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The Expanding Model of The Conformally Flat Non-Static Spherically Symmetric Perfect Fluid Distributions

Manjusha (Hajare) Borkar

71manjuborkar@gmail.com

Assistant Professor, Department of Mathematics, Kamla Nehru Mahavidyalaya, Nagpur.

Abstract : The explicit expressions for pressure, density expansion, rotation, sheer and non vanishing component of flow vector have been obtained. In this this note ,we have discussed the various geometrical and physical properties of spherically symmetric metric space time obtained by **Roy and Bali [5]** by considering the triples of orthogonal unit vectorsα_i, β_i. Here we connect all the physical quantities obtained by **Roy and Bali [5]** with mathematical quantities ρ^a in **Borkar and Hajare [3].** Further it is noticed that the model is expanding, rotating, shearing but non geodetic.

Introduction:

Takeno (1966) [1] developed the theory of SSST on the group of invariant quantities containing two orthogonal unit vectors and deduced its geometrical and physical properties. The investigation into spherically symmetric space time is one of the important problem in general theory of relativity, because the many space times dealt with in relativistic theory are spherical symmetric in nature. Many authors have investigated the physical, geometrical and mathematical properties of spherical symmetric space time. For the sake of convenience we recall the definition from [1],

Definition: A space-time V_4 with metric tensor g_{ij} is said to be spherically symmetric if the following conditions are satisfied.

1. The curvature tensor V_4 satisfies the equation,

$$K_{ijlm} = -\rho^1 \alpha_{[i}\alpha_{[l}\beta_{j]}\beta_{m]} - \rho^2 g_{[i[l}\alpha_{j]}\alpha_{m]} + \rho^3 g_{[i[l}\beta_{j]}\beta_{m]} + \rho^4 g_{[i[l}g_{j]m]} + \rho^5 g_{[i[l}Q_{j]m]} \tag{1}$$

Whereα_i , β_i are mutually orthogonal unit vector satisfying

$$\alpha_{j|i} = \sigma\alpha_i\beta_j + \kappa(g_{ij} + \alpha_i\alpha_j - \beta_i\beta_j) + \overline{\sigma}\beta_i\beta_j, \tag{2}$$

$$\beta_{j|i} = \overline{\sigma}\beta_i\alpha_j + \overline{\kappa}(g_{ij} + \alpha_i\alpha_j - \beta_i\beta_j) + \sigma\alpha_i\alpha_j , \tag{3}$$

$$\alpha_i\alpha^i = -1, \quad \beta_i\beta^i = 1, \quad \alpha_i\beta^i = 0 \tag{4}$$

$$Q_{ij} = \alpha_{(i}\beta_{j)}.$$

Here $\alpha_{j|i}$ is the usual Riemannian covariant derivative of α_j and the notation ( ) and [ ] stand for usual symmetric and antisymmetric relations respectively. These equations determine the scalars ρ^a, a = 1,2 …, 5,σ,$\overline{\sigma}$,κ, $\overline{\kappa}$.

2. One of the five scalarsρ^1,…, ρ^4 and the scalars curvature $K = K_{ij}^{\cdot\cdot ji}$ is such that its gradient is a linear combination of $\alpha_i$and β_i.

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3. $\rho^4, \kappa \ and \ \overline{\kappa}$ satisfy
   $\rho^4 - 2(\kappa^2 - \overline{\kappa}^2) \neq 0$.

The signature of the metric tensor is given by (-,-,-, +).

The quantities α_i,β_i,ρ^a,a = 1,…., 5,σ,$\overline{\sigma}$,κ, $\overline{\kappa}$ are called the analytical invariants (or characteristic quantities) of the spherically symmetric space time.

The analytical invariants of orthogonal spherically symmetric space time have been studied by **Borkar and Hajare [3]** by assuming the various triples of orthogonal unit vectorsα_i, β_i. In this note an attempt has been made to discuss the various geometrical and physical properties of the model obtained by **Roy and Bali [5]** which is known as conformally flat non-static spherically symmetric perfect fluid distribution in general relativity purely on mathematical ground through the various triples of orthogonal unit vectors α_i,β_i. These triples are playing an important role in analyzing the spacetime and in the discussion of physical and geometrical properties of space time. In our discussion we studied it through the theory of triples of orthogonal unit vectors developed by **Borkar and Hajare [3].**

We consider the metric of **Roy and Bali [5],**

$$ds^2 = \frac{16m^2}{(\phi(\xi) - nt)^2}(dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2 - dt^2). \tag{5}$$

The pressure and density for model (5) are given by

$$8\pi p = \frac{(\phi - nt)^2}{16m^2}\left[(1 - 4m\xi)\left(\frac{2\phi''}{(\phi - nt)} - \frac{3\phi'^2}{(\phi - nt)^2}\right) - \frac{12m\phi'}{\phi - nt} - \frac{6n(2mt+1)\phi'}{(\phi - nt)^2} - \frac{3n^2}{(\phi - nt)^2}\right] - \Lambda \tag{6}$$

and

$$8\pi\epsilon = \frac{(\phi - nt)^2}{16m^2}\left[\frac{3(1-4m\xi)\phi'^2}{(\phi - nt)^2} + \frac{12m\phi'}{\phi - nt} + \frac{6n(2mt+1)\phi'}{(\phi - nt)^2} + \frac{3n^2}{(\phi - nt)^2}\right] + \Lambda \tag{7}$$

Where $\xi = mr^2 - mt^2 - t$ and the prime indicates differentiation with respect to its arguments.

The non-vanishing component of flow vector are given by,

$$v_1 = \frac{8m^2 r}{(\phi - nt)\sqrt{1-4m\xi}} \quad \text{and} \quad v_4 = \frac{-4m(2mt+1)}{(\phi - nt)\sqrt{1-4m\xi}} \tag{8}$$

It satisfy the reality condition

i) $(\epsilon + p) > 0$
ii) $(\epsilon + 3p) > 0$

The non-vanishing component of the vector $\dot{v_i} = v_{i|j}v^j$ are,

$$\dot{v}_1 = \frac{2mrn(2mt+1)}{(\phi - nt)(1-4m\xi)} and \dot{v}_4 = \frac{-n(2mt+1)^2}{(\phi - nt)(1-4m\xi)} \tag{9}$$

The expression for shear σ_{ij}, rotation ω_{ij} and expansion θ', calculated for the flow vector v^i

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are given by,

$$\sigma_{11} = \frac{8mn(2mt+1)\{1-4m\xi+4m^2r^2\}}{(\phi-nt)^2(1-4m\xi)^{3/2}} \tag{10}$$

$$\sigma_{22} = \frac{8mnr^2(2mt+1)}{(\phi-nt)^2\sqrt{1-4m\xi}} \tag{11}$$

$$\sigma_{33} = \frac{8mnr^2sin^2\theta(2mt+1)}{(\phi-nt)^2\sqrt{1-4m\xi}} \tag{12}$$

$$\sigma_{44} = \frac{4mn(2mt+1)\{2(2mt+1)^2-(1-4m\xi)\}}{(\phi-nt)^2(1-4m\xi)^{3/2}} \tag{13}$$

$$\sigma_{14} = -\frac{4m^2rn\{4(2mt+1)^2+1-4m\xi\}}{(\phi-nt)^2(1-4m\xi)^{3/2}} \tag{14}$$

$$\omega_{14} = \frac{4m^2rn}{(\phi-nt)^2\sqrt{1-4m\xi}} \tag{15}$$

And

$$\theta' = \frac{3(\phi-nt)}{2\sqrt{1-4m\xi}} + \frac{3\phi'\sqrt{1-4m\xi}}{4m} - \frac{3n(2mt+1)}{4m\sqrt{1-4m\xi}} \tag{16}$$

The other components of rotation tensor ω_{ij} and shear tensor σ_{ij} vanish.

In this note we connect all the physical quantities of model (5) obtained by **Roy and Bali [5]** with the mathematical quantity ρ^a given in **Borkar and Hajare [3]** on the basis of the triples of orthogonal unit vectors $\alpha_i$and β_i and discuss geometrical and physical properties of space time.

II. Characteristic Quantities:

We are studying the space time (5) in relation to the triples corresponding to the triples $(\alpha_1, \alpha_3, \beta_2)$.

For $(\alpha_1, \alpha_3, \beta_2)$:
We deduced the value of ρ^a, a = 1,2,….,5 from the component of curvature tensor K_{ijlm} in view of the definition of SSST in [3] for the line element (5).

$$\rho^2 = -4K_{4143}\frac{(1+r^2sin^2\theta)(\phi-nt)^4}{16^2m^4r^2sin^2\theta} \tag{17}$$

$$\rho^1+\rho^2 = 4K_{2123}\frac{(1+r^2sin^2\theta)(\phi-nt)^4}{16^2m^4r^4sin^2\theta} \tag{18}$$

$$\rho^2+2\rho^4 = 4K_{1313}\frac{(\phi-nt)^4}{16^2m^4r^2sin^2\theta} \tag{19}$$

$$\rho^3+2\rho^4 = -4K_{2424}\frac{(\phi-nt)^4}{16^2m^4r^2} \tag{20}$$

$$\rho^5\ \frac{16^2m^4r^2sin^2\theta}{(\phi-nt)^4}\sqrt{\frac{-1}{1+r^2sin^2\theta}} = 8\left[K_{1213}\ ,\ \frac{K_{3132}}{r^2sin^2\theta}, -K_{4142}\ , -K_{4243}\right] \tag{21}$$

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III. Some Physical properties:

For the space time (5) we calculate the physical quantities for the triple $(\alpha_1, \alpha_3, \beta_2)$ using mathematical values $\rho^a, a = 1, \ldots, 5$ as follows,

The equation for pressure and density for the line element (5) from equation (6) and (7), we get,

$$8\pi p = \frac{r\,sin\theta}{2}\sqrt{\frac{\rho^2 + 2\rho^4}{K_{1313}}}\left[(1-4m\xi)\left(\frac{2\phi''}{(\phi-nt)} - \frac{3\phi'^2}{(\phi-nt)^2}\right) - \frac{12m\phi'}{\phi-nt} - \frac{6n(2mt+1)\phi'}{(\phi-nt)^2} - \frac{3n^2}{(\phi-nt)^2}\right] - \Lambda \quad (22)$$

And

$$8\pi\epsilon = \frac{r\,sin\theta}{2}\sqrt{\frac{\rho^2 + 2\rho^4}{K_{1313}}}\left[\frac{3(1-4m\xi)\phi'^2}{(\phi-nt)^2} + \frac{12m\phi'}{\phi-nt} + \frac{6n(2mt+1)\phi'}{(\phi-nt)^2} + \frac{3n^2}{(\phi-nt)^2}\right] + \Lambda \quad (23)$$

respectively.

The non-vanishing component of flow vector from equation (8) and (9), we get,

$$\nu_1 = \left[\frac{4K_{1313}}{(\rho^2 + 2\rho^4)r^2sin^2\theta}\right]^{\frac{1}{4}}\frac{2mr}{\sqrt{1-4m\xi}} \quad \text{and} \quad \nu_4 = -\left[\frac{4K_{1313}}{(\rho^2 + 2\rho^4)r^2sin^2\theta}\right]^{\frac{1}{4}}\frac{(2mt+1)}{\sqrt{1-4m\xi}} \quad (24)$$

The non-vanishing component of the vector $\dot{\nu_i} = \nu_{i|j}\nu^j$ from equation (10) and (11)

$$\dot{\nu_1} = \frac{1}{2}\left[\frac{4K_{1313}}{(\rho^2 + 2\rho^4)r^2sin^2\theta}\right]^{\frac{1}{4}}\frac{rn(2mt+1)}{(1-4m\xi)} \quad \text{and} \quad \dot{\nu_4} = -\frac{1}{4m}\left[\frac{4K_{1313}}{(\rho^2 + 2\rho^4)r^2sin^2\theta}\right]^{\frac{1}{4}}\frac{n(2mt+1)^2}{(1-4m\xi)} \quad (25)$$

(Therefore the flow is non-geodetic in general).

The expression for shear σ_{ij}, rotation ω_{ij} and expansionθ' are given by,

And

$$\sigma_{11} = \frac{n(2mt+1)\{1-4m\xi + 4m^2r^2\}}{mr\sin\theta(1-4m\xi)^{3/2}}\sqrt{\frac{K_{1313}}{(\rho^2 + 2\rho^4)}} \quad (26)$$

$$\sigma_{22} = \frac{nr(2mt+1)}{m\sin\theta}\sqrt{\frac{K_{1313}}{(\rho^2 + 2\rho^4)(1-4m\xi)}} \quad (27)$$

$$\sigma_{33} = \frac{nrsin\theta(2mt+1)}{m}\sqrt{\frac{K_{1313}}{(\rho^2 + 2\rho^4)(1-4m\xi)}} \quad (28)$$

$$\sigma_{44} = \frac{n(2mt+1)\{2(2mt+1)^2 - (1-4m\xi)\}}{2mr\sin\theta(1-4m\xi)^{3/2}}\sqrt{\frac{K_{1313}}{(\rho^2 + 2\rho^4)}} \quad (29)$$

$$\sigma_{14} = -\frac{n\{4(2mt+1)^2 + 1-4m\xi\}}{2sin\theta(1-4m\xi)^{3/2}}\sqrt{\frac{K_{1313}}{(\rho^2 + 2\rho^4)}} \quad (30)$$

$$\omega_{14} = \frac{n}{2\sin\theta}\sqrt{\frac{K_{1313}}{(\rho^2 + 2\rho^4)(1-4m\xi)}} \quad (31)$$

and

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$$\theta'=\frac{6m}{\sqrt{1-4m\xi}}\left[\frac{(\rho^2+2\rho^4)r^2\sin^2\theta}{4\mathrm{K}_{1313}}\right]^{\frac{1}{4}}+\frac{3\phi'\sqrt{1-4m\xi}}{4m}-\frac{3n(2mt+1)}{4m\sqrt{1-4m\xi}} \tag{32}$$

The other components of rotation tensor ω_{ij} and shear tensor σ_{ij} vanish. Hence the model is expanding, rotating, shearing but non-geodetic in general. The metric (5) cannot represent the dust distribution or disordered radiation unless n = 0. Clearly the component $\dot{\nu_1}$ and $\dot{\nu_4}$ vanish when $n = 0$. We observed that the physical properties for the triple $(\alpha_1, \alpha_3, \beta_4)$ are same as the triple $(\alpha_1, \alpha_3, \beta_2)$

By adopting the similar procedure, we deduced the result for other triples.

We observed that there is relationship among them in the following manner:

For $(\alpha_2, \beta_1, \beta_3) \Leftrightarrow (\alpha_1, \alpha_3, \beta_2)$and $(\alpha_2, \beta_1, \beta_3) \Leftrightarrow (\alpha_1, \alpha_3, \beta_2)$,the scalars$\rho^1, \rho^4, \rho^5$ does not change their values, $\rho^2 \Leftrightarrow \rho^3$.

Our result agreed with the result of Roy and Bali [5].

The non-vanishing components of Ricci tensorK_{ij}, satisfies the relation by [1,3].
For $(\alpha_1, \alpha_3, \beta_2)$:

$$\delta=\frac{(\phi-nt)^2}{16m^2}(\gamma\alpha_1^2-\mathrm{K}_{11})=-\frac{(\phi-nt)^2}{16m^2r^2}(\mathrm{K}_{22}+\lambda\beta_2^2)=\frac{(\phi-nt)^2}{16m^2r^2\sin^2\theta}(\gamma\alpha_3^2-\mathrm{K}_{33})=-\frac{(\phi-nt)^2}{16m^2}\mathrm{K}_{44}.$$

And
For $(\alpha_1, \alpha_3, \beta_2)$:

$$\delta=\frac{(\phi-nt)^2}{16m^2}(\gamma\alpha_1^2-\mathrm{K}_{11})=-\frac{(\phi-nt)^2}{16m^2r^2}\mathrm{K}_{22}=\frac{(\phi-nt)^2}{16m^2r^2\sin^2\theta}(\gamma\alpha_3^2-\mathrm{K}_{33})=-\frac{(\phi-nt)^2}{16m^2}(\mathrm{K}_{44}-\lambda\beta_4^2).$$

$$where\delta=\frac{1}{4}(\rho^2+\rho^3+6\rho^4),\ \gamma=\frac{1}{4}(\rho^1+2\rho^2),\ \lambda=\frac{1}{4}(\rho^1+2\rho^3).$$

The values of the scalars σ,$\bar{\sigma}$,κ, $\bar{\kappa}$ for $(\alpha_1, \alpha_3, \beta_2)$ are,

$$\sigma=-\frac{(\phi-nt)}{8mr}\frac{\partial}{\partial\theta}\left[\log\left(\frac{16m^2r^2\sin^2\theta}{(\phi-nt)^2}\right)\right] \tag{33}$$

$$\bar{\sigma}=-\frac{(\phi-nt)\sqrt{(1+r^2\sin^2\theta)}}{8mr\sin\theta\sqrt{-1}}\frac{\partial}{\partial r}\left[\log\left(\frac{16m^2r^2}{(\phi-nt)^2}\right)\right] \tag{34}$$

$$\kappa=\frac{(\phi-nt)^3\sqrt{(1+r^2\sin^2\theta)}}{128m^3r^3\sin^3\theta}\frac{\partial}{\partial r}\left(\frac{16m^2r^2\sin^2\theta}{(\phi-nt)^2}\right) \tag{35}$$

$$\bar{\kappa}=\frac{(\phi-nt)^3}{128m^3r^3\sin^2\theta}\frac{\partial}{\partial\theta}\left(\frac{16m^2r^2\sin^2\theta}{(\phi-nt)^2}\right) \tag{36}$$

The scalars σ,$\bar{\sigma}$,κ, $\bar{\kappa}$ corresponding to $(\alpha_r, \beta_p, \beta_q)$ takes the values ,$i\,\bar{\sigma}, -i\,\sigma, i\bar{\kappa}, -i\kappa$ of $(\alpha_p, \alpha_q, \beta_r)$ respectively.

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The values of the scalars σ,$\overline{\sigma}$,κ, $\overline{\kappa}$ for $(\alpha_1, \alpha_3, \beta_2)$ are,

$$\sigma = -\frac{(\phi-nt)}{8m\sqrt{-1}}\frac{\partial}{\partial t}\left[\log\left(\frac{16m^2}{(\phi-nt)^2}\right)\right] \tag{37}$$

$$\bar{\sigma} = -\frac{(\phi-nt)\sqrt{(1+r^2sin^2\theta)}}{8mr\sin\theta\sqrt{-1}}\frac{\partial}{\partial r}\left[\log\left(\frac{16m^2}{(\phi-nt)^2}\right)\right] \tag{38}$$

$$\kappa = \frac{(\phi-nt)^3\sqrt{-(1+r^2sin^2\theta)}}{128m^3r^3sin^3\theta}\frac{\partial}{\partial r}\left(\frac{16m^2r^2sin^2\theta}{(\phi-nt)^2}\right) \tag{39}$$

$$\bar{\kappa} = -\frac{(\phi-nt)^3\sqrt{-1}}{128m^3}\frac{\partial}{\partial t}\left(\frac{16m^2}{(\phi-nt)^2}\right) \tag{40}$$

The scalars σ,$\overline{\sigma}$,κ, $\overline{\kappa}$ corresponding to $(\alpha_4, \beta_1, \beta_3)$ takes the values ,$i\,\overline{\sigma}, -i\,\sigma, i\overline{\kappa}, \; -i\kappa$ of $(\alpha_1, \alpha_3, \beta_4)$ respectively.

Conclusion:

Here we discussed the various geometrical and physical properties of spherically symmetric metric space time obtained by Roy and Bali [5] by considering the triples of orthogonal unit vectorsα_i, β_i. The explicit expressions for pressure, density expansion, rotation, sheer and non vanishing component of flow vector have been obtained. Further it is noticed that the model is expanding, rotating, shearing but non geodetic in general.

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