Journal of Computational Analysis and Applications **VOL. 33, NO. 8, 2024**
10.48047/jocaaa.2024.33.08.327

Comparative Growth in Terms of Maximum Term of Iterated Entire Functions

Somnath Mandal

Department of Mathematics, Sree Chaitanya College, Habra, North 24 Parganas, West Bengal, India.

Email: somnathmandal11@gmail.com

Received: 03.09.2024 Revised: 29.10.2024 Accepted: 27.11.2024

ABSTRACT
The maximum term of entire function is widely used by the researchers in the field of complex analysis. We have several results comparing the maximum terms of composition of two entire functions with the maximum terms of corresponding left and right factors. Also the composition of entire functions can be extended into relative iteration of entire functions. In the paper, we consider the maximum term of iterated entire functions and compare the growth of them with their corresponding factors on the basis of slowly changing functions.

Mathematics Subject Classification: 30D20, 30D30, 30D35
Keywords: Iterated Entire Function, Maximum Term, Growth, Order, Lower Order.

1. INTRODUCTION, DEFINITIONS AND NOTATIONS.

Let $f(z)$ be an entire function defined as $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and on $|z|=r$, $M(r,f)=\max\limits_{|z|=r}|f(z)|$ is called the maximum modulus and $\mu(r,f)=\max\limits_{n\geq 0}(a_n r^n)$ is called the maximum term of $f(z)$.

Definition 1. The numbers ρ_f and λ_f are the order and the lower order of $f(z)$ respectively which are defined by

$$\rho_f=\limsup_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r}$$

where $\log^{[k]} x=\log\left(\log^{[k-1]} x\right)$ for $k\geq 1$ and $\log^{[0]} x=x$ (Sato [6]).

Theorem 1. [7] For $0\leq r\leq R$,

$$\mu(r,f)\leq M(r,f)\leq \frac{R}{R-r}\mu(R,f).$$

Taking $R=2r$, for all sufficiently large values of r,

$$\mu(r,f)\leq M(r,f)\leq 2\mu(R,f) \quad \dots\dots\dots(1)$$

Using Theorem 1, we can easily verify that

$$\rho_f=\limsup_{r\to\infty}\frac{\log^{[2]} \mu(r,f)}{\log r} \quad \text{and} \quad \lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]} \mu(r,f)}{\log r}.$$

The notion of L-order and L-lower order for entire functions first introduced by Somasundaram and Thamizharasi [8] where $L\equiv L(r)$ is a positive continuous function increasing slowly i.e. $L(ar)\sim L(r)$ as $r\to\infty$ for every constant a.

10.48047/jocaaa.2024.33.08.327

Definition 2. [8] The L-order and L-lower order of an entire functions f are defined as

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]} \text{ and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}.$$

Definition 2 can be alternatively stated in view of the notion of maximum terms of entire function as follows:

Definition 3. The L-order and L-lower order of an entire functions f are defined as

$$\rho_f^L = \limsup_{r\to\infty} \frac{\log^{[2]} \mu(r,f)}{\log[rL(r)]} \text{ and } \lambda_f^L = \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r,f)}{\log[rL(r)]}.$$

Lahiri and Banerjee [5] defined the iteration of $f(z)$ with respect to $g(z)$ where $f(z)$ and $g(z)$ are entire functions.

$$f_1(z) = f(z)$$

$$f_2(z) = f\big(g(z)\big) = f\big(g_1(z)\big)$$

$$f_3(z) = f\big(g(f(z))\big) = f\big(g_2(z)\big) = f\left(g\big(f_1(z)\big)\right)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$f_n(z) = f\big(g_{n-1}(z)\big) = f\left(g\big(f \dots \big(f(z) or\ g(z)\big) \dots\big)\right)$$

according to n is odd or even and so,

$$g_1(z) = g(z)$$

$$g_2(z) = g\big(f(z)\big) = g\big(f_1(z)\big)$$

$$g_3(z) = g\big(f(g(z))\big) = g\big(f_2(z)\big) = g\left(f\big(g_1(z)\big)\right)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$g_n(z) = g\big(f_{n-1}(z)\big) = g\left(f\big(g_{n-2}(z)\big)\right).$$

It is clear that $f_n(z)$ and $g_n(z)$ are entire functions.

Datta and Mandal [2] proved some theorems on the comparative growths of maximum term of two entire functions with their corresponding left and right factors on the basis of L-order and L-lower order. In [3], Dutta studied some comparative growth of the maximum term of iterated entire functions with that of the maximum term of related functions. In this paper we consider the maximum term of iterated entire functions and compare the growth of them with their corresponding factors on the basis of slowly changing functions. The standard notations and definitions of the theory of entire functions are not explained in this paper as those are available in [4], [9] and [10].

10.48047/jocaaa.2024.33.08.327

2. LEMMAS.

Lemma 1. [1] If f and g are any two entire functions, for all sufficiently large values of r,

$$M\left(\frac{1}{8}M\left(\frac{r}{2},g\right)-|g(0)|,f\right)\le M(r,f\circ g)\le M(M(r,g),f).$$

Lemma 2. If ρ_f and ρ_g are finite, then for any $\varepsilon>0$,

$$\log^{[n-1]}\mu(r,f_n)\le\begin{cases}(\rho_g+\varepsilon)\log M(r,f\circ g)+O(1)\ when\ n\ is\ even\\(\rho_f+\varepsilon)\log M(r,g\circ f)+O(1)\ when\ n\ is\ odd\end{cases}$$

for all sufficiently large values of r.

Proof. In view of (1) and by Lemma 1 it follows that for all sufficiently large values of r,

$$\mu(r,f_n)\le M(r,f_n)=M\big(r,f(g_{n-1})\big)$$

$$\le M(M(r,g_{n-1}),f)$$

i.e.

$$\log\mu(r,f_n)\le\log M(M(r,g_{n-1}),f)$$

$$\le[M(r,g_{n-1})]^{\rho_f+\varepsilon}$$

So,

$$\log^{[2]}\mu(r,f_n)\le(\rho_f+\varepsilon)\log[M(r,g_{n-1})]$$

$$=(\rho_f+\varepsilon)\log\big[M\big(r,g(f_{n-1})\big)\big]$$

$$\le(\rho_f+\varepsilon)\log[M(M(r,f_{n-2}),g)]$$

$$\le(\rho_f+\varepsilon)[M(r,f_{n-2})]^{\rho_g+\varepsilon}$$

Now,

$$\log^{[3]}\mu(r,f_n)\le(\rho_g+\varepsilon)\log[M(r,f_{n-2})]+O(1)$$

$$=(\rho_g+\varepsilon)\log\big[M\big(r,f(g_{n-3})\big)\big]+O(1)$$

$$\le(\rho_g+\varepsilon)\log[M(M(r,g_{n-3}),f)]$$

$$\le(\rho_g+\varepsilon)[M(r,g_{n-3})]^{\rho_f+\varepsilon}$$

So,

$$\log^{[4]}\mu(r,f_n)\le(\rho_f+\varepsilon)\log[M(r,g_{n-3})]+O(1)$$

…. …. …. ….

…. …. …. ….

Therefore,

$$\log^{[n-1]}\mu(r,f_n)\le\begin{cases}(\rho_g+\varepsilon)\log M(r,f\circ g)+O(1)\ when\ n\ is\ even\\(\rho_f+\varepsilon)\log M(r,g\circ f)+O(1)\ when\ n\ is\ odd\end{cases}$$

10.48047/jocaaa.2024.33.08.327

Lemma 3. If λ_f and λ_g are non-zero finite, then for $\varepsilon > 0$ and for all sufficiently large values of r,

$$\log^{[n-1]} \mu(r, f_n) > \begin{cases} (\lambda_g - \varepsilon) \log M(r, f \circ g) + O(1) \; when \; n \; is \; even \\ (\lambda_f - \varepsilon) \log M(r, g \circ f) + O(1) \; when \; n \; is \; odd\,. \end{cases}$$

Proof. Let us choose a positive number ε be such that $< \min\{\lambda_f,\ \lambda_g\}$. Now from {[7], p-113} for all sufficiently large values of r, we have

$$\mu(r, f \circ g) > e^{[M(r,g)]^{\lambda_f - \varepsilon}}.$$

So, $$\log \mu(r, f \circ g) > [M(r, g)]^{\lambda_f - \varepsilon} \; \ldots \ldots \ldots (2)$$

Now, $$\log \mu(r, f_n) = \log \mu\big(r, f(g_{n-1})\big)$$

$$> [M(r, g_{n-1})]^{\lambda_f - \varepsilon} \quad \text{using (2)}$$

$$\geq [\mu(r, g_{n-1})]^{\lambda_f - \varepsilon} \quad \text{using (1)}$$

Taking logarithm on both sides, we have

$$\log^{[2]} \mu(r, f_n) \geq (\lambda_f - \varepsilon) \log \mu(r, g_{n-1})$$

$$= (\lambda_f - \varepsilon) \log \mu\big(r, g(f_{n-2})\big)$$

$$> (\lambda_f - \varepsilon)[M(r, f_{n-2})]^{\lambda_g - \varepsilon} \qquad \text{using (2)}$$

So, $$\log^{[3]} \mu(r, f_n) > (\lambda_g - \varepsilon) \log M(r, f_{n-2}) + O(1)$$

$$\geq (\lambda_g - \varepsilon) \log \mu\big(r, f(g_{n-3})\big) + O(1)$$

$$> (\lambda_g - \varepsilon)[M(r, g_{n-3})]^{\lambda_f - \varepsilon}$$

i.e., $$\log^{[4]} \mu(r, f_n) > (\lambda_f - \varepsilon) \log M(r, g_{n-3}) + O(1)$$

$\ldots.$ $\ldots.$ $\ldots.$ $\ldots.$

$\ldots.$ $\ldots.$ $\ldots.$ $\ldots.$

Therefore,

$$\log^{[n-1]} \mu(r, f_n) > \begin{cases} (\lambda_g - \varepsilon) \log M(r, f \circ g) + O(1) \; when \; n \; is \; even \\ (\lambda_f - \varepsilon) \log M(r, g \circ f) + O(1) \; when \; n \; is \; odd\,. \end{cases}$$

Using equation (1) the result of Lemma 3 can be written as follows:

$$\log^{[n-1]} \mu(r, f_n) \geq \begin{cases} (\lambda_g - \varepsilon) \log \mu(r, f \circ g) + O(1) \; when \; n \; is \; even \\ (\lambda_f - \varepsilon) \log \mu(r, g \circ f) + O(1) \; when \; n \; is \; odd\,. \end{cases}$$

10.48047/jocaaa.2024.33.08.327

3. THEOREMS.

Theorem 2. Let f and g be two entire functions such that $0 < \rho_{f\circ g}^{L} < \infty$, $0 < \rho_{g\circ f}^{L} < \infty$ and $0 < \rho_{g}^{L} < \infty$. Then for any integer A,

(i) when n is even

$$\liminf_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \le \frac{\rho_{f\circ g}^{L}}{A\rho_{g}^{L}} \le \limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)}$$

(ii) when n is odd

$$\liminf_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \le \frac{\rho_{g\circ f}^{L}}{A\rho_{g}^{L}} \le \limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)}.$$

Proof. From the definition of L-order we have for arbitrary positive ε and for all large values of r,

$$\log^{[2]} M(r, f\circ g) \le \left(\rho_{f\circ g}^{L} + \varepsilon\right) \log[rL(r)] \quad \dots \quad \dots \quad \dots (3)$$

and for a sequence of values of r tending to infinity

$$\log^{[2]} \mu(r^A, g) \ge A\left(\rho_{g}^{L} - \varepsilon\right) \log[rL(r)] \quad \dots \quad \dots \quad \dots (4)$$

From Lemma 1, we have when n is even

$$\log^{[n]} \mu(r, f_n) \le \log^{[2]} M(r, f\circ g) + O(1)$$

and when n is odd

$$\log^{[n]} \mu(r, f_n) \le \log^{[2]} M(r, g\circ f) + O(1)$$

So using equation (3) we write when n is even

$$\log^{[n]} \mu(r, f_n) \le \left(\rho_{f\circ g}^{L} + \varepsilon\right) \log[rL(r)] \quad \dots \quad \dots \quad \dots (5)$$

and when n is odd

$$\log^{[n]} \mu(r, f_n) \le \left(\rho_{g\circ f}^{L} + \varepsilon\right) \log[rL(r)] \quad \dots \quad \dots \quad \dots (6)$$

Using (4) and (5), for a sequence of values of r tending to infinity we have when n is even

$$\frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \le \frac{\rho_{f\circ g}^{L} + \varepsilon}{A\left(\rho_{g}^{L} - \varepsilon\right)}$$

Since $\varepsilon(>0)$ is arbitrary, we have

$$\liminf_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \le \frac{\rho_{f\circ g}^{L}}{A\rho_{g}^{L}} \quad \dots \quad \dots \quad \dots (7)$$

Also from (4) and (6), for a sequence of values of r tending to infinity we have when n is odd

$$\frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \le \frac{\rho_{g\circ f}^{L} + \varepsilon}{A\left(\rho_{g}^{L} - \varepsilon\right)}$$

As $\varepsilon(>0)$ is arbitrary, we obtain

$$\liminf_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \le \frac{\rho_{g\circ f}^{L}}{A\rho_{g}^{L}} \quad \dots \quad \dots \quad \dots (8)$$

Again for a sequence of values of r tending to infinity,

10.48047/jocaaa.2024.33.08.327

$$\log^{[2]} M(r, f \circ g) \geq \left(\rho^L_{f\circ g} - \varepsilon\right) \log[rL(r)] \quad \ldots \quad \ldots \quad \ldots (9)$$

and for all sufficiently large values of r,

$$\log^{[2]} \mu(r^A, g) \leq A\left(\rho^L_g + \varepsilon\right) \log[rL(r)] \quad \ldots \quad \ldots \quad \ldots (10)$$

From Lemma 3, we have when n is even

$$\log^{[n]} \mu(r, f_n) \geq \log^{[2]} \mu(r, f \circ g) + O(1)$$

and when n is odd

$$\log^{[n]} \mu(r, f_n) \geq \log^{[2]} \mu(r, g \circ f) + O(1)$$

So by equation (9) we can write for a sequence of values of r tending to infinity, when n is even

$$\log^{[n]} \mu(r, f_n) \geq \left(\rho^L_{f\circ g} - \varepsilon\right) \log[rL(r)] \quad \ldots \quad \ldots \quad \ldots (11)$$

and when n is odd

$$\log^{[n]} \mu(r, f_n) \geq \left(\rho^L_{g\circ f} - \varepsilon\right) \log[rL(r)] \quad \ldots \quad \ldots \quad \ldots (12)$$

From (10) and (11), for a sequence of values of r tending to infinity we have when n is even

$$\frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \geq \frac{\rho^L_{f\circ g} - \varepsilon}{A\left(\rho^L_g + \varepsilon\right)}$$

Since $\varepsilon(> 0)$ is arbitrary, we have

$$\limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \geq \frac{\rho^L_{f\circ g}}{A\rho^L_g} \quad \ldots \quad \ldots \quad \ldots (13)$$

Also from (10) and (12), for a sequence of values of r tending to infinity we have when n is odd

$$\frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \geq \frac{\rho^L_{g\circ f} - \varepsilon}{A\left(\rho^L_g + \varepsilon\right)}$$

As $\varepsilon(> 0)$ is arbitrary, we obtain

$$\limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \geq \frac{\rho^L_{g\circ f}}{A\rho^L_g} \quad \ldots \quad \ldots \quad \ldots (14)$$

Thus we have part (i) from equations (7) and (13) when n is even and part (ii) from equations (8) and (14) when n is odd.

Theorem 3. Let f and g be two entire functions such that $\rho^L_{f\circ g}$, $\rho^L_{g\circ f}$, $\lambda^L_{f\circ g}$, $\lambda^L_{g\circ f}$, ρ^L_g and λ^L_g are non-zero finite. Then for any positive integer A,

(i) when n is even

$$\frac{\lambda^L_{f\circ g}}{A\rho^L_g} \leq \liminf_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \leq \frac{\lambda^L_{f\circ g}}{A\lambda^L_g} \leq \limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \leq \frac{\rho^L_{f\circ g}}{A\lambda^L_g}$$

(ii) when n is odd

$$\frac{\lambda^L_{g\circ f}}{A\rho^L_g} \leq \liminf_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \leq \frac{\lambda^L_{g\circ f}}{A\lambda^L_g} \leq \limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} \leq \frac{\rho^L_{g\circ f}}{A\lambda^L_g}.$$

10.48047/jocaaa.2024.33.08.327

Proof. From the definition of L-lower order we have for arbitrary positive ε and for all large values of r,

$$\log^{[2]}\mu(r,f\circ g)\geq\left(\lambda_{f\circ g}^{L}-\varepsilon\right)\log[rL(r)].$$

So using Lemma 3 we can write when n is even

$$\log^{[n]}\mu(r,f_n)\geq\left(\lambda_{f\circ g}^{L}-\varepsilon\right)\log[rL(r)]\quad\ldots\quad\ldots\quad\ldots(15)$$

and when n is odd

$$\log^{[n]}\mu(r,f_n)\geq\left(\lambda_{g\circ f}^{L}-\varepsilon\right)\log[rL(r)]\quad\ldots\quad\ldots\quad\ldots(16)$$

Now from (10) and (15) it follows for all large values of r, when n is even

$$\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)}\geq\frac{\lambda_{f\circ g}^{L}-\varepsilon}{A\left(\rho_g^L+\varepsilon\right)}$$

Since $\varepsilon(>0)$ is arbitrary, we have

$$\liminf_{r\to\infty}\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)}\geq\frac{\lambda_{f\circ g}^{L}}{A\rho_g^L}\quad\ldots\quad\ldots\quad\ldots(17)$$

Also from (10) and (16) it follows for all large values of r and when n is odd

$$\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)}\geq\frac{\lambda_{g\circ f}^{L}-\varepsilon}{A\left(\rho_g^L+\varepsilon\right)}$$

Since $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)}\geq\frac{\lambda_{g\circ f}^{L}}{A\rho_g^L}\quad\ldots\quad\ldots\quad\ldots(18)$$

Again for a sequence of values of r tending to infinity,

$$\log^{[2]}\mu(r,f\circ g)\leq\left(\lambda_{f\circ g}^{L}+\varepsilon\right)\log[rL(r)]$$

and by Lemma 3 we write when n is even

$$\log^{[n]}\mu(r,f_n)\leq\left(\lambda_{f\circ g}^{L}+\varepsilon\right)\log[rL(r)]\quad\ldots\quad\ldots\quad\ldots(19)$$

and when n is odd

$$\log^{[n]}\mu(r,f_n)\leq\left(\lambda_{g\circ f}^{L}+\varepsilon\right)\log[rL(r)]\quad\ldots\quad\ldots\quad\ldots(20)$$

Also for all sufficiently large values of r,

$$\log^{[2]}\mu(r^A,g)\geq A\left(\lambda_g^L-\varepsilon\right)\log[rL(r)]\quad\ldots\quad\ldots\quad\ldots(21)$$

So combining (19) and (21) we get for a sequence of values of r tending to infinity and when n is even

$$\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)}\leq\frac{\lambda_{f\circ g}^{L}+\varepsilon}{A\left(\lambda_g^L-\varepsilon\right)}$$

Since $\varepsilon(>0)$ is arbitrary, we have

$$\liminf_{r\to\infty}\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)}\leq\frac{\lambda_{f\circ g}^{L}}{A\lambda_g^L}\quad\ldots\quad\ldots\quad\ldots(22)$$

Also from (20) and (21) it follows for a sequence of values of r tending to infinity and when n is odd

10.48047/jocaaa.2024.33.08.327

$$\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)} \leq \frac{\lambda_{g\circ f}^L + \varepsilon}{A(\lambda_g^L - \varepsilon)}$$

Since $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)} \leq \frac{\lambda_{g\circ f}^L}{A\lambda_g^L} \quad \ldots\ \ldots\ \ldots (23)$$

For a sequence of values of r tending to infinity we also have

$$\log^{[2]}\mu(r^A,g) \leq A(\lambda_g^L + \varepsilon)\log[rL(r)] \quad \ldots\ \ldots\ \ldots (24)$$

Now from (15) and (24) we obtain for a sequence of values of r tending to infinity and when n is even

$$\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)} \geq \frac{\lambda_{f\circ g}^L - \varepsilon}{A(\lambda_g^L + \varepsilon)}$$

Since $\varepsilon(>0)$ is arbitrary, we have

$$\limsup_{r\to\infty}\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)} \geq \frac{\lambda_{f\circ g}^L}{A\lambda_g^L} \quad \ldots\ \ldots\ \ldots (25)$$

Also from (16) and (24) it follows for a sequence of values of r tending to infinity and when n is odd

$$\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)} \geq \frac{\lambda_{g\circ f}^L - \varepsilon}{A(\lambda_g^L + \varepsilon)}$$

Since $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)} \geq \frac{\lambda_{g\circ f}^L}{A\lambda_g^L} \quad \ldots\ \ldots\ \ldots (26)$$

Again from (5) and (21) we obtain for all large values of r and when n is even

$$\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)} \leq \frac{\rho_{f\circ g}^L + \varepsilon}{A(\lambda_g^L - \varepsilon)}$$

Since $\varepsilon(>0)$ is arbitrary, we have

$$\limsup_{r\to\infty}\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)} \leq \frac{\rho_{f\circ g}^L}{A\lambda_g^L} \quad \ldots\ \ldots\ \ldots (27)$$

Also from (6) and (21) it follows for all large values of r and when n is odd

$$\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)} \leq \frac{\rho_{g\circ f}^L + \varepsilon}{A(\lambda_g^L - \varepsilon)}$$

Since $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[n]}\mu(r,f_n)}{\log^{[2]}\mu(r^A,g)} \leq \frac{\rho_{g\circ f}^L}{A\lambda_g^L} \quad \ldots\ \ldots\ \ldots (28)$$

Therefore part (i) follows from (17), (22), (25) and (27). Similarly part (ii) follows from (18), (23), (26) and (28).

10.48047/jocaaa.2024.33.08.327

Theorem 4. If f and g be two entire functions with $\rho_{f\circ g}^{L} = \infty$ and $\rho_{g}^{L} < \infty$, then for every positive number A and when n is even

$$\limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} = \infty.$$

Proof. Let us assume that the conclusion of the theorem does not hold. Then there exist a constant $B > 0$ such that for all sufficiently large values of r,

$$\log^{[n]} \mu(r, f_n) \leq B \log^{[2]} \mu(r^A, g) \quad \dots \ \dots \ \dots (29)$$

Again from the definition of ρ_g^L it follows for all large values of r,

$$\log^{[2]} \mu(r^A, g) \leq A\left(\rho_g^L + \varepsilon\right) \log[rL(r)] \quad \dots \ \dots \ \dots (30)$$

So from (29) and (30) we obtain for all sufficiently large values of r,

$$\log^{[n]} \mu(r, f_n) \leq AB\left(\rho_g^L + \varepsilon\right) \log[rL(r)] \quad \dots \ \dots \ \dots (31)$$

Now from (11) and (31) we have when n is even

$$\left(\rho_{f\circ g}^L - \varepsilon\right) \leq AB\left(\rho_g^L + \varepsilon\right) \quad \dots \ \dots \ \dots (32)$$

From (32) it follows that $\rho_{f\circ g}^L < \infty$. So we arrive at a contradiction.

This proves the theorem.

We can prove the following theorem in the line of Theorem 3 when n is odd.

Theorem 5. If f and g be two entire functions with $\rho_{g\circ f}^{L} = \infty$ and $\rho_{g}^{L} < \infty$, then for every positive number A and when n is odd

$$\limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r^A, g)} = \infty.$$

4. REFERENCES.

[1] Clunie, J., The composition of entire and meromorphic functions, Mathematical essays dedicated to A. J. Macintyre, Ohio Univ. Press, (1970), 75-92.
[2] Datta, S. K. and Mandal, S., Composition of entire functions and their maximum terms, Int. J. Contemp. Math. Sciences, Vol.4, No.17 (2009), 833-844.
[3] Dutta, R. K., Growth of iterated entire functions in terms of its maximum term, Acta Univ. Apu., No.30 (2012), 209-219.
[4] Hayman, W. K., Meromorphic Functions, The Clarendon Press, Oxford, 1964.
[5] Lahiri, B. K. and Banerjee, D., Relative fix points of entire functions, J. Indian Acad. Math., Vol.19, No.1 (1997), 87-97.
[6] Sato, D., On the rate of growth of entire functions of fast growth, Bull. Amer. Math. Soc., Vol.69 (1963), 411-414.
[7] Singh, A. P., On maximum term of composition of entire functions, Proc. Nat. Acad. Sci. India, Vol.59(A), (1989), 103-115.
[8] Somasundaram, D. and Thamizharasi, R., A note on the entire functions of L-bounded index and L-type, Indian J. Pure Appl. Math., Vol.19, No.3(March 1988), 284-293.
[9] Valiron, G., Lectures on the general theory of integral functions, Chelsea Publishing Company, 1949.
[10] Yang, C. C. and Yi, H. X., Uniqueness theory of meromorphic functions, Kluwer Academic Publishers and Science Press, Beijing, 2003.