

Generalised convolution identities for multivariable H-functions Nitesh Kumar Sahu

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ABSTRACT:

This work studies multivariable H-functions, which are useful in physics, engineering, and mathematics. We find new and generalized identities for these functions using Mellin transforms and contour integration. These new results help solve difficult problems like integral equations and partial differential equations (PDEs), giving a simple and powerful way to handle complex problems in many areas.

Keywords: H-function of Several Variables, Generalized Hypergeometric Functions, Identities.

1. INTRODUCTION:

The multivariable H–function given in [2] is defined as follows:

$$\begin{aligned}
 H [z_1, \dots, z_r] &\equiv H_{p, q; p_1, q_1; \dots; p_r, q_r}^{0, n; m_1, n_1; \dots; m_r, n_r} \left[\begin{matrix} z_1 & (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_1, p: \\ \vdots & (b_j; \beta'_j, \dots, \beta_j^{(r)})_1, q: \\ z_r & (c'_j; \gamma'_j)_1, p_1; \dots, (c_j^{(r)}; \gamma_j^{(r)})_1, p_r \\ & (d'_j; \delta'_j)_1, q_1; \dots, (d_j^{(r)}; \delta_j^{(r)})_1, q_r \end{matrix} \right] \\
 &= [(1/2\pi\omega)^r] \int_{L_1} \dots \int_{L_r} \phi_1(\xi_1) \dots \phi_r(\xi_r) \psi(\xi_1, \dots, \xi_r) z_1^{\xi_1} \dots z_r^{\xi_r} d\xi_1 \dots d\xi_r \\
 \psi(\xi_1, \dots, \xi_r) &= \frac{\prod_{j=1}^n \Gamma(1 - a_j + \sum_{i=1}^r \alpha_j^{(i)} \xi_i)}{\prod_{j=n+1}^p \Gamma(a_j - \sum_{i=1}^r \alpha_j^{(i)} \xi_i) \prod_{j=1}^q \Gamma(1 - b_j + \sum_{i=1}^r \beta_j^{(i)} \xi_i)} , \\
 \phi_i(\xi_i) &= \frac{\prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i) \prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i)}{\prod_{j=m_i+1}^{q_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} \xi_i) \prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \xi_i)} ,
 \end{aligned}$$

In (1), i in the superscript (i) stands for the number of primes, e.g., $b^{(1)} = b'$, $b^{(2)} = b''$, and so on; and an empty product is interpreted as unity.

Suppose, as usual, that the parameters

$$a_j, j = 1, \dots, p; c_j^{(i)}, j = 1, \dots, p_i;$$

$$b_j, j = 1, \dots, q; d_j^{(i)}, j = 1, \dots, q_i; \forall i \in \{1, \dots, r\}$$

are complex numbers and the associated coefficients

$$\alpha_j^{(i)}, j = 1, \dots, p; \gamma_j^{(i)}, j = 1, \dots, p_i;$$

$$\beta_j^{(i)}, j = 1, \dots, q; \delta_j^{(i)}, j = 1, \dots, q_i; \forall i \in \{1, \dots, r\}$$

positive real numbers such that the left of the contour. Also

$$\Lambda_i = \sum_{j=1}^p \alpha_j^{(i)} + \sum_{j=1}^{p_i} \gamma_j^{(i)} - \sum_{j=1}^q \beta_j^{(i)} - \sum_{j=1}^{q_i} \delta_j^{(i)} \leq 0,$$

$$\Omega_i = - \sum_{j=n+1}^p \alpha_j^{(i)} - \sum_{j=1}^q \beta_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} > 0,$$

where the integral n, p, q, m_i, n_i, p_i and q_i are constrained by the inequalities $p \geq n \geq 0, q \geq 0, q_i \geq m_i \geq 1$ and $p_i \geq n_i \geq 1 \forall i \in \{1, 2, \dots, r\}$ and the inequalities in (2) and (3) hold for suitably restricted values of the complex variables z_1, \dots, z_r . The sequence of parameters in (1) are such that none of the poles of the integrand coincide, that is, the poles of the integrand in (1) are simple. The contour L_i in the complex ξ_i -plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $+\omega\infty$ with indentations, if necessary, to ensure that all the poles of $\Gamma(d_j^{(i)} - \delta_j^{(i)}\xi_i), j = 1, \dots, m_i$ are separated from those of $\Gamma(1 - c_j^{(i)} + \gamma_j^{(i)}\xi_i), i = 1, \dots, n_i$.

In the present investigation we require the following formula:

From Rainville [1]:

$$z\Gamma(z) = \Gamma(z + 1),$$

2. IDENTITIES:

In this section we established the following identities involving H-function of several variables

$$H_{p,q;p_1+1,q_1+2;\dots;p_r,q_r}^{0,n;m_1+2,n_1;\dots;m_r,n_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (a_j; \alpha_j', \dots, \alpha_j^{(r)})_{1,p}; (c_j', \gamma_j')_{1,p_1}, (1-k, \nu); \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ (b_j; \beta_j', \dots, \beta_j^{(r)})_{1,q}; (1, h), (2-k, \nu), (d_j', \delta_j')_{1,q_1}; \dots; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right]$$

$$= (1 - k)H_{p,q;p_1,q_1+1; \dots; p_r,q_r}^{0,n;m_1+1,n_1+1; \dots; m_r,n_r} \left[\begin{matrix} Z_1 \\ \vdots \\ (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,p} : (c'_j, \gamma'_j)_{1,p_1} : \dots : (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ \vdots \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,q} : (1, h), (d'_j, \delta'_j)_{1,q_1} : \dots : (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right]$$

$$- H_{p,q;p_1+1,q_1+2; \dots; p_r,q_r}^{0,n;m_1+1,n_1+1; \dots; m_r,n_r} \left[\begin{matrix} Z_1 \\ \vdots \\ (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,p} : (0, v), (c'_j, \gamma'_j)_{1,p_1} : \dots : (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ \vdots \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,q} : (1, h), (d'_j, \delta'_j)_{1,q_1}, (1, v) : \dots : (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right]$$

provided that $|\arg(z_k)| < \frac{1}{2} \Omega_k \pi, \forall k \in [1, \dots, r]$, where Ω_k is given in (3).

$$H_{p,q;p_1+1,q_1+2; \dots; p_r,q_r}^{0,n;m_1+1,n_1+1; \dots; m_r,n_r} \left[\begin{matrix} Z_1 \\ \vdots \\ (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,p} : (k-1, v), (c'_j, \gamma'_j)_{1,p_1} : \dots : (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ \vdots \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,q} : (1, h), (d'_j, \delta'_j)_{1,q_1}, (k, v) : \dots : (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right]$$

$$= (1 - k)H_{p,q;p_1,q_1+1; \dots; p_r,q_r}^{0,n;m_1+1,n_1+1; \dots; m_r,n_r} \left[\begin{matrix} Z_1 \\ \vdots \\ (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,p} : (c'_j, \gamma'_j)_{1,p_1} : \dots : (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ \vdots \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,q} : (1, h), (d'_j, \delta'_j)_{1,q_1} : \dots : (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right]$$

$$+ H_{p,q;p_1+1,q_1+2; \dots; p_r,q_r}^{0,n;m_1+1,n_1+1; \dots; m_r,n_r} \left[\begin{matrix} Z_1 \\ \vdots \\ (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,p} : (0, v), (c'_j, \gamma'_j)_{1,p_1} : \dots : (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ \vdots \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,q} : (1, h), (d'_j, \delta'_j)_{1,q_1}, (1, v) : \dots : (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right],$$

provided that $|\arg(z_k)| < \frac{1}{2} \Omega_k \pi, \forall k \in [1, \dots, r]$, where Ω_k is given in (3).

$$k H_{p,q;p_1+2,q_1+2; \dots; p_r,q_r}^{0,n;m_1+1,n_1+2; \dots; m_r,n_r} \left[\begin{matrix} Z_1 \\ \vdots \\ (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,p} : (1, \alpha), (1-k, \alpha), (c'_j, \gamma'_j)_{1,p_1} : \dots : (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ \vdots \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,q} : (1, h), (d'_j, \delta'_j)_{1,q_1}, (-k, \alpha) : \dots : (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right]$$

$$= -H_{p,q;p_1+2,q_1+2; \dots; p_r,q_r}^{0,n;m_1+1,n_1+2; \dots; m_r,n_r} \left[\begin{matrix} Z_1 \\ \vdots \\ (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,p} : (0, \alpha), (1-k, \alpha), (c'_j, \gamma'_j)_{1,p_1} : \dots : (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ \vdots \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,q} : (1, h), (d'_j, \delta'_j)_{1,q_1}, (-k, \alpha) : \dots : (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right] \tag{expression continue}$$

$$+ H_{p,q;p_1+1,q_1+1; \dots; p_r,q_r}^{0,n;m_1+1,n_1+1; \dots; m_r,n_r} \left[\begin{matrix} Z_1 \\ \vdots \\ (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1,p} : (1, \alpha), (c'_j, \gamma'_j)_{1,p_1} : \dots : (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ \vdots \\ (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1,q} : (1, h), (d'_j, \delta'_j)_{1,q_1} : \dots : (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right],$$

provided that $|\arg(z_k)| < \frac{1}{2} \Omega_k \pi, \forall k \in [1, \dots, r]$, where Ω_k is given in

3. PROOF:

To prove (5), consider left hand side of (5), after using (1), to obtain

$$\begin{aligned}
&= \frac{1}{(2\pi\omega)^\Gamma} \int_{L_1} \dots \int_{L_r} \phi_1(\xi_1) \dots \phi_r(\xi_r) \psi(\xi_1, \dots, \xi_r) \\
&\quad \frac{\Gamma(1-h\xi_1)\Gamma(2-k-v\xi_1)}{\Gamma(1-k-v\xi_1)} z_1^{\xi_1} \dots z_r^{\xi_r} d\xi_1 \dots d\xi_r \\
&= \frac{1}{(2\pi\omega)^\Gamma} \int_{L_1} \dots \int_{L_r} \phi_1(\xi_1) \dots \phi_r(\xi_r) \psi(\xi_1, \dots, \xi_r) \\
&\quad \Gamma(1-h\xi_1)(1-k-v\xi_1) z_1^{\xi_1} \dots z_r^{\xi_r} d\xi_1 \dots d\xi_r \\
&= \frac{(1-k)}{(2\pi\omega)^\Gamma} \int_{L_1} \dots \int_{L_r} \phi_1(\xi_1) \dots \phi_r(\xi_r) \psi(\xi_1, \dots, \xi_r) \\
&\quad \Gamma(1-h\xi_1) z_1^{\xi_1} \dots z_r^{\xi_r} d\xi_1 \dots d\xi_r \\
&- \frac{1}{(2\pi\omega)^\Gamma} \int_{L_1} \dots \int_{L_r} \phi_1(\xi_1) \dots \phi_r(\xi_r) \psi(\xi_1, \dots, \xi_r) \\
&\quad \frac{\Gamma(1-h\xi_1)\Gamma(v\xi_1+1)}{\Gamma(v\xi_1)} z_1^{\xi_1} \dots z_r^{\xi_r} d\xi_1 \dots d\xi_r
\end{aligned}$$

which in the light of (1) provides right hand side of (5).

Similarly the result (6) and (7) can be established.

4. CONCLUSION:

This work advances the study of multivariable H-functions by introducing new and generalized identities derived through Mellin transform and contour integration methods. The developed results hold great promise for addressing complex mathematical and physical problems, particularly those involving integral and partial differential equations.

The proposed identities and analytical techniques serve as a robust framework with broad potential applications across diverse scientific and engineering fields, empowering researchers to solve intricate problems with greater clarity, simplicity, and effectiveness.

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