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The Chromatic Restrained Domination on Cartesian Product with Complete Bipartite and Wheel Graphs

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Abstract

Let $G = (V, E)$ be a graph. A subset D of $V(G)$ is said to be a chromatic restrained dominating set (or crd-set) of G if D is a restrained dominating set of G and $\chi(< D >) = \chi(G)$. The minimum cardinality taken over all minimal chromatic restrained dominating sets is called the chromatic restrained domination number of G and is denoted by $\gamma_r^c(G)$. In this paper, the chromatic restrained domination number on the cartesian product of certain standard graphs with complete bipartite graphs and wheel graphs were obtained.

Keywords: Domination, Restrained Domination, Chromatic Number, Cartesian Product

AMS Subject Classification: 05C15, 05C69

1. INTRODUCTION

All the graphs $G = (V, E) = (n, m)$ considered here are simple, finite and undirected, with neither loops nor multiple edges. For $D \subseteq V$, the subgraph induced by D is denoted by $\langle D \rangle$. A $k-$vertex coloring of a graph is an assignment of $k-$colors to its vertices. The coloring is proper if no two adjacent vertices are assigned the same color. A coloring in which $k-$ colors are used is a $k-$coloring. A graph is $k-$colorable if it has a proper $k-$coloring. The minimum k for which a graph G is $k-$colorable is called its *chromatic number*, denoted by $\chi(G)$. Graph Theory terminologies which are not defined here can be seen in [2] and [6].

A set $D \subseteq V$ of vertices in a graph G is called a dominating set if every vertex $u \in V$ is either an element of D or is adjacent to an element of D. The minimum cardinality taken over all minimal dominating sets is called the *domination number* of G and is denoted by $\gamma(G)$. A set $D \subseteq V$ is a restrained dominating set if every vertex in $V - D$ is adjacent to a vertex in D and another vertex in $V - D$ [3]. The minimal cardinality taken over all minimal restrained dominating sets is

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called the *restrained domination number* of G and is denoted by $\gamma_r(G)$. A set D is a γ_r −set if D is a restrained dominating set of cardinality $\gamma_r(G)$.

The *cartesian product* of simple graphs G and H is the graph $G \square H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$ or, $v_1v_2 \in E(H)$ and $u_1 = u_2$. Several domination parameters including power domination, total domination, integer domination, split domination and so on were analysed on the cartesian product of graphs.

A set $D \subseteq V$ is a *chromatic preserving set* or a cp-set if $\chi(< D >) = \chi(G)$ and the minimum cardinality taken over all cp-set in G is called the *chromatic preserving number* or cp-number of G, denoted by $\text{cpn}(G)$. This concept was introduced by T. N. Janakiraman and M. Poobalaranjani in 2005 [5]. They also defined dom-chromatic sets of a graph G. A subset D of V is said to be a *dom-chromatic set* (or dc-set) if D is a dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality taken over all minimal dom-chromatic sets in G is called the *dom-chromatic number* and is denoted by $\gamma_{ch}(G)$ [4]. Numerous researchers including S. Balamurugan et al., worked on dom-chromatic numbers with various domination parameters [1]. In this paper, the chromatic restrained domination number on the cartesian product of some standard graphs are obtained.

2. MAIN RESULTS

In this section, we obtained the chromatic restrained domination number for the cartesian product of some graphs.

Definition 2.1 Let $G = (V, E)$ be a graph. A subset D of V is said to be a *chromatic restrained dominating set* (or *crd*-set) if D is a restrained dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality taken over all minimal chromatic restrained dominating sets is called the *chromatic restrained domination number* of G and is denoted by $\gamma_r^c\,(G)$.

Theorem 2.2 For any $m, r, s \geq 2$,

$$\gamma_r^c(K_m \square K_{r,s}) = \begin{cases} m + min\{r, s\} - 1 & if \;\; r, s \leq m \;\; or \;\; r \leq m \;\; or \;\; s \leq m \\ 2m & if \;\; r, s > m \end{cases}.$$

Proof. Let $V(K_m) = \{u_1, u_2, u_3, \dots, u_m\}$ and $V(K_{r,s}) = \{v_1, v_2, v_3, \dots, v_r, v_{r+1}, v_{r+2}, \dots, v_{r+s}\}$

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$= V_1 \cup V_2$ where $V_1 = \{v_1, v_2, v_3, \dots, v_r\}, V_2 = \{v_{r+1}, v_{r+2}, v_{r+3}, \dots, v_{r+s}\}$. Then $V(K_m \square K_{r,s}) = \{(u_i, v_j)/1 \leq i \leq m, 1 \leq j \leq r+s\}$ with cardinality $(r+s)m$. Also, $\chi(K_m \square K_{r,s}) = max\{\chi(K_m), \chi(K_{r,s})\} = max\{m, 2\} = m$ and $K_m \square K_{r,s}$ contains m rows and s columns where the induced subgraph formed by the vertices of each column is a complete graph on m vertices. Thus, every chromatic restrained dominating set of $K_m \square K_{r,s}$ must contain all the vertices of at least one column. Assume $r \leq s$.

Case (i): $r, s > m$

Let $D = \{(u_i, v_r)/1 \leq i \leq m\}$ where the vertices of D belongs to the r^{th} column of $K_m \square K_{r,s}$ and $\langle D \rangle$ is a complete graph on m vertices. Thus, $\chi(\langle D \rangle) = m = \chi(K_m \square K_{r,s})$. But D is not a restrained dominating set of $K_m \square K_{r,s}$, since the first $r-1$ columns of $K_m \square K_{r,s}$ is not adjacent to any vertex of D. Since $r - 1 \geq m$, consider $D_1 = D \cup \{(u_i, v_{r+1})/1 \leq i \leq m\}$ where all the vertices belonging to first $r-1$ columns is adjacent to a vertex of $\{(u_i, v_{r+1})/1 \leq i \leq m\}$. Clearly, D_1 is a restrained dominaing set and $\chi(\langle D_1 \rangle) = m = \chi(K_m \square K_{r,s})$. Thus, D_1 is a chromatic restrained dominating set of $K_m \square K_{r,s}$ and $\gamma_r^c(K_m \square K_{r,s}) \leq |D_1| = 2m$. Furthermore, every chromatic restrained set must contain all the vertices of at least one column. If γ_r^c −set contains a column of vertices among the first r columns, then all the vertices of $(r+1)^{th}$ to $(r+s)^{th}$ column is adjacent to it. Since $r, s > m$, choosing any one of the columns among $(r+1)^{th}$ to $(r+s)^{th}$ column dominates all the remaining vertices among the first r columns. Thus, a minimum γ_r^c −set is obtained and $\gamma_r^c(K_m \square K_{r,s}) \geq 2m$. Therefore, $\gamma_r^c(K_m \square K_{r,s}) = 2m$.

Case (ii): $r, s \leq m$ or $r \leq m$ or $s \leq m$

Since $r \leq s$, $s \leq m$ is not possible. Consider $D = \{(u_i, v_r)/1 \leq i \leq m\}$ where $\chi(\langle D \rangle) = m = \chi(K_m \square K_{r,s})$, but D is not a restrained dominating set. Clearly, D dominates all the vertices belonging to $(r+1)^{th}$ to $(r+s)^{th}$ columns and the remaining $r-1$ columns is not adjacent to any of the vertices in D. Since $r \leq m$, $r - 1 < m$. Then choosing the first vertex of each $r-1$ columns gives a minimum chromatic restrained dominating set. Thus, $\gamma_r^c(K_m \square K_{r,s}) \leq 2m = m + r - 1 = m + min\{r, s\} - 1$. Moreover, any chromatic restrained dominating set of $K_m \square K_{r,s}$ with $r, s \leq m$ or $r \leq m$ must contain at least one column of vertices among the first r columns together with $r-1$ vertices and so, $\gamma_r^c(K_m \square K_{r,s}) \geq m + r - 1 = m + min\{r, s\} - 1$. Therefore, $\gamma_r^c(K_m \square K_{r,s}) = m + min\{r, s\} - 1$.

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Observation 2.3 $\gamma_r^c(K_3 \square W_s) = \begin{cases} 3 \ if \ s \ is \ odd \\ s \ if \ s \ is \ even \end{cases}$, where $s \geq 4$.

Theorem 2.4 For any $r, s \geq 4$, $\gamma_r^c(K_r \square W_s) = r$.

Proof. Let $V(K_r) = \{u_1, u_2, u_3, \ldots, u_r\}$ and $V(W_s) = \{v_0, v_1, v_2, \ldots, v_{s-1}\}$ where v_0 is the full degree vertex of W_s. Clearly, $V(K_r \square W_s) = \{(u_i, v_j)/1 \leq i \leq r, 0 \leq j \leq s-1\}$ and $|V(K_r \square W_s)| = rs$. Furthermore, $K_r \square W_s$ contains r rows and s columns where each i represents a row and each j represents a column. Since $\chi(K_r \square W_s) = max\{\chi(K_r), \chi(W_s)\}$, $\chi(K_r \square W_s) = r$. Let $D = \{(u_1, v_0), (u_2, v_0), \ldots, (u_r, v_0)\}$ where all the elements of D belongs to the entire first column of $K_r \square W_s$. Clearly, D is a restrained dominating set since the first element of each row which is adjacent to the remaining vertices in the row is in D and the elements of $V - D$ are also adjacent to another element in $V - D$. Also, $\chi(\langle D \rangle) = r$ as $\langle D \rangle$ is a complete graph on r vertices. Therefore, D is a chromatic restrained dominating set of $K_r \square W_s$ and $\gamma_r^c(K_r \square W_s) \leq |D| = r$. As $\chi(K_r \square W_s) = r$, any chromatic restrained dominating set must contain at least r elements and so $\chi(K_r \square W_s) \geq r$. Therefore, $\gamma_r^c(K_r \square W_s) = r$.

Theorem 2.5 For any $r \geq 3, s \geq 4$, $\gamma_r^c(P_r \square W_s) = \begin{cases} r+s-3 & if \ s \ is \ even \\ r+2 & if \ s \ is \ odd \end{cases}$.

Proof. Consider $V(P_r) = \{u_1, u_2, \ldots, u_r\}$ and $V(W_s) = \{v_0, v_1, v_2, \ldots, v_{s-1}\}$, where v_0 is the full degree vertex of W_s. Clearly, $V(P_r \square W_s) = \{(u_i, v_j)/1 \leq i \leq r, 0 \leq j \leq s-1\}$ and denote the rows of $P_r \square W_s$ by $V_1, V_2, \ldots, V_r$. Let $D = \{(u_1, v_0), (u_2, v_0), \ldots, (u_r, v_0)\}$, where $< D >= P_r$. As D contains the first vertex of each row which is adjacent to all the remaining elements of the row, D is a restrained dominating set. Also, D is minimal. Thus, $\gamma_r(P_r \square W_s) = r$. Since $\chi(P_r \square W_s) = \begin{cases} 3 & if \ s \ is \ odd \\ 4 & if \ s \ is \ even \end{cases}$ and $\chi(< D >) = 2 \neq \chi(P_r \square W_s)$, D is not a chromatic restrained dominating set of $P_r \square W_s$.

Case (i): s is even

Then $\chi(P_r \square W_s) = 4$ and so, any γ_r^c −set must contain $< W_s >$. Consider $D_1 = \{(u_2, v_0), (u_2, v_1), (u_2, v_2), \ldots, (u_2, v_{s-1})\} \cup \{(u_4, v_0), (u_5, v_0), \ldots, (u_r, v_0)\}$. Clearly, $\{(u_2, v_0), \ldots, (u_2, v_{s-1})\}$ is adjacent to all the vertices of V_1 and V_3 and each (u_i, v_0),$4 \leq i \leq r$ is adjacent to the remaining elements of V_i. In addition, $< V \setminus D_1 >$ is connected and $\chi(< D_1 >) = 4 = \chi(P_r \square W_s)$. This implies that, D_1 is a chromatic restrained dominating set of $P_r \square W_s$and

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$\gamma_r^c(P_r \square W_s) \leq r + s - 3$. For any γ_r^c −set with vertices of $< W_s >$, removal of any vertex (u_i, v_j), for a fixed i and $0 \leq j \leq s - 1$ does not forms a cp −set. Thus, $< W_s >$ is present in each γ_r^c −set with vertices (u_i, v_j) for a f ixed $i, i \neq 1, r, 0 \leq j \leq s - 1$ which are adjacent to the vertices of two rows. For the remaining $r - 3$ rows with a full degree vertex in each $< V_i >$, any γ_r^c −set must contain a minimum of $r - 3$ vertices. This implies that, $\gamma_r^c(P_r \square W_s) < r + s - 3$ is impossible. Hence, $\gamma_r^c(P_r \square W_s) = r + s - 3$.

Case (ii): s is odd

As $\chi(P_r \square W_s) = 3$, any γ_r^c −set of $P_r \square W_s$ must contain an induced 3 −cycle. Consider $D_2 = D \cup \{(u_1, v_1), (u_1, v_2)\}$. Since $(u_1, v_0) \in D$, $\{(u_1, v_0), (u_1, v_1), (u_1, v_2)\}$ forms a 3 −cycle and so, $\chi(< D_2 >) = 3 = \chi(P_r \square W_s)$. Clearly, D_2 is a restrained dominating set. This implies that, D_2 is a chromatic restrained dominating set of $P_r \square W_s$ with $\gamma_r^c(P_r \square W_s) \leq r + 2$. Suppose there exists a chromatic restrained dominating set S such that $|S| < r + 2$. Then $|D| < |S| < |D2|$, which implies that $r < |S| < r + 2$. Thus, the only possibility for $|S|$ is $r + 1$. But there does not exists a γ_r^c −set with cardinality $r + 1$ and so, $|S| < r + 2$ is impossible. Therefore, $\gamma_r^c(P_r \square W_s) = r + 2$.

Theorem 2.6 For any $r \geq 3, s \geq 4, \gamma_r^c(C_r \square W_s) = \begin{cases} r & if\ r, s\ is\ odd \\ r + 2 & if\ r\ is\ even, s\ is\ odd \\ r + s - 3 & if\ r\ is\ odd\ or\ even, s\ is\ even. \end{cases}$

Proof. Let $V(C_r) = \{u_1, u_2, \ldots, u_r\}$ and $V(W_s) = \{v_0, v_1, v_2, \ldots, v_{s-1}\}$ where v_0 is the full degree vertex of W_s. Then $V(C_r \square W_s) = \{(u_i, v_j)/1 \leq i \leq r, 0 \leq j \leq s - 1\}$ and $C_r \square W_s$contains r rows and s columns, where each row contains $< W_s >$ and each column contains $< C_r >$. Clearly, $D = \{(u_1, v_0), (u_2, v_0), \ldots, (u_r, v_0)\}$ is a restrained dominating set, as each (u_i, v_0),$1 \leq i \leq r$ is adjacent to all the elements of the particular row with (u_i, v_0). Then D is a minimal restrained dominating set of $C_r \square W_s$and $\gamma_r(C_r \square W_s) = r$. Since $< D >$ forms a cycle $C_r, \chi(< D >) = \begin{cases} 2 & if\ r\ is\ even \\ 3 & if\ r\ is\ odd. \end{cases}$

Case (i): r, s is odd

Then $\chi(C_r \square W_s) = 3$. Since r is odd, $\chi(< D >) = 3 = \chi(C_r \square W_s)$. This implies that, D is a chromatic restrained dominating set of $C_r \square W_s$ and so, $\gamma_r^c(C_r \square W_s) = r$.

Case (ii): r is even and s is odd

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Clearly, $\chi(C_r \square W_s) = 3$ and $\chi(< D >) = 2$. As $\chi(< D >) \neq \chi(C_r \square W_s)$, D is not a chromatic restrained dominating set. Let $D_1 = D \cup \{(u_1, v_1), (u_1, v_2)\}$. Then, $\{(u_1, v_0), (u_1, v_1), (u_1, v_2)\} \subset D1$ forms a $3-$cycle and so, $\chi(< D1 >) = 3$. Also, D_1 is a restrained dominating set. Thus, D_1 is a chromatic restrained dominating set and $\gamma_r^c(C_r \square W_s) \leq |D| + 2 = r + 2$. In addition, removal of any vertex from D_1 does not forms a restrained dominating set, or else, is not a $cp-$set. Therefore, D_1 is a minimal chromatic restrained dominating set and $\gamma_r^c(C_r \square W_s) = r + 2$.

Case (iii): r is odd or even and s is even

Then, $\chi(C_r \square W_s) = 4$ and $\chi(< D >) = 2 \neq \chi(C_r \square W_s)$. Thus, D is not a chromatic restrained dominating set of $C_r \square W_s$. As $\chi(W_s) = 4$, any γ_r^c-set of $C_r \square W_s$ must contain all the vertices of a row, which is adjacent to all the vertices of another two rows. From the remaining $r-3$ rows, choosing the first vertex of each row forms a γ_r^c-set. Thus, $D_2 = \{(u_2, v_0), (u_2, v_1), (u_2, v_2), \dots, (u_2, v_{s-1})\} \cup \{(u_4, v_0), (u_5, v_0), \dots, (u_r, v_0)\}$ and $\gamma_r^c(C_r \square W_s) \leq |D_2| = r + s - 3$. Also, for any vertex $(u_2, v_j), D_2 - \{(u_2, v_j)\}$ for some j does not forms a $cp-$set for $C_r \square W_s$, and for any vertex $(u_i, v_0), 4 \leq i \leq r$ for some i, D_2 is not a restrained dominating set. Hence, D_2 is a minimal chromatic restrained dominating set of $C_r \square W_s$ and $\gamma_r^c(C_r \square W_s) = r + s - 3$.

Result 2.7

(i) If $m \geq s$, then $\gamma_r^c(K_{1,m} \square K_{r,s}) = r + s, r \leq s$.

(ii) If $m < s$, then $\gamma_r^c(K_{1,m} \square K_{r,s}) = \begin{cases} m + r & if\ m + 1 \geq r - 1 \\ 2(m+1) & if\ m + 1 \leq r - 1. \end{cases}$

Theorem 2.8 Let $G = K_{1,r} \square W_s$. Then for any $r \geq 2$ and $s \geq 4$,

(i) if $r + 1 \geq s, \gamma_r^c(K_{1,r} \square W_s) = s$.

(ii) if $r + 1 < s$ and s is even, $\gamma_r^c(K_{1,r} \square W_s) = s$.

(iii) if $r + 1 < s$ and s is odd, $\gamma_r^c(K_{1,r} \square W_s) = \begin{cases} r + 3 & if\ \ s > r + 3 \\ s & if\ \ s \leq r + 3 \end{cases}$.

Proof. Let $V(K_{1,r}) = \{u_0, u_1, u_2, u_3, \dots, u_r\}$ and $V(W_s) = \{v_0, v_1, v_2, v_3, \dots, v_{s-1}\}$ where u_0 is the full degree vertex of $K_{1,r}$ and v_0 is the full degree vertex of W_s. Then $V(K_{1,r} \square W_s) = \{(u_i, v_j)/0 \leq i \leq r, 0 \leq j \leq s - 1\}$ and $|V(K_{1,r} \square W_s)| = (r+1)s$. Since $\chi(K_{1,r}) = 2$ and

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$\chi(W_s) = \begin{cases} 3 \; if \;\; s \;\; is \;\; odd \\ 4 \; if \;\; s \;\; is \;\; even \end{cases}$, $\chi(K_{1,r} \,\square\, W_s) = max\{\chi(K_{1,r}), \chi(W_s)\} = \chi(W_s)$. Clearly, $K_{1,r} \,\square\, W_s$ contains $r+1$ rows and s columns where each row can be colored with 3 or 4 colors according as s is odd or even and each column can be colored with 2 colors.

Case (i): $r + 1 \geq s$

Let $D_1 = \{(u_0, v_j)/0 \leq j \leq s-1\}$ and $D_2 = \{(u_i, v_0)/0 \leq i \leq r\}$ where $\langle D_1 \rangle$ contains all the elements in the first row and $\langle D_2 \rangle$ contains all the elements in the first column. Clearly, D_1 and D_2 are restrained dominating sets. But $\chi(\langle D_2 \rangle) = 2 \neq \chi(K_{1,r} \,\square\, W_s)$. Therefore, D_2 is not a chromatic restrained dominating set of $K_{1,r} \,\square\, W_s$. On the other hand, $\chi(\langle D_1 \rangle) = 3$ or 4 according as s is odd or even. Therefore, D_1 is the minimum chromatic restrained dominating set of $K_{1,r} \,\square\, W_s$ and $\gamma_r^c(K_{1,r} \,\square\, W_s) \leq |D_1| = s$. Since $r + 1 \geq s$ and $\chi(K_{1,r}) < \chi(W_s)$, any minimum chromatic restrained dominating set must contain all the vertices of the first row. Therefore, $\gamma_r^c(K_{1,r} \,\square\, W_s) \geq s$. Hence, $\gamma_r^c(K_{1,r} \,\square\, W_s) = s$.

Case (ii): $r + 1 < s$

Then the number of rows in $K_{1,r} \,\square\, W_s$ is less than the number of columns in $K_{1,r} \,\square\, W_s$.

Subcase (i): s is even

Then $\chi(W_s) = 4$ and so $\chi(K_{1,r} \,\square\, W_s) = 4$. Consider $D_1 = \{(u_0, v_j)/0 \leq j \leq s-1\}$ where D_1 contains all the elements belonging to the first row of $K_{1,r} \,\square\, W_s$. Since D_1 contains first element of each column which is adjacent to all the vertices in $V(K_{1,r} \,\square\, W_s) \backslash D_1$, D_1 is a restrained dominating set of $K_{1,r} \,\square\, W_s$. Clearly, $\chi(\langle D_1 \rangle) = 4 = \chi(K_{1,r} \,\square\, W_s)$. Therefore, D_1 is a unique minimum chromatic restrained dominating set of $K_{1,r} \,\square\, W_s$ and so $\gamma_r^c(K_{1,r} \,\square\, W_s) = |D_1| = s$.

Subcase (ii): s is odd

Then $\chi(W_s) = 3$ and so $\chi(K_{1,r} \,\square\, W_s) = 3$. This implies that, any minimum chromatic restrained dominating set must contain an odd cycle on three vertices. It is evident that, the first three vertices of each row forms a $3-$cycle. If the remaining $s - 3$ vertices of the first row is less than or equal to r, then clearly, $\gamma_r^c(K_{1,r} \,\square\, W_s) = s$. Assume $s > r + 3$. Let $D_3 = \{(u_i, v_0), (u_0, v_1), (u_0, v_2)/ 0 \leq i \leq r\}$ and $|D_3| = r + 3$. Since D_3 contains first element of each row, D_3 is a restrained dominating set of $K_{1,r} \,\square\, W_s$. Also $\langle\{(u_0, v_0), (u_0, v_1), (u_0, v_2)\}\rangle$ of D_3 forms a $3-$cycle. Thus, $\chi(\langle D_3 \rangle) = 3 = \chi(K_{1,r} \,\square\, W_s)$. Therefore, D_3 is a chromatic restrained dominating set of $K_{1,r} \,\square\, W_s$ and $\gamma_r^c(K_{1,r} \,\square\, W_s) \leq |D_3| = r + 3$. Since $s - 3 > r$, any chromatic restrained dominating set

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must contain a 3 −cycle together with the remaining vertices of the first column. Thus, $\gamma_r^c(K_{1,r} \square W_s) \geq r+3$. Therefore, $\gamma_r^c(K_{1,r} \square W_s) = r+3$. If D_1 is considered, then D_1 is also a chromatic restrained dominating set of $K_{1,r} \square W_s$ with cardinality s. As $s > r+3$ and $\gamma_r^c(K_{1,r} \square W_s)$ must be minimum, $\gamma_r^c(K_{1,r} \square W_s) = r+3$.

Theorem 2.9 Let $r, s \geq 4$. Then

$$\gamma_r^c(W_r \square W_s) = \begin{cases} min\{r,s\} \ if \ both \ r\ ,s \ is \ odd \ or \ even \\ \quad s \qquad if \ r \ is \ odd \ and \ s \ is \ even \\ \quad r \qquad if \ r \ is \ even \ and \ s \ is \ odd \end{cases}.$$

Proof. Let $V(W_r) = \{u_0, u_1, u_2, \ldots, u_{r-1}\}$ and $V(W_s) = \{v_0, v_1, v_2, \ldots, v_{s-1}\}$ where u_0 and v_0 are the full degree vertices of W_r and W_s. Then $V(W_r \square W_s) = \{(u_i, v_j)/0 \leq i \leq r-1, 0 \leq j \leq s-1\}$. Clearly, $W_r \square W_s$ contains r rows and s columns and so, $|V(W_r \square W_s)| = rs$.

Case (i): Both r, s is odd or even

If r, s is odd, then $\chi(W_r \square W_s) = max\{\chi(W_r), \chi(W_s)\} = max\{3,3\} = 3$ and if both r, s is even, then $\chi(W_r \square W_s) = max\{\chi(W_r), \chi(W_s)\} = max\{4,4\} = 4$.

Subcase (i): r, s is odd

Let $D_1 = \{(u_i, v_0)/0 \leq i \leq r-1\}$ when $r < s$ and all the elements of D_1 belongs to the entire first column of $W_r \square W_s$. Also, let $D_2 = \{(u_0, v_j)/0 \leq j \leq s-1\}$ when $r > s$ and all the elements of D_2 belongs to the entire first row of $W_r \square W_s$. Then, $|D_1| = r = min\{r,s\}$ and $|D_2| = s = min\{r,s\}$. Clearly, D_1 and D_2 are restrained dominating sets of $W_r \square W_s$ since D_1 contains first element of each row which dominates all the remaining elements of that particular row and D_2 contains first element of each column which is adjacent to all the remaining elements of that particular column. Also, every vertex in $V \backslash D_1$ and $V \backslash D_2$ is adjacent to another vertex in $V \backslash D_1$ and $V \backslash D_2$. Furthermore, $\chi(\langle D_1 \rangle) = \chi(\langle D_2 \rangle) = 3 = \chi(W_r \square W_s)$, as both D_1 and D_2 contains even cycle together with a vertex adjacent to all the vertices of the cycle. Thus, D_1 and D_2 are chromatic restrained dominating sets of $W_r \square W_s$ and $\gamma_r^c(W_r \square W_s) \leq min\{r,s\}$. Since any chromatic restrained dominating set must contain all the vertices which belongs to either the first row or first column according as r or s is minimum, $\gamma_r^c(W_r \square W_s) \geq min\{r,s\}$. Therefore, $\gamma_r^c(W_r \square W_s) = min\{r,s\}$.

Subcase (ii): r, s is even

Consider D_1 and D_2. Then D_1 and D_2 are restrained dominating sets of $W_r \square W_s$. Also, $\chi(\langle D_1 \rangle) =$

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$\chi(\langle D_2\rangle) = 4 = \chi(W_r \square W_s)$, as both D_1 and D_2 contains an odd cycle together with a vertex adjacent to all the vertices of the cycle. Therefore, D_1 and D_2 are chromatic restrained dominating sets of $W_r \square W_s$ and $\gamma_r^c(W_r \square W_s) \leq min\{r, s\}$. Since any chromatic restrained dominating set of $W_r \square W_s$ must contain all the vertices which belongs to either the first row or first column according as either r or s is minimum, $\gamma_r^c(W_r \square W_s) \geq min\{r, s\}$. Therefore, $\gamma_r^c(W_r \square W_s) = min\{r, s\}$.

Case (ii): r is odd and s is even.

Then $\chi(W_r) = 3$ and $\chi(W_s) = 4$. This implies that, $\chi(W_r \square W_s) = max\{\chi(W_r), \chi(W_s)\} = max\{3,4\} = 4$. Let $D_3 = \{(u_i, v_0)/0 \leq i \leq r-1\}$ where all the elements of D_3 belongs to the first column of $W_r \square W_s$. Since D_3 contains first element of each row, D_3 is a restrained dominating set of $W_r \square W_s$. But $\chi(\langle D_3\rangle) = 3 \neq \chi(W_r \square W_s)$. Thus, D_3 is not a chromatic restrained dominating set of $W_r \square W_s$. Let $D_4 = \{(u_0, v_j)/0 \leq j \leq s-1\}$ where all the elements of D_4 belongs to the entire first row of $W_r \square W_s$. Then, D_4 is a restrained dominating set and $\chi(\langle D_4\rangle) = 4 = \chi(W_r \square W_s)$, since $\langle D_4\rangle$ contains an odd cycle on $s-1$ vertices together with a vertex adjacent to all the $s-1$ vertices of the odd cycle. Therefore, D_4 is a chromatic restrained dominating set of $W_r \square W_s$ and $\gamma_r^c(W_r \square W_s) \leq |D_4| = s$. Since $\chi(W_r \square W_s) = 4$, any chromatic restrained dominating set of $W_r \square W_s$ must contain all the vertices of the smallest odd cycle together with a vertex adjacent to all the vertices of the odd cycle. As any smallest odd cycle of $W_r \square W_s$ contains $s-1$ vertices, $\gamma_r^c(W_r \square W_s) \geq s$. Therefore, $\gamma_r^c(W_r \square W_s) = s$.

Similarly, if r is even and s is odd, then $\gamma_r^c(W_r \square W_s) = r$.

3. CONCLUSION

In this paper, we have determined the chromatic restrained domination number for the cartesian product of certain standard graphs with complete bipartite and wheel graphs. An encouraging direction for future research is to analyse the bounds on the chromatic restrained domination number for the cartesian product of two graphs and to characterise the extremal graphs that represents the upper and the lower limit of the chromatic restrained domination number.

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