ModularChromaticNumberofSnowGraphsofSomeCycle Related Graph and Its Extended Snow Graph

P.Sumathi[1],S.Tamilselvi[2]*

[1]DepartmentofMathematics,C.KandaswamiNaiduCollegeforMen,UniversityofMadras AnnaNagar, Chennai 600 102,Tamilnadu,India, Email: sumathipaul@yahoo.com
[2]DepartmentofMathematics,SchoolofArtsandScience,Vinayakamission'sChennaiCampus, Vinayaka mission's research foundation(Deemed to be University), Chennai 603104, Tamilnadu, India, Email:rat31323@gmail.com
*CorrespondingAuthor

Received:19.04.2024 Revised:25.05.2024 Accepted:10.05.2024

ABSTRACT
For a graph $G = (V(G),E(G))$ a coloring $C:V(G) \rightarrow \mathbb{Z}_k$, $k \geq 2$ (not a proper coloring) called the modular coloring if for each pair of neighboring vertices $\mathcal{S}(u) \neq \mathcal{S}(v)$. Theminimum k for which G has anmodular k -coloringiscalledmodularchromaticnumberofG.Themodulark-coloringofagraphisdenotedby $m_c(G)$.Inthis study we examine the modular chromatic number for some snow graph and extended snow graph of cycle related graph.

Keywords: Snow graph, extended snow graph, wheel graph, gear graph, friendship graph, generalized fan graph, flower graph.

INTRODUCTION

There are many real-world applications for graph theory. That's when a graph coloring comes into play. One of the research's emerging topics is modular coloring. We consider the finite undirected connected graph. F. Okamoto, E. Salehi, and P.Zhang in 2010. In [1,4], a modular k- coloring, $k \geq 2$ of a graph without isolated vertices is a coloring of the vertices of G with the element in $\mathbb{Z}_k$(where adjacent vertices may be colored the same) having the property that for every two adjacent vertices of G,The sum of the colors of the neighbors are different in $\mathbb{Z}_k$. The minimum k for which G has a modular k- coloring is called modular chromatic number $m_c(G)$.

LetGbeanygraphandHisadoubleclawgraph.Let$\{v_1,v_2,\ldots,v_\mu\}$aretheexternalvertex(exceptthe center vertex) of a graphG, take μ copies of H and the root vertex of the double clawgraph joined withthe each external node G is called snow graph and it is denoted by $ST_{dc}(G_\mu)$ where μ is the number of external vertices of graph G and T_{dc}denote the double claw tree graph.

LetGbethecyclicgraphandHisanyγ−arytreegraph.Let$\{v_1,v_2,\ldots,v_\mu\}$aretheexternalvertex(except the center vertex)of a cyclic graph, take μ copies of H and the root vertex of the γ−arytree joined with the external node of G by an edge is called extended snow graph denoted by $ST_\gamma(G_\mu)$ where μ,γ is the number of external vertices of cyclic graph G and height of n-ary tree graph T respectively.

In [2] P. Sumathi, S. Tamilselvi, determined the modular chromatic number of certain cyclic graphs. In [3], we obtain the modular chromatic number of the inflated graphs of the wheel, gear, fan, friendship, and flower graphs. In [4], also we examined the modular coloring of the corona product of a generalized Jahangir graph. In [5] N. Paramaguru, R. Sampathkumar, investigatedmodular colorings of join of two special graphs. In [6], T. Nicholas, G. R, Sanma, discussed the modular colorings of cycle related graphs. In [7],R.Rajarajachozhan,R.Sampathkumar,foundthemodularcoloringofthecartesianproducts$K_m\square K_n$, $K_m\square C_n$,And $K_m\square P_n$. In [8], T. Nicholas, G. R, Sanma, found the modular colorings of circular Halin graphsof level two.In [9] Sanma. G. R, P. Maya obtained the Modular coloring and switching in some planargraphs.

Mainresults

Let G be a non-trivial undirected connected graph. In this paperwe introduced two new graph structures, snow graph $ST_{dc}(G_\mu)$ and extended snow graph $ST_\gamma(G_\mu)$ of cycle related graphs. The following theorem results that the snow graph and its extended snow graph of cycle related graph admits modular coloring.

Theorem1. Foranyinteger$\mu\geq3$,themodularchromaticnumberofasnowgraphofwheelgraph

$mc(ST_{dc}(W_{\mu}))=\begin{cases}3 & \text{when}\mu\text{is even}\\ 4 & \text{when}\mu\text{isodd}\end{cases}$.

Proof: Theconstructionofthesnowgraphof W_{μ}isdescribedas,letW_{μ}bea wheelgraph onμverticesand HisanydoubleclawtreeT_{dc}.EachexternalvertexofW_{μ}exceptthecentervertexisattachedwiththeroot vertexoftheT_{dc}isjoinedbyanedgeiscalledsnowgraphofwheelgraph.Let$V(ST_{dc}(W_{\mu}))=c_0\cup v_{\alpha}\cup v_{\alpha\beta}$where$1\leq\alpha\leq\mu$and$1\leq\beta\leq6$andlet$E(ST_{dc}(W_{\mu}))=c_0v_{\alpha}\cup v'_{\alpha}:1\leq\alpha\leq\mu\cup v'_{\alpha\beta}:1\leq\alpha\leq\mu$and$1\leq\beta\leq6$.Themodularcoloringof$ST_{dc}(W_{\mu})$isdefinedbythefollowingtwocases

Case1: When μisodd,

Coloringof$ST_{dc}(W_{\mu})$is

$C(c_0)=0,C(v_{\alpha})=0$forall α.

Whenαrangesfrom1to$\mu-1$andβrangesfrom 1to6

Foralloddα; $C(v_{\alpha\beta})=\begin{cases}1: & \beta=2\\ 0: & \text{otherwise}\end{cases}$,

Forallevenα;$C(v_{\alpha\beta})=\begin{cases}2: & \beta=2\\ 0: & \text{otherwise}\end{cases}$

When;$C(v_{\alpha\beta})=\begin{cases}3: & \beta=2\\ 0: & \text{otherwise}\end{cases}$.

Modularcoloringof$ST_{dc}(W_{\mu})$is

$\mathcal{S}(c_0)=0,\mathcal{S}(v_{\alpha})=\begin{cases}1: & 1\leq\alpha\leq\mu-1,\text{where}\alpha\text{ isodd}\\ 2: & 1\leq\alpha\leq\mu-1,\text{where}\alpha\text{iseven}\\ 3: & \alpha=\mu\end{cases}$

Whenαrangesfrom1to$\mu-1$andβrangesfrom 1to6

Foralloddα; $\mathcal{S}(v_{\alpha\beta})=\begin{cases}0: & \beta=1,2,3\\ 1: & \text{otherwise}\end{cases}$, foralleven$\alpha$; $\mathcal{S}(v_{\alpha\beta})=\begin{cases}0: & \beta=1,2,3\\ 2: & \text{otherwise}\end{cases}$.

When$\alpha=\mu$;$\mathcal{S}(v_{\alpha\beta})=\begin{cases}0: & \beta=1,2,3\\ 3: & \text{otherwise}\end{cases}$.

Case2: Whenμiseven,

Coloringof$ST_{dc}(W_{\mu})$is

$C(c_0)=0,C(v_{\alpha})=0$forall α.

Whenαrangesfrom1toμandβrangesfrom1to6

Foralloddα; $C(v_{\alpha\beta})=\begin{cases}1: & \beta=2\\ 0: & \text{otherwise}\end{cases}$,forall even$\alpha$;$C(v_{\alpha\beta})=\begin{cases}2: & \beta=2\\ 0: & \text{otherwise}\end{cases}$.

Modularcoloringof$ST_{dc}(W_{\mu})$is

$\mathcal{S}(c_0)=0,\mathcal{S}(v_{\alpha})=\begin{cases}1: & \text{foralloddα}\\ 2: & \text{foralloddα}\end{cases}$

Whenαrangesfrom1to$\mu-1$andβrangesfrom 1to6

Foralloddα; $\mathcal{S}(v_{\alpha\beta})=\begin{cases}0: & \beta=1,2,3\\ 1: & \text{otherwise}\end{cases}$, foralleven$\alpha$; $\mathcal{S}(v_{\alpha\beta})=\begin{cases}0: & \beta=1,2,3\\ 2: & \text{otherwise}\end{cases}$.

Fromtheabovecasesthegraph$ST_{dc}(W_{\mu})$is4modularcolorable,thereforethemodularcoloring of$m_c(ST_{dc}(W_{\mu}))=4$.

Theorem2. Foranyinteger$\mu\geq3$,themodularchromaticnumberofasnowgraphofgeargraph $mc(ST_{dc}(G_{\mu}))=2$.

Proof: The structure of the snow graph ofG_{μ}, let the graph G_{μ}be a gear graph with μ vertices then take μ copiesofdc,eachexternalvertexofG_{μ}isaffixedbyonedc,let$V(ST_{dc}(G_{\mu}))=c_0\cup v_{\alpha}\cup v_{\alpha\beta}$where $1\leq\alpha\leq\mu$ and$1\leq\beta\leq6$andlet$E(ST_{dc}(G_{\mu}))=v'_{\alpha}:1\leq\alpha\leq2\mu\cup v'_{\alpha\beta}:1\leq\alpha\leq\mu$and$1\leq\beta\leq6$.

Coloringof $ST_{dc}(G_{\mu})$is$C(c_0)=0$,whenαrangesfrom1toμ;$C(v_{\alpha})=\begin{cases}0: & \text{foralloddα}\\ 1: & \text{forallevenα}\end{cases}$.

Whenαrangesfrom1toμandβrangesfrom 1to6

Foralloddα; $C(v_{\alpha\beta})=\begin{cases}1: & \beta=2\\ 0: & \text{otherwise}\end{cases}$,forall even$\alpha$;$C(v_{\alpha\beta})=0$.

Modularcoloringof$ST_{dc}(G_{\mu})$is

$\mathcal{S}(c_0)=0,\mathcal{S}(v_{\alpha})=\begin{cases}1: & \text{foralloddα}\\ 0: & \text{foralloddα}\end{cases}$

Whenαrangesfrom1to$\mu-1$andβrangesfrom 1to6

Foralloddα;$\mathcal{S}(v_{\alpha\beta})=\begin{cases}0: & \beta=1,2,3\\ 1: & \text{otherwise}\end{cases}$, foralleven$\alpha$; $\mathcal{S}(v_{\alpha\beta})=\begin{cases}1: & \beta=1,2,3\\ 0: & \text{otherwise}\end{cases}$.

FromtheabovecasesthegraphST$_{dc}$(G$_\mu$)ismodular2-colorable,thereforethemodularcoloringof $m_c(ST_{dc}(G_\mu))=2$.

Theorem3. Foranyintegerμ≥3,themodularchromaticnumberofasnowgraphoffriendshipgraph $mc(ST_{dc}(F_\mu)) = 3$.

Proof: TheconstructionofthesnowgraphofF$_\mu$isdescribedas,letF$_\mu$beafriendshipgraphonμvertices thentakeμcopiesofdc,eachexternalvertexofF$_\mu$isaffixedbyonedc,let$V(ST_{dc}(F_\mu))=c_0 \cup v_\alpha \cup v_{\alpha\beta}$where$1\le\alpha\le2\mu$ and$1\le\beta\le6$andlet$E(ST_{dc}(F_\mu))=c_0v'_\alpha:1\le\alpha\le2\mu \cup v'_\alpha:1\le\alpha\le\mu \cup v'_{\alpha\beta}:1\le\alpha\le\mu$and$1\le\beta\le6$.

Coloringof $ST_{dc}(F_\mu)$is, $C(c_0) = 0$,when αrangesfrom 1to2μ;$C(v_\alpha) = 0$. When α ranges from 1 to 2μ and β ranges from 1 to 6,

Foralloddα; $C(v_{\alpha\beta})=\begin{cases}1: & \beta=2\\ 0:\text{otherwise}\end{cases}$,forall evenα;$C(v_{\alpha\beta})=\begin{cases}2: & \beta=2\\ 0:\text{otherwise}\end{cases}$.

Modularcoloringof$ST_{dc}(F_\mu)$is

$\mathcal{S}(c_0)=0,\mathcal{S}(v_\alpha)=\begin{cases}1:\text{foralloddα}\\ 2:\text{foralleven}\alpha\end{cases}$

Whenαrangesfrom1to2μandβrangesfrom 1to6

Foralloddα;$\mathcal{S}(v_{\alpha\beta})=\begin{cases}0:\beta=1,2,3\\ 1:\text{otherwise}\end{cases}$, foralleven α; $\mathcal{S}(v_{\alpha\beta})=\begin{cases}0:\beta=1,2,3\\ 2:\text{otherwise}\end{cases}$.

ThegraphST$_{dc}$(F$_\mu$)ismodular3-colorable,thereforethemodularcoloringof$m_c(ST_{dc}(F_\mu))=3$.

Theorem 4. For any integer μ ≥ 3,the modular chromatic number of a snow graph of generalized fan graph$mc(ST_{dc}(GF_\mu)) = 3$.

Proof: AsnowgraphofageneralisedfangraphST$_{dc}$(GF$_\mu$)isconstructedbytakingageneralisedfangraph $GF_\mu=\overline{K_m}\oplus P_n$andμ copies double claw graph (dc), since all the vertex of the generalized fan graphGF$_\mu$is external vertices, each vertex of the GF$_\mu$attached by the double claw graph. The vertex set of ST$_{dc}$(GF$_\mu$) isdefinedas$v_\alpha\cup v_{\alpha\beta}$suchthat$1 \le \alpha \le \mu = m + n$ and $1 \le \beta \le 6$.considerbi-vertexsets$V_1(GF_\mu)$isa vertexsetof K$_m$complement,and $V_2(GF_\mu)$isa vertexsetofa pathgraph P$_n$.Thegraph ST$_{dc}$(GF$_\mu$)contains K$_3$asaninducedsubgraphso,itisminimum3colorsarenecessarytocolorthegraph,thatimplies $m_c(ST_{dc}(GF_\mu)) \le 3$. The graph'scoloring is accomplished based on the vertex sets, like$C(V_1(GF_\mu)) = 1$and $C(V_2(GF_\mu))=0$.Thedcgraphaffixedwiththevertexset$V_1(GF_\mu)$iscoloredinawaythat;$C(v_{\alpha\beta})=0$forallβ.Forthedcgraphthatisattachedwith$V_2(GF_\mu)$,thecoloringisgivenby,whenαisodd; $C(v_{\alpha\beta})=\begin{cases}1:\beta=1\text{to}3\\ 0:\text{otherwise}\end{cases}$,whenαiseven;$C(v_{\alpha\beta})=\begin{cases}1: & \beta=2\\ 0:\text{otherwise}\end{cases}$.Itresultsthemodularcoloringis $\mathcal{S}(V_1(GF_\mu))=0$andwhenαrangesfrom1ton$\mathcal{S}(V_2(GF_\mu))=\begin{cases}1:\text{whenαisodd}\\ 2:\text{whenαiseven}\end{cases}$.Themodularchromatic numberof$v_{\alpha\beta}$thatisconnectedto$V_1(GF_\mu)$is$\mathcal{S}(v_{\alpha\beta})=\begin{cases}1:\beta=1\text{ to}3\\ 0:\text{otherwise}\end{cases}$,similarlywhichisconnectedto $V_2(GF_\mu)$is$\mathcal{S}(GF_\mu)=\begin{cases}0:\beta=1\text{ to}3\\ 1:\text{otherwise}\end{cases}$.Thisisvividlyproducetheresultthatthemodularcoloringis $m_c(ST_{dc}(GF_\mu))=3$.

Note: The previously mentioned theorem applies to the modular chromatic number of the snow graph of the fan graph.

Theorem5. Foranyintegerμ≥3,themodularchromaticnumberofasnowgraphofflower graph$mc(ST_{dc}(Fl_\mu)) = 3$ or 4

Proof: TheconstructionofthesnowgraphofFl$_\mu$isdescribedas,letFl$_\mu$beaflowergraphonμvertices then takeμcopiesofdc,eachexternalvertex ofFl$_\mu$isaffixedbyonedc,let$V(ST_{dc}(Fl_\mu))=v_\alpha\cup v_{\alpha\beta}$where $1\le\alpha\le2\mu$and $1\le\beta\le6$ and let $E(ST_{dc}(Fl_\mu))=c_0v'_\alpha:1\le\alpha\le2\mu\cup v'_\alpha:1\le\alpha\le\mu\cup v'_{\alpha\beta}:1\le\alpha\le\mu$ and $1 \le \beta \le 6$.

ColoringofST$_{dc}$(Fl$_\mu$)is,

$C(c_0)=0$,whenαrangesfrom 1to2μ;$C(v_\alpha)=0$.

Whenαrangesfrom1to2μexcept $\alpha=2\mu-1$andβrangesfrom1to 6, Forallodd αwhereα≡2,3(mod 4);$C(v_{\alpha\beta})=\begin{cases}2: & \beta=2\\ 0: & \text{otherwise}\end{cases}$.

Foralloddαwhere≡0,1(mod4);$C(v_{\alpha\beta})=\begin{cases}1: & \beta=2\\ 0: & \text{otherwise}\end{cases}$.

Whenα=2μ −1;$C(v_{\alpha\beta})=\begin{cases}3: & \beta=2\\ 0: & \text{otherwise}\end{cases}$.

ThemodularcoloringofST$_{dc}$(Fl$_{\mu}$)is, is

$\mathcal{S}(c_0) = 0$,

whenαrangesfrom1to2μexcept2μ−1, ifα ≡ 0,1(mod 4) ; $\mathcal{S}(v_\alpha) = 1$,

ifα≡2,3(mod4); $\mathcal{S}(v_\alpha)=2$,

ifα=μ −1;$\mathcal{S}(v_\alpha)=3$.

Whenαrangesfrom1to2μexcept 2μ −1andβrangesfrom1to6, For all α where α ≡ 0,1 (mod 4); $\mathcal{S}(v_{\alpha\beta}) =\begin{cases}0:\beta= 1,2,3\\ 1:\text{otherwise}\end{cases}$,

Forall αwhere α≡2,3(mod4);$\mathcal{S}(v_{\alpha\beta})=\begin{cases}0:\beta=1,2,3\\ 2:\text{otherwise}\end{cases}$,

When α=μ − 1;$\mathcal{S}(v_{\alpha\beta})=\begin{cases}0:\beta=1,2,3\\ 3:\text{otherwise}\end{cases}$.

FromtheabovecasesthegraphST$_{dc}$(Fl$_{\mu}$)ismodular3-colorable,thereforethemodularcoloringof $m_c(ST_{dc}(Fl_\mu))=3$.

Extendedsnowgraph

Theorem6. IfG=W$_{\mu}$andHisanγ−arygraph,thenST$_{\gamma}$(W$_{\mu}$)ismodular3or4- colorable.

Proof: Let H be a γ-ary graph with γ- children and G be a graph of W$_{\mu}$with μ vertices. Then, since every vertex in the wheel graph W$_{\mu}$is an external vertex, the extended snow graph of wheelW$_{\mu}$is created by taking μ copies of the H = γ-ary graph and attaching them to the μ vertices of the wheelgraph by an edge. The$V[ST_\gamma(W_\mu)]=\{v_o\cup v_\alpha\cup v^{h\alpha}_\beta \ni \alpha\to 1$ toμ;h→1tonandβ→1to$\sum_{i=1}^{h-1}\gamma^i\,\forall$ α,γ,n,i∈ℕ} andthe $E[ST_\gamma(W_\mu)]=\{e_\alpha\cup v'_\alpha\cup v''_\alpha\cup e^{h\alpha}_\beta\ni\alpha\to 1$toμ;h→1tonandβ→1 to$\sum_{i=1}^{h}\gamma^i\,\forall$ μ,γ∈ℕ}.The following twocasesdealsthetheorem.ThemodularcoloringofST$_{\gamma}$(W$_{\mu}$)isdefinedbyaninjectivemapping $C(C_\mu):v_\alpha\cup v_{\alpha\beta}\to\mathbb{Z}_k;k\geq 2$.

Case 1: When μ is odd,

ThecoloringofST$_{\gamma}$(W$_{\mu}$)is,

$C(v_0)=0,C(v_\alpha)=0$forallα.

Themodularcoloringoftheaboveverticesis

$\mathcal{S}(c_0)=0,\mathcal{S}(v_\alpha)=\begin{cases}1 & \text{if}1\leq\alpha\leq\mu-1,\text{where}\alpha\text{isodd}\\ 2 & \text{if}1\leq\alpha\leq\mu,\text{where}\alpha\text{iseven}\\ 3 & \text{if}\alpha=\mu,\text{where}\mu\text{isodd}\end{cases}$

Coloringoftheγ-araygraphofST$_{\gamma}$(W$_{\mu}$)ispartitionedintothreesubcasesasfollows

Subcase1: Whenγ≡0(mod4)

Whenαrangesfrom1toμ−1andforallβ

Forall oddα;$C(v^{\alpha h}_\beta)=\begin{cases}0 & \text{if}1\leq h\leq n,\text{where}h\text{is even}\\ 1 & \text{if}1\leq h\leq n,\text{where}h\text{ isodd}\end{cases}$,

Forallevenα;$C(v^{\alpha h}_\beta)=\begin{cases}0 & \text{if}1\leq h\leq n,\text{where}h\text{is even}\\ 1 & \text{if}2\leq h\leq n,\text{where}h\text{ isodd}\\ 2 & \text{if}h=1\end{cases}$,

Whenα=μ;$C(v^{\alpha h}_\beta)=\begin{cases}0 & \text{if}1\leq h\leq n,\text{where}h\text{is even}\\ 1 & \text{if}2\leq h\leq n,\text{where}h\text{ isodd}\\ 3 & \text{if}h=1\end{cases}$

ModularcoloringofST$_{\gamma}$(W$_{\mu}$)is

Whenαrangesfrom1toμ−1andforallβ

Foralloddα;$\mathcal{S}(v^{\alpha h}_\beta)=\begin{cases}0 & \text{if}1\leq h\leq n,\text{where}h\text{isodd}\\ 1 & \text{if}1\leq h\leq n,\text{where}h\text{ iseven}\end{cases}$.

For allevenα;$\mathcal{S}(v^{\alpha h}_\beta)=\begin{cases}0 & \text{if}1\leq h\leq n,\text{where}h\text{isodd}\\ 1 & \text{if}4\leq h\leq n,\text{where}h\text{ iseven}\\ 2 & \text{if}h=2\end{cases}$.

Whenα=μ;$\mathcal{S}(v^{\alpha h}_\beta)=\begin{cases}0 & \text{if}1\leq h\leq n,\text{where}h\text{ isodd}\\ 1 & \text{if}\leq h\leq n,\text{where}h\text{is even}\\ 3 & \text{if}h=2\end{cases}$.

Subcase2: When $\gamma \equiv 1 (\mathrm{mod}\ 4)$

When α ranges from 1 to $\mu-1$ and for all β

For all odd α; $C(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{ where } h \text{ is even} \\ 1 & \text{if } 1 \le h \le n, \text{ where } h \text{ is odd} \end{cases}$,

For all even α; $C(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{ where } h \text{ is even} \\ 1 & \text{if } 2 \le h \le n, \text{ where } h \text{ is odd} \\ 2 & \text{if } h=1 \end{cases}$,

When $\alpha=\mu$; $C(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{ where } h \text{ is even} \\ 1 & \text{if } 2 \le h \le n, h \equiv 1 (\mathrm{mod}\ 4) \text{ where } h \text{ is odd} \\ 2 & \text{if } 2 \le h \le n, h \equiv 3 (\mathrm{mod}\ 4) \text{ where } h \text{ is odd} \\ 3 & \text{if } h=1 \end{cases}$

Modular coloring of $ST_{\gamma}(W_{\mu})$ is

When α ranges from 1 to $\mu-1$ and for all β

For all odd α; $\mathcal{S}(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{ where } h \text{ is odd} \\ 2 & \text{if } 1 \le h \le n, \text{ where } h \text{ is even} \end{cases}$.

For all even α; $\mathcal{S}(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{ where } h \text{ is odd} \\ 2 & \text{if } 4 \le h \le n, \text{ where } h \text{ is even} \\ 3 & \text{if } h=2 \end{cases}$.

When $\alpha=\mu$; $\mathcal{S}(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{ where } h \text{ is odd} \\ 1 & h=2, n; n \equiv 2 (\mathrm{mod}\ 4) \text{ where } h \text{ is even} \\ 2 & h=n, n \equiv 0 (\mathrm{mod}\ 4), \text{ where } h \text{ is even} \\ 3 & \text{if } 4 \le h \le n-1, \text{ where } h \text{ is even} \end{cases}$

Subcase3: When $\gamma \equiv 2 (\mathrm{mod}\ 4)$

when α ranges from 1 to $\mu-1$ and for all β

For all odd α; $C(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{ where } h \text{ is even} \\ 1 & \text{if } 1 \le h \le n, \text{ where } h \text{ is odd} \end{cases}$,

For all even α; $C(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{ where } h \text{ is even} \\ 2 & \text{if } 1 \le h \le n, \text{ where } h \text{ is odd} \end{cases}$.

When $\alpha=\mu$; $C(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{ where } h \text{ is even} \\ 1 & \text{if } 2 \le h \le n, \text{ where } h \text{ is odd} \\ 3 & \text{if } h=1 \end{cases}$.

Modular coloring of $ST_{\gamma}(W_{\mu})$ is

When α ranges from 1 to $\mu-1$ and for all β

For all odd α; $\mathcal{S}(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{ where } h \text{ is odd} \\ 3 & \text{if } 1 \le h \le n, \text{ where } h \text{ is even} \end{cases}$.

For all even α; $\mathcal{S}(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{ where } h \text{ is odd} \\ 2 & \text{if } 1 \le h \le n, \text{ where } h \text{ is even} \end{cases}$.

When $\alpha=\mu$; $\mathcal{S}(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n-1, \text{ where } h \text{ is odd} \\ 1 & \text{if } h=2 \text{ and } h=n, h \text{ is even} \\ 3 & \text{if } 4 \le h \le n, \text{ where } h \text{ is even} \end{cases}$.

Subcase4: When $\gamma \equiv 3 (\mathrm{mod}\ 4)$

When α ranges from 1 to $\mu-1$ and for all β

For all odd α; $C(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n-1, \text{ where } h \text{ is even} \\ 1 & \text{if } h \equiv 1 (\mathrm{mod}\ 4), \text{ where } h \text{ is odd}, \\ 2 & \text{if } h \equiv 3 (\mathrm{mod}\ 4), \text{ where } h \text{ is odd} \end{cases}$

For all even α; $C(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n-1, \text{ where } h \text{ is even} \\ 1 & \text{if } h \equiv 3 (\mathrm{mod}\ 4), \text{ where } h \text{ is odd}, \\ 2 & \text{if } h \equiv 1 (\mathrm{mod}\ 4), \text{ where } h \text{ is odd} \end{cases}$

When $\alpha=\mu$; $C(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{ where } h \text{ is even} \\ 1 & \text{if } 1 \le h \le n; h \equiv 3 (\mathrm{mod}\ 4), \text{ where } h \text{ is odd} \\ 2 & \text{if } 5 \le h \le n; h \equiv 1 (\mathrm{mod}\ 4), \text{ where } h \text{ is odd} \\ 3 & \text{if } h=1 \end{cases}$.

Modular coloring of $ST_{\gamma}(W_{\mu})$ is

When α ranges from 1 to $\mu-1$ and for all β For all odd α;

$$\mathcal{S}(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{ where } h \text{ is odd} \\ 1 & \text{if } 1 \le h \le n-1; h \equiv 0 (\mathrm{mod}\ 4), \text{ and } h=n, h \equiv 2 (\mathrm{mod}\ 4) \\ 2 & \text{if } h=n, h \equiv 0 (\mathrm{mod}\ 4) \\ 3 & \text{if } 1 \le h \le n-1; h \equiv 2 (\mathrm{mod}\ 4), \text{ where } h \text{ is even} \end{cases},$$

For all even α;

$$\mathcal{S}(v^{\alpha h}_{\beta})=\begin{cases}0 & \text{if} 1\le h\le n, \text{where} h \text{ is odd}\\ 1 & \text{if} 1\le h\le n-1; h\equiv 2(\text{mod}4), \text{and} h=n; h\equiv 0(\text{mod}4)\\ 2 & \text{if} h=n; h\equiv 2(\text{mod}4)\\ 3 & \text{if} 1\le h\le n-1; h\equiv 2(\text{mod } 4), \text{where} n\ h \text{ is even}\end{cases}.$$

When $\alpha=\mu$;

$$\mathcal{S}(v^{\alpha h}_{\beta})=\begin{cases}0 & \text{if} 1\le h\le n, \text{where} h \text{ is odd}\\ 1 & \text{if} 1\le h\le n-1; h\equiv 2(\text{mod}4), \text{and} h=n; h\equiv 0(\text{mod}4)\\ 2 & \text{if} h=2 \text{ and } h=n, h\equiv 2(\text{mod}4)\\ 3 & \text{if} 1\le h\le n-1; h\equiv 0(\text{mod}4), \text{where} h \text{ is even}\end{cases}.$$

Thus from the above results $ST_{\gamma}(W_{\mu})$ is 4 modular colorable for μ- is odd, therefore the modular coloring of $m_c(ST_{\gamma}(W_{\mu}))=4$.

Case2: When μ is even,

$C(v_0)=0, C(v_{\alpha})=0$ for all α.

Themodularcoloringoftheaboveverticesare

$$\mathcal{S}(c_0)=0, \mathcal{S}(v_{\alpha})=\begin{cases}1 & \text{if} 1\le\alpha\le\mu, \text{where} \alpha \text{ is odd}\\ 2 & \text{if } 1\le\alpha\le\mu, \text{where} \alpha \text{ is even}\end{cases}$$

Coloringof $ST_{\gamma}(W_{\mu})$ issplittedasthreesubcasesasfollows

Subcase1: When $\gamma\equiv 0(\text{mod}3)$

When α rangesfrom1to μ andforall β

$$\text{For all odd}\alpha; C(v^{\alpha h}_{\beta})=\begin{cases}0 & \text{if} 1\le h\le n, \text{where} h \text{ is even}\\ 1 & \text{if} 1\le h\le n, \text{where} h \text{ is odd}\end{cases},$$

$$\text{For all even}\alpha; C(v^{\alpha h}_{\beta})=\begin{cases}0 & \text{if} 1\le h\le n, \text{where} h \text{ is even}\\ 1 & \text{if} 2\le h\le n, \text{where} h \text{ is odd},\\ 2 & \text{if} h=1\end{cases}$$

Modularcoloringof $ST_{\gamma}(W_{\mu})$ is

$$\mathcal{S}(c_0)=0, \mathcal{S}(v_{\alpha})=\begin{cases}1 & \text{if} 1\le h\le n, \text{where} h \text{ is odd}\\ 2 & \text{if } 1\le h\le n, \text{where} h \text{ is even}\end{cases},$$

When α rangesfrom1to μ andforall β

$$\text{For all odd}\alpha; \mathcal{S}(v^{\alpha h}_{\beta})=\begin{cases}0 & \text{if} 1\le h\le n, \text{where} h \text{ is odd}\\ 1 & \text{if} 1\le h\le n, \text{where} h \text{ is even}\end{cases},$$

$$\text{For all even}\alpha; C(v^{\alpha h}_{\beta})=\begin{cases}0 & \text{if} 1\le h\le n, \text{where} h \text{ is odd}\\ 1 & \text{if} 4\le h\le n, \text{where} h \text{ is even}.\\ 2 & \text{if} h=2\end{cases}$$

Subcase2: When $\gamma\equiv 1(\text{mod}3)$

When α rangesfrom1to μ andforall β

$$\text{For all odd}\alpha; C(v^{\alpha h}_{\beta})=\begin{cases}0 & \text{if} 1\le h\le n, \text{where} h \text{ is even}\\ 1 & \text{if} 1\le h\le n, \text{where} h \text{ is odd}\end{cases},$$

$$\text{For all even}\alpha; C(v^{\alpha h}_{\beta})=\begin{cases}0 & \text{if} 1\le h\le n, \text{where} h \text{ is even}\\ 2 & \text{if} 1\le h\le n, \text{where} h \text{ is odd}\end{cases},$$

Modularcoloringof $ST_{\gamma}(W_{\mu})$ is

When α rangesfrom1to μ andforall β

$$\text{For all odd}\alpha; \mathcal{S}(v^{\alpha h}_{\beta})=\begin{cases}0 & \text{if} 1\le h\le n, \text{where} h \text{ is odd}\\ 1 & \text{if} h=n, \text{where} n \text{ is even}\\ 2 & \text{if} 1\le h\le n-1, \text{where} h \text{ is even}\end{cases},$$

$$\text{For all even}\alpha; \mathcal{S}(v^{\alpha h}_{\beta})=\begin{cases}0 & \text{if} 1\le h\le n, \text{where} h \text{ is odd}\\ 1 & \text{if} 1\le h\le n-1, \text{where} h \text{ is even}.\\ 2 & \text{if} h=n, \text{where} n \text{ is even}\end{cases}$$

Subcase3: When $\gamma\equiv 2(\text{mod}3)$

When α rangesfrom1to μ andforall β

$$\text{For all odd}\alpha; C(v^{\alpha h}_{\beta})=\begin{cases}0 & \text{if} 1\le h\le n, \text{where} h \text{ is even}\\ 1 & \text{if} h\equiv 1(\text{mod}4), \text{where} n \text{ is odd},\\ 2 & \text{if} h\equiv 3(\text{mod}4), \text{where} h \text{ is odd}\end{cases}$$

$$\text{For all even}\alpha; C(v^{\alpha h}_{\beta})=\begin{cases}0 & \text{if} 1\le h\le n, \text{where} h \text{ is even}\\ 1 & \text{if} h\equiv 3(\text{mod}4), \text{where} n \text{ is odd},\\ 2 & \text{if} h\equiv 1(\text{mod}4), \text{where} h \text{ is odd}\end{cases}$$

Modularcoloringof $ST_{\gamma}(W_{\mu})$ is

When α rangesfrom1to μ andforall β

For all odd α; $\mathcal{S}(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1\leq h\leq n, \text{where } h \text{ is odd} \\ 1 & \text{if } 1 \leq h \leq n; h \equiv 0 (\text{mod} 4), \text{where } n \text{ is even and } h = n, \text{where } n \equiv 2 (\text{mod} 4) \\ 2 & \text{if } 1\leq h\leq n\ h\equiv 2 (\text{mod} 4), \text{where } h \text{ is even and } h=n, \text{where } n\equiv 0 (\text{mod} 4) \end{cases}$,

For all even α; $C(v^{\alpha h}_{\beta}) = \begin{cases} 0 & \text{if } 1\leq h\leq n, \text{where } h \text{ is odd} \\ 1 & \text{if } 1 \leq h \leq n; h \equiv 2 (\text{mod} 4), \text{where } n \text{ is even and } h = n, \text{where } n \equiv 0 (\text{mod} 4) \\ 2 & \text{if } 1\leq h\leq n\ h\equiv 0 (\text{mod} 4), \text{where } h \text{ is even and } h=n, \text{where } n\equiv 2 (\text{mod} 4) \end{cases}$.

From the above cases the graph $ST_\gamma(W_\mu)$ is 3 modular colorable for μ- is even, therefore the modular coloring of $m_c(ST_\gamma(W_\mu)) = 3$.

Theorem7. If $G=G_\mu$ and H is an $\gamma-$ary graph, then $ST_\gamma(G_\mu)$ is modular 2 -colorable.

Proof: Let H be a γ-ary graph with γ- children and G be a graph of G_μ with μ vertices. Then, since every vertex in the gear graph W_μ is an external vertex, the extended snowgraph of gear G_μ is created by taking μ copies of the H=γ-ary graph and attaching them to the μ vertices of the gear graph by an edge. The $V[ST_\gamma(G_\mu)]=\{v_0\cup v_\alpha\cup v^{h\alpha}_{\beta} \ni \alpha\to 1 \text{ to } 2\mu; h\to 1 \text{ to } n \text{ and } \beta\to 1 \text{ to } \sum_{i=1}^{h-1}\gamma^i \forall \alpha,\gamma,n,i\in\mathbb{N}\}$ and the $E[ST_\gamma(G_\mu)]=\{e_\alpha\cup v_\alpha\cup v_\alpha\cup e_{\frac{\beta}{2}} \ni \alpha\to 1 \text{ to } 2\mu; h\to 1 \text{ to } n \text{ and } \beta\to 1 \text{ to } \sum_{i=1}^{h}\gamma^i\ \forall \mu,\gamma\in\mathbb{N}\}$. The following two cases deals the theorem. The modular coloring of $ST_\gamma(G_\mu)$ is defined by an injective mapping $C(C_\mu):V \to \mathbb{Z}_k;\ k \geq 2$. The coloring of $ST_\gamma(G_\mu)$ is,

Case1: When $\mu\geq 3$, γ is odd

Subcase1: $h\equiv 0 (\text{mod} 4)$

The coloring of $ST_\gamma(G_\mu)$ is given by

$C(v_0)=1, C(v_\alpha)=0 \forall\alpha$,

When α ranges from 1 to 2μ, for all β, and $1\leq h\leq n$

For each odd α; $C(v^{h\alpha}_{\beta}) = \begin{cases} 0 & \text{if } h \text{ is even and } h\equiv 1 (\text{mod} 4); \text{where } h \text{ is odd} \\ 1 & \text{if } h\equiv 3 (\text{mod } 4); \text{where } h \text{ is odd} \end{cases}$

For each even α; $C(v^{h\alpha}_{\beta}) = \begin{cases} 0 & \text{if } h \text{ is odd and } h\equiv 0 \ (\text{mod } 4); \text{where } h \text{ is even} \\ 1 & \text{if } h\equiv 2 (\text{mod } 4); \text{where } h \text{ is even} \end{cases}$.

The modular coloring is

$\mathcal{S}(v_0)=0, \mathcal{S}(v_\alpha)=\begin{cases} 0 & \text{if } \alpha \text{ is even} \\ 1 & \text{if } \alpha \text{ is odd} \end{cases} \forall\alpha$,

When α ranges from 1 to 2μ, for all β, and $1\leq h\leq n$

For each odd α; $\mathcal{S}(v^{h\alpha}_{\beta}) = \begin{cases} 0 & \text{if } \alpha \text{ is odd} \\ 1 & \text{if } \alpha \text{ is even} \end{cases}$,

For each even α; $\mathcal{S}(v^{h\alpha}_{\beta}) = \begin{cases} 0 & \text{if } \alpha \text{ is even} \\ 1 & \text{if } \alpha \text{ is odd} \end{cases}$.

Subcase2: Except $h\equiv 0 (\text{mod} 4)$

The coloring of $ST_\gamma(G_\mu)$ is given by

$C(v_0)=0, C(v_\alpha)=\begin{cases} 0 & \text{if } \alpha \text{ is even} \\ 1 & \text{if } \alpha \text{ is odd} \end{cases}$,

When α ranges from 1 to 2μ, for all β, and $1\leq h\leq n$

For each odd α; $C(v^{h\alpha}_{\beta}) = \begin{cases} 0 & \text{if } h\equiv 1 (\text{mod } 4); \text{where } h \text{ is odd} \\ 1 & \text{if } h \text{ is even and } h\equiv 3 (\text{mod} 4); \text{where } h \text{ is odd} \end{cases}$,

For each even α; $C(v^{h\alpha}_{\beta}) = \begin{cases} 0 & \text{if } h \text{ is odd and } h\equiv 2 \ (\text{mod } 4); \text{where } h \text{ is even} \\ 1 & \text{if } h\equiv 0 (\text{mod } 4); \text{where } h \text{ is even} \end{cases}$.

The above applications reflect the modular coloring is

$\mathcal{S}(v_0)=0, \mathcal{S}(v_\alpha)=\begin{cases} 0 & \text{if } \alpha \text{ is odd} \\ 1 & \text{if } \alpha \text{ is even} \end{cases}$,

When α ranges from 1 to 2μ, for all β, and $1\leq h\leq n$

For each odd α; $\mathcal{S}(v^{h\alpha}_{\beta}) = \begin{cases} 0 & \text{if } h \text{ is odd} \\ 1 & \text{if } h \text{ is even} \end{cases}$,

For each even α; $\mathcal{S}(v^{h\alpha}_{\beta}) = \begin{cases} 0 & \text{if } h \text{ is even} \\ 1 & \text{if } h \text{ is odd} \end{cases}$

Case2: When $\mu\geq 3$, γ is even

The coloring of $ST_\gamma(G_\mu)$ is given by

$C(v_0)=0, C(v_\alpha)=\begin{cases} 0 & \text{if } \alpha \text{ is even} \\ 1 & \text{if } \alpha \text{ is odd} \end{cases}$, Whenαranges from1to2μ,forallβ,and1≤h≤n

Foreachoddα; $C(v^{h\alpha}_{\beta})=\begin{cases}0 & \text{ifhisodd}\\ 1 & \text{if hiseven}\end{cases}$,

Foreachevenα; $C(v^{h\alpha}_{\beta})=\begin{cases}0 & \text{if hiseven}\\ 1 & \text{ifh isodd}\end{cases}$

Theaboveapplicationsreflectthemodularcoloringis

$\mathcal{S}(v_0)=0, \mathcal{S}(v_\alpha)=\begin{cases}0 & \text{ifαiseven}\\ 1 & \text{ifαis odd}\end{cases}$,

Whenαranges from1to2μ,forallβ,and1≤h≤n

Foreachoddα; $\mathcal{S}(v^{h\alpha}_{\beta})=\begin{cases}0 & \text{ifhisodd}\\ 1 & \text{ifh iseven}\end{cases}$,

Foreachevenα; $\mathcal{S}(v^{h\alpha}_{\beta})=\begin{cases}0 & \text{ifh iseven}\\ 1 & \text{ifhisodd}\end{cases}$.

Thustheabovecaseisleadsto$ST_\gamma(G_\mu)$is3modularcolorable,thereforethemodularcoloringof $m_c(ST_\gamma(G_\mu))=3$.

Theorem8. If G=$F_\mu$andHisanγ−arygraph,then$ST_\gamma(F_\mu)$ismodular3 -colorable.

Proof: Let H be a γ-ary graph with γ- children and G be a graph of $F_\mu$with μ vertices. Then, since every vertex in the friendship graph $F_\mu$is an external vertex except the center vertex, the extended snow graphof friendship graph$F_\mu$is created by taking μ copies of the H = γ-ary graph and attaching them to the μ verticesofthefriendshipgraphbyanedge.The$V[ST_\gamma(F_\mu)]=\{v_o \cup v_\alpha \cup v^{h\alpha}_{\beta} \ni \alpha\to 1 \text{ to} 2\mu; h\to 1 \text{ton} \text{and} \beta\to 1 \text{to} \sum_{i=1}^{h-1}\gamma^i \forall\ \alpha,\gamma,n,i\in\mathbb{N}\}$and the $E[ST_\gamma(F_\mu)]=\{e_{2\alpha}\cup v_\alpha v_{\alpha+1} \forall \text{odd}\alpha \cup e^{h\alpha}_{\beta} \ni\alpha\to 1 \text{ to}\mu; h\to 1\text{ton} \text{and}\beta\to 1\text{to}\sum_{i=1}^{h}\gamma^i\forall\mu,\gamma\in\mathbb{N}\}$.Thefollowingthreecasesdealsthetheorem.Themodular coloringof$ST_\gamma(F_\mu)$isdefinedbyaninjectivemapping$C:V\to\mathbb{Z}_k;k\geq 2$.Thecoloringof $ST_\gamma(F_\mu)$is, $C(v_\alpha) = 0$;whereαrangesfrom0to2μ. Thecoloringpatternsandthemodularcoloringofthe graph$ST_\gamma(F_\mu)$are followed from the case 2 of theorem 6.

Thegraph$ST_\gamma(F_\mu)$is3modularcolorable,thereforethemodularcoloringof$m_c(ST_\gamma(F_\mu))=3$.

Theorem9. IfG=$\bar{k}_m \oplus P_n$andHisanγ−arygraph,then$ST_\gamma(\bar{k}_m \oplus P_n)$ismodular3-colorable.

Proof: Let H be a γ-ary graph with γ- children and G be a graph of $\bar{k}_m \oplus P_l$with μ vertices. Then, since every vertex in the generalized fan graph $\bar{k}_m \oplus P_l$is an external vertex, the extended snow graph of generalized fan graph$\bar{k}_m \oplus P_l$is created by taking μ = m+l copies of the H = γ-ary graph and attaching themtothe μverticesofthegeneralizedfangraphbyanedge.The$V[ST_\gamma(\bar{k}_m \oplus P_l)]=\{u_\alpha \ni\alpha\to 1 \text{tom} \cup v_\alpha \ni \alpha\to 1\text{tol} \cup v^{h\alpha}_{\beta} \ni h\to 1\text{ton} \text{and} \beta\to 1\text{to}\sum_{i=1}^{h-1}\gamma^i \forall \alpha,\gamma,n,i\in\mathbb{N}\}$andthe$E[ST_\gamma(\bar{k}_m \oplus P_n)]=\{u'_\alpha \ni\alpha\to 1 \text{ tom} \cup v'_\alpha \ni\alpha\to 1\text{tol} \cup e^{h\alpha}_{\beta} \ni h\to 1\text{ton} \text{and}\beta\to 1 \text{ to}\sum_{i=1}^{h}\gamma^i \forall\ \mu,\gamma\in\mathbb{N}\}$.Thefollowingthreecasesdeals thetheorem.Themodularcoloringof $ST_\gamma(\bar{k}_m \oplus P_l)$isdefinedbyaninjectivemapping$C:V \to \mathbb{Z}_k;k \geq 2$.the coloring of$ST_\gamma(\bar{k}_m \oplus P_l)$ is given by

$$C(u_\alpha) = 0 \forall \alpha \text{rangeasfrom1tom}$$

$$C(v_\alpha) = \begin{cases}1 & \text{if } \alpha \text{ is odd}\\ 2 & \text{if } \alpha \text{ is even}\end{cases}$$

Case1:whenγ≡0(mod3)

Whenαrangesfrom 1tom,1≤h≤n,and$1\leq\beta\leq\sum_{i=1}^{h}\gamma^i$

$$C(v^{h\alpha}_{\beta})=\begin{cases}0 & \text{if}1\leq h\leq n;h \text{ isodd}\\ 1 & \text{if}4\leq h\leq n;h \text{ iseven}\\ 2 & \text{if}h=2\text{and}\beta=1\end{cases}$$

themodularcoloringis

$$\mathcal{S}(v^{h\alpha}_{\beta})=\begin{cases}0 & \text{if}1\leq h\leq n;h\text{is even}\\ 1 & \text{if}1\leq h\leq n;h\text{isodd,except}1\leq\beta\leq\gamma\text{in}h=3\\ 2 & \text{if}1\leq\beta\leq\gamma\text{in}h=3\end{cases}$$

Whenαrangesfrom 1tol,1≤h≤n,and$1\leq\beta\leq\sum_{i=1}^{h}\gamma^i$

Thecoloringoftheverticesof$ST_\gamma(\bar{k}_m \oplus P_l)$isfollowedfromsubcase1ofcase2inthetheorem6.

Case2:whenγ≡1(mod3)

Whenαrangesfrom1tom,1≤h≤n,and $1\leq\beta\leq\sum_{i=1}^{h}\gamma^i$

$$C(v^{h\alpha}_{\beta})=\begin{cases}0 & \text{ifhisodd}\\ 1 & \text{ifh iseven}\end{cases}$$

themodularcoloringis

$$\mathcal{S}(v_\beta^{h\alpha})=\begin{cases}0 & \text{if } h \text{ is even}\\ 1 & \text{if } h \text{ is odd}\end{cases}$$

When α ranges from 1 to l, $1\le h\le n$, and $1\le\beta\le\sum_{i=1}^{h}\gamma^i$

Coloring of the vertices $v_\beta^{h\alpha}$ is described in the same way of coloring of subcase 1 of case 2 in the theorem 6.

Case 3: when $\gamma\equiv 2 \pmod 3$

When α ranges from 1 to m, $1\le h\le n$, and $1\le\beta\le\sum_{i=1}^{h}\gamma^i$

For $1\le h\le n$; where h is odd, $C(v_\beta^{h\alpha})=0$

For $h=2$; $C(v_\beta^{h\alpha})=1$,

For $h\ge 4$; where h is even, the subsequent of the vertex (h) of the vertex $(h-2)$ which is colored by 1 then the coloring is given by $C(v_\beta^{h\alpha})=\begin{cases}2 & h\equiv 1 \pmod\gamma\\ 1 & \text{otherwise}\end{cases}$, that is colored by 2 then the coloring is given by $C(v_\beta^{h\alpha}) = 1$.

The modular coloring is

$$C(v_\beta^{h\alpha})=\begin{cases}0 & \text{if } 1 \le h \le n; \text{ where } h \text{ is even}\\ 1 & \text{if } 3\le h\le n; \text{ where } h \text{ is odd}\\ 2 & \text{if } h=1\end{cases}$$

When α ranges from 1 to l, $1\le h\le n$, and $1\le\beta\le\sum_{i=1}^{h}\gamma^i$.

Coloring of $C(v_\beta^{h\alpha})$ described in the same way of coloring of subcase 2 of case 2 in the theorem 6.

The above results that, the graph $ST_\gamma(\bar{k}_m\oplus P_l)$ is modular-3 colorable, thus the $m_c(ST_\gamma(\bar{k}_m\oplus P_l))=3$.

Theorem 10. If $G=Fl_\mu$ and H is an $\gamma-$ary graph, then $ST_\gamma(Fl_\mu)$ is modular 3-colorable.

Proof: Let H be a γ-ary graph with γ- children and G be a graph of Fl_μ with μ vertices. Then, since every vertex in the flower graph Fl_μ is an external vertex except the center vertex, the extended snow graph of flower graph Fl_μ is created by taking μ copies of the H= γ-ary graph and attaching them to the μ vertices of the flower graph by an edge. The $V[ST_\gamma(Fl_\mu)]=\{v_o\cup v_\alpha\cup v_\beta^{h\alpha}\ni\alpha\to 1 \text{ to } 2\mu; h\to 1 \text{ to } n \text{ and } \beta\to 1 \text{ to } \sum_{i=1}^{h-1}\gamma^i\,\forall\alpha,\gamma,n,i\in\mathbb{N}\}$ and the $E[ST_\gamma(Fl_\mu)]=\{e_{2\alpha}\cup v_\alpha v_{\alpha+1}\forall \text{odd}\,\alpha\cup e_\beta^{h\alpha}\ni\alpha\to 1 \text{ to } \mu; h\to 1 \text{ to } n \text{ and } \beta\to 1 \text{ to } \sum_{i=1}^{h}\gamma^i\,\forall\mu,\gamma\in\mathbb{N}\}$. The following three cases deals the theorem. The modular coloring of $ST_\gamma(Fl_\mu)$ is defined by an injective mapping $C:V \to \mathbb{Z}_k; k \ge 2$. The coloring of $ST_\gamma(W_\mu)$ is,

Case 1: When μ is odd,

$C(v_0) = 0, C(v_\alpha) = 0$ for all α ranges from 1 to 2μ. The modular coloring of the above vertices is

$$\mathcal{S}(c_0)=0,\ \mathcal{S}(v_\alpha)=\begin{cases}1 & \text{if } 1\le\alpha\le\mu-1 \text{ and } \mu+1\le\alpha\le 2\mu, \text{ where } \alpha \text{ is odd; } \alpha=2\mu, \text{ where } \mu \text{ is even}\\ 2 & \text{if } 1\le\alpha\le\mu-1, \text{ and } \mu+1\le\alpha\le 2\mu; \text{ where } \alpha \text{ is even}\\ 3 & \text{if } \alpha=\mu, \text{ where } \mu \text{ is odd}\end{cases}$$

Coloring of the γ-aray graph of $ST_\gamma(W_\mu)$ is partitioned into three subcases as follows

Subcase 1: When $\gamma\equiv 0 \pmod 4$

When α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha=2\mu$, α is even; $C(v_\beta^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{ where } h \text{ is even}\\ 1 & \text{if } 1\le h\le n, \text{ where } h \text{ is odd}\end{cases}$,

For all even α except $\alpha=2\mu$; $C(v_\beta^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{ where } h \text{ is even}\\ 1 & \text{if } 2\le h\le n, \text{ where } h \text{ is odd,}\\ 2 & \text{if } h=1\end{cases}$

When $\alpha=\mu$; $C(v_\beta^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{ where } h \text{ is even}\\ 1 & \text{if } 2\le h\le n, \text{ where } h \text{ is odd}\\ 3 & \text{if } h=1\end{cases}$

Modular coloring of $ST_\gamma(W_\mu)$ is

when α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha=2\mu$, α is even; $\mathcal{S}(v_\beta^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{ where } h \text{ is odd}\\ 1 & \text{if } 1\le h\le n, \text{ where } h \text{ is even}\end{cases}$,

For all even α except $\alpha=2\mu$; $\mathcal{S}(v_\beta^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{ where } h \text{ is odd}\\ 1 & \text{if } 4\le h\le n, \text{ where } h \text{ is even.}\\ 2 & \text{if } h=2\end{cases}$

When $\alpha=\mu$; $\mathcal{S}(v_\beta^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{ where } h \text{ is odd}\\ 1 & \text{if} \le h\le n, \text{ where } h \text{ is even.}\\ 3 & \text{if } h=2\end{cases}$

Subcase 2: When $\gamma\equiv 1 \pmod 4$

When α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha=2\mu$, α is even; $C(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n, \text{ where } h \text{ is even} \\ 1 & \text{if } 1\le h\le n, \text{ where } h \text{ is odd} \end{cases}$,

For all even α except $\alpha=2\mu$; $C(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n, \text{ where } h \text{ is even} \\ 1 & \text{if } 2\le h\le n, \text{ where } h \text{ is odd,} \\ 2 & \text{if } h=1 \end{cases}$

When $\alpha=\mu$; $C(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n, \text{ where } h \text{ is even} \\ 1 & \text{if } 2\le h\le n, h\equiv 1 \pmod 4 \text{ where } h \text{ is odd} \\ 2 & \text{if } 2\le h\le n, h\equiv 3 \pmod 4 \text{ where } h \text{ is odd} \\ 3 & \text{if } h=1 \end{cases}$

Modular coloring of $ST_\gamma(W_\mu)$ is

When α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha=2\mu$, α is even; $\mathcal{S}(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n, \text{ where } h \text{ is odd} \\ 2 & \text{if } 1\le h\le n, \text{ where } h \text{ is even} \end{cases}$,

For all even α except $=2\mu$; $\mathcal{S}(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n, \text{ where } h \text{ is odd} \\ 2 & \text{if } 4\le h\le n, \text{ where } h \text{ is even.} \\ 3 & \text{if } h=2 \end{cases}$

When $\alpha=\mu$; $\mathcal{S}(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n, \text{ where } h \text{ is odd} \\ 1 & h=2, n; n\equiv 2 \pmod 4 \text{ where } h \text{ is even} \\ 2 & h=n, n\equiv 0 \pmod 4, \text{ where } h \text{ is even} \\ 3 & \text{if } 4\le h\le n-1, \text{ where } h \text{ is even} \end{cases}$

Subcase 3: When $\gamma\equiv 2 \pmod 4$

When α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha=2\mu$, α is even; $C(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n, \text{ where } h \text{ is even} \\ 1 & \text{if } 1\le h\le n, \text{ where } h \text{ is odd} \end{cases}$,

For all even α; $C(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n, \text{ where } h \text{ is even} \\ 2 & \text{if } 1\le h\le n, \text{ where } h \text{ is odd} \end{cases}$,

For all even α except $\alpha=2\mu$; $C(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n, \text{ where } h \text{ is even} \\ 1 & \text{if } 2\le h\le n, \text{ where } h \text{ is odd.} \\ 3 & \text{if } h=1 \end{cases}$

Modular coloring of $ST_\gamma(W_\mu)$ is

When α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha=2\mu$, α is even; $\mathcal{S}(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n, \text{ where } h \text{ is odd} \\ 3 & \text{if } 1\le h\le n, \text{ where } h \text{ is even} \end{cases}$,

For all even α except $\alpha=2\mu$; $\mathcal{S}(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n, \text{ where } h \text{ is odd} \\ 2 & \text{if } 1\le h\le n, \text{ where } h \text{ is even} \end{cases}$.

When $\alpha=\mu$; $\mathcal{S}(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n-1, \text{ where } h \text{ is odd} \\ 1 & \text{if } h=2 \text{ and } h=n, h \text{ is even} \\ 3 & \text{if } 4\le h\le n, \text{ where } h \text{ is even} \end{cases}$.

Subcase 4: When $\gamma\equiv 3 \pmod 4$

When α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha=2\mu$, α is even; $C(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n-1, \text{ where } h \text{ is even} \\ 1 & \text{if } h\equiv 1 \pmod 4, \text{ where } h \text{ is odd,} \\ 2 & \text{if } h\equiv 3 \pmod 4, \text{ where } h \text{ is odd} \end{cases}$

For all even α except $\alpha=2\mu$; $C(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n-1, \text{ where } h \text{ is even} \\ 1 & \text{if } h\equiv 3 \pmod 4, \text{ where } h \text{ is odd,} \\ 2 & \text{if } h\equiv 1 \pmod 4, \text{ where } h \text{ is odd} \end{cases}$

When $\alpha=\mu$; $C(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n, \text{ where } h \text{ is even} \\ 1 & \text{if } 1\le h\le n; h\equiv 3 \pmod 4, \text{ where } h \text{ is odd} \\ 2 & \text{if } 5\le h\le n; h\equiv 1 \pmod 4, \text{ where } h \text{ is odd} \\ 3 & \text{if } h=1 \end{cases}$.

Modular coloring of $ST_\gamma(W_\mu)$ is

When α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha=2\mu$, α is even;

$$\mathcal{S}(v_\beta^{\alpha h})=\begin{cases} 0 & \text{if } 1\le h\le n, \text{ where } h \text{ is odd} \\ 1 & \text{if } 1\le h\le n-1; h\equiv 0 \pmod 4, \text{ and } h=n, h\equiv 2 \pmod 4 \\ 2 & \text{if } h=n, h\equiv 0 \pmod 4 \\ 3 & \text{if } 1\le h\le n-1; h\equiv 2 \pmod 4, \text{ where } h \text{ is even} \end{cases},$$

For all even α except $\alpha=2\mu$;

$$\mathcal{S}(v_{\beta}^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{where } h \text{ is odd}\\ 1 & \text{if } 1\le h\le n-1; h\equiv 2(\text{mod}4), \text{and } h=n; h\equiv 0(\text{mod}4)\\ 2 & \text{if } h=n; h\equiv 2(\text{mod}4)\\ 3 & \text{if } 1\le h\le n-1; h\equiv 2(\text{mod}4), \text{where } nh \text{ is even}\end{cases}.$$

When $\alpha=\mu$;

$$\mathcal{S}(v_{\beta}^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{where } h \text{ is odd}\\ 1 & \text{if } 1\le h\le n-1; h\equiv 2(\text{mod}4), \text{and } h=n; h\equiv 0(\text{mod}4)\\ 2 & \text{if } h=2 \text{ and } h=n, h\equiv 2(\text{mod}4)\\ 3 & \text{if } 1\le h\le n-1; h\equiv 0(\text{mod}4), \text{where } h \text{ is even}\end{cases}.$$

Thus from the above results $ST_{\gamma}(W_{\mu})$ is 4 modular colorable for μ- is odd, therefore the modular coloring of $m_c(ST_{\gamma}(W_{\mu}))=4$.

Case2: When μ is even,

$C(v_0)=0, C(v_{\alpha})=0$ for all α ranges from 1 to 2μ. The modular coloring of the above vertices are

$$\mathcal{S}(c_0)=0, \mathcal{S}(v_{\alpha})=\begin{cases}1 & \text{if } 1\le\alpha\le\mu, \text{where } \alpha \text{ is odd}\\ 2 & \text{if } 1\le\alpha\le\mu, \text{where } \alpha \text{ is even}\end{cases}$$

Coloring of $ST_{\gamma}(W_{\mu})$ is splitted as three subcases as follows

Subcase1: When $\gamma\equiv 0(\text{mod}3)$

When α ranges from 1 to 2μ and for all β

For all $1\le\alpha\le\mu$, where α is even and ; $\mu+1\le\alpha\le 2\mu$, where α is odd

$$C(v_{\beta}^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{where } h \text{ is even}\\ 1 & \text{if } 1\le h\le n, \text{where } h \text{ is odd}\end{cases}$$

For all $1\le\alpha\le\mu$, where α is odd and; $\mu+1\le\alpha\le 2\mu$, where α is even

$$C(v_{\beta}^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{where } h \text{ is even}\\ 1 & \text{if } 2\le h\le n, \text{where } h \text{ is odd},\\ 2 & \text{if } h=1\end{cases}$$

Modular coloring of $ST_{\gamma}(W_{\mu})$ is

$$\mathcal{S}(c_0)=0, \mathcal{S}(v_{\alpha})=\begin{cases}1 & \text{if } 1\le h\le n, \text{where } h \text{ is odd}\\ 2 & \text{if } 1\le h\le n, \text{where } h \text{ is even}\end{cases}$$

When α ranges from 1 to 2μ and for all β

For all $1\le\alpha\le\mu$, where α is even and; $\mu+1\le\alpha\le 2\mu$, where α is odd

$$\mathcal{S}(v_{\beta}^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{where } h \text{ is odd}\\ 1 & \text{if } 1\le h\le n, \text{where } h \text{ is even}\end{cases},$$

For all $1\le\alpha\le\mu$, where α is odd and; $\mu+1\le\alpha\le 2\mu$, where α is even

$$C(v_{\beta}^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{where } h \text{ is odd}\\ 1 & \text{if } 4\le h\le n, \text{where } h \text{ is even}.\\ 2 & \text{if } h=2\end{cases}$$

Subcase2: When $\gamma\equiv 1(\text{mod}3)$

When α ranges from 1 to 2μ and for all β

For all $1\le\alpha\le\mu$, where α is even and; $\mu+1\le\alpha\le 2\mu$, where α is odd

$$C(v_{\beta}^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{where } h \text{ is even}\\ 1 & \text{if } 1\le h\le n, \text{where } h \text{ is odd}\end{cases}$$

For all $1\le\alpha\le\mu$, where α is odd and; $\mu+1\le\alpha\le 2\mu$, where α is even

$$C(v_{\beta}^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{where } h \text{ is even}\\ 2 & \text{if } 1\le h\le n, \text{where } h \text{ is odd}\end{cases}$$

Modular coloring of $ST_{\gamma}(W_{\mu})$ is

When α ranges from 1 to 2μ and for all β

For all $1\le\alpha\le\mu$, where α is even and; $\mu+1\le\alpha\le 2\mu$, where α is odd

$$\mathcal{S}(v_{\beta}^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{where } h \text{ is odd}\\ 1 & \text{if } h=n, \text{where } n \text{ is even}\\ 2 & \text{if } 1\le h\le n-1, \text{where } h \text{ is even}\end{cases},$$

For all $1\le\alpha\le\mu$, where α is odd and; $\mu+1\le\alpha\le 2\mu$, where α is even

$$\mathcal{S}(v_{\beta}^{\alpha h})=\begin{cases}0 & \text{if } 1\le h\le n, \text{where } h \text{ is odd}\\ 1 & \text{if } 1\le h\le n-1, \text{where } h \text{ is even}.\\ 2 & \text{if } h=n, \text{where } n \text{ is even}\end{cases}$$

Subcase3: When $\gamma\equiv 2(\text{mod}3)$

When α ranges from 1 to 2μ and for all β

For all $1\le\alpha\le\mu$, where α is even and; $\mu+1\le\alpha\le 2\mu$, where α is odd

$$C(v_{\beta}^{\alpha h})=\begin{cases} 0 & \text{if } 1 \le h \le n,\text{where h is even} \\ 1 & \text{if } h\equiv 1(\text{mod}4),\text{where n is odd,} \\ 2 & \text{if } h\equiv 3(\text{mod}4),\text{where h is odd} \end{cases}$$

Forall$1\le\alpha\le\mu$,whereαisoddand;$\mu+1\le\alpha\le 2\mu$,where αiseven

$$C(v_{\beta}^{\alpha h})=\begin{cases} 0 & \text{if } 1 \le h \le n,\text{where h is even} \\ 1 & \text{if } h\equiv 3(\text{mod}4),\text{where n is odd,} \\ 2 & \text{if } h\equiv 1(\text{mod}4),\text{where h is odd} \end{cases}$$

Modularcoloringof$ST_{\gamma}(W_{\mu})$is

Whenαrangesfrom1toμandforallβ

Forall$1\le\alpha\le\mu$,whereαisevenand;$\mu+1\le\alpha\le 2\mu$,where αisodd

$$\mathcal{S}(v_{\beta}^{\alpha h})=\begin{cases} 0 & \text{if} 1\le h\le n,\text{where h is odd} \\ 1 & \text{if} 1\le h\le n; h\equiv 0(\text{mod}4),\text{where n is even} \\ & \text{and } h=n,\text{where } n\equiv 2(\text{mod}4) \\ 2 & \text{if} 1\le h\le n h\equiv 2(\text{mod}4),\text{where h is even} \\ & \text{and } h=n,\text{where } n\equiv 0(\text{mod}4) \end{cases},$$

Forall$1\le\alpha\le\mu$,whereαisoddand;$\mu+1\le\alpha\le 2\mu$,where αiseven

$$C(v_{\beta}^{\alpha h})=\begin{cases} 0 & \text{if} 1\le h\le n,\text{where h is odd} \\ 1 & \text{if} 1\le h\le n; h\equiv 2(\text{mod}4),\text{where n is even} \\ & \text{and } h=n,\text{where } n\equiv 0(\text{mod}4) \\ 2 & \text{if} 1\le h\le n h\equiv 0(\text{mod}4),\text{where h is even} \\ & \text{and } h=n,\text{where } n\equiv 2(\text{mod}4) \end{cases}.$$

From the above cases the graph $ST_{\gamma}(W_{\mu})$ is 3 modular colorable for μ- is even, therefore the modular coloring of $m_c(ST_{\gamma}(W_{\mu})) = 3$.

CONCLUSION

Inthis paper, wedeveloped two new graphs calledsnow graph and extendedsnow graph,and inaddition to that, we examined the modular coloring for some famous graphs, such as snow graph and its extended snow graph of wheel, gear, friendship, generalized fan, and flower graph. It results in the graph being modular - K colorable, and the modular chromatic number is obtained. The application of the result is extended to the traffic signal of a busy road with huge passengers, and in the future we can develop many applications related to this work using various graph structures.

REFERENCES

[1] F.Okamoto,E.Salehi,P.Zhang.ACheckerboardproblemandmodularcoloringsofgraph, Bull.Inst.combin.appl58(2010),29-47

[2] P. Sumathi, S. Tamilselvi, Modular chromatic number of certain cyclic graphs. Journal of algebraic statistics, Volume 13, No. 3, 2022, p. 4459-4466.

[3] P.Sumathi,S.Tamilselvi,Modularcoloringoninflatedgraphs.Industrialengineeringjournal,ISSN 0970-2555 Volume 52, Issue 4,No. 2,april 2023.

[4] Kacheru, G. "The role of AI-Powered Telemedicine software in healthcare during the COVID-19 Pandemic." Turkish Journal of Computer and Mathematics Education (TURCOMAT) 11.3 (2020).

[5] P. Sumathi, S. Tamilselvi, The modular chromatic number of the corona product of a generalized Jahangirgraph.Indianjournalofscienceand technology, Year:2023,Volume:16,Issue:46,Pages: 4309-4327. DOI: 10.17485/IJST/v16i46.2028.

[6] N. Paramaguru, R. Sampathkumar, Modular Colorings Of Join Of Two Special Graphs. Electronic Journal of Graph Theory and Applications 2 (2) (2014), 139–149 https://dx.doi.org/10.5614/ejgta.2014.2.2.6

[7] T. Nicholas, G. R, Sanma, Modular Colorings Of Cycle Related Graphs, Global Journal of Pure and Applied Mathematics, 13 (7) (2017), 3779-3788 © Research India Publications http://www.ripublication.com

[8] Kacheru, Goutham. "The Future of Cyber Defence: Predictive Security with Artificial Intelligence." system 2: 4.

[9] R.Rajarajachozhan,R.Sampathkumar,ModularColoringOfTheCartesianProducts$K_m\square K_n$, $K_m\square C_n$, And $K_m\square P_n$, Discrete mathematics, Algorithms and Applications / Vol. 09, No. 06, 1750075 (2017) https://doi.org/10.1142/S1793830917500756

[10] T.Nicolas, Sanma, G.R Modular colorings of circular Halin graphs of level two, Asian journal of Mathematics and Computer Research 17(1): 48-55, 2017.

[11] Sanma. G. R, P. Maya Modular Coloring and Switching In Some Planar Graphs, Ratio Mathematica. Volume 45, 2023, DOI: http://dx.doi.org/10.23755/rm.v45i0.984