ON THE GREEN FUNCTIONS TO ITERATED BELTRAMI OPERATORS ON n-DIMENSIONAL SPHERES

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ABSTRACT. A method for computation of iterated Beltrami operator on an n-dimensional sphere is proposed. It is based on integration of the Poisson kernel.

1. INTRODUCTION

Green functions with respect to iterated Beltrami operators $(\Delta_{E^n} + \lambda)^m$ on the two-dimensional sphere have been studied in detail e.g. by Freeden [3, 4] and Backus [2]. In [5] the authors introduce iterated operators

$$\partial_{0,\dots,K} := \partial_0 \dots\dots\dots \partial_K$$

for

$$\partial_k := \Delta_{E^n} - (\Delta_{E^n})^\wedge(k), \tag{1}$$

where

$$(\Delta_{E^n})^\wedge(k) = -k(k+1)$$

is the k^{th} eigenvalue to the Laplace-Beltrami operator on the two-dimensional sphere. Using the Green function with respect to the iterated Beltrami operator they generalize the Green second fundamental theorem, see [5, Theorem 4.2.4]. Further, they study their properties in spherical spline setting corresponding to iterated Beltrami deivatives [5, Chapter 6].

In the present paper this definition is generalized to the n-dimensional sphere. I propose an alternative method (to the one udes in [5] or other textbooks of Willi Freeden and his co-workers like [6]) of computing an explicit representation of the Green function with respect to operators being products of (1) (with $(\Delta_{E^n})^\wedge(k)$ adapted to the n-dimensional case).

2. PRELIMINARIES

A square integrable function f over the n-dimensional unit sphere $S^n \subseteq \mathbb{R}^{n+1}$, $n \geq 2$, with the rotation-invariant measure $d\sigma$ normalized such that

$$\Sigma_n = \int_{S^n} d\sigma(\mathsf{x}) = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)},$$

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can be represented as a Fourier series in terms of the hyperspherical harmonics,

$$f = \sum_{l=0}^{\infty} \sum_{k \in \mathcal{M}_{n-1}(l)} a_l^k(f)\, Y_l^k, \tag{2}$$

where $\mathsf{M}_{n-1}(l)$ denotes the set of sequences $k = (k_0, k_1, \ldots, k_{n-1})$ in $\mathbb{N}^{n-1} \times \mathbb{Z}$ such that l k_0 k_1 $\ldots$ k_{n-1} and $a^k(f)$ are the Fourier coefficients of f. The hyperspherical harmonics of degree l and order k are given by

$$Y_l^k(\mathsf{x}) = A_l^k \prod_{\tau=1}^{n-1} C_{k_{\tau-1}-k_\tau}^{\frac{n-\tau}{2}+k_\tau}(\cos\vartheta_\tau)\sin^{k_\tau}\vartheta_\tau \cdot e^{\pm i k_{n-1}\varphi} \tag{3}$$

for some constants A_l^k. Here, $(\vartheta_1, \ldots, \vartheta_{n-1}, \phi)$ are the hyperspherical coordinates of $\mathsf{x} \in \mathsf{S}^n$,

$$\begin{aligned} x_1 &= \cos\vartheta_1, \\ x_2 &= \sin\vartheta_1 \cos\vartheta_2, \\ &\ldots \\ x_n &= \sin\vartheta_1 \ldots \sin\vartheta_{n-2} \sin\vartheta_{n-1} \cos\phi, \\ x_{n+1} &= \sin\vartheta_1 \ldots \sin\vartheta_{n-2} \sin\vartheta_{n-1} \sin\phi, \end{aligned}$$

and C_κ^K are the Gegenbauer polynomials of degree κ and order K. The set of degree l hyperspherical harmonics is denoted by H_l.

Zonal (rotation-invariant) functions are those depending only on the first hyperspherical coordinate $\vartheta = \vartheta_1$. Unless it leads to misunderstandings, we identify them with functions of ϑ or $t = \cos\vartheta$. A zonal L^1-function f has the following Gegenbauer expansion

$$f(t) = \sum_{l=0}^{\infty} \widehat{f}(l)\, \mathcal{C}_l^\lambda(t), \qquad t = \cos\vartheta, \tag{4}$$

where $f(l)$ are the Gegenbauer coefficients of f and λ is related to the space dimension by

$$\lambda = \frac{n-1}{2}.$$

For $f, g \in \mathsf{L}^1(\mathsf{S}^n)$, g zonal, their convolution $f * g$ is defined by

$$(f * g)(\mathsf{y}) = \frac{1}{\Sigma_n} \int_{S^n} f(\mathsf{x})\, \tau_{\mathsf{y}} g(\mathsf{x})\, d\sigma(\mathsf{x}), \qquad \tau_{\mathsf{y}} g(\mathsf{x}) = g(\mathsf{x} \cdot \mathsf{y}), \tag{5}$$

and for $f \in \mathsf{L}^2(\mathsf{S}^n)$ it is equal to

$$f * g = \sum_{l=0}^{\infty} \sum_{k \in \mathcal{M}_{n-1}(l)} \frac{\lambda}{\lambda + l}\, a_l^k(f)\, \widehat{g}(l)\, Y_l^k.$$

If f is a zonal function, then

$$\widehat{f * g}(l) = \frac{\lambda}{\lambda + l} \widehat{f}(l)\, \widehat{g}(l). \tag{6}$$

The Laplace-Beltrami operator Δ_{E^n} on the sphere is the tangential part of the Laplace operator in R^{n+1},

$$\Delta f = R^{-n}\frac{\partial}{\partial R}\left(R^n\frac{\partial f}{\partial R}\right) + \frac{1}{R^2}\Delta_{S^n} f, \tag{7}$$

where $R \geq 0$ is the radial distance of $\mathbf{x} \in R^{n+1}$ in the hyperspherical coordinates, see [9, Chapter II, Proposition 3.3]. It satisfies the following relation [1, formula (3.8)]:

$$\Delta_{S^n} f = \frac{1}{(1-t^2)^{\frac{n-2}{2}}}\frac{\partial}{\partial t}\left[(1-t^2)^{\frac{n}{2}}\cdot\frac{\partial f}{\partial t}\right] + \frac{1}{1-t^2}\Delta_{S^{n-1}} f. \tag{8}$$

In both cases, S^{n-1} denotes the $(n-1)$-dimensional unit sphere with hyperspherical coordinates $(\vartheta_2, \ldots, \vartheta_{n-1}, \phi)$. The hyperspherical harmonics are the eigenfunctions of Δ_{E^n}, i.e.,

$$\Delta_{S^n} Y_l^k = -l(n+l-1)Y_l^k, \tag{9}$$

see [9, Chapter II, Theorem 4.1].

Denote by D^n the interior of the sphere, $D^n = \{\mathbf{x} \in R^{n+1} : |\mathbf{x}| < 1\}$. Let $\xi \in D^n$ with $r = |\xi|$ and $\mathbf{x} \in S^n$. The Poisson kernel for the sphere is given by

$$p^{\lambda}(\xi, \mathbf{x}) = p_r^{\lambda}(x) = \frac{1}{\Sigma_n}\cdot\frac{1-|\xi|^2}{|\xi-\mathbf{x}|^n} = \frac{1}{\Sigma_n}\cdot\frac{1-r^2}{(1-2r\cos\vartheta+r^2)^{(n+1)/2}} \tag{10}$$

where ϑ denotes the angle between the vectors ξ and $\mathbf{x}$. It is given as a series

$$p^{\lambda}(\xi, \mathbf{x}) = \frac{1}{\Sigma_n}\cdot\sum_{l=0}^{\infty} r^l\cdot\frac{\lambda+l}{\lambda}C_l^{\lambda}(\cos\vartheta) \tag{11}$$

and it is a harmonic function of the variable ξ,

$$\Delta_{\xi}\, p^{\lambda}(\xi, \mathbf{x}) = 0.$$

Since the Gegenbauer polynomials C_l^{λ} over the interval $[-1, 1]$ are bounded by

$$\left|C_l^{\lambda}(\cos\vartheta)\right| \leq (n+l-2)^{n-2} \tag{12}$$

uniformly in ϑ (compare [10, Theorem 7.33.1]), the series in (11) is uniformly (in ϑ) convergent for each $r \in [0, 1)$.

3. Iterated Beltrami operator

Green functions with respect to the iterated Beltrami operators on the 2-dimensional sphere are introduced and discussed in [6, Section 5]. The definition can be generalized to the case of an n-dimensional sphere in the following way.

Definition 3.1. *Let the differential operator* ∂_k, $k \in N_0$, *be given by*

$$\partial_k := \Delta_{E^n} - (\Delta_{E^n})\hat{}(k),$$

where

$$(\Delta_{E^n})\hat{}(k) = -k(n+k-1)$$

is the k^{th} eigenvalue to the Laplace-Beltrami operator. $G(\partial_0, x \cdot y)$ *is the Green function to the Laplace-Beltrami operator and the Green functions with respect to the iterated operators*

$$\partial_{0,\dots,K} := \partial_0\partial_1 \dots \partial_K, \qquad K \in \mathbb{N},$$

are defined inductively by the following convolutions

$$G(\partial_{0,\dots,K}; x \cdot y) = \frac{1}{\Sigma_n} \int_{E^n} G(\partial_{0,\dots,K-1}; x \cdot z)\, G(\partial_K; z \cdot y)\, d\sigma(z), \qquad K = 1, 2, \dots .$$

The Green functions to $\partial_{0,\dots,K}$ are given explicitely as series.

Lemma 3.2. *For $K \in \mathbb{N}_0$ and $t \in [-1, 1)$,*

$$G(\partial_{0,\dots,K}; t) = \sum_{l=K+1}^{\infty} \left[\prod_{k=0}^{K} \frac{1}{k(n+k-1) - l(n+l-1)}\right] \cdot \frac{\lambda + l}{\lambda} \mathcal{C}_l^\lambda(t). \tag{13}$$

Proof. According to [7, Theorem 1 together with Remark 2 p. 2],

$$G(\partial_k; t) = \sum_{l=0,\, l\neq k}^{\infty} \frac{1}{k(n+k-1) - l(n+l-1)} \cdot \frac{\lambda + l}{\lambda} \mathcal{C}_l^\lambda(t).$$

Thus, (13) holds for $K = 0$. For $K \geq 1$, representation (13) is obtained recursively via (6).

Remark 3.3. *The difference in the normalization of the Green functions $G(\partial_{0,\dots,K}, t)$ between the present paper and [6] lies in the fact that convolution in the case of a two-dimensional sphere in the textbook of Freeden and Schreiner is defined by integral of a product of fuctions over the sphere, whereas in formulas (5) factor $\frac{1}{\Sigma_n}$ is used.*

The aim of this paper is to obtain an integral representation of the Green functions. For a specific set of parameters (dimension n and order K), a closed formula can be computed.

Theorem 3.4. *Let $G(\partial_{0,\dots,K}; t)$ be the Green function to the operator $\partial_{0,\dots,K}$ on an n-dimensional sphere. For $t \in [-1, 1)$ it is given explicitely by*

$$G(\partial_{0,\dots,K}; t) = \int_0^1 \left(\sum_{k=0}^{K} A_k r^{-k-1} + B_k r^{n+k}\right) \cdot \left(\Sigma_n p_r^\lambda(t) - \sum_{l=0}^{K} \frac{\lambda + l}{\lambda} r^l \mathcal{C}_l^\lambda(t)\right) dr,$$

where A_k and B_k are the coefficients of the partial fraction decomposition

$$\prod_{k=0}^{K} \frac{-1}{(X-k)(X+n+k-1)} = \sum_{k=0}^{K} \frac{A_k}{X-k} + \sum_{k=0}^{K} \frac{B_k}{X+n+k-1}. \tag{14}$$

Proof. Set

$$G_r(\partial_{0,\dots,K}; \circ) := \sum_{k=0}^{K} \left(A_k r^{-k-1} + B_k r^{n+k}\right) \left(\Sigma_n \cdot p_r^\lambda(t) - \sum_{l=0}^{K} \frac{\lambda + l}{\lambda} r^l \mathcal{C}_l^\lambda(t)\right) \tag{15}$$

$$= \sum_{k=0}^{K} A_k \sum_{l=K+1}^{\infty} \frac{\lambda + l}{\lambda} r^{l-k-1} \mathcal{C}_l^\lambda + \sum_{k=0}^{K} B_k \sum_{l=K+1}^{\infty} \frac{\lambda + l}{\lambda} r^{l+n+k} \mathcal{C}_l^\lambda .$$

Now, for $l + \alpha > 0$,

$$\int_0^1 r^{l+\alpha-1}\,dr = \frac{1}{l+\alpha}.$$

By (12), the series $\sum_{K+1}^{\infty} \frac{\lambda+l}{\lambda} r^{l+\alpha-1} C_l^{\lambda}(t)$ is absolutely convergent for each $r \in [0, 1)$ and $K + \alpha \geq 0$. Hence, when computing $\int_0^1 G_r\,dr$, the order of integration and summation may be changed and

$$\int_0^1 G_r\,dr = \sum_{l=K+1}^{\infty}\left(\sum_{k=0}^{K}\frac{A_k}{l-k} + \sum_{k=0}^{K}\frac{B_k}{l+n+k-1}\right)\frac{\lambda+l}{\lambda}C_l^{\lambda}. \tag{16}$$

The denominator of the fraction on the left-hand-side of (14) is equal to

$$X(n + X - 1) - k(n + k - 1),$$

hence, the right-hand-side of (16) is equal to (13).

Example 3.5. *Let $n = 2$ and $K = 1$. In this case,*

$$\prod_{k=0}^{1}\frac{1}{k(k+1) - l(l+1)} = \frac{1}{(l-1)l(l+1)(l+2)} = \frac{1}{6(l-1)} - \frac{1}{2l} + \frac{1}{2(l+1)} - \frac{1}{6(l+2)}$$

and

$$P_r(t) := \sum_{l=2}^{\infty}\frac{\lambda+l}{\lambda} r^l C_l^{\lambda}(t) = \frac{1-r^2}{(1-2rt+r^2)^{3/2}} - 1 - 3rt.$$

Further,

$$G_r(\partial_{0,1};t) := \int \left(\frac{P_r(t)}{6r^2} - \frac{P_r(t)}{2r} + \frac{P_r(t)}{2} - \frac{r\,P_r(t)}{6}\right)dr$$

$$= \frac{1}{6}\left(\frac{1}{r} - 3r(1-3t) + \frac{1}{2}r^2(1-9t) + r^3 t - \frac{(1-r)[1 + r(4-6t) + r^2]}{r\sqrt{1-2rt+r^2}}\right.$$

$$\left. + 3(1-t)\ln\frac{1 - rt + \sqrt{1-2rt+r^2}}{r - t + \sqrt{1-2rt+r^2}}\right) + C$$

and

$$G(\partial_{0,1};t) = G_1(\partial_{0,1};t) - \lim_{r\to 0} G_r(\partial_{0,1};t) = \frac{3 + t + 6(1-t)\ln\frac{1-t}{2}}{12}.$$

This result coincides with the one obtained by Freeden and Schreiner in [6, Lemma 5.25] *(up to the constant $\frac{1}{4\pi}$).*

Remark 3.6. *Function $G(\partial_{0,1};t)$ from Example 3.5 does not satisfy the equation*

$$(\Delta_{E^2} + 1\cdot 2)\,G(\partial_{0,1};t) = G(\partial_0;t). \tag{17}$$

Using (8)*, one obtains*

$$1 + \frac{3t}{2} + \ln\frac{1-t}{2},$$

$n=2$	$G(\partial_{0,1};t)=\frac{3+t}{12}+\frac{1-t}{2}\ln\left(\frac{1-t}{2}\right)$ $G(\partial_{0,1,2};t)=\frac{41-10t-43t^2}{960}+\frac{(1-t)^2}{16}\ln\left(\frac{1-t}{2}\right)$ $G(\partial_{0,\dots,3};t)=\frac{301-447t-273t^2+451t^3}{120960}+\frac{(1-t)^3}{288}\ln\left(\frac{1-t}{2}\right)$ $G(\partial_{0,\dots,4};t)=\frac{1749-4596t+1614t^2+4524t^3-3371t^4}{23224320}+\frac{(1-t)^4}{9216}\ln\left(\frac{1-t}{2}\right)$
$n=3$	$G(\partial_{0,1};t)=\frac{7+13t}{36}-\frac{(1-t)(1+2t)(\pi-\vartheta)}{6\sqrt{1-t^2}}$ $G(\partial_{0,1,2};t;t)=\frac{373+136t-642t^2}{7200}-\frac{(1-t)^2(2+3t)(\pi-\vartheta)}{60\sqrt{1-t^2}}$ $G(\partial_{0,\dots,3};t;t)=\frac{1343-1340t-2334t^2+2484t^3}{352800}-\frac{(1-t)^3(3+4t)(\pi-\vartheta)}{1260\sqrt{1-t^2}}$ $G(\partial_{0,\dots,4};t;t)=\frac{63379-136256t-16914t^2+214336t^3-127460t^4}{457228800}-\frac{(1-t)^4(4+5t)(\pi-\vartheta)}{45360\sqrt{1-t^2}}$
$n=4$	$G(\partial_{0,1};t)=\frac{5-111t}{720}+\frac{1-3t}{12}\ln\left(\frac{1-t}{2}\right)$ $G(\partial_{0,1,2};t)=-\frac{5+189t+66t^2}{10080}-\frac{(1-t)t}{24}\ln\left(\frac{1-t}{2}\right)$ $G(\partial_{0,\dots,3};t)=-\frac{1509+7527t-3681t^2-8035t^3}{4354560}-\frac{(1-t)^2(1+5t)}{1728}\ln\left(\frac{1-t}{2}\right)$ $G(\partial_{0,\dots,4};t)=-\frac{19143+31416t-99570t^2-24992t^3+81507t^4}{766402560}-\frac{(1-t)^3(1+3t)}{27648}\ln\left(\frac{1-t}{2}\right)$
$n=5$	$G(\partial_{0,1};t;t)=\frac{93+251t+128t^2}{1200(1+t)}-\frac{(3+6t-6t^2-8t^3)(\pi-\vartheta)}{40(1+t)\sqrt{1-t^2}}$ $G(\partial_{0,1,2};t;t)=\frac{1927+1259t-6763t^2-5675t^3}{117600(1+t)}-\frac{(1-t)(6+3t-24t^2-20t^3)(\pi-\vartheta)}{560(1+t)\sqrt{1-t^2}}$ $G(\partial_{0,\dots,3};t;t)=\frac{7357-20396t-47748t^2+29455t^3+46930t^4}{9525600(1+t)}-\frac{(1-t)^2(7-12t-60t^2-40t^3)(\pi-\vartheta)}{15120(1+t)\sqrt{1-t^2}}$ $G(\partial_{0,\dots,4};t;t)=\frac{703049-9799287t-1774856t^2+26525240t^3-374520t^4-17285240t^5}{73766246400(1+t)}$ $-\frac{(1-t)^3(4-45t-120t^2-70t^3)(\pi-\vartheta)}{665280(1+t)\sqrt{1-t^2}}$

for the left hand-side of (17)*, whereas* $G(\partial_0;t)$ *equals* [7, Table 1]

$$1+\ln\frac{1-t}{2}.$$

The difference between these expressions is equal to the 1^{st} *summand in* (13) *in the representation of* $G(\partial_0;t)$*. Statements given by formulas (5.158) or (5.159) in* [6] *should be corrected taking respect to the different summation range in the series representation of the kernels whereas formula* [6, (5.163)] *(respectively* [5, (4.2.8)]*) reflects this property.*

Table 3 gives the expressions for $G(\partial_{0,\dots,K};t)$ for $n=2,3,4,5$ and $K=1,2,3,4$.

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