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The Outer Connected Domination Value in Graphs and Middle Graph of a Graph

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Abstract

Let $G = (V, E)$ be a graph and $M(G)$ be a middle graph, a set $S \subseteq V$ is outer connected dominating set of G if S is a dominating set and $G[V - S]$ is connected. The outer connected domination number $\overline{\gamma}_c(G)$ is the smallest cardinality of a outer connected dominating sets of G. A outer connected dominating set of G of smallest cardinality is called $\overline{\gamma}_c$ - set of G. The total number of $\overline{\gamma}_c$- sets of G is denoted by $\overline{\tau}_c(G)$. We define the outer connected domination value of each vertex $v \in V(G)$, as the number of $\overline{\gamma}_c$- sets of G to which v belongs. In this paper, we study the outer connected domination value for standard graphs and the outer connected domination value in middle graph of a graph.

Keywords: dominating set, outer connected dominating set, middle graph of a graph, domination value, connected domination value.
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1. Introduction

Domination in graphs is a fundamental concept with significant theoretical and practical implications. It is extensively studied in [3,7]. While research has extensively explored dominating sets, domination numbers, and related parameters, the local analysis of domination remains relatively under explored. A variation of standard dominating set, Joanna Cyman introduce the concepts of outer connected domination in 2007 [11], where not only does the set dominate all vertices in the graph, but also the set of non-dominated vertices (the outer vertices) must form a connected subgraph. This includes the introduction of concepts like outer connected domination number[1,10]. After this Connected domination and Total connected domination was introduced by E. Sampathkumar and H.B. Walikar in 1979.Further it was extented in[5]. After this complementary connected domination number was introduced by V.R.Kulli and Janakiraman they called the parameter as non split domination number[12]. Further it is studied in[13,14]. Although domination and other related concepts has been studied extensively. Mynhardt's in [9] initiated that a vertex of a tree with domination value 0 or the total number of dominating set. This initiated the study of vertex-centric domination values [4]. Even a casual chess player knows that controlling the chessboard's center squares is crucial, especially during the first and middle stages of the game. For this reason, the center square has a higher total domination value. This leads Kang, paving the way to found total domination value in

10.48047/jocaaa.2024.33.08.310

graphs in 2014 [11] and Angsuman Das [2] introduced the concept of domination value to connected domination value of a vertex in a graph in 2023.

Let $G = (V(G), E(G))$ be a simple, connected graph with order $|V(G)|$ and size$|E(G)|$. A subgraph H of a graph G whose vertex set and edge set are the subsets of the graph G. For $S \subseteq V(G)$, We denote $\langle S \rangle$ the subgraph of G induced by S. A vertex having degree one is called end vertex. A vertex is called central vertex if its eccentricity is equal to the radius of the graph. The distance between two vertices in a graph is the number of edges in a shortest path (also called a graph geodesic) connecting them. The longest distance between any two vertices is called diameter of G.

A set $S \subseteq V$ is called a *dominating set* of G if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

A subset S of vertices in a graph G is called *outer connected dominating set* of G if S is a dominating set and $G[V - S]$ is connected. The minimum cardinality of a outer connected dominating set of G is its outer connected domination number and is denoted by $\overline{\gamma}_c(G)$. A minimum outer connected dominating set $\overline{\gamma}_c(G)$ is said to be a $\overline{\gamma}_c$ -set of G.

The middle graph of a graph G denoted by $M(G)$ is defined as follows the vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set $M(G)$ are adjacent in $M(G)$ in the case one of the following condition holds.

(i) x, y are in $E(G)$ and x, y is adjacent in G.

(ii) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G.

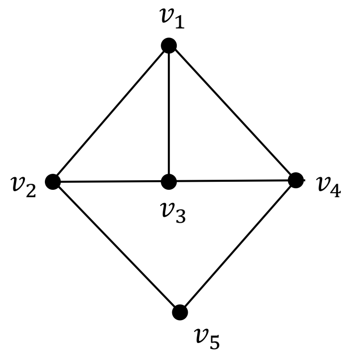
For $S \subseteq M(G)$, let $M(G)$[S] denoted the induced subgraph of S in $M(G)$.

In this the paper, we denote by C_n, P_n, K_n $B_{r,s}$, $K_{r,s}$, W_n, S_n, $F_n$and bull graph on n vertices, respectively. We introduce a new parameter outer connected domination value and outer connected domination value in middle graph of a graph.

2. Basic properties of outer connected domination value.

Definition 2.1. The total number of $\overline{\gamma}_c$- sets of G is denoted by $\overline{\tau}_c(G)$. We define the outer connected domination value of each vertex $v \in V(G)$, as the number of $\overline{\gamma}_c$- sets of G to which v belongs.

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{v_1, v_2\}$, $S_2 = \{v_1, v_4\}, S_3 = \{v_1, v_5\}, S_4 = \{v_2, v_3\}, S_5 = \{v_2, v_5\}, S_6 = \{v_3, v_4\}, S_7 = \{v_3, v_5\}, S_8 = \{v_4, v_5\}$ are the minimum outer connected dominating sets of G, So that $\overline{\gamma}_c(G) = 2$. Also, the total number of $\overline{\gamma}_c$- sets of G is eight, So that $\overline{\tau}_c(G) = 8$. So that $OCDV_G(v) = 3, v \in \{v_1, v_2, v_3, v_4\}$, and $OCDV_G(v) = 4 , v \in v_5$



G

Figure 2.1

10.48047/jocaaa.2024.33.08.310

Observation 2.2. $\sum_{v\in V(G)} OCDV_G(v) = \overline{\tau}_c(G).\overline{\gamma}_c(G)$.

Theorem 2.3. For any graph G and for any vertex $v \in V(G), 0 \leq OCDV_G(v) \leq \overline{\tau}_c(G)$.

Theorem 2.4. For any graph G and for any vertex $v \in V(G), 0 \leq OCDV_G(v) \leq \overline{\gamma}_c(G)$.

Theorem 2.5. Let G be a connected graph

(i) If $OCDV_G(v) = 0$ then no $\overline{\gamma}_c$- set of G contain the vertex v.
(ii) If $OCDV_G(v) = 1$ then unique $\overline{\gamma}_c$- set of G contain the vertex v.
(iii) If $OCDV_G(v) = n$ then n number of $\overline{\gamma}_c$ - set of G contain the vertex v.

Observation 1.5. Let G be a connected graph $\overline{\tau}_c(G) = 1$ if and only if G has a unique minimum outer connected dominating set.

3. OUTER CONNECTED DOMINATION VALUE IN GRAPHS

Theorem 3.1. For the Path $G = P_n$ $(n \geq 2)$, $\overline{\gamma}_c(G) = \begin{cases} 1 & if\, n = 2 \\ 2 & if\, n = 3 \\ n-2 & if n \geq 4 \end{cases}$, $\overline{\tau}_c(G) = \begin{cases} 2 & if\, n = 2 \\ 3 & if\, n = 3 \\ n-3 & if n \geq 4 \end{cases}$ and $OCDV_G(v) = \begin{cases} n-3 & if\ \ v \in \{v_1, v_n\} \\ n-4 & if\ \ v \in \{v_2, v_{n-1}\} \\ n-5 & if\ \ v \in \{v_3, v_4, \dots, v_{n-2}\} \end{cases}$.

Proof. Let $V(P_n)$: $v_1, v_2, \dots, v_{n-1}, v_n$ be the cycle of order n.

Case(i): Let $n = 2$, then $S_1 = \{v_1\}$ and $S_2 = \{v_2\}$ are the only two $\overline{\gamma}_c$-sets of G so that $\overline{\gamma}_c(G) = 1, \overline{\tau}_c(G) = 2$ and $OCDV_G(v) = 1,\ v \in \{v_{1,}, v_2\}$.

Case(ii): Let $n = 3$, then $S_1 = \{v_1, v_2\}$, $S_2 = \{v_1, v_3\}$ and $S_3 = \{v_2, v_3\}$ are the only tthree $\overline{\gamma}_c$-sets of G so that $\overline{\gamma}_c(G) = 3, \overline{\tau}_c(G) = 3$ and $OCDV_G(v) = 1,\ v \in \{v_{1,}, v_2, v_3\}$.

Case(iii): Let $n \geq 4$ and $S_1 = \{v_1, v_4, v_5, \dots, v_n\}$ be a dominating set of G and $G[V - S]$ is connected. Therefore S_1 is a minimum outer connected dominating set of G and $\overline{\gamma}_c(P_n) \leq n - 2$. We prove that $\overline{\gamma}_c(P_n) = n - 2$. On the contrary suppose that $\overline{\gamma}_c(P_n) \leq n - 1$. Then there exists a $\overline{\gamma}_c$-set S' of G such that $|S'| \leq n - 1$. Let x_0 be a vertex of G of $x_0 \in S$ and $x_0 \notin S'$. Then either x_0 is not dominate by any element S' or $G[V - S']$ is not connected. Which is a contradiction. Therefore $\overline{\gamma}_c(P_n) = n - 2$. Similarly we can prove that $S_2 = \{v_1, v_2, v_5, \dots, v_n\}, S_3 = \{v_1, v_2, v_3, v_6 \dots, v_n\}, \dots, S_{n-4} = \{v_1, v_2, v_3, \dots, v_{n-4}, v_{n-1}, v_n\}, S_{n-3} = \{v_1, v_2, v_3, \dots, v_{n-4},, v_{n-3}, v_n\}$ are also the $\overline{\gamma}_c$-set of G and so $\overline{\gamma}_c(P_n) = n - 2$. Also, there are $(n - 3)$ $\overline{\gamma}_c$- sets of G. Hence $\overline{\tau}_c(G) = n - 3$. Here v_1 present in $(n - 3)$ $\overline{\gamma}_c$- set of G, v_2 present in $(n - 4)$ $\overline{\gamma}_c$- set of G, v_3 present in $(n - 4)$ $\overline{\gamma}_c$-set of G ..., v_{n-2} present in $(n - 5)$ $\overline{\gamma}_c$- set of G, v_{n-1} present in $(n - 4)$ $\overline{\gamma}_c$-set of G, v_n present in $(n - 5)$ $\overline{\gamma}_c$- set of G. Therefore the $OCDV_G(v) = n - 3,\ v \in \{v_{1,}, v_n\}$, $OCDV_G(v) = n - 4,\ v \in \{v_{2,}, v_{n-1}\}$ and $OCDV_G(v) = n - 5,\ v \in \{v_{3,}, v_4, \dots, v_{n-2}\}$. ■

Theorem 3.2. For the cycle $G = C_n$, $(n \geq 3)$, $\overline{\gamma}_c(G) = n - 2$, $\overline{\tau}_c(G) = n$ and $OCDV_G(v) = n - 2, \forall\ v \in V(C_n)$.

Proof. Let $V(C_n)$: $v_1, v_2, \dots, v_{n-1}, v_n$ be the cycle of order n. Let $S_1 =$

10.48047/jocaaa.2024.33.08.310

$\{v_1, v_2, v_3, \dots, v_{n-2}\}$ be a dominating set of G and $G[V-S]$ is connected. Therefore S_1 is a minimum outer connected dominating set of G and $\overline{\gamma}_c(C_n) \leq n-2$. We prove that $\overline{\gamma}_c(C_n) = n-2$. On the contrary suppose that $\overline{\gamma}_c(C_n) \leq n-1$. Then there exists a $\overline{\gamma}_c$-set S' of G such that $|S'| \leq n-1$. Let x_0 be a vertex of G of $x_0 \in S$ and $x_0 \notin S'$. Then either x_0 is not dominate by any element S' or $G[V-S']$ is not connected. Which is a contradiction. Therefore $\overline{\gamma}_c(C_n) = n-2$. Similarly we can prove that $S_2 = \{v_2, v_3, v_4, \dots, v_{n-1}\}, S_3 = \{v_3, v_4, v_5, \dots, v_n\}, \dots, S_n = \{v_n, v_1, v_2, \dots, v_{n-3}\}$ are also the $\overline{\gamma}_c$-set of G and so $\overline{\gamma}_c(P_n) = n$. Also, there are n $\overline{\gamma}_c$- sets of G. Hence $\overline{\tau}_c(G) = n$. Here v_1 present in $(n-2)$ $\overline{\gamma}_c$- set of G, v_2 present in $(n-2)$ $\overline{\gamma}_c$- set of $G, \dots, v_n$ present in $(n-2)$ $\overline{\gamma}_c$- set of G. Therefore the outer-connected domination value of each vertex is $n-2$. Therefore the $OCDV_G(v) = n-2, \forall\, v \in V(C_n)$. ■

Theorem 3.3. For the complete graph $G = K_n$, $(n \geq 2)$, $\overline{\gamma}_c(G) = 1$, $\overline{\tau}_c(G) = n$ and $OCDV_G(v) = 1, \forall\, v \in V(K_n)$.

Proof. Let $V(K_n)$: $v_1, v_2, \dots, v_n$. Here any singleton subset of G is a $\overline{\gamma}_c$-sets of G. Hence $\overline{\gamma}_c(G) = 1$. Also there are n number of $\overline{\gamma}_c$- set in G. Hence $\overline{\tau}_c(G) = n$. Here each vertex present exactly once in $\overline{\gamma}_c$- set of G $OCDV_G(v) = 1, \forall v \in V(K_n)$. ■

Theorem 3.4. For the star graph $G = K_{1,n-1}, (n \geq 3), \overline{\gamma}_c(G) = n-1$, $\overline{\tau}_c(G) = n$ and $OCDV_G(v) = n-1, \forall\, v \in V(K_{1,n-1})$.

Proof. Let $V(K_{1,n-1})$: $x, v_1, v_2, \dots, v_{n-1}$. Let x be the central vertex and $v_1, v_2, \dots, v_{n-1}$ be the set of all end vertices of G. Let $S_1 = \{x, v_2, v_3, \dots, v_{n-1}\}$, be a dominating set of G and $G[V-S]$ is connected. Therefore S_1 is a minimum outer connected dominating set of G and $\overline{\gamma}_c(C_n) \leq n-1$. We prove that $\overline{\gamma}_c(C_n) = n-1$. On the contrary suppose that $\overline{\gamma}_c(C_n) \leq n-2$. Then there exists a $\overline{\gamma}_c$-set S' of G such that $|S'| \leq n-2$. Let x_0 be a vertex of G and $x_0 \in S$ and $x_0 \notin S'$. Then either x_0 is not dominate by any element S' or $G[V-S']$ is not connected. Which is a contradiction. Therefore $\overline{\gamma}_c(C_n) = n-2$. Similarly we can prove that $S_2 = \{x, v_1, v_3, \dots, v_{n-1}\}, \dots, S_{n-2} = \{x, v_1, v_2, \dots, v_{n-2}\}, S_{n-1} = \{v_1, v_2, \dots, v_{n-1}\}$ are also the $\overline{\gamma}_c$-sets of G and so $\overline{\gamma}_c(K_{1,n-1}) = n-2$. Also, there are $(n-1)$ $\overline{\gamma}_c$-set of G. Hence $\overline{\tau}_c(G) = n-1$. Here x present in exactly $(n-1)$ $\overline{\gamma}_c$-sets. Hence $OCDV_G(x) = n-1$. Here v_1 present in $(n-1)\overline{\gamma}_c$-set of G, v_2 present in $(n-1)$ $\overline{\gamma}_c$-set of $G, \dots, v_n$ present in $(n-1)$ $\overline{\gamma}_c$-set of G. Therefore the outer-connected domination value of each vertex is $n-1$, So that $OCDV_G(v) = n-1, \forall\, v \in V(K_{1,n-1})$. ■

Theorem 3.5. For the complete bipartite graph $G = K_{r,s}, (2 \leq r \leq s)$, $\overline{\gamma}_c(G) = 2$, $\overline{\tau}_c(G) = r.s$ and $OCDV_G(v) = \begin{cases} s & if\ v \in X \\ r & if\ v \in Y. \end{cases}$

Proof. Let $X = \{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_s\}$ be the partite sets of G. Then $S_{ij} = \{(x_i, y_j)\,/(1 \leq i \leq s), ((1 \leq j \leq s)\}$ are the $\overline{\gamma}_c$-set of G with cardinality two. Hence $\overline{\gamma}_c(G) = 2$. Also, that there are $(r.s)$ $\overline{\gamma}_c$- set of G. Hence $\overline{\tau}_c(G) = r.s$. Here x_1 present in s $\overline{\gamma}_c$-set of G, x_2 present in s $\overline{\gamma}_c$-set of G, …,x_n present in s $\overline{\gamma}_c$-set of G. Similarly, y_1 present in r $\overline{\gamma}_c$-set of G, y_2 present in r $\overline{\gamma}_c$-set of G, …, y_n present in r $\overline{\gamma}_c$-set of G. Therefore $OCDV_G(v) = s\ if\ v \in X$ and $OCDV_G(v) = r\ if\ v \in Y$. ■

Theorem 3.6. For the Bistar graph $G = B_{m,n}$, $\overline{\gamma}_c(G) = m+n$, then $\overline{\tau}_c(G) = 1$ and

10.48047/jocaaa.2024.33.08.310

$OCDV_G(v) = \begin{cases} 1 & if \;\; v \; is \; an \; end \; vertex \\ 0 & otherwise \end{cases}$

Proof. Let $V(B_{m,n}) = \{u, v, u_i, v_j / where \;\; 1 \le i \le m, 1 \le j \le n\}$ and $E(B_{m,n}) = \{uv, uu_i, vv_j / where \;\; 1 \le i \le m, 1 \le j \le n\}$. Then $S = \{u_i, v_j \; / \; where \;\; 1 \le i \le m,$ $1 \le j \le n\}$ be the dominating set of G and $G[V - S]$ is connected. Therefore, S_1 is a minimum outer connected dominating set of G and $\overline{\gamma}_c(B_{m,n}) \le m + n$. We prove that $\overline{\gamma}_c(B_{m,n}) = m + n$. On the contrary suppose that $\overline{\gamma}_c(B_{m,n}) \le m + n - 1$. Then there exists a $\overline{\gamma}_c$-set S' of G such that $|S'| \le m + n - 1$. Let x_0 be a vertex of G and $x_0 \in S$ and $x_0 \notin S'$. Then either x_0 is not dominate by any element S' or $G[V - S']$ is not connected. Which is a contradiction. $\overline{\gamma}_c(B_{m,n}) = m + n$. Also S is the only minimum outer connected dominating set of G. So that $\overline{\gamma}_c(G) = m + n$ and also $\overline{\tau}_c(G) = 1$. All u_i's and v_i's present once in the $\overline{\gamma}_c$-set of G. Thus the outer connected domination value of each u_i $(1 \le i \le m)$ and v_j $(1 \le j \le n)$ is 1. Also, the outer connected domination value of u is zero and v is zero. ■

Observation 3.7. Let $G = K_{1,n-1}$ is a wounded spider and x be the universal vertex of G and $u_1, u_2, \ldots, u_{n-2}$ be the subdivisions then $\overline{\gamma}_c(G) = \Delta(G)$ then $\overline{\tau}_c(G) = 1$ and $0 \le OCDV_G(v) \le 1$. ■

Theorem 3.8. Let G be a connected graph without cut vertices $\Delta(G) = n - 1$ then $0 \le OCDV_G(v) \le 1$.

Proof. By the definition of outer connected domination value, we have $OCDV_G(v) \ge 0$. Since G contains no cut vertices, $\overline{\gamma}_c(G) = 1$. Therefore, by theorem 2.4. $0 \le OCDV_G(v) \le 1$. ■

Theorem 3.9. For the wheel graph $G = W_n, (n \ge 4), \overline{\gamma}_c(G) = 1,\; \overline{\tau}_c(G) = \begin{cases} 4 \; if \; n = 4 \\ 1 \; if \; n \ge 5 \end{cases}$. If $n = 4$ then $OCDV_G(v)$ =1. If $n \ge 5$ then $OCDV_G(v) = \begin{cases} 1 & if \; v \; is \; the \; central \; vertex \\ 0 & otherwise \end{cases}$.

Proof. Let $V(W_n)$: $x, v_1, v_2, \ldots, v_{n-1}$ be the vertex set of G and x be the central vertex.

Case(i). If $n = 4$, W_4 is isomorphic to K_4 . Then by Theorem 2.3 any singleton subset $\overline{\gamma}_c(G) = 1$, $\overline{\tau}_c(G) = 4$ and $OCDV_G(v) = 1$.

Case(i). If $n \ge 5$, then G contains exactly one universal vertex which is not a cut vertex, $\overline{\gamma}_c(G) = 1$. There are only one $\overline{\gamma}_c$- set of G. Therefore $\overline{\tau}_c(G) = 1$. Then by Theorem 2.8. $0 \le OCDV_G(v) \le 1$. ■

Theorem 3.10. For the helm graph $G = H_n$ $(n \ge 3)$, $\overline{\gamma}_c(G) = n + 1$, $\overline{\tau}_c(G) = n +$ 1 and $OCDV_G(v) = \begin{cases} 1 & if \;\; v \; is \; a \; central \;\; vertex \; or \; cut \; vertex \\ n + 1 & if \; v \; is \; an \; end \; vertex. \end{cases}$.

Proof. Let $V(H_n)$: $x, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n$ and x is the central vertex adjacent to $v_1, v_2, \ldots, v_n$ which are the cut vertices and $w_1, w_2, \ldots, w_n$ are the end vertices. Let $S_1 = \{x, w_1, w_2, \ldots, w_n\}$ be the dominating set of G and $G[V - S]$ is connected. Therefore S_1 is a minimum outer connected dominating set of G and $\overline{\gamma}_c(G) \le n + 1$. We prove that $\overline{\gamma}_c(G) = n + 1$. On the contrary suppose that $\overline{\gamma}_c(G) \le n$. Then there exists a $\overline{\gamma}_c$-set S' of G such that $|S'| \le n$. Let x_0 be a vertex of G of $x_0 \in S$ and $x_0 \notin S'$. Then either x_0 is not dominate by any element S' or $G[V - S']$ is not connected. Which is a contradiction. Therefore $\overline{\gamma}_c(G) = n + 1$. Similarly we can prove that $S_2 = \{v_1, w_1, w_2, \ldots, w_n\}, S_2 =$

10.48047/jocaaa.2024.33.08.310

$\{v_2, w_1, w_2, \ldots, w_n\} \ldots, S_{n-1} = \{v_{n-1}, w_1, w_2, \ldots, w_n\}, S_{n+1} = \{v_n, w_1, w_2, \ldots, w_n\}$ are also the $\overline{\gamma}_c$-set of G and $\overline{\gamma}_c(G) = n+1$ and $\overline{\tau}_c(G) = n+1$. Here the vertices $\{x, v_1, v_2, \ldots, v_n\}$ is present in exactly one $\overline{\gamma}_c$-set and $\{w_1, w_2, \ldots, w_n\}$ present all the $(n+1)$ $\overline{\gamma}_c$-set of G. Hence $OCDV_G(v) = 1$ if $v \in \{x, v_1, v_2, \ldots, v_n\}$ and $OCDV_G(v) = n + 1\ if\ v \in \{w_1, w_2, \ldots, w_n\}$. ■

Theorem 3.11. For the sunlet graph $G = S_n, (n \geq 3)$, $\overline{\gamma}_c(G) = n$, $\overline{\tau}_c(G) = 1$ and $OCDV_G(v) = \begin{cases} 1 & if\ \ v\ is\ an\ end\ vertex \\ 0 & otherwise \end{cases}$.

Proof. Let $V(S_n): v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n$ be the vertex set of G and $w_1, w_2, \ldots, w_n$ be the end vertices. Let $S = \{w_1, w_2, \ldots, w_n\}$ be the dominating set of G and $G[V-S]$ is connected. Therefore S is a minimum outer connected dominating set of G and $\overline{\gamma}_c(G) \leq n$. We prove that $\overline{\gamma}_c(G) = n$. On the contrary suppose that $\overline{\gamma}_c(G) \leq n-1$. Then there exists a $\overline{\gamma}_c$-set S' of G such that $|S'| \leq n-1$. Let x_0 be a vertex of G of $x_0 \in S$ and $x_0 \notin S'$. Then either x_0 is not dominate by any element S' or $G[V-S']$ is not connected. Which is a contradiction. Therefore $\overline{\gamma}_c(G) = n$. So $\overline{\gamma}_c(G) = n$. There are only one $\overline{\gamma}_c$- set G. Therefore $\overline{\tau}_c(G) = 1$. Here the vertices $\{w_1, w_2, \ldots, w_n\}$ is present exactly once in $\overline{\gamma}_c$-set of G .Thus the outer connected domination value for the pendent vertices is one and zero for all the other vertices. ■

Theorem 3.12. For the bull graph G, $\overline{\gamma}_c(G) = n-2$, $\overline{\tau}_c(G) = n-2$ and $OCDV_G(v) = \begin{cases} 3 & if\ \ v\ is\ an\ end\ vertices \\ 1 & otherwise \end{cases}$

Proof. Let G be a bull graph with $V(G): v_1, v_2, v_3, v_4, v_5$ and $E(G): v_1v_2, v_2v_3,, v_2v_4, v_3v_4, v_4v_5$. Then $S_1 = \{v_1, v_3, v_5\}, S_2 = \{v_1, v_4, v_5\}$ and $S_3 = \{v_1, v_2, v_5\}$ are the $\overline{\gamma}_c$-set of G with cardinality 3. So that $\overline{\gamma}_c(G) = 3$. There are three $\overline{\gamma}_c$-sets of G. Therefore $\overline{\tau}_c(G) = 3$. Thus the outer connected domination value for the end vertices is three and one for the other vertices. ■

Theorem 3.13. For the graph $G = F_n (n \geq 2)$, $\overline{\gamma}_c(G) = n$, $\overline{\tau}_c(G) = 4$ and $OCDV_G(v) = \begin{cases} 2 & if\ v \in \{v_1, v_2, \ldots, v_{2n}\} \\ 0 & if\ v\ is\ a\ cut\ vertex \end{cases}$.

Proof. Let F_n be the friendship graph with $V(F_n): \{v_0, v_1, v_2, \ldots, v_{2n}\}$ and $E(F_n): \{v_0v_1, v_0v_2, \ldots, v_0v_{2n}\} \cup \{v_1v_2, v_3v_4, \ldots, v_{2n-1}v_{2n}\}$. Let $S_1 = \{v_1, v_3, v_5, \ldots, v_{2n-1}\}$ be the dominating set of G and $G[V-S]$ is connected. Therefore S is a minimum outer connected dominating set of G and $\overline{\gamma}_c(G) \leq n$. We prove that $\overline{\gamma}_c(G) = n$. On the contrary suppose that $\overline{\gamma}_c(G) \leq n-1$. Then there exists a $\overline{\gamma}_c$-set S' of G such that $|S'| \leq n-1$. Let x_0 be a vertex of G of $x_0 \in S$ and $x_0 \notin S'$. Then either x_0 is not dominate by any element S' or $G[V-S']$ is not connected. Which is a contradiction. Therefore $\overline{\gamma}_c(G) = n$. Similarly we can prove $S_2 = \{v_1, v_4, v_6, \ldots, v_{2n}\}$, $S_3 = \{v_2, v_3, v_5, \ldots, v_{2n-1}\}$, $S_4 = \{v_2, v_4, v_6, \ldots, v_{2n}\}$ are also $\overline{\gamma}_c$- set G. So that there are four $\overline{\gamma}_c$- set G. Therefore $\overline{\tau}_c(G) = 4$. Here the vertices $\{v_1, v_2, \ldots, v_{2n}\}$ present exactly twice in the four $\overline{\gamma}_c$- set G. Thus, the outer connected domination value for the cut vertex is 0 and two for the other vertices. ■

4. OUTER CONNECTED DOMINATION VALUE IN MIDDLE GRAPH OF A GRAPH

Theorem 4.1. For the graph $G = P_n$, $n \geq 3$, $\overline{\gamma}_c(M(G)) = n$, $\overline{\tau}_c(M(G)) = 4$ and

10.48047/jocaaa.2024.33.08.310

$$OCDV_{M(G)}(v) = \begin{cases} 4 & if\ v\ \in \{v_1, v_3, \dots, v_{n-2}, v_n\} \\ 2 & if\ v\ \in \{v_2, v_{n-1}, f_1, f_{n-1}\} \\ 0 & otherwise \end{cases}$$

Proof. Let P_n be the path on n vertices with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$. Then without loss of generality we assume that $f_i = \{v_iv_{i+1} | 1 \le i \le n-1\}$. Then $V(M(P_n)) = \{v_i, f_j | 1 \le i \le n, 1 \le j \le n-1\}$ and $E(M(P_n)) = 3n-4$. Let $S_1 = \{v_1, v_3, v_4, v_5, \dots, v_n\} \cup \{f_1\}$ be a dominating set of $M(G)$ and $M(G)[V-S]$ is connected. Therefore S_1 is a minimum outer connected dominating set of $M(P_n)$ and $\overline{\gamma}_c(M(P_n)) \le n$. We prove that $\overline{\gamma}_c(M(P_n)) = n$. On the contrary suppose that $\overline{\gamma}_c(M(P_n)) \le n-2$. Then there exists a $\overline{\gamma}_c$-set S' of $M(G)$ such that $|S'| \le n-1$. Let x_0 be a vertex of $M(G)$ such that $x_0 \in S$ and $x_0 \notin S'$. Then either x_0 is not dominate by any element S' or $M(G)[V-S']$ is not connected. Which is a contradiction. Therefore $\overline{\gamma}_c(M(P_n)) = n$. Similarly we can prove that $,S_2 = \{v_1, v_2, \dots, \dots, v_{n-2}\} \cup \{f_{n-1}\}, S_3 = \{v_1, v_3, v_4, \dots, v_{n-2}\} \cup \{f_1, f_{n-1}\}, S_4 = \{v_1, v_2, v_3, v_5, \dots, v_n\}$ are also $\overline{\gamma}_c$-set of $M(G)$ with $\overline{\gamma}_c(M(P_n)) = n-1$. Here the vertices $\{v_1, v_3, \dots, v_{n-2}, v_n\}$ present all the four $\overline{\gamma}_c$- sets of G. It is noted that $\{v_2, v_{n-1}, f_1, f_{n-1}\}$ present two $\overline{\gamma}_c$- sets of G and $\{f_2, f_3, \dots, f_{n-2}, f_n\}$ does not present in $\overline{\gamma}_c$- sets of G so that $OCDV_{M(G)}(v) = 4, v\ \in \{v_1, v_3, \dots, v_{n-2}, v_n\}$, $OCDV_{M(G)}(v) = 0, v\ \in \{f_2, f_3, \dots, f_{n-2}, f_n\}$. ■

Theorem 4.2. For the graph $G = C_n,\ n \ge 3, \overline{\gamma}_c(M(G)) = n-1,\ \overline{\tau}_c(M(G)) = n$. If $v \in M(G)$, $OCDV_{M(G)}(v) = \begin{cases} n-2 & if\ v\ \in \{v_1, v_2, \dots, v_{n-1}\} \\ 1 & otherwise \end{cases}$

Proof. Let $G = C_n$ be the cycle on n vertices with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{v_1v_2, v_3v_4, \dots, v_{n-1}v_n\}$ Then without loss of generality we assume that $f_i = \{v_iv_{i+1} | 1 \le i \le n-1\}$. Then $V(M(C_n)) = \{v_i, f_j | 1 \le i \le n, 1 \le j \le n-1\}$ $E(M(C_n)) = 3$. Let $S_1 = \{v_3, v_4, v_5, \dots, v_n\} \cup \{f_1\}$ be a dominating set of $M(G)$ and $M(G)[V-S]$ is connected. Therefore S_1 is a minimum outer connected dominating set of $M(C_n)$ and $\overline{\gamma}_c(M(C_n)) \le n-1$. We prove that $\overline{\gamma}_c(M(C_n)) = n-1$. On the contrary suppose that $\overline{\gamma}_c(M(C_n)) \le n-2$. Then there exists a $\overline{\gamma}_c$-set S' of $M(G)$ such that $|S'| \le n-2$. Let x_0 be a vertex of $M(G)$ of $x_0 \in S$ and $x_0 \notin S'$. Then either x_0 is not dominate by any element S' or $M(G)[V-S']$ is not connected. Which is a contradiction. Therefore $\overline{\gamma}_c(M(C_n)) = n$. Similarly we can prove that $S_2 = \{v_1, v_4, v_5, \dots, v_n\} \cup \{f_2\}$, $S_3 = \{v_1, v_2, v_5, v_6, \dots, v_n\} \cup \{f_3\}, \dots, S_{n-1} = \{v_1, v_2, \dots, v_{n-2}\} \cup \{f_{n-1}\}$, $S\ _n = \{v_2, v_3, \dots, v_{n-1}\} \cup \{f_n\}$ are also $\overline{\gamma}_c$- set of $M(G)$ and $\overline{\gamma}_c(M(G)) = n-1$ and $\overline{\tau}_c(M(G)) = n$. Here the vertices $\{v_1, v_2, \dots, v_{n-1}\}$ present $(n-2\ )$ $\overline{\gamma}_c$- set of $M(G)$ and also noted that $\{f_1, f_2, \dots, f_n\}$ presents exactly once in $\overline{\gamma}_c$- set of $M(G)$. Therefore $OCDV_{M(G)}(v) = n-2$, $v \in \{v_1, v_2, \dots, v_{n-1}\}$ and $OCDV_{M(G)}(v) = 1, v \in \{u_1, u_2, \dots, u_n\}$ ■

Theorem 4.3. For the graph $G = K_{1,n-1}, n \ge 4$, $\overline{\gamma}_c(M(G)) = n$, $\overline{\tau}_c(M(G)) = n$, and $OCDV_{M(G)}(v) = \begin{cases} n & if\ v\ \in \{v_1, v_2, \dots, v_{n-1}\} \\ 1 & if\ v\ \in \{x_1, u_1, u_2, \dots, u_n\} \end{cases}$.

Proof. Let $G = K_{1,n-1}$ be the star graph on n vertices. Then $V(K_1) = x$ and $\{v_1, v_2, \dots, v_{n-1}\}$ are the end vertices of $K_{1,n-1}$ and $u_i = xv_i,\ 1 \le i \le n-1$. Then

10.48047/jocaaa.2024.33.08.310

$V(M(K_{n-1})) = 2n-1$ and $E(M(K_{n-1})) = \frac{n^2+3n}{2}$. Let $S_1 = \{v_1, v_2, \dots, v_{n-1}\} \cup \{u_1\}$ be a dominating set of $M(G)$ and $M(G)[V-S]$ is connected. Therefore S_1 is a minimum outer connected dominating set of $M(K_{n-1})$ and $\overline{\gamma}_c\left(M(K_{1,n-1})\right) \le n$. We prove that $\overline{\gamma}_c\left(M(K_{1,n-1})\right) = n$. On the contrary suppose that $\overline{\gamma}_c\left(M(K_{1,n-1})\right) \le n-1$. Then there exists a $\overline{\gamma}_c$-set S' of $M(G)$ such that $|S'| \le n+1$. Let x_0 be a vertex of $M(G)$ of $x_0 \in S$ and $x_0 \notin S'$. Then either x_0 is not dominate by any element S' or $M(G)[V-S']$ is not connected. Which is a contradiction. Therefore $\overline{\gamma}_c\left(M(K_{1,n-1})\right) = n$. Similarly we can prove that $S_2 = \{v_1, v_2, \dots, v_{n-1}\} \cup \{u_2\}$, $S_3 = \{v_1, v_2, \dots, v_{n-1}\} \cup \{u_3\}, \dots S_{n-1} = \{v_1, v_2, \dots, v_{n-1}\} \cup \{u_{n-1}\}, S_n = \{v_1, v_2, \dots, v_{n-1}\} \cup \{x\}$ are also $\overline{\gamma}_c$-set of $M(G)$ and $\overline{\gamma}_c(M(G)) = n$. Here the vertices $\{v_1, v_2, \dots, v_{n-1}\}$ present all the n $\overline{\gamma}_c$-set of $M(G)$ and also noted $\{u_1, u_2, \dots, u_{n-1}, x\}$ present exactly once in $\overline{\gamma}_c$- set of $M(G)$. Hence $OCDV_{M(G)}(v) = n$, $v \in \{v_1, v_2, \dots, v_{n-1}\}$ and $OCDV_{M(G)}(v) = 1, v \in \{u_1, u_2, \dots, u_{n-1}, x\}$. ■

Theorem 4.4. Let G be the graph obtained from $K_{1,n-1}$ by subdividing each edge exactly once. Then $n \ge 3$, $\overline{\gamma}_c(M(G)) = 2n-1$, $\overline{\tau}_c(M(G)) = 3$. If $v \in M(G)$,

$$OCDV_{M(G)}(v) = \begin{cases} 3 & if\ v \in \{u_i, x\}, 1 \le i \le n-1 \\ 2 & otherwise \end{cases}.$$

Proof. Let $K_{1,n-1}$ be the star graph with $V(K_1) = x$ and $\{v_1, v_2, \dots, v_{n-1}\}$ be the end vertices. Let G be a graph obtained from $K_{1,n-1}$ with vertex set $\{x, v_1, v_2, \dots, v_{n-1}, u_1, u_2, \dots, u_{n-1}\}$ and edge set $\{xv_1, xv_2, \dots, xv_n, v_1u_1, v_2u_2,,,,, v_2u_2, \dots, v_{n-1}u_{n-1}\}$. Without loss of generality let us assume that $e_j = xv_j, (1 \le j \le n-1)$ and $f_k = v_ku_k, (1 \le k \le n-1)$ Then $V\left(M(K_{1,n-1,n-1})\right) = \{v_0, v_i, u_i, e_j, f_k / (1 \le i \le n-1)(1 \le j \le n-1)\}$. $E\left(M(K_{1,n-1,m-1})\right) = (n-1)(n+3)$. Let $S_1 = \{u_1, u_2, \dots, u_{n-1}, v_1, v_2, \dots, v_{n-1}, v_0\}$ be a dominating set of G and $M(G)[V-S]$ is connected. Therefore S_1 is a minimum outer connected dominating set of $M(K_{1,n-1,n-1})$ and $\overline{\gamma}_c(G) \le 2n-1$. We prove that $\overline{\gamma}_c(M(G)) = 2n-1$. On the contrary suppose that $\overline{\gamma}_c\left(M(K_{1,n-1,m-1})\right) \le 2n$. Then there exists a $\overline{\gamma}_c$-set S' of $M(G)$ such that $|S'| \le 2n$. Let x_0 be a vertex of $M(G)$ of $x_0 \in S$ and $x_0 \notin S'$. Then either x_0 is not dominate by any element S' or $M(G)[V-S']$ is not connected. Which is a contradiction. Therefore $\overline{\gamma}_c(M(G)) = 2n-1$. Similarly we can prove that $S_2 = \{u_1, u_2, \dots, u_{n-1}, e_1, e_2, \dots, e_{n-1}, v_0\}, S_3 = \{u_1, u_2, \dots, u_{n-1}, f_1, f_2, \dots, f_{n-1}, v_0\}$ are also $\overline{\gamma}_c$-set of G . Here the vertices $\{v_0. u_1, u_2, \dots, u_{n-1}\}$ present in three $\overline{\gamma}_c$- sets of $M(G)$ and it is also noted that $\{v_1, v_2, \dots, v_{n-1}\}$, $\{e_1, e_2, \dots, e_{n-1}\}$, $\{f_1, f_2, \dots, f_{n-1}\}$ present exactly once in $\overline{\gamma}_c$- sets of $M(G)$. Hence $OCDV_{M(G)}(v) = 3, v \in \{v_0, u_1, u_2,,.., u_{n-1}\}$, $OCDV_{M(G)}(v) = 2, v \in \{v_1, v_2, \dots, v_{n-1}, e_1, e_2, \dots, e_{n-1}, f_1, f_2, \dots, f_{n-1}\}$. ■

Theorem 4.5. For the graph $G = S_n$, $(n \ge 3)$, $\overline{\gamma}_c(M(S_n)) = 2n$, $\overline{\tau}_c(M(S_n)) = 2$ and

$$OCDV_{M(G)}(v) = \begin{cases} 2 & if\ v = u_i, 1 \le i \le n \\ 1 & if\ v = v_i, f_i, 1 \le i \le n \\ 0 & otherwise \end{cases}.$$

Proof. Let C_n be the cycle graph on n vertices with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge

10.48047/jocaaa.2024.33.08.310

set $\{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$. Let S_n be a graph obtained from C_n with vertex set $\{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$ and edge set $\{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, u_1v_1, u_2v_2, \dots, u_nv_n\}$. Without loss of generality let us assume that $f_j = \{u_iv_i|1 \le i \le n\}, (1 \le j \le n)$ and $w_j = \{v_iv_{i+1}|1 \le i \le n\}, (1 \le j \le n)$. Then $V(M(S_n)) = \{v_i, u_i, f_j, w_j|1 \le i \le n, 1 \le j \le n\}$ and $E(M(S_n)) = 7n$. Let $S_1 = \{v_1, v_2, \dots, v_n\} \cup \{u_1, u_2, \dots, u_n\}$ be a dominating set of $M(G)$ and $M(G)[V - S]$ is connected. Therefore S_1 is a minimum outer connected dominating set of $M(G)$ and $\overline{\gamma}_c(M(G)) \le 2n$. We prove that $\overline{\gamma}_c(M(G)) = 2n$. On the contrary suppose that $\overline{\gamma}_c(M(G)) \le 2n - 1$. Then there exists a $\overline{\gamma}_c$-set S' of $M(G)$ such that $|S'| \le 2n - 1$. Let x_0 be a vertex of $M(G)$ such that $x_0 \in S$ and $x_0 \notin S'$. Then either x_0 is not dominate by any element S' or $M(G)[V - S']$ is not connected. Which is a contradiction. Therefore $\overline{\gamma}_c(M(G)) = 2n$. Similarly we can prove that $S_2 = \{u_1, u_2, u_3, \dots, u_n\} \cup \{f_1, f_2, \dots, f_n\}$ is also a $\overline{\gamma}_c$- set of $M(G)$ with $\overline{\gamma}_c(M(G)) = 2n$. Here the vertices $\{u_1, u_2, \dots, u_n\}$ presents in two $\overline{\gamma}_c$- sets of $M(G)$ and also noted that $\{v_1, v_2, \dots, v_n, f_1, f_2, \dots, f_n\}$ present exactly once in $\overline{\gamma}_c$- sets of $M(G)$. Hence $OCDV_{M(G)}(v) = 2, v \in \{u_1, u_2, \dots, u_n\}, OCDV_{M(G)}(v) = 1, v \in \{v_1, v_2, \dots, v_n, f_1, f_2, \dots, f_n\}$ and $OCDV_{M(G)}(v) = 0, v \in \{w_1, w_2, \dots, w_n\}$ ■

Theorem 4.6. For the graph $G = F_n$ with $n \ge 2$, $\overline{\gamma}_c(M(F_n)) = n + 1$, $\overline{\tau}_c(M(F_n)) = 1$ and $OCDV_{M(G)}(v) = \begin{cases} 1 & if\ \ v = v_0, w_k, (1 \le k \le n) \\ 0 & if\ v = v_i, e_j, (1 \le i \le n, 1 \le j \le n) \end{cases}$

Proof. Let F_n be the friendship graph with vertex set $\{v_0, v_1, v_2, \dots, v_{2n}\}$ and edge set $\{v_0v_1, v_0v_2, \dots, v_0v_{2n}\} \cup \{v_1v_2, v_3v_4, \dots, v_{2n-1}v_{2n}\}$. Without loss of generality assume that $e_j = v_0v_i$ and $w_k = v_iv_{i+1}$ such that $1 \le i \le n, 1 \le j \le n, 1 \le k \le n$. Then $V(M(F_n)) = \{v_i, e_j, w_k|1 \le i \le n, 1 \le j \le n, 1 \le k \le n\}$ and $E(M(F_n)) = 7n$. Let $S_1 = \{v_0, w_1, w_2, \dots, w_n\}$ be a dominating set of $M(G)$ and $M(G)[V - S]$ is connected. Therefore S_1 is a minimum outer connected dominating set of $M(G)$ and $\overline{\gamma}_c(M(G)) \le n + 1$. We prove that $\overline{\gamma}_c(M(G)) = n + 1$. On the contrary suppose that $\overline{\gamma}_c(M(G)) \le n$. Then there exists a $\overline{\gamma}_c$-set S' of $M(G)$ such that $|S'| \le n$. Let x_0 be a vertex of $M(G)$ such that $x_0 \in S$ and $x_0 \notin S'$. Then either x_0 is not dominate by any element S' or $M(G)[V - S']$ is not connected. Which is a contradiction. Therefore $\overline{\gamma}_c(M(G)) = 2n$. Clearly S_1 is a unique minimum outer connected dominating set of $M(G)$ so that $\overline{\tau}_c(M(G)) = 1$. Here the vertices $\{v_0, w_1, w_2, \dots, w_n\}$ present exactly once in $M(G)$ and also noted that $\{v_1, v_2, \dots, v_{2n}, e_1, e_2, \dots, e_n\}$ does not present in $\overline{\gamma}_c$-set of $M(G)$. Hence $OCDV_{M(G)}(v) = 1, v \in \{v_0, w_1, w_2, \dots, w_n\}, OCDV_{M(G)}(v) = 0, v \in \{v_1, v_2, \dots, v_{2n}, e_1, e_2, \dots, e_n\}$. ■

5. Conclusion

In this article we explore the concept of the outer connected domination value in graph and outer connected domination value in middle graph of a graph. Further, we extend this concept to some other distance related parameters in graphs.

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