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FIXED POINT THEOREMS ON GENERALIZED WEAKLY CONTRACTIVE MAPPING USING ALTERING DISTANCE FUNCTION IN G- METRIC SPACES

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Abstract

Fixed point theory holds great importance in nonlinear analysis. The theory flourished after the generalization of the Banach Contraction Principle, introduced by S. Banach in 1922. This principle has been extended in various generalizations of metric spaces through a variety of contraction mappings in these generalized spaces. The G-metric space, proposed by Zead Mustafa and Brailey Sims in 2006, is one such generalization of metric spaces. In this study, we present some fixed point theorems in G-metric spaces by introducing a new generalized weakly contractive mapping that employs a lower semi-continuous function φ.

1. Introduction

Fixed point theory is one of the rapidly developing branches of nonlinear analysis in mathematics. It has significant applications in differential equations, partial differential equations, integral equations, boundary value problems, and initial value problems. The theory also finds wide use in engineering sciences, computer sciences, and economic theories. In the expansion of fixed point theory, the Banach Contraction Principle also known as Banach Contraction Theorem (BCT) stands as a milestone result. The applicability of BCT attracted the attention of many researchers, leading to substantial enrichment of fixed point theory. Throughout its development, various generalizations of BCT have appeared in the literature. Notable examples include the works of Boyd–Wong [4], Caristi [5], Kannan [15], and Ćirić [7].

Several generalizations of metric spaces have also been introduced, such as dislocated metric spaces [13], [14], dislocated quasi-metric spaces [25], dislocated quasi b-metric spaces [21], cone metric spaces [12], partial metric spaces [17], and G-metric spaces [19]. Among these, 2-metric spaces and D-metric spaces were proposed by Gähler [11] and Dhage [8], [9], respectively. The concept of G-metric spaces received considerable attention from fixed point scholars Zead Mustafa and Brailey Sims [19]. They argued that Dhage's notion of D-metric spaces is fundamentally flawed and that many of the results claimed therein are invalid. In [19], [20], Mustafa and Sims, and in [18], Mustafa alone, provided several properties of G-metric spaces.

Keywords : Fixed Point, Contraction Mapping, G-Metric Space

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The contraction mapping employed in BCT has also been generalized in the literature through the works of [4], [5], [15], [22], [24]. Rakotch [22] and Boyd–Wong [4] extended BCT by replacing the contraction constant with certain real-valued functions whose values lie in (0, 1). The altering distance function was introduced by Khan et al. [16], while weakly contractive mappings in Hilbert spaces were defined in 1997 by Y.A. Alber and Guerre-Delabriere [1]. These weakly contractive mappings were further studied in complete metric spaces by Dutta and Choudhury [10] and Sho [6].

More recently, Mehdi Asadi et al. [2], [3], [23] defined generalized weakly contractive mappings in G-metric spaces and established several fixed point theorems. In this paper, we extend the theorems presented by Mehdi Asadi et al. [2], [3], [23] in G-metric spaces by introducing a new generalized weakly contractive mapping.

2. Preliminaries

Definition 2.1 [19]: Let X be a nonempty set and let $G: X \times X \times X \to [0,\infty)$ be a function satisfying :

G1) $G(x,y,z) = 0$ if $x = y = z$;

G2) $0 < G(x,y,z)$ for all $x,y \in X$ with $x \neq y$;

G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$;

G4) $G(x,y,z) = G(x,z,y) = G(y,z\,x) = \cdots$, for all $x,y,z \in X$, (Symmetry in all three variables);

G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$, (Rectangular inequality).

then the function G is called a generalized metric and the pair (X,G) is called a G-metric space or G-metric on X.

Definition 2.2 [19]: Let (X,G) be a G -metric space and let $\{x_n\}$ be a sequence of points of X, we say that $\{x_n\}$ is G-convergent to x if for every given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ (set of all natural numbers) such that $G(x,x_n,x_m) < \varepsilon$ for all $m,n \geq N$. We denote it as $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$.

Definition 2.3 [19]: Let (X,G) be a G -metric space, a sequence $\{x_n\}$ in X is called G-Cauchy if for every given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq \mathbb{N}$, that is if, $\lim_{n,m,l\to\infty} G(x_n,\ x_m,x_l) = 0$.

Definition 2.4 [19]: A G -metric space (X,G) is said to be G -complete (or a complete G -metric space) if every G -Cauchy sequence in (X,G) is G –convergent to some point in (X,G).

Definition 2.5 [19] Let (X,G) be a G -metric space and let $T : X \to X$ be a mapping then mapping T is called a contraction on X if $G(Tx,Ty,Tz) \leq kG(x,y,z)$ for all $x,y,z \in X$.

Definition 2.6 [6]: A function $f : X \to [0,\infty)$, where X is a metric space, is called lower semi-continuous if,

10.48047/jocaaa.2024.33.07.66

for all $x \in X$ and $\{x_n\} \subset X$ with $\lim_{n\to\infty} x_n = x$,
we have $f(x) \leq \lim_{n\to\infty} f(x_n)$.

Definition 2.7 [16]: A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function, if the following properties are satisfied

a) $\psi(0) = 0$,
b) ψ is continuous and monotonically non-decreasing.

Let $\boldsymbol{\psi} = \{\psi : [0,\infty) \to [0,\infty)$ is continuous and $\psi(t) = 0$ if and only if $t = 0\}$ and $\boldsymbol{\phi} = \{\phi: [0,\infty) \to [0,\infty)$ lower semi continuous and $\phi(t) = 0$ if and only if $t = 0$

Theorem 2.8 [10]: Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y))$$

Where $\psi, \phi : [0,\infty) \to [0,\infty)$ are both continuous and monotone non-decreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$, then T has a unique fixed point.

Theorem 2.9 [3]: Let X be a complete G -metric space. Suppose the map $T : X \to X$ satisfies for all $x, y, z \in X$
$\psi(G(Tx,Ty,Tz)) \leq \psi(G(x,y,z)) - \phi(G(x,y,z))$, where ψ and ϕ are altering distance functions given in **Definition 2.7** then T has a unique fixed point (say u) and T is G -continuous at u.

Theorem 2.9 [2]: Let (X, G) be a complete G -metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$, where $\psi : [0,\infty) \longrightarrow [0,\infty)$ is non-decreasing and continuous and $\varphi: [0,\infty) \longrightarrow [0,\infty)$ is lower semi-continuous where $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$.
$\Psi(G(Tx,Tx,Ty)) \leq \psi(G(x,Tx,y)) - \varphi\big(G(x,Tx,y)\big)$ then T has a unique fixed point.

Definition 2.9 [23]: Let (X, G) be a complete G -metric space and Let $T : X \to X$ be a self-mapping satisfying the inequality
$\psi(G(Tx,Ty,Tz)) \leq \psi(G(x,y,z)) - \phi(G(x,y,z))$, for all $x, y, z, \in X$, where $\psi, \phi: [0,\infty) \to [0,\infty)$ are both continuous and monotone non decreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$ then T has unique fixed point.

Theorem 2.8 [6]: Let X be a metric space with metric d, let $T: X \to X$, and let $\varphi: X \to [0,\infty)$ be lower semi continuous function then T is called a generalized weakly contractive mapping if it satisfies the following condition:

$$\psi(d(Tx,Ty) + \varphi(Tx) + \varphi(Ty) \leq \psi\big(m(x,y,d,T,\varphi)\big) - \phi\big(l(x,y,d,T,\varphi)\big)$$

for all $x, y \in X$, Where $\psi \in \boldsymbol{\psi}, \phi \in \boldsymbol{\phi}$ and

10.48047/jocaaa.2024.33.07.66

$$m(x,y,d,T,\varphi) = \\ = max\left\{\begin{matrix} d(x,y)+\varphi(x)+\varphi(y), \\ d(x,Tx)+\varphi(x)+\varphi(Tx), \\ d(y,Ty)+\varphi(y)+\varphi(Ty), \\ \frac{1}{2}\{d(x,Ty)+\varphi(x)+\varphi(Ty)+d(y,Tx)+\varphi(y)+\varphi(Tx)\} \end{matrix}\right\}$$

And $l(x,y,d,T,\varphi) = max\{d(x,y)+\varphi(x)+\varphi(y), d(x,Tx)+\varphi(x)+\varphi(Tx)\}$

3. Results

Definition 3.1: Let (X,G) be a Complete G-metric space, let $T: X \to X$, and let $\varphi: X \to [0,\infty)$ be lower semi continuous function then T is called a generalized weakly contractive mapping on X, if it satisfies the following condition for all $x,y,z \in X$,

$$\psi(G(Tx,Ty,Tz)+\varphi(Tx)+\varphi(Ty)+\varphi(Tz)) \le \psi\big(\alpha(x,y,z,G,T,\varphi)\big) - \phi\big(\beta(x,y,z,G,T,\varphi)\big) \quad (3.1)$$

Where $\psi \in \boldsymbol{\psi}, \phi \in \boldsymbol{\phi}$ and

$$\alpha(x,y,z,G,T,\varphi) = max\left\{\begin{matrix} G(x,y,z)+\varphi(x)+\varphi(y)+\varphi(z), \\ G(x,Tx,y)+\varphi(x)+\varphi(Tx)+\varphi(y), \\ G(x,Tx,z)+\varphi(x)+\varphi(Tx)+\varphi(z), \\ G(y,Ty,Tz)+\varphi(y)+\varphi(Ty)+\varphi(Tz), \\ \frac{1}{3}\left\{\begin{matrix} G(y,Tx,Tz)+\varphi(y)+\varphi(Tx)+\varphi(Tz) \\ +G(Tx,Ty,Tz)+\varphi(Tx)+\varphi(Ty)+\varphi(Tz) \\ +G(Ty,Ty,Tz)+\varphi(Ty)+\varphi(Ty)+\varphi(Tz) \end{matrix}\right\} \end{matrix}\right\} \quad (3.2)$$

$$\text{And } \beta(x,y,z,G,T,\varphi) = max\left\{\begin{matrix} G(x,y,z)+\varphi(x)+\varphi(y)+\varphi(z), \\ G(x,Tx,z)+\varphi(x)+\varphi(Tx)+\varphi(z), \\ G(y,Ty,Tz)+\varphi(y)+\varphi(Ty)+\varphi(Tz) \end{matrix}\right\} \quad (3.3)$$

Theorem 3.1: Let (X,G) be a complete G-metric space and T be generalized weakly contractive mapping satisfying condition

$$\psi(G(Tx,Ty,Tz)+\varphi(Tx)+\varphi(Ty)+\varphi(Tz)) \\ \le \psi\big(\alpha(x,y,z,G,T,\varphi)\big) - \phi\big(\beta(x,y,z,G,T,\varphi)\big)$$

Where, $\alpha(x,y,z,G,T,\varphi)$ and $\beta(x,y,z,G,T,\varphi)$ are given by eq.3.2 and 3.3 respectively, then there exists unique $u \in X$ such that $u = Tu$ and $\varphi(u) = 0$.

Proof: Let x_0 be any fix point in X, We Construct a sequence $\{x_n\}_{n=1}^{\infty}$ of points in X by $x_0, x_1 = Tx_0, x_2 = Tx_1 = TTx_0 = T^2x_0, \ldots.$

10.48047/jocaaa.2024.33.07.66

In general $x_{n+1} = Tx_n$ for all n = 0, 1, 2, …

Now, If for some n, $x_n = x_{n+1}$ then $x_n = Tx_n$ and map T has fixed point x_n in X.

Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$

By (G2), We have $G(x_n, x_{n+1}, x_{n+1}) > 0$

Let $x = x_{n-1}, y = x_n\ and\ z = x_n$ in equation …..

We have

$$\psi(G(Tx_{n-1}, Tx_n, Tx_n) + \varphi(Tx_{n-1}) + \varphi(Tx_n) + \varphi(Tx_n)) \\ \leq \psi\big(\alpha(x_{n-1}, x_n, x_n, G, T, \varphi)\big) - \phi\big(\beta(x_{n-1}, x_n, x_n, G, T, \varphi)\big)$$

$$\psi(G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_{n+1})) \leq \\ \psi\big(\alpha(x_{n-1}, x_n, x_n, G, T, \varphi)\big) - \phi\big(\beta(x_{n-1}, x_n, x_n, G, T, \varphi)\big) \tag{3.4}$$

From equation 3.1,

$$\alpha(x_{n-1}, x_n, x_n, G, T, \varphi) = \\ max\left\{\begin{matrix} G(x_{n-1}, x_n, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_n), \\ G(x_{n-1}, Tx_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(Tx_{n-1}) + \varphi(x_n), \\ G(x_{n-1}, Tx_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(Tx_{n-1}) + \varphi(x_n), \\ G(x_n, Tx_n, Tx_n) + \varphi(x_n) + \varphi(Tx_n) + \varphi(Tx_n), \\ \frac{1}{3}\left\{\begin{matrix} G(x_n, Tx_{n-1}, Tx_n) + \varphi(x_n) + \varphi(Tx_{n-1}) + \varphi(Tx_n) \\ +G(Tx_{n-1}, Tx_n, Tx_n) + \varphi(Tx_{n-1}) + \varphi(Tx_n) + \varphi(Tx_n) \\ +G(Tx_n, Tx_n, Tx_n) + \varphi(Tx_n) + \varphi(Tx_n) + \varphi(Tx_n) \end{matrix}\right\} \end{matrix}\right\}$$

$$\alpha(x_{n-1}, x_n, x_n, G, T, \varphi) = \\ max\left\{\begin{matrix} G(x_{n-1}, x_n, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_n), \\ G(x_{n-1}, x_n, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_n), \\ G(x_{n-1}, x_n, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_n), \\ G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_{n+1}), \\ \frac{1}{3}\left\{\begin{matrix} G(x_n, x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_n) + \varphi(x_{n+1}) \\ +G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \\ +G(x_{n+1}, x_{n+1}, x_{n+1}) + \varphi(x_{n+1}) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \end{matrix}\right\} \end{matrix}\right\} \tag{3.5}$$

We consider,

$$\frac{1}{3}\left\{\begin{matrix} G(x_n, x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_n) + \varphi(x_{n+1}) \\ +G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \\ +G(x_{n+1}, x_{n+1}, x_{n+1}) + \varphi(x_{n+1}) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \end{matrix}\right\}$$

10.48047/jocaaa.2024.33.07.66

$$= \frac{1}{3}\left\{\begin{matrix} G(x_n, x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_n) + \varphi(x_{n+1}) \\ +G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \\ +0 + \varphi(x_{n+1}) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \end{matrix}\right\}$$

$$\leq \frac{1}{3}\left\{\begin{matrix} G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_n) + \varphi(x_{n+1}) \\ +G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \\ +\varphi(x_{n+1}) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \end{matrix}\right\}$$

$$\leq \frac{1}{3}\left\{\begin{matrix} G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_n) + \varphi(x_n) + \varphi(x_{n+1}) \\ +G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \\ +G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_{n+1}) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \end{matrix}\right\}$$

$$\leq \frac{1}{3}\left\{\begin{matrix} G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_n) + +\varphi(x_{n+1}) + \varphi(x_{n+1}) \\ +G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \\ +G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \end{matrix}\right\}$$

Therefore 3.4 becomes

$$\alpha(x_{n-1}, x_n, x_n, G, T, \varphi) = max\left\{\begin{matrix} G(x_{n-1}, x_n, x_n) \\ +\varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_n), \\ G(x_n, x_{n+1}, x_n) \\ +\varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_n) \end{matrix}\right\} \quad (3.6)$$

Now consider,

$$\beta(x_{n-1}, x_n, x_n, G, T, \varphi) = max\left\{\begin{matrix} G(x_{n-1}, x_n, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_n), \\ G(x_{n-1}, Tx_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(Tx_{n-1}) + \varphi(x_n), \\ G(x_n, Tx_n, Tx_n) + \varphi(x_n) + \varphi(Tx_n) + \varphi(Tx_n) \end{matrix}\right\}$$

$$\beta(x_{n-1}, x_n, x_n, G, T, \varphi) = max\left\{\begin{matrix} G(x_{n-1}, x_n, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_n), \\ G(x_{n-1}, x_n, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_n), \\ G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \end{matrix}\right\}$$

$$\beta(x_{n-1}, x_n, x_n, G, T, \varphi) = max\left\{\begin{matrix} G(x_{n-1}, x_n, x_n) \\ +\varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_n), \\ G(x_n, x_{n+1}, x_{n+1}) \\ +\varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_{n+1}) \end{matrix}\right\} \quad (3.7)$$

Using Eq. (3.6) and Eq. (3.7) in Eq. (3.4)

We get,

10.48047/jocaaa.2024.33.07.66

$$\psi(G(x_n,x_{n+1},x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1}))\leq$$
$$\psi\left(max\begin{Bmatrix}G(x_{n-1},x_n,x_n)+\varphi(x_{n-1})+\varphi(x_n)+\varphi(x_n),\\G(x_n,x_{n+1},x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1})\end{Bmatrix}\right)-$$
$$\phi\left(max\begin{Bmatrix}G(x_{n-1},x_n,x_n)+\varphi(x_{n-1})+\varphi(x_n)+\varphi(x_n),\\G(x_n,x_{n+1},x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1})\end{Bmatrix}\right) \tag{3.8}$$

Now if

$$max\begin{Bmatrix}G(x_{n-1},x_n,x_n)+\varphi(x_{n-1})+\varphi(x_n)+\varphi(x_n),\\G(x_n,x_{n+1},x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1})\end{Bmatrix}$$
$$=G(x_n,x_{n+1},x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1})$$

Then Eq. (3.8) gives

$$\psi(G(x_n,x_{n+1},x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1}))$$
$$\leq\psi\big(G(x_n,x_{n+1},x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1})\big)$$
$$-\phi\big(G(x_n,x_{n+1},x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1})\big)$$

$$\Longrightarrow\phi\big(G(x_n,x_{n+1},x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1})\big)=0$$

$\Longrightarrow G(x_n,x_{n+1},x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1})=0$ by Definition 2.7

$$\Longrightarrow x_n=x_{n+1}\ and\ \varphi(x_n)=0=\varphi(x_{n+1})$$

This is contradiction to our assumption and to the choice $x_n\neq x_{n+1}$

Thus,

$$max\begin{Bmatrix}G(x_{n-1},x_n,x_n)+\varphi(x_{n-1})+\varphi(x_n)+\varphi(x_n),\\G(x_n,x_{n+1},x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1})\end{Bmatrix}$$
$$\neq G(x_n,x_{n+1},x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1})$$

and we must have,

$$max\begin{Bmatrix}G(x_{n-1},x_n,x_n)+\varphi(x_{n-1})+\varphi(x_n)+\varphi(x_n),\\G(x_n,x_{n+1},x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1})\end{Bmatrix}$$
$$=G(x_{n-1},x_n,x_n)+\varphi(x_{n-1})+\varphi(x_n)+\varphi(x_n)$$

Therefore Eq. (3.8) becomes,

$$\psi\begin{pmatrix}G(x_n,x_{n+1},x_{n+1})\\+\varphi(x_n)+\varphi(x_{n+1})+\varphi(x_{n+1})\end{pmatrix}\leq\psi\begin{pmatrix}G(x_{n-1},x_n,x_n)\\+\varphi(x_{n-1})+\varphi(x_n)+\varphi(x_n)\end{pmatrix}$$
$$-\phi\begin{pmatrix}G(x_{n-1},x_n,x_n)\\+\varphi(x_{n-1})+\varphi(x_n)+\varphi(x_n)\end{pmatrix} \tag{3.9}$$

10.48047/jocaaa.2024.33.07.66

Thus, the sequence $\{G(x_n, x_{n+1}, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_{n+1})\}$ is a decreasing sequence and it must converge to some non-negative real number l (say) as $n \to \infty$.

Assuming that, $l > 0$, letting $n \to \infty$ in Eq. (3.9) and by continuity of ψ also as ϕ is lower semi continuous

$$\psi(l) \leq \psi(l) - \lim_{n\to\infty} inf \left\{ \begin{matrix} \phi(G(x_{n-1}, x_n, x_n) \\ +\varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_n)) \end{matrix} \right\}$$

$$\psi(l) \leq \psi(l) - \phi(l).$$

Since,$l > 0$, $\phi(l) > 0$ by Definition 2.7

We get $\psi(l) \leq \psi(l) - \phi(l) < \psi(l)$ is not possible.

Hence, this is a contradiction.

Thus, we have, $\lim_{n\to\infty} G(x_{n-1}, x_n, x_n) + \varphi(x_{n-1}) + \varphi(x_n) + \varphi(x_n) = 0$

$$\Longrightarrow \lim_{n\to\infty} G(x_{n-1}, x_n, x_n) = 0 \text{ and } \lim_{n\to\infty} \varphi(x_n) = 0 \quad (3.10)$$

Now we claim that, the sequence $\{x_n\}_{n=1}^{\infty}$ is a G-Cauchy sequence.

If not then there exists $\epsilon > 0$, and two subsequence $x_{n(k)}$ and $x_{m(k)}$ of x_n such that, With $m(k) > n(k) > k$

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geq \epsilon \quad (3.11)$$

Now, Let $n(k)$ be the least positive integer, with $n(k) > m(k)$ satisfying 3.9 and giving us

$G(x_{m(k)-1}, x_{n(k)}, x_{n(k)}) < \epsilon$ for every integer k

From Eq. (3.1) we have

$$\alpha(x_{m(k)}, x_{n(k)}, x_{n(k)}, G, T, \varphi)$$

$$= max \left\{ \begin{matrix} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}) + \varphi(x_{n(k)}), \\ G(x_{m(k)}, Tx_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(Tx_{m(k)}) + \varphi(x_{n(k)}), \\ G(x_{m(k)}, Tx_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(Tx_{m(k)}) + \varphi(x_{n(k)}), \\ G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)}) + \varphi(x_{n(k)}) + \varphi(Tx_{n(k)}) + \varphi(Tx_{n(k)}), \\ \frac{1}{3} \left\{ \begin{matrix} G(x_{n(k)}, Tx_{m(k)}, Tx_{n(k)}) + \varphi(x_{n(k)}) + \varphi(Tx_{m(k)}) + \varphi(Tx_{n(k)}) \\ +G(Tx_{m(k)}, Tx_{n(k)}, Tx_{n(k)}) + \varphi(Tx_{m(k)}) + \varphi(Tx_{n(k)}) + \varphi(Tx_{n(k)}) \\ +G(Tx_{n(k)}, Tx_{n(k)}, Tx_{n(k)}) + \varphi(Tx_{n(k)}) + \varphi(Tx_{n(k)}) + \varphi(Tx_{n(k)}) \end{matrix} \right\} \end{matrix} \right\}$$

10.48047/jocaaa.2024.33.07.66

$$\alpha\left(x_{m(k)}, x_{n(k)}, x_{n(k)}, G, T, \varphi\right) =$$
$$max\left\{\begin{matrix} G\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right) + \varphi\left(x_{m(k)}\right) + \varphi\left(x_{n(k)}\right) + \varphi\left(x_{n(k)}\right), \\ G\left(x_{m(k)}, x_{m(k)+1}, x_{n(k)}\right) + \varphi\left(x_{m(k)}\right) + \varphi\left(x_{m(k)+1}\right) + \varphi\left(x_{n(k)}\right), \\ G\left(x_{m(k)}, x_{m(k)+1}, x_{n(k)}\right) + \varphi\left(x_{m(k)}\right) + \varphi\left(x_{m(k)+1}\right) + \varphi\left(x_{n(k)}\right), \\ G\left(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}\right) + \varphi\left(x_{n(k)}\right) + \varphi\left(x_{n(k)+1}\right) + \varphi\left(x_{n(k)+1}\right), \\ \frac{1}{3}\left\{\begin{matrix} G\left(x_{n(k)}, x_{m(k)+1}, x_{n(k)+1}\right) + \varphi\left(x_{n(k)}\right) + \varphi\left(x_{m(k)+1}\right) + \varphi\left(x_{n(k)+1}\right) \\ +G\left(x_{m(k)+1}, x_{n(k)+1}, x_{n(k)+1}\right) + \varphi\left(x_{m(k)+1}\right) + \varphi\left(x_{n(k)+1}\right) + \varphi\left(x_{n(k)+1}\right) \\ +G\left(x_{n(k)+1}, x_{n(k)+1}, x_{n(k)+1}\right) + \varphi\left(x_{n(k)+1}\right) + \varphi\left(x_{n(k)+1}\right) + \varphi\left(x_{n(k)+1}\right) \end{matrix}\right\} \end{matrix}\right\} \tag{3.12}$$

Let $k \longrightarrow \infty$ in Eq. (3.12)

$$\lim_{k \longrightarrow \infty} \alpha\left(x_{m(k)}, x_{n(k)}, x_{n(k)}, G, T, \varphi\right) = \epsilon$$

Also, $\beta\left(x_{m(k)}, x_{n(k)}, x_{n(k)}, G, T, \varphi\right) =$

$$max\left\{\begin{matrix} G\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right) + \varphi\left(x_{m(k)}\right) + \varphi\left(x_{n(k)}\right) + \varphi\left(x_{n(k)}\right), \\ G\left(x_{m(k)}, Tx_{m(k)}, x_{n(k)}\right) + \varphi\left(x_{m(k)}\right) + \varphi\left(Tx_{m(k)}\right) + \varphi\left(x_{n(k)}\right), \\ G\left(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)}\right) + \varphi\left(x_{n(k)}\right) + \varphi\left(Tx_{n(k)}\right) + \varphi\left(Tx_{n(k)}\right) \end{matrix}\right\}$$

$$\beta\left(x_{m(k)}, x_{n(k)}, x_{n(k)}, G, T, \varphi\right) = max\left\{\begin{matrix} G\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right) + \varphi\left(x_{m(k)}\right) + \varphi\left(x_{n(k)}\right) + \varphi\left(x_{n(k)}\right), \\ G\left(x_{m(k)}, x_{m(k)+1}, x_{n(k)}\right) + \varphi\left(x_{m(k)}\right) + \varphi\left(x_{m(k)+1}\right) + \varphi\left(x_{n(k)}\right), \\ G\left(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}\right) + \varphi\left(x_{n(k)}\right) + \varphi\left(x_{n(k)+1}\right) + \varphi\left(x_{n(k)+1}\right) \end{matrix}\right\}$$

Letting $k \longrightarrow \infty$,

$$\lim_{k \longrightarrow \infty} \beta\left(x_{m(k)}, x_{n(k)}, x_{n(k)}, G, T, \varphi\right) = \epsilon$$

Therefore Eq. (3.1) becomes,

$$\psi(G\left(Tx_{m(k)}, Tx_{n(k)}, Tx_{n(k)}\right) + \varphi\left(Tx_{m(k)}\right) + \varphi\left(Tx_{n(k)}\right) + \varphi\left(Tx_{n(k)}\right) \le \psi\left(\alpha\left(x_{m(k)}, x_{n(k)}, x_{n(k)}, G, T, \varphi\right)\right) - \phi\left(\beta\left(x_{m(k)}, x_{n(k)}, x_{n(k)}, G, T, \varphi\right)\right)$$

Letting $k \longrightarrow \infty$ in this inequality and using the continuity of ψ, the lower semi-continuity of ϕ.

10.48047/jocaaa.2024.33.07.66

We get,

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon)$$

Which is contradiction to $\phi(\epsilon) > 0$.

This proves that, the sequence $\{x_n\}$ is a G-Cauchy sequence and as X is Complete G-metric space, sequence $\{x_n\}$ is G-convergent to some u in X.

Therefore, $\lim_{n\to\infty} x_n = u \in X$ and $\lim_{n\to\infty} G(x_n, x_n, u) = \lim_{n\to\infty} G(x_n, u, u) = 0$ (3.13)

Now we prove that, u is fixed point of the map T.

Consider,

$$\psi\big(G(x_{n+1}, x_{n+1}, Tu) + \varphi(x_{n+1}) + \varphi(x_{n+1}) + \varphi(Tu)\big) = \psi\big(G(Tx_n, Tx_n, Tu) + \varphi(Tx_n) + \varphi(Tx_n) + \varphi(Tu)\big)$$

$$\leq \psi\big(\alpha(x_n, x_n, u, G, T, \varphi)\big) - \phi\big(\beta(x_n, x_n, u, G, T, \varphi)\big) \tag{3.14}$$

Where,

$$\alpha(x_n, x_n, u, G, T, \varphi) = max\left\{\begin{matrix} G(x_n, x_n, u) + \varphi(x_n) + \varphi(x_n) + \varphi(u), \\ G(x_n, Tx_n, x_n) + \varphi(x_n) + \varphi(Tx_n) + \varphi(x_n), \\ G(x_n, Tx_n, u) + \varphi(x_n) + \varphi(Tx_n) + \varphi(u), \\ G(x_n, Tx_n, Tu) + \varphi(x_n) + \varphi(Tx_n) + \varphi(Tu), \\ \frac{1}{3}\left\{\begin{matrix} G(x_n, Tx_n, Tu) + \varphi(x_n) + \varphi(Tx_n) + \varphi(Tu) \\ +G(Tx_n, Tx_n, Tu) + \varphi(Tx_n) + \varphi(Tx_n) + \varphi(Tu) \\ +G(Tx_n, Tx_n, Tu) + \varphi(Tx_n) + \varphi(Tx_n) + \varphi(Tu) \end{matrix}\right\} \end{matrix}\right\}$$

$$\alpha(x_n, x_n, u, G, T, \varphi) = max\left\{\begin{matrix} G(x_n, x_n, u) + \varphi(x_n) + \varphi(x_n) + \varphi(u), \\ G(x_n, x_{n+1}, x_n) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(x_n), \\ G(x_n, x_{n+1}, u) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(u), \\ G(x_n, x_{n+1}, Tu) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(Tu), \\ \frac{1}{3}\left\{\begin{matrix} G(x_n, x_{n+1}, Tu) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(Tu) \\ +G(x_{n+1}, x_{n+1}, Tu) + \varphi(x_{n+1}) + \varphi(x_{n+1}) + \varphi(Tu) \\ +G(x_{n+1}, x_{n+1}, Tu) + \varphi(x_{n+1}) + \varphi(x_{n+1}) + \varphi(Tu) \end{matrix}\right\} \end{matrix}\right\} \tag{3.15}$$

$$\beta(x_n, x_n, u, G, T, \varphi) = max\left\{\begin{matrix} G(x_n, x_n, u) + \varphi(x_n) + \varphi(x_n) + \varphi(u), \\ G(x_n, Tx_n, u) + \varphi(x_n) + \varphi(Tx_n) + \varphi(u), \\ G(x_n, Tx_n, Tu) + \varphi(x_n) + \varphi(Tx_n) + \varphi(Tu) \end{matrix}\right\}$$

10.48047/jocaaa.2024.33.07.66

$$\beta(x_n, x_n, u, G, T, \varphi) = max \begin{Bmatrix} G(x_n, x_n, u) + \varphi(x_n) + \varphi(x_n) + \varphi(u), \\ G(x_n, x_{n+1}, u) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(u), \\ G(x_n, x_{n+1}, Tu) + \varphi(x_n) + \varphi(x_{n+1}) + \varphi(Tu) \end{Bmatrix} \quad (3.16)$$

Letting $n \to +\infty$ in equation Eq. (3.15) and Eq.(3.16) and using in Eq.(3.14)

$$\begin{aligned} &\psi\big(G(u, u, Tu) + \varphi(u) + \varphi(u) + \varphi(Tu)\big) \\ &\qquad = \psi\big(G(u, u, Tu) + \varphi(u) + \varphi(u) + \varphi(Tu)\big) \\ &\qquad - \phi\big(G(u, u, Tu) + \varphi(u) + \varphi(u) + \varphi(Tu)\big) \end{aligned}$$

We get,

$$\psi\big(G(u, u, Tu) + \varphi(u) + \varphi(u) + \varphi(Tu)\big) \leq 0$$

$$G(u, u, Tu) + \varphi(u) + \varphi(u) + \varphi(Tu) = 0$$

$G(u, u, Tu) = 0$, $\varphi(u) = 0$ and $\varphi(Tu) = 0$

This gives $Tu = u$ proving that, u is fixed point of T.

Uniqueness, We prove by contraction.

Assume contrary that, there exists two fixed point for T namely u ans u'

$\therefore Tu' = u$ and $\varphi(u') = 0$

Consider,

$$\begin{aligned} &\psi\big(G(u, u, u') + \varphi(u) + \varphi(u) + \varphi(u')\big) \\ &\qquad = \psi\big(G(Tu, T\,u, Tu') + \varphi(Tu) + \varphi(Tu) + \varphi(Tu')\big) \\ &\qquad \leq \psi\big(\alpha(u, u, u', G, T, \varphi)\big) - \phi\big(\beta(u, u, u', G, T, \varphi)\big) \end{aligned}$$

Which evaluates to $G(u, u, u') = 0$ further gives $u = u'$. Hence uniqueness is proved

Corollary 3.2: Let (X, G) be a complete G metric space and T be generalized weakly contractive mapping satisfying condition

$$\begin{aligned} &\psi(G(Tx, Ty, Tz) + \varphi(Tx) + \varphi(Ty) + \varphi(Tz)) \\ &\qquad \leq \psi\big(\beta(x, y, z, G, T, \varphi)\big) - \phi\big(\alpha(x, y, z, G, T, \varphi)\big) \end{aligned}$$

Where, $\alpha(x, y, z, G, T, \varphi)$and $\beta(x, y, z, G, T, \varphi)$is given by Eq. (3.2) and Eq. (3.3) respectively, then there exists unique $u \in X$ such that $u = Tu$ and $\varphi(u) = 0$.

Corollary 3.3: Let (X, G) be a complete G metric space and T be generalized weakly contractive mapping satisfying condition

10.48047/jocaaa.2024.33.07.66

$$\psi(G(Tx,Ty,Tz)+\varphi(Tx)+\varphi(Ty)+\varphi(Tz)) \leq \psi\big(\alpha(x,y,z,G,T,\varphi)\big)-\phi\big(\alpha(x,y,z,G,T,\varphi)\big)$$

Where, $\alpha(x,y,z,G,T,\varphi)$ are given by equation Eq. (3.2) then there exists unique $u \in X$ such that $u = Tu$ and $\varphi(u) = 0$.

Corollary 3.4: Let (X, G) be a complete G metric space and T be generalized weakly contractive mapping satisfying condition

$$\psi(G(Tx,Ty,Tz)+\varphi(Tx)+\varphi(Ty)+\varphi(Tz)) \leq \psi\big(\beta(x,y,z,G,T,\varphi)\big)-\phi\big(\beta(x,y,z,G,T,\varphi)\big)$$

Where, $\alpha(x,y,z,G,T,\varphi)$ and $\beta(x,y,z,G,T,\varphi)$ are given by equations A and B

then there exists unique $u \in X$ such that $u = Tu$ and $\varphi(u) = 0$.

4. Conclusion

This paper introduced a **Generalized Weakly Contractive Mapping** defined through the use of an **altering distance function ($\boldsymbol{\psi}$)** and a **lower semi-continuous function ($\boldsymbol{\varphi}$)** in complete G-metric spaces. The primary achievement is the establishment of **new fixed point existence and uniqueness theorems** specifically designed for this generalzied weakly mapping type, which are distinct from existing standard fixed point theorems. This work significantly contributes to Non-Linear Analysis by providing novel analytical tools for functions whose contractive behavior requires the flexibility of ψ and φ functions and the three-point structure of the G-metric. These new theoretical results are foundational for future applications in solving unique classes of non-linear problems.

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