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Development of High-Order Adaptive Spectral–Galerkin Schemes for Multi-Dimensional Nonlinear Fractional Differential Equations

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Abstract

This paper presents an adaptive symmetric pseudo-spectral fractional operator for temporal approximation and a high-order adaptive spectral–Galerkin framework with Legendre polynomial bases for spatial discretization that offers better accuracy, unconditional stability, and less memory. The anticipated fifth-order precision in time and spectral convergence in space are confirmed by numerical tests on multi-dimensional nonlinear fractional Burgers and Fisher equations. The suggested strategy offers a dependable and scalable way to solve complicated fractional systems in higher dimensions, and it exhibits outstanding efficiency and durability.

Keywords: Fractional differential equations; Spectral–Galerkin method; Adaptive pseudo-spectral operator; High-order numerical scheme; Nonlinear multi-dimensional systems.

1. Introduction

In order to describe anomalous diffusion, viscoelasticity, and nonlocal transport processes that are not well represented by traditional integer-order formulations, fractional differential equations (FDEs) have become a potent mathematical model [1, 2, 3]. Realistic modeling of phenomena where the current state depends on the system's whole history is made possible by the fractional operators, which incorporate memory and hereditary effects. Despite these benefits, there are significant difficulties in numerically solving nonlinear and multi-dimensional FDEs. Consequently, creating precise, stable, and computationally efficient numerical solvers with Nystrom methods for such equations remains an open and active research area in numerical analysis [4].

A wide range of numerical techniques has been developed for fractional equations. Early approaches, including Grünwald–Letnikov and L1-type finite difference schemes, are conceptually simple but limited to first or second order of accuracy and often unstable over long integration times. Spectral and pseudo-spectral methods, based on Chebyshev or Legendre

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polynomials, later improved accuracy for one-dimensional integral equations problems by exploiting global polynomial bases with exponential convergence properties [5, 6, 7].

However, their extension to multi-dimensional and nonlinear systems significantly increases computational cost and complexity. Hybrid methods combining finite differences with spectral or finite element formulations with adaptive schemes have been proposed [8, 9] but they typically achieve only moderate accuracy and are not easily generalizable. In particular, few existing studies address high-order adaptive algorithms for nonlinear fractional equations in three or more dimensions, leaving a critical research gap [10, 11]. Euler discretization methods for the numerical solution of nonlinear functional equations of Urysohn type is considered in [12] and also, Sawi transform of high order differential equations is introduced in [15].

2. Mathematical Model and Numerical Formulation

2.1 Governing Equation

We consider a general form of a multi-dimensional nonlinear time-fractional differential equation defined on a bounded spatial domain [13, 14]

$$\Omega \subset \mathbb{R}^d, \qquad d = 2,3.$$

with temporal interval $t \epsilon [0, T]$:

$$\begin{cases} D_t^{\alpha} u(x,t) = \nabla.(k(x)\nabla u(x,t)) + f(u,x,t), & x \in \Omega, t > 0, \\ u(x,0) = g_0(x), & x \in \Omega, \\ u(x,t) = g_b(x), & x \in \partial\Omega, \end{cases}$$

where D_t^{α} denotes the Caputo fractional derivative of order $\alpha \in (0,1)$:

$$D_t^{\alpha} u(x.t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u(x,\tau)}{\partial \tau} d\tau.$$

Here, $k(x) > 0$ is the diffusion coefficient, $f(u,x,t)$ a nonlinear source term, g_0 the initial condition, and g_b the prescribed boundary condition.

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This formulation encompasses many well-known models, such as the fractional Burgers, fractional Fisher–KPP, and fractional reaction–diffusion equations, which are often used to describe transport and pattern formation in complex media.

2.2 Temporal Discretization: Adaptive Symmetric Pseudo-Spectral Operator

The fractional time derivative is approximated using a high-order adaptive symmetric pseudo-spectral (ASPS) operator.

Let the temporal domain $[0, T]$ be partitioned into N_t subintervals with nodes

$$t_j = j\Delta t, \qquad j = 0,1, \dots, N_t, \qquad \Delta t = \frac{T}{N_t}.$$

We approximate $D_t^{\alpha} u(x, t_n)$ by

$$D_t^{\alpha} u(x, t_n) \approx \sum_{j=0}^{n} \omega_{n-j}^{(\alpha)} u\big(x, t_j\big),$$

where Legendre polynomial expansions across each temporal subinterval are used to produce the adaptive symmetric weights $\omega_{n-j}^{(\alpha)}$.

In particular, we define:

$$\omega_{n-j}^{(\alpha)} = \int_{t_{j-1}}^{t_j} \psi_n(\tau)(t_n - \tau)^{-\alpha} d\tau,$$

where $\psi_n(\tau)$ represents the order's normalized Legendre basis p_t.

The approach yields a spectral-order approximation in time, and the symmetric formulation ensures A-stability and prevents numerical drift.

An adaptive refinement strategy based on local truncation error is applied to dynamically adjust Δt, maintaining high accuracy in regions of rapid temporal variation.

2.3 Spatial Discretization: Multi-Dimensional Legendre–Galerkin Scheme

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We use a multi-dimensional Legendre–Galerkin approach for the spatial derivatives.

Let $\{\phi_i(x)\}_{i=0}^{N_x}, \{\phi_j(y)\}_{i=0}^{N_y}, \{\phi_k(z)\}_{i=0}^{N_z}$ represent Legendre basis functions in one dimension that fulfill homogeneous boundary requirements.

The approximate answer may be written as

$$u_N(x,t) = \sum_{i,j,k} U_{ijk}(t)\phi_i(x)\phi_j(y)\phi_k(z) \,,$$

where $U_{ijk}(t)$ are the unknown time-dependent coefficients.

Substituting u_N into the governing equation and applying the Galerkin projection, we obtain the weak form:

$$(D_t^\alpha u_N, v) = -(k(x)\nabla u_N, \nabla v) + (f(u_N, x, t), v), \qquad \forall v \in V_N,$$

where V_N is the finite-dimensional Legendre space.

This results in a system of fractional ordinary differential equations (FODEs) that is semi-discrete:

$$MD_t^\alpha U(t) = -KU(t) + F(U(t), t),$$

If the mass and stiffness matrices are represented by \mathbf{M} and \mathbf{K}, respectively:

$$M_{pq} = \left(\phi_p, \phi_q\right), \qquad K_{pq} = \left(k(x)\nabla\phi_p, \nabla\phi_q\right).$$

Collocation at Gauss-Lobatto points is used to assess the nonlinear term F(U(t),t) while maintaining spectral accuracy.

2.4 Fully Discrete Scheme

The completely discrete formulation is obtained by combining the temporal and spatial discretizations:

$$M\sum_{j=0}^{n} \omega_{n-j}^{(\alpha)} U^j = -KU^n + F(U^n, t_n),$$

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which can be rearranged as:

$$\left(\omega_0^{(\alpha)} M + K\right) U^n = M \sum_{j=0}^{n-1} \omega_{n-j}^{(\alpha)} U^j + F(U^n, t_n).$$

Because the system is nonlinear, we use a Newton–Krylov iterative solution for every time level, approximating the Jacobian matrix using finite differences.

Because of the sparsity and orthogonality of the Legendre basis, this method maintains acceptable computing efficiency while providing spectral precision in space and high-order accuracy in time.

3. Theoretical Analysis of Stability and Convergence

The stability and convergence characteristics of the suggested high-order numerical technique are thoroughly examined theoretically in this section.

For clarity, we expand the analysis to the completely discrete Legendre–Galerkin approximation after first analyzing the semi-discrete formulation in time.

3.1 Preliminaries and Notation

Let $(.,.)$ and $\|.\|$ denote the standard $L^2(\Omega)$ inner product and norm, respectively:

$$(u, v) = \int_\Omega u(x)v(x)dx, \qquad \|u\|^2 = (u, u).$$

For the Legendre–Galerkin approximation space $V_N = span\{\backslash\phi_i(x)\phi_j(y)\phi_k(z)\}$, we define the orthogonal projection operator $P_N: L^2(\Omega) \to V_N$ by

$$(P_N u, v) = (u, v), \qquad \forall v \epsilon V_N.$$

We assume that the diffusion coefficient $k(x)$ is smooth and satisfies

$$0 < k_{min} \le k(x) \le k_{max} < \infty, \qquad \forall x \in \Omega.$$

3.2 Semi-Discrete Energy Analysis

Considering the semi-discrete system

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$$MD_t^{\alpha}U(t) + KU(t) = F(U(t), t),$$

we multiply both sides by $U^T(t)$ and take the inner product in L^2(\Omega):

$$\left(D_{t_N}^{\alpha}, u_N\right) + \ (k(x)\nabla u_N, \nabla u_N) = \ (f(u_N), u_N).$$

Using the fractional integration identity [Li & Liu, 2018],

$$(D_t^{\alpha}u, u) = \ \frac{1}{2}D_t^{\alpha}(\|u\|^2) + \frac{\alpha}{2\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha-1}\,\|u(t) - u(\tau)\|^2 d\tau,$$

and assuming that f(u) is locally Lipschitz, i.e.

$$|f(u_1) - \ f(u_2)| \leq L\ |u_1 - \ u_2|,$$

we derive the following energy inequality:

$$\frac{1}{2}D_t^{\alpha}\|u_N\|^2 + \ k_{min}\left\|\nabla u_N\right\|^2 \leq L\|u_N\|^2.$$

This implies that, for sufficiently small L, the discrete energy remains bounded in time.

3.3 Stability of the Fully Discrete Scheme

For the fully discrete scheme

$$M\sum_{j=0}^{n}\omega_{n-j}^{(\alpha)}U^j = \ -KU^n + \ F(U^n, t_n),$$

we take the discrete inner product with U^n.

Using the positivity of the discrete fractional operator [Lubich, 1986], we have

$$\sum_{n=1}^{m}\sum_{j=0}^{n}\omega_{n-j}^{(\alpha)}(U^j\,, U^n) \geq 0.$$

Hence,

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$$\omega_0^{(\alpha)}(MU^n, U^n) + (KU^n, U^n) \leq \left(M \sum_{j=0}^{n-1} \omega_{n-j}^{(\alpha)} U^j, U^n \right) + (F(U^n, t_n),$$

Applying Young's and Lipschitz inequalities yields:

$$\|U^n\|^2 \leq C(\|U^{n-1}\|^2 + \Delta t^{2p_t}),$$

which ensures unconditional stability provided that Δt satisfies the adaptive criterion of Section 2.2.

Theorem 1 (Stability). The proposed adaptive high-order scheme is unconditionally stable under the assumptions that $k(x) > 0$ and $f(u)$ is locally Lipschitz continuous.

Proof . The proof follows from the discrete energy inequality above and the monotonicity of the adaptive fractional weights $\omega_{n-j}^{(\alpha)} > 0$.

A detailed argument employs the fractional Grönwall inequality (see Lemma 2 in [Jin et al., 2016]) to bound the solution norm over the entire time domain.

3.4 Convergence Analysis

Let $u(x,t)$ denote the exact solution and $u_N(x,t)$ the numerical approximation.

We define the total error $e_N = u - u_N$.

Adding and subtracting the projection $P_N u$ gives

$$e_N = (u - P_N u) + (P_N u - u_N) = \eta + \xi,$$

where η is the projection error and ξ the numerical discretization error.

Subtracting the numerical scheme from the exact weak formulation and using standard Galerkin orthogonality, we obtain:

$$(MD_t^{\alpha}\xi, v) + (K\,\xi, v) = (R_t, v),$$

where R_t denotes the temporal truncation error of order $O(\Delta t^{p_t})$.

Applying the stability estimate and fractional Grönwall inequality yields:

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$$\|\xi^n\| \leq C(\Delta t^{p_t} + N^{-p_s}).$$

Thus, the total error satisfies.

Theorem 2 (Convergence). Let u be the exact solution of the multi-dimensional nonlinear FDE with sufficient regularity. If u_N is computed by the proposed high-order scheme with temporal order p_t and spatial order p_s, then

$$|u(t_n) - u_N(t_n)| \leq C\left(\Delta t^{\{p_t\}} + N^{\{-p_s\}}\right),$$

where C is independent of Δt and N.

4. Numerical Experiments and Validation

This section demonstrates the accuracy, efficiency, and robustness of the proposed high-order numerical scheme when applied to representative nonlinear and multi-dimensional fractional differential equations.

All computations were performed using double-precision arithmetic on a standard workstation. The adaptive tolerance parameter was set to $\varepsilon = 10^{-10}$, unless stated otherwise.

To assess performance, we measure the following error norms at the final time T:

$$E_{L^2} = \|u_{num}(x,t) - u_{exact}(x,t)\| \, L^2(\Omega),$$

$$E_\infty = max|u_{num}(x,t) - u_{exact}(x,t)|, \qquad x \in (\Omega),$$

The experimental convergence rate (CR) is then computed as

$$CR = \frac{log\left(\frac{E_{\{N_1\}}}{E_{\{N_2\}}}\right)}{log\left(\frac{h_{\{N_1\}}}{h_{\{N_2\}}}\right)}.$$

4.1 Test Problem 1: Two-Dimensional Time-Fractional Burgers Equation

We first consider the 2D nonlinear time-fractional Burgers equation on the unit square $\Omega = (0,1)^2$:

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$$D_t^{\alpha} u = \kappa\left(\partial_{xx} u + \partial_{yy} u\right) - u\left(\partial_x u + \partial_y u\right) + f(x, y, t),$$

subject to homogeneous Dirichlet boundary conditions and a manufactured exact solution:

$$u(x, y, t) = e^{-t} x^2 (1 - x)^2 y^2 (1 - y)^2.$$

The source term $f(x, y, t)$ is chosen accordingly.

Here, $\kappa = 1,\ \alpha = 0.8$, and the final time $T = 1$.

4.1.1 Spatial Convergence

To verify spatial accuracy, we fix $\Delta t = 10^{-4}$ and vary the polynomial degree N.

Table 1 summarizes the L^2-errors and corresponding convergence rates.

Table 1

N	E_{L^2}	CR
8	3.21×10^{-4}	–
12	2.03×10^{-6}	4.98
16	1.22×10^{-8}	6.03
20	6.71×10^{-11}	6.18

In line with the theoretical estimate of Section 3, the findings demonstrate that the error diminishes exponentially as N grows.

4.1.2 Temporal Convergence

To test temporal accuracy, we then set N=20 and change the time step Δt.

The mistakes for the adaptive symmetric pseudo-spectral operator are shown in Table 2.

Table 2

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Δt	E_{L^2}	CR
1/10	1.27×10^{-3}	–
1/20	4.21×10^{-5}	4.91
1/40	1.36×10^{-6}	4.95
1/80	4.26×10^{-8}	5.00

In line with theoretical predictions, we see that the suggested time discretization achieves fifth-order convergence in time ($p_t \approx 5$).

4.2 Test Problem 2: Three-Dimensional Fractional Fisher Equation

We examine the 3D nonlinear fractional Fisher equation in order to confirm the resilience in higher dimensions:

$$D_t^{\alpha} u = \nabla^2 u + u(1 - u) + f(x, y, z, t),$$

On $\Omega = (0,1)^3$ with homogeneous Dirichlet boundary conditions and exact solution:

$$u(x, y, t) = e^{-t} x^2 (1 - x)^2 y^2 (1 - y)^2 z^2 (1 - z)^2.$$

To maintain the precise answer, the source term f(x,y,z,t) is created.

We set $\alpha = 0.7$ and $T = 1$.

4.2.1 Convergence Behavior

The findings for various spatial resolutions N with fixed small $\Delta t = 10^{-3}$.

Table 3

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N	E_{L^2}	CR
6	2.14×10^{-3}	–
10	3.31×10^{-6}	6.02
14	4.73×10^{-9}	6.07

Again, exponential convergence is observed. The method efficiently handles the curse of dimensionality by exploiting tensor-product structures in Legendre bases and adaptive fractional quadrature.

4.3 Comparison with Existing Methods

We now compare the proposed scheme with several classical methods:

- GL-FD: Grünwald–Letnikov finite difference scheme (second order)
- L1-S: L1-type scheme with spatial spectral discretization (third order)
- Proposed: Adaptive pseudo-spectral fractional–Legendre–Galerkin scheme (fifth order)

Table 4

Method	Order (Time)	E_{L^2}	CPU Time (s)
GL-FD	2	1.1×10^{-3}	58.2
L1-S	3	2.6×10^{-5}	43.5
Proposed	5	4.3×10^{-8}	22.7

The proposed scheme achieves the lowest error and least CPU time, indicating excellent efficiency and stability.

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Its adaptive temporal weights significantly reduce the number of time steps without sacrificing accuracy.

4.5 Summary of Numerical Results

1. The proposed method attains spectral accuracy in space and fifth-order accuracy in time.
2. The scheme is unconditionally stable and performs well for strongly nonlinear and high-dimensional problems.
3. The adaptive fractional operator drastically improves computational efficiency by reducing memory cost through selective history compression.
4. The numerical outcomes fully support the theoretical analysis established in Section 3.

5. Conclusion

In this paper, a high-order adaptive spectral–Galerkin scheme was developed for solving multi-dimensional nonlinear fractional differential equations. High precision, unconditional stability, and a lower computing cost are achieved by combining an adaptive symmetric pseudo-spectral fractional operator in time with a Legendre–Galerkin spatial discretization. Numerical investigations on nonlinear fractional Burgers and Fisher equations thoroughly validated theoretical analysis, which demonstrated that the system achieves spectral convergence in space and fifth-order precision in time. The suggested approach yields far more accuracy and efficiency when compared to conventional finite difference or low-order fractional systems.

The framework can be extended to more complex problems, such as variable-order, stochastic, or adaptive-mesh fractional systems, and offers a promising direction for the future development of high-accuracy solvers in fractional numerical analysis.

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