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ON HUB DOMINATION NUMBER OF CARTESIAN PRODUCT OF GRAPHS

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ABSTRACT

For a graph G, a set $H \subseteq V$ is a hub dominating set if every vertex u in $V - H$ is adjacent to a vertex v in H and any two non-adjacent vertices $v, w \in V - H$ has a path in G in which all the internal vertices of the path must be in H. The least cardinality taken over all hub dominating set H of vertices in graph G is known as the hub domination number. We denote the hub domination number of G by $\gamma_h(G)$.

Keywords: Hub dominating set, Hub domination number, Cartesian product, line graph.

AMS Subject Classification: 05C38, 05C69, 05C07.

1. INTRODUCTION

All the graphs considered in this context are finite, undirected and loop-free. Graph theoretical terms are defined as in Harary [2]. The concept of dominating sets in graph theory was first formally investigated in the early 1960's. Ore introduced the terms dominating set and domination number in his work [3]. The notion of hubs in graphs was first proposed by Walsh in 2006, expanding the toolkit for network analysis and optimization [5]. Hub domination has evolved by combining graph domination and hub concepts. In this paper we studied the hub domination number of the cartesian product of various graphs.

2. PRELIMINARIES

Definition 2.1 The Cartesian product of G and H is a graph, denoted as $G \square H$, whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent precisely if $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$. Thus,

$$V(G \square H) = \{(g, h) : g \in V(G) \text{ and } h \in V(H)\}$$

$$E(G \square H) = \{(g, h)(g', h') : g = g', hh' \in E(H) \text{ or } gg' \in E(G), h = h'\}.$$

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Definition 2.2 The line graph of a graph $G$is the graph $L(G)$ with the edges of G as its vertices and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident in G.

3. HUB DOMINATION NUMBER

Definition 3.1 For a graph G, a set $H \subseteq V$ is a hub dominating set if every vertex u in $V - H$ is adjacent to a vertex v in H and any two non-adjacent vertices $v, w \in V - H$ has a path in G in which all the internal vertices of the path must be in H. The least cardinality taken over all hub dominating set H of vertices in graph G is known as the hub domination number. We denote the hub domination number of G by $\gamma_h(G)$.

Theorem 3.2 If G_1 is a path graph P_m or cycle graph C_m and G_2 is a path graph P_n or cycle graph C_n then for any $m, n > 2$,

$$\gamma_h(G_1 \square G_2) = \begin{cases} mn - 2m \ \ for\ m > n \\ mn - 2n \ \ for\ m < n \\ n^2 - 2n \ \ for\ m = n \end{cases}$$

Proof:

Let $G_1 \square G_2$ be the graph with vertex set $V = V_1 \times V_2$ where V_1 is the vertex set of P_m or C_m and V_2 is the vertex set of P_n or C_n.

Case i: $m > n$

We claim, $H = \{(u_i, v_j) \in V_1 \times V_2 : i = 1,2, \dots, m; 2 \leq j \leq n-1\}$ is a hub dominating set of $G_1 \square G_2$. Now, $V - H = \{(u_i, v_j) \in V_1 \times V_2 : i = 1,2, \dots, m; j = 1, n\}$. Clearly the vertices $(u_i, v_1) \in V - H$ are adjacent to $(u_i, v_2) \in H$ and $(u_i, v_n) \in V - H$ are adjacent to $(u_i, v_{n-1}) \in H$.

Subcase i: For any pair of non-adjacent vertices $(u_i, v_1), (u_i, v_n) \in V - H$ where $i = 1,2, \dots, m$.

Then $(u_i, v_1), (u_i, v_n) \in V - H$ has a path $(u_i, v_1)(u_i, v_2) \dots (u_i, v_{n-1})(u_i, v_n)$ in $G_1 \square G_2$ whose internal vertices are in H.

Subcase ii: For any pair of non-adjacent vertices $(u_i, v_1), (u_k, v_1) \in V - H$ where $i, k = 1,2, \dots, m$ and $i \neq k$.

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Then $(u_i, v_1), (u_k, v_1) \in V - H$ has a path $(u_i, v_1)(u_i, v_2)(u_{i+1}, v_2) \ldots (u_k, v_2)(u_k, v_1)$ in $G_1 \square G_2$ whose internal vertices are in H.

Subcase iii: For any pair of non-adjacent vertices $(u_i, v_n), (u_k, v_n) \in V - H$ where $i, k = 1,2, \ldots, m$ and $i \neq k$.

Then $(u_i, v_n), (u_k, v_n) \in V - H$ has a path $(u_i, v_n)(u_i, v_{n-1})(u_{i+1}, v_{n-1}) \ldots (u_k, v_{n-1})(u_k, v_n)$ in $G_1 \square G_2$ whose internal vertices are in H.

Subcase iv: For any pair of non-adjacent vertices $(u_i, v_1), (u_k, v_n) \in V - H$ where $i, k = 1,2, \ldots, m$ and $i < k$.

Then $(u_i, v_1), (u_k, v_n) \in V - H$ has a path $(u_i, v_1)(u_i, v_2) \ldots (u_i, v_{n-1}) (u_{i+1}, v_{n-1}) \ldots (u_{k-1}, v_{n-1})(u_k, v_{n-1})(u_k, v_n)$ in $G_1 \square G_2$ whose internal vertices are in H.

Subcase v: For any pair of non-adjacent vertices $(u_i, v_1), (u_k, v_n) \in V - H$ where $i, k = 1,2, \ldots, m$ and $i > k$.

Then $(u_i, v_1), (u_k, v_n) \in V - H$ has a path $(u_i, v_1)(u_i, v_2) \ldots (u_i, v_{n-1}) (u_{i-1}, v_{n-1}) \ldots (u_{k+1}, v_{n-1})(u_k, v_{n-1})(u_k, v_n)$ in $G_1 \square G_2$ whose internal vertices are in H.

Hence H is a hub dominating set.

$$|H| = |V_1| \, (|V_2| - 2)$$

$$= m(n-2)$$

$$= mn - 2m$$

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Case ii: $m < n$

We claim, $H = \{(u_i , v_j) \in V_1 \times V_2 : 2 \leq i \leq m-1, \ j = 1,2, \dots , n\}$ is a hub dominating set of $G_1 \square G_2$. Now, $V - H = \{(u_i , v_j) \in V_1 \times V_2 : i = 1, m; j = 1,2, \dots , n\}$. Clearly the vertices $(u_1, v_j) \in V - H$ are adjacent to $(u_2 , v_j) \in H$ and $(u_m, v_j) \in V - H$ are adjacent to $(u_{m-1} , v_j) \in H$.

Subcase i: For any pair of non-adjacent vertices $(u_1 , v_j) , (u_m , v_j) \in V - H$ where $j = 1,2, \dots , n$.

Then $(u_1 , v_j) , (u_m , v_j) \in V - H$ has a path $(u_1 , v_j)(u_2 , v_j) \dots (u_{m-1} , v_j)(u_m , v_j)$ in $G_1 \square G_2$ whose internal vertices are in H.

Subcase ii: For any pair of non-adjacent vertices $(u_1 , v_j) , (u_1 , v_k) \in V - H$ where $j , k = 1,2, \dots , n$ and $j \neq k$.

Then $(u_1 , v_j) , (u_1 , v_k) \in V - H$ has a path $(u_1 , v_j)(u_2 , v_j)(u_2 , v_{j+1})$ $\dots (u_2 , v_k)(u_1 , v_k)$ in $G_1 \square G_2$ whose internal vertices are in H.

Subcase iii: For any pair of non-adjacent vertices $(u_m , v_j) , (u_m , v_k) \in V - H$ where $j , k = 1,2, \dots , n$ and $j \neq k$.

Then $(u_m , v_j) , (u_m , v_k) \in V - H$ has a path $(u_m , v_j) \ (u_{m-1} , v_j)(u_{m-1} , v_{j+1}) \dots$

$(u_{m-1} , v_k)(u_m , v_k)$ in $G_1 \square G_2$ whose the internal vertices are in H.

Subcase iv: For any pair of non-adjacent vertices $(u_1 , v_j) , (u_m , v_k) \in V - H$ where $j , k = 1,2, \dots , n$ and $j < k$.

Then $(u_1 , v_j) , (u_m , v_k) \in V - H$ has a path $(u_1 , v_j)(u_2 , v_j) \dots (u_{m-1} , v_j)$ $(u_{m-1} , v_{j+1}) \dots (u_{m-1} , v_{k-1})(u_{m-1} , v_k)(u_m , v_k)$ in $G_1 \square G_2$ whose internal vertices are in H.

Subcase v: For any pair of non-adjacent vertices $(u_1 , v_j) , (u_m , v_k) \in V - H$ where $j , k = 1,2, \dots , n$ and $j > k$.

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Then $(u_1, v_j), (u_m, v_k) \in V - H$ has a path $(u_1, v_j)(u_2, v_j) \dots (u_{m-1}, v_j)$ $(u_{m-1}, v_{j-1}) \dots (u_{m-1}, v_{k+1})(u_{m-1}, v_k)(u_m, v_k)$ in $G_1 \square G_2$ whose internal vertices are in H.

Hence H is a hub dominating set.

$$|H| = |V_2|\,(|V_1| - 2)$$

$$= n(m-2)$$

$$= mn - 2n$$

Case iii: $m = n$

From case i and case ii we get,

$$|H| = n \times n - 2n$$

$$= n^2 - 2n$$

Hence, $\gamma_h(G_1 \square G_2) = \begin{cases} mn - 2m \ \ for\ m > n \\ mn - 2n \ \ for\ m < n \\ n^2 - 2n \ \ for\ m = n \end{cases}$ where G_1 is a path graph P_m or cycle graph C_m and G_2 is a path graph P_n or cycle graph C_n for any $m, n > 2$.

Theorem 3.3 If G is a path graph P_n or cycle graph C_n then $\gamma_h(G \square K_m) = n$, where K_m is the complete graph with m vertices.

Proof:

Let $G \square K_m$ be the graph with vertex set $V = V_1 \times V_2$ where V_1 is the vertex set of P_n or C_n and V_2 is the vertex set of K_m. We claim $H = \{(u_i, v_j) \in V_1 \times V_2 : 1 \le i \le n, j = 1\}$ is a hub dominating set of $G \square K_m$.Now, $V - H = \{(u_i, v_j) \in V_1 \times V_2 : 1 \le i \le n,\ 2 \le j \le m\}$. Clearly, every vertex $(u_i, v_j) \in V - H$ is adjacent to $(u_i, v_1) \in H$. Hence, $(u_i, v_1) \in H$ dominates all the vertices $(u_i, v_j) \in V - H$.

Case i : For any pair of non-adjacent vertices $(u_i, v_j), (u_i, v_k) \in V - H$ where $1 \le i \le n$, $2 \le j, k \le m, j \ne k$.

Then $(u_i, v_j), (u_i, v_k) \in V - H$ has a path $(u_i, v_j)(u_i, v_1)(u_i, v_k)$ in $G \square K_m$ whose internal vertices are in H.

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Case ii: For any pair of non-adjacent vertices $(u_i, v_j), (u_k, v_l) \in V - H$ where $1 \leq i, k \leq n$, $i < k$ and $2 \leq j, l \leq m$.

Then $(u_i, v_j), (u_k, v_l) \in V - H$ has a path $(u_i, v_j)(u_i, v_1)(u_{i+1}, v_1)$ ... $(u_k, v_1)(u_k, v_l)$ in $G \square K_m$ whose internal vertices are in H.

Case iii: For any pair of non-adjacent vertices $(u_i, v_j), (u_k, v_l) \in V - H$ where $1 \leq i, k \leq n$, $i > k$ and $2 \leq j, l \leq m$.

Then $(u_i, v_j), (u_k, v_l) \in V - H$ has a path $(u_i, v_j)(u_i, v_1)(u_{i-1}, v_1)$ ... $(u_k, v_1)(u_k, v_l)$ in $G \square K_m$ whose internal vertices are in H.

Clearly, H is a hub dominating set with n elements. Hence, $\gamma_h(G \square K_m) = n$, where G is P_n or C_n.

Theorem 3.4 If G is the cycle graph C_n then for any > 3 ,

$$\gamma_h(L(P_m) \square G) = \begin{cases} mn - 2m - n + 2 & for\ m > n \\ mn - 3n & for\ m \leq n \end{cases}.$$

Proof:

Let $L(P_m) \square G$ be the graph with vertex set $V = V_1 \times V_2$ where V_1 is the vertex set of $L(P_m)$ with $m - 1$ vertices and V_2 is the vertex set of C_n with n vertices.

Case i: $m > n$

We claim, $H = \{(u_i, v_j) \in V_1 \times V_2 : i = 1,2, \dots, m-1; 2 \leq j \leq n-1\}$ is a hub dominating set of $L(P_m) \square G$. Now, $V - H = \{(u_i, v_j) \in V_1 \times V_2 : i = 1,2, \dots, m-1; j = 1, n\}$. Clearly the vertices $(u_i, v_1) \in V - H$ are adjacent to $(u_i, v_2) \in H$ and $(u_i, v_n) \in V - H$ are adjacent to $(u_i, v_{n-1}) \in H$.

Subcase i: For any pair of non-adjacent vertices $(u_i, v_1), (u_k, v_1) \in V - H$ where $i, k = 1,2, \dots, m-1$ and $i \neq k$.

Then $(u_i, v_1), (u_k, v_1) \in V - H$ has a path $(u_i, v_1)(u_i, v_2)(u_{i+1}, v_2)$ ... $(u_k, v_2)(u_k, v_1)$ in $L(P_m) \square G$ whose internal vertices are in H.

Subcase ii: For any pair of non-adjacent vertices $(u_i, v_n), (u_k, v_n) \in V - H$ where $i, k = 1,2, \dots, m-1$ and $i \neq k$.

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Then $(u_i, v_n), (u_k, v_n) \in V - H$ has a path $(u_i, v_n)(u_i, v_{n-1})(u_{i+1}, v_{n-1}) \dots (u_k, v_{n-1})(u_k, v_n)$ in $L(P_m) \square G$ whose internal vertices are in H.

Subcase iii: For any pair of non-adjacent vertices $(u_i, v_1), (u_k, v_n) \in V - H$ where $i, k = 1,2, \dots, m-1$ and $i < k$.

Then $(u_i, v_1), (u_k, v_n) \in V - H$ has a path $(u_i, v_1)(u_i, v_2) \dots (u_i, v_{n-1})(u_{i+1}, v_{n-1}) \dots (u_{k-1}, v_{n-1})(u_k, v_{n-1})(u_k, v_n)$ in $L(P_m) \square G$ whose internal vertices are in H.

Subcase iv: For any pair of non-adjacent vertices $(u_i, v_1), (u_k, v_n) \in V - H$ where $i, k = 1,2, \dots, m-1$ and $i > k$.

Then $(u_i, v_1), (u_k, v_n) \in V - H$ has a path $(u_i, v_1)(u_i, v_2) \dots (u_i, v_{n-1})(u_{i-1}, v_{n-1}) \dots (u_{k+1}, v_{n-1})(u_k, v_{n-1})(u_k, v_n)$ in $L(P_m) \square G$ whose internal vertices are in H.

Hence H is a hub dominating set.

$$|H| = |V_1| (|V_2| - 2)$$

$$= (m-1)(n-2)$$

$$= mn - 2m - n + 2$$

Case ii: $m \leq n$

We claim, $H = \{(u_i, v_j) \in V_1 \times V_2 : 2 \leq i \leq m-2,\ j = 1,2, \dots, n\}$ is a hub dominating set of $L(P_m) \square G$. Now, $V - H = \{(u_i, v_j) \in V_1 \times V_2 : i = 1, m-1; j = 1,2, \dots, n\}$. Clearly the vertices $(u_1, v_j) \in V - H$ are adjacent to $(u_2, v_j) \in H$ and $(u_{m-1}, v_j) \in V - H$ are adjacent to $(u_{m-2}, v_j) \in H$.

Subcase i: For any pair of non-adjacent vertices $(u_1, v_j), (u_{m-1}, v_j) \in V - H$ where $j = 1,2, \dots, n$.

Then $(u_1, v_j), (u_{m-1}, v_j) \in V - H$ has a path $(u_1, v_j)(u_2, v_j) \dots (u_{m-2}, v_j)(u_{m-1}, v_j)$ in $L(P_m) \square G$ whose internal vertices are in H.

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Subcase ii: For any pair of non-adjacent vertices $(u_1, v_j), (u_1, v_k) \in V - H$ where $j, k = 1,2, \dots, n$ and $j \neq k$.

Then $(u_1, v_j), (u_1, v_k) \in V - H$ has a path $(u_1, v_j)(u_2, v_j)(u_2, v_{j+1}) \dots (u_2, v_k)(u_1, v_k)$ in $L(P_m) \square G$ whose internal vertices are in H.

Subcase iii: For any pair of non-adjacent vertices $(u_{m-1}, v_j), (u_{m-1}, v_k) \in V - H$ where $j, k = 1,2, \dots, n$ and $j \neq k$.

Then $(u_{m-1}, v_j), (u_{m-1}, v_k) \in V - H$ has a path $(u_{m-1}, v_j)(u_{m-2}, v_j)(u_{m-2}, v_{j+1}) \dots (u_{m-2}, v_k)(u_{m-1}, v_k)$ in $L(P_m) \square G$ whose internal vertices are in H.

Subcase iv: For any pair of non-adjacent vertices $(u_1, v_j), (u_{m-1}, v_k) \in V - H$ where $j, k = 1,2, \dots, n$ and $j < k$.

Then $(u_1, v_j), (u_{m-1}, v_k) \in V - H$ has a path $(u_1, v_j)(u_2, v_j) \dots (u_{m-2}, v_j)(u_{m-2}, v_{j+1}) \dots (u_{m-2}, v_{k-1})(u_{m-2}, v_k)(u_{m-1}, v_k)$ in $L(P_m) \square G$ whose internal vertices are in H.

Subcase v: For any pair of non-adjacent vertices $(u_1, v_j), (u_{m-1}, v_k) \in V - H$ where $j, k = 1,2, \dots, n$ and $j > k$.

Then $(u_1, v_j), (u_{m-1}, v_k) \in V - H$ has a path $(u_1, v_j)(u_2, v_j) \dots (u_{m-2}, v_j)(u_{m-2}, v_{j-1}) \dots (u_{m-2}, v_{k+1})(u_{m-2}, v_k)(u_{m-1}, v_k)$ in $L(P_m) \square G$ whose internal vertices are in H.

Hence H is a hub dominating set.

$$|H| = |V_2| \, (|V_1| - 2)$$

$$= n(m - 1 - 2)$$

$$= n(m - 3)$$

$$= mn - 3n$$

Thus, $\gamma_h(L(P_m) \square G) = \begin{cases} mn - 2m - n + 2 & for\ m > n \\ mn - 3n & for\ m \leq n \end{cases}$ where G is C_n, $\forall\, m > 3$.

Theorem 3.5 $\gamma_h(L(P_m) \square K_n) = m - 1$ where $L(P_m)$ is the line graph of the path graph P_m and K_n is the complete graph with n vertices.

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Proof:

Let G be the cartesian product graph $L(P_m) \square K_n$ with vertex set $V = V_1 \times V_2$ where V_1 is the vertex set of $L(P_m)$ with $m-1$ vertices and V_2 is the vertex set of K_n with n vertices. We claim $H = \{(u_i, v_j) \in V_1 \times V_2 : 1 \le i \le m-1, j = 1\}$ is a hub dominating set of G. Now, $V - H = \{(u_i, v_j) \in V_1 \times V_2 : 1 \le i \le m-1, \ 2 \le j \le n\}$. Clearly, every vertex $(u_i, v_j) \in V - H$ is adjacent to $(u_i, v_1) \in H$. Hence, $(u_i, v_1) \in H$ dominates all the vertices $(u_i, v_j) \in V - H$.

Case i: For any pair of non-adjacent vertices $(u_i, v_j), (u_i, v_k) \in V - H$ where $1 \le i \le m-1, 2 \le j, k \le n, j \ne k$.

Then $(u_i, v_j), (u_i, v_k) \in V - H$ has a path $(u_i, v_j)(u_i, v_1)(u_i, v_k)$ in G whose internal vertices are in H.

Case ii: For any pair of non-adjacent vertices $(u_i, v_j), (u_k, v_l) \in V - H$ where $1 \le i, k \le m-1, i < k$ and $2 \le j, l \le n$.

Then $(u_i, v_j), (u_k, v_l) \in V - H$ has a path $(u_i, v_j)(u_i, v_1)(u_{i+1}, v_1) \ldots (u_k, v_1)(u_k, v_l)$ in G whose internal vertices are in H.

Case iii: For any pair of non-adjacent vertices $(u_i, v_j), (u_k, v_l) \in V - H$ where $1 \le i, k \le m-1, i > k$ and $2 \le j, l \le n$.

Then $(u_i, v_j), (u_k, v_l) \in V - H$ has a path $(u_i, v_j)(u_i, v_1)(u_{i-1}, v_1) \ldots (u_k, v_1)(u_k, v_l)$ in G whose internal vertices are in H.

Clearly, H is a hub dominating set with $m-1$ elements. Hence, $\gamma_h(L(P_m) \square K_n) = m-1$.

Theorem 3.6 For all $n > 3$, $\gamma_h(P_n \square L(P_n)) = n^2 - 3n$ where $L(P_n)$ is the line graph of the path graph P_n.

Proof:

Let G be the cartesian product graph $P_n \square L(P_n)$ with vertex set $V = V_1 \times V_2$, where V_1 is the vertex set of P_n with n vertices and V_2 is the vertex set of $L(P_n)$ with $n-1$ vertices. We claim, $H = \{(u_i, v_j) \in V_1 \times V_2 : i = 1,2, \ldots, n, 2 \le j \le n-2\}$ is a hub dominating set

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of G. Now, $V - H = \{(u_i, v_j) \in V_1 \times V_2 : i = 1,2,\ldots,n,\ j = 1, n-1\}$. Clearly the vertices $(u_i, v_1) \in V - H$ are adjacent to $(u_i, v_2) \in H$ and $(u_i, v_{n-1}) \in V - H$ are adjacent to $(u_i, v_{n-2}) \in H$.

Case i: For any pair of non-adjacent vertices $(u_i, v_1), (u_i, v_{n-1}) \in V - H$ where $i = 1,2,\ldots,n$.

Then $(u_i, v_1), (u_i, v_{n-1}) \in V - H$ has a path $(u_i, v_1)(u_i, v_2) \ldots (u_i, v_{n-2})$ (u_i, v_{n-1}) in G whose internal vertices are in H.

Case ii: For any pair of non-adjacent vertices $(u_i, v_1), (u_k, v_1) \in V - H$ where $i, k = 1,2,\ldots,n$ and $i \neq k$.

Then $(u_i, v_1), (u_k, v_1) \in V - H$ has a path $(u_i, v_1)(u_i, v_2)(u_{i+1}, v_2) \ldots$ $(u_k, v_2)(u_k, v_1)$ in G whose internal vertices are in H.

Case iii: For any pair of non-adjacent vertices $(u_i, v_{n-1}), (u_k, v_{n-1}) \in V - H$ where $i, k = 1,2,\ldots,n$ and $i \neq k$.

Then $(u_i, v_{n-1}), (u_k, v_{n-1}) \in V - H$ has a path $(u_i, v_{n-1})(u_i, v_{n-2})$ $(u_{i+1}, v_{n-2}) \ldots (u_k, v_{n-2})(u_k, v_{n-1})$ in G whose internal vertices are in H.

Case iv: For any pair of non-adjacent vertices $(u_i, v_1), (u_k, v_{n-1}) \in V - H$ where $i, k = 1,2,\ldots,n$ and $i < k$.

Then $(u_i, v_1), (u_k, v_{n-1}) \in V - H$ has a path $(u_i, v_1)(u_i, v_2) \ldots (u_i, v_{n-2})$ $(u_{i+1}, v_{n-2}) \ldots (u_{k-1}, v_{n-2})(u_k, v_{n-2})(u_k, v_{n-1})$ in G whose internal vertices are in H.

Case v: For any pair of non-adjacent vertices $(u_i, v_1), (u_k, v_{n-1}) \in V - H$ where $i, k = 1,2,\ldots,n$ and $i > k$.

Then $(u_i, v_1), (u_k, v_{n-1}) \in V - H$ has a path $(u_i, v_1)(u_i, v_2) \ldots (u_i, v_{n-2})$ $(u_{i-1}, v_{n-2}) \ldots (u_{k+1}, v_{n-2})(u_k, v_{n-2})(u_k, v_{n-1})$ in G whose internal vertices are in H.

Hence H is a hub dominating set.

$$|H| = |V_1|\,(|V_2| - 2)$$

$$= n(n - 1 - 2)$$

$$= n(n - 3)$$

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$$= n^2 - 3n$$

Thus, $\gamma_h(P_n \square L(P_n)) = n^2 - 3n \; \forall \; n > 3$.

CONCLUSION

In this paper. we have found the hub domination number of cartesian product of various graphs. The topics of further research on finding includes the suggestion on finding the bounds on hub domination number of graphs and also finding the hub domination number for line graphs, middle graphs, jump graphs, etc.

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