in [0, 4]$, therefore $\Psi(\vartheta)$ is increasing on $\vartheta \in [0, 4]$. Therefore, the maximum value of Ψ is $\frac{1}{2}$ on $\vartheta = 4$. Hence, we obtain

$$|A_2^2 - A_3^2| \leq \frac{1}{2}.$$

This completes the proof of Theorem 9.

Theorem 10. If a function $f \in \mathcal{R}_{sinh}^*$. Let F given by (1.6) be the inverse function of f , then

$$\mathcal{T}_3(1) := |1 - 2A_2^2 + 2A_2^2A_3 - A_3^2| \leq \frac{5}{3}.$$

Proof. Since $\in \mathcal{R}_{sinh}^*$ given in the form (1.1), then

$$\mathcal{T}_3(1) := |1 - 2A_2^2 + 2A_2^2A_3 - A_3^2|.$$

Taking modulus and applying the triangle inequality, we get

$$|\mathcal{T}_3(1)| \leq 1 + 2|A_2|^2 + |A_3||A_3 - 2A_2^2|.$$

Using (2.12), (2.13) and applying Theorem 4, we get

$$|\mathcal{T}_3(1)| \leq 1 + \frac{1}{2} + \frac{1}{6} = \frac{5}{3}.$$

This completes the proof of Theorem 10.

5. Logarithmic coefficient

Now, next we examine the logarithmic coefficients for the function $f \in \mathcal{R}_{sinh}^*$.

Theorem 11. If a function $f \in \mathcal{R}_{sinh}^*$ given in the form (1.1), then

$$|\gamma_1| \leq \frac{1}{4}, \quad (5.1)$$

$$|\gamma_2| \leq \frac{1}{6} \quad (5.2)$$

and

$$|\gamma_3| \leq \frac{1}{8}. \quad (5.3)$$

Proof. Substituting (2.6), (2.7) and (2.8) into (1.3), (1.4) and (1.5), we get

$$\gamma_1 = \frac{1}{8}q_1, \quad (5.4)$$

$$\gamma_2 = \frac{1}{12} \left(q_2 - \frac{11}{16}q_1^2 \right) \quad (5.5)$$

and

$$\gamma_3 = \frac{1}{16} \left(q_3 - \frac{4}{3}q_1q_2 + \frac{1}{2}q_1^3 \right). \quad (5.6)$$

Now by using Lemma 1 and Lemma 2, in (5.4) and (5.5) gives (5.1) and (5.2), respectively.

Now, from (5.6), we have

$$|\gamma_3| = \frac{1}{2} |q_3 - 2bq_2q_1 + dq_1^3|. \quad (5.7)$$

where

$$b = \frac{2}{3} \quad \text{and} \quad d = \frac{1}{2}.$$

We can see that

$$\frac{2}{9} = b(2b - 1) \leq d \leq b.$$

Therefore, by using Lemma 3, in (5.7) we get (5.3). This completes the proof of Theorem 11.

Now, next we examine the logarithmic coefficients for the inverse function $f^{-1} \equiv F$.

Theorem 12. If a function $f \in \mathcal{R}_{sinh}^*$. Let F given by (1.6) be the inverse function of f , then

$$|\Gamma_1| \leq \frac{1}{4}, \tag{5.8}$$

$$|\Gamma_2| \leq \frac{3}{8} \tag{5.9}$$

and

$$|\Gamma_3| \leq \frac{37}{48}.$$

Proof. Substituting (2.15), (2.16) and (2.17) into (1.7), (1.8) and (1.9), we get

$$\Gamma_1 = -\frac{1}{4}q_1, \tag{5.10}$$

$$\Gamma_2 = -\frac{1}{6}\left(q_2 - \frac{17}{16}q_1^2\right) \tag{5.11}$$

and

$$\Gamma_3 = -\frac{1}{384}(24q_3 + 56q_1q_2 + 31q_1^3). \tag{5.12}$$

Now, using Lemma 1 in (5.10) and Lemma 2 in (5.11), we get (5.8) and (5.9), respectively. Let us assume that $q_1 = q \in [0, 2]$. Now by using Lemma 4, to express q_2 and q_3 in term of $q_1 = q$ in (5.12), we get

$$\Gamma_3 = -\frac{1}{384}[67q^3 + 44qy(4 - q^2) - 8qy^2(4 - q^2) + 16(4 - q^2)(1 - |y|^2)x].$$

Let us assume that $|y| = \delta \in [0, 1]$ and $|x| \leq 1$, we get

$$|\Gamma_3| \leq \frac{1}{384}[67q^3 + 44q\delta(4 - q^2) - 8q\delta^2(4 - q^2) + 16(4 - q^2)(1 - \delta^2)x].$$

$$:= \Phi(\delta, q).$$

Differentiating $\Phi(\delta, q)$ with respect to δ , we get

$$\frac{\partial \Phi}{\partial \delta} = \frac{1}{384} [44q(4 - q^2) + 16q\delta(4 - q^2) - 32(4 - q^2)\delta] \geq 0.$$

As $\frac{\partial \Phi}{\partial \delta} \geq 0$ on $\delta \in [0, 1]$, therefore $\Phi(\delta, q)$ is increasing on $\delta \in [0, 1]$ and hence $\Phi(\delta, q) \leq \Phi(1, q)$. Putting $\delta = 1$, we get

$$\Psi(q) := \Phi(1, q) := \frac{1}{384} [208q - 15q^3].$$

Differentiating $\Psi(\vartheta)$ with respect to q , we get

$$\Psi'(q) = \frac{1}{576} [208q - 45q^2] > 0.$$

As $\Psi'(q) > 0$ on $q \in [0, 2]$, therefore $\Psi(q)$ is increasing on $\vartheta \in [0, 2]$. Therefore, the maximum value of Ψ is $\frac{37}{48}$ on $\vartheta = 2$. Hence, we obtain

$$|\Gamma_3| \leq \frac{37}{48}.$$

This completes the proof of Theorem 12.

6. Krushkal Inequality

The following theorems confirm the validity of the inequality (1.10) for the instances where $n = 4$ and $r = 1$.

Theorem 13. If a function $f \in \mathcal{R}_{sinh}^*$ given in the form (1.1), then

$$|a_4 - a_2^3| \leq \frac{1}{4}. \tag{6.1}$$

Proof. From (2.6) and (2.8), we get

$$|a_4 - a_2^3| = \frac{1}{8} |q_3 - 2bq_1q_2 + dq_1^3|, \tag{6.2}$$

where

$$b = \frac{1}{2} \quad \text{and} \quad d = \frac{1}{6}.$$

We can see that

$$0 = b(2b - 1) \leq \frac{1}{6} \leq \frac{1}{2}.$$

Therefore, by using Lemma 3, in (6.2), we get (6.1). This completes the proof of Theorem 13.

Theorem 14. If a function $f \in \mathcal{R}_{sinh}^*$. Let F given by (1.6) be the inverse function of f , then

$$|A_4 - A_2^3| \leq 1. \quad (6.3)$$

Proof. From (2.15) and (2.17), we get

$$|A_4 - A_2^3| = \frac{1}{8} |q_3 - 2bq_1q_2 + dq_1^3|, \quad (6.4)$$

where

$$b = \frac{17}{24} \quad \text{and} \quad d = \frac{41}{64}.$$

We can see that

$$\frac{85}{288} = b(2b - 1) \leq \frac{41}{64} \leq \frac{17}{24}.$$

Therefore, by using Lemma 3, in (6.4), we get (6.3). This completes the proof of Theorem 14.

Concluding remarks and observations

In this paper, the authors present a new family of bounded turning functions associated with the sine hyperbolic function and obtained sharp upper bounds of some of the initial coefficients, including the Fekete-Szegő inequality, Hankel determinant, Toeplitz determinant, logarithmic coefficient and Krushkal inequality. The study also suggests that by employing q -calculus for values of $0 < q < 1$, along with functions that exhibit bounded boundary and bounded radius rotation, one can define a functional class. The results obtained in this study open new avenues for future research.

Author Contributions

Conceptualization: P.S. and S.S.; original draft preparation: D.S.P. and P.S.; writing—review and editing: P.S. and S.S.; investigation: D.S.P., P.S. and S.S. All authors have read and agreed to the published version of the manuscript.

Data Availability Statement

No data were used to support this study.

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Conflict of interest

The authors declare no conflicts of interest.

References

- [1] R. M. Ali, Coefficients of the inverse of strongly starlike functions, *Bull. Malays. Math. Sci. Soc.* (2) 26 (2003), no. 1, 63–71.
- [2] A. Alotaibi, M. Arif, M. A. Alghamdi and S. Hussain, Starlikeness associated with cosine hyperbolic function, *Mathematics*, 8(2020), 1118.
- [3] M. Arif, M. Raza, H. Tang, S. Hussain and H. Khan, Hankel determinant of order three for familiar subsets of analytic functions related with sine function, *Open Math.* 17 (2019), no. 1, 1615–1630.
- [4] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, in *Mathematical analysis and its applications (Kuwait, 1985)*, 53–60, *KFAS Proc. Ser.*, 3, Pergamon, Oxford.
- [5] D. Breaz, P. Sharma, S. Sivasubramanian, and S.M. El-Deeb, On a new class of bi-close-to-convex functions with bounded boundary rotation. *Mathematics*, 11(20) (2023), 4376.
- [6] C. Caratheodory, Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werten nicht annehmen, *Math. Ann.* 64 (1907), no. 1, 95–115.
- [7] N. E. Cho, B. Kowalczyk, O.S. Kwon, A. Lecko and Y.J. Sim, The bounds of some determinants for starlike functions of order α , *Bull. Malays. Math. Sci. Soc.* 41 (2018), no. 1, 523–535.
- [8] R. Ehrenborg, The Hankel determinant of exponential polynomials, *Amer. Math. Monthly* 107 (2000), no. 6, 557–560.
- [9] P. Goel and S. S. Kumar, Certain class of starlike functions associated with modified sigmoid function, *Bull. Malays. Math. Sci. Soc.* 43 (2020), no. 1, 957–991.
- [10] A. W. GOODMAN, *Univalent Functions*, Mariner, Tampa, Florida, 1983.
- [11] W. Janowski, Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math.* 23 (1970/71), 159–177.
- [12] M. G. Khan, N. E. Cho, T. G. Shaba, B. Ahemad and W. K. Mashwani, Coefficient functionals for a class of bounded turning functions related to modified sigmoid function, *AIMS Math.* 7 (2022), no. 2, 3133–3149.
- [13] S. K. Krushkal, A short geometric proof of the Zalcman and Bieberach conjectures, *arXiv.*,

<https://arxiv.org/abs/1408.1948>.

- [14] O. S. Kwon, A. Lecko and Y. J. Sim, The bound of the Hankel determinant of the third kind for starlike functions, *Bull. Malays. Math. Sci. Soc.* 42 (2019), no. 2, 767–780.
- [15] R. J. Libera and E. J. Zlotkiewicz, Early coefficients of the inverse of a regular convex function, *Proc. Amer. Math. Soc.* 85 (1982), no. 2, 225–230.
- [16] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, *Conf. Proc. Lecture Notes Anal., I*, Int. Press, Cambridge, MA.
- [17] I. M. Milin, *Univalent functions and orthonormal systems* Russian, Izdat, “Nauka”, Moscow, 1971.
- [18] K. I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation, *Rev. Roumaine Math. Pures Appl.* 28 (1983), no. 8, 731–739.
- [19] C. Pommerenke, On the coefficients and Hankel determinants of univalent functions, *J. London Math. Soc.* 41 (1966), 111–122.
- [20] C. Pommerenke, On the Hankel determinants of univalent functions, *Mathematika* 14 (1967), 108–112.
- [21] V. Radhika, S. Sivasubramanian, G. Murugusundaramoorthy and J.M. Jahangiri, Toeplitz matrices whose elements are the coefficients of functions with bounded boundary rotation, *J. Complex Anal.* 2016, Art. ID 4960704, 4 pp.
- [22] A. Riaz, M. Raza and D. K. Thomas, Hankel determinants for starlike and convex functions associated with sigmoid functions, *Forum Math.* 34 (2022), no. 1, 137–156.
- [23] M. S. Robertson, Certain classes of starlike functions, *Michigan Mathematical Journal*, 32 (1985), no. 2, 135–140.
- [24] K. Sharma, N. K. Jain and V. Ravichandran, Starlike functions associated with a cardioid, *Afr. Mat.* 27 (2016), no. 5-6, 923–939.
- [25] P. Sharma, A. Alharbi, S. Sivasubramanian and S.M. El-Deeb, On Ozaki close-to-convex functions with bounded boundary rotation, *Symmetry* 2024, 16, 839.
- [26] P. Sharma, A. Murugan and S. Sivasubramanian, Some coefficient bounds of certain subclasses of bi-univalent functions associated with lemniscate of Bernoulli, *Rom. J. Math. Comput. Sci.* 15 (2025), no. 1, 59–73.

- [27] P. Sharma, S. Sivasubramanian, G. Murugusundaramoorthy and N. E. Cho, On a new class of concave bi-univalent functions associated with bounded boundary rotation, *Mathematics* 2025, 13, 370.
- [28] P. Sharma, S. Sivasubramanian and N. E. Cho, Initial coefficient bounds for certain new subclasses of bi-univalent functions with bounded boundary rotation, *AIMS Math.* 8 (2023), no. 12, 29535–29554.
- [29] P. Sharma, S. Sivasubramanian and N. E. Cho, initial coefficient bounds for certain new subclasses of bi-bazilevic functions and exponentially bi-convex functions with bounded boundary rotation. *Axioms.* 2024; 13(1):25.
- [30] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.* 23 (2010), no. 10, 1188–1192.
- [31] H. M. Srivastava, G. Kaur and G. Singh, Estimates of the fourth Hankel determinant for a class of analytic functions with bounded turnings involving cardioid domains, *J. Nonlinear Convex Anal.* 22 (2021), no. 3, 511–526.
- [32] D. K. Thomas and S. Abdul Halim, Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions, *Bull. Malays. Math. Sci. Soc.* 40 (2017), no. 4, 1781–1790.
- [33] Z. G. Wang et al., On the third and fourth Hankel determinants for a subclass of analytic functions, *Bull. Malays. Math. Sci. Soc.* 45 (2022), no. 1, 323–359.
- [34] P. Zaprawa, Third Hankel determinants for subclasses of univalent functions, *Mediterr. J. Math.* 14 (2017), no. 1, Paper No. 19, 10 pp.
- [35] P. Zaprawa, M. Obradović and N. Tuneski, Third Hankel determinant for univalent starlike functions, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* 115 (2021), no. 2, Paper No. 49, 6 pp