On the Negative Pell Equation $y^2 = 23x^2 - 11$

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ABSTRACT: The various integer solutions of the binary quadratic equation, shown by the negative pellian equation $y^2 = 23x^2 - 11$, are studied. There are also some intriguing relationships between the solutions provided. Other options for hyperbolas, parabolas, and the unique Pythagorean triangle have been solved.

KEYWORDS: Binary quadratic, hyperbola, parabola, integral solutions, Pell equation.

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Introduction:

Mathematicians find the Pell equation intriguing due to its apparent simplicity and its linkages to a wide range of theoretical issues. Consequently, Pell equation chapters are included in nearly all beginning number theory courses; we suggest Positive Pell equation $a^2 - Db^2 = 1$ (I)

Is being considered, nevertheless has an unlimited number of non-trivial solutions (a, b) for any non-square natural integer D; however, this is not the case for the negative Pell equation $a^2 - Db^2 = -1$. (II)

The following critical points about positive and negative Pell equations are stated by theory.

(I) There is a least positive solution to equation (1), known as the basic solution (a_0, b_0), where variables $a > 0$ and $b > 0$ have their least positive integer values. When equation (II) is solvable, the same can be said; in this case, its basic solution is shown by the notation (a_0, b_0).

(II) The basic solution (a_0, b_0) of (I) for a single fixed D value yields all (infinitely many) of the expression's rational and irrational components.

$$a + b\sqrt{D} = \left(a_0 + b_0\sqrt{D}\right)^k \quad k = 1,2,3 \ldots\ldots$$

$a^2 - Db^2 = -1$ with x; y ∈ Z (1)for square free numbers d ∈ Z >1.

When Euler examined the solvability of this problem, he inadvertently connected the related equation $a^2 - Db^2 = 1$ to the name of the English mathematician John Pell (1611-1685). Long before Euler's time, several mathematicians had previously investigated the difficulty of finding nontrivial solutions to the Pell equation itself, and Fermat, who presented it as a challenge to the English mathematicians in 1657, recognized that it was solvable for every non square D > 1. [Weil 1984] provides a comprehensive overview of the equation's lengthy history.

The solvability of the negative Pell equation we are addressing here turns out to be a far more complex process than the comparatively simple solution in the case of the Pell equation itself. The goal of this work is to develop a class of positive integers $D \equiv D(u_n, u_{n+1,m})$ represented by u_n, u_{n+1} and fourth-degree polynomials of m such that $u_0 = u_1 = 1$ and $u_{n+2} = 3u_{n+1} - u_n$ for $n \in \mathbb{N} \cup \{0\}$ fulfills the negative Pell equation $a^2 - Db^2 = -1$.Conversely, a unique equation $a^2 - k(k+4)m^2b^2 = -1$, where $k, m \in \mathbb{N}$, likewise piques our curiosity. It has been determined what conditions are sufficient and necessary for this equation to be solvable in integers. Prior to outlining the primary concepts and findings, we review the background of the Pell equation and a few relevant studies.

A well-known issue with the Pell equation's nontrivial integer solutions

$$a^2 - Db^2 = c \qquad \text{(A)}$$

has a history spanning several centuries, in which D is often assumed to be a positive square- free integer and c to be a part of $Z - \{0\}$. Named after the mathematician John Pell, equation (A) is a particular example of binary quadratic Diophantine equations

$A_{11}x^2 + A_{12}xy + A_{22}y^2 + B_1x + B_2y = c$ with $A_{11}A_{22} /= 0$ and $A^2 - 4A_{11}A_{22} > 0$. One characteristic of

(A) is that it must have an unlimited number of different integer solutions if it has an integer

solution (x, y) with $xy \neq 0$. In fact, it is well known (established by Lagrange in 1768 that $x^2 - dy^2 = 1$ has nontrivial integer solutions for every square-free integer $d \geq 2$, and that all of its integer solutions can be produced by its fundamental solution $(x,y) = (x_0, y_0)$ (ie.,x_0, y_0 є N and $x_0 + y_0 = \min\{x+y: x,y \in N$ and (x,y) solves (A)} Assume that the integer answer to (A) is $(x,y) = (a_0, b_0)$.Then, if n is in N, One may verify that $(x,y) = (a_n, b_n)$ likewise resolves (A)

By establishing the recurrence relations and applying the Binomial Theorem, these solutions may be immediately found. Additionally, even if (A) can be solved in integers x and y, it might not be simple to determine its basic answer. One well-known example is $x^2-991y^2=1$. Its fundamental solution given as follows is quite huge

$(x,y)=(379516400906811930638014896080,12055735790331359447442538767)$.

On the other hand,(A) may be unsolvable for some D and c.

For example, congruence modulo 3 may be used to quickly demonstrate that $x^2-2y^2 = 3$ is unsolvable in integers x and y. Therefore, the situation for the solvability of (A) gets more complex when $c \neq 1$. A quite interesting question immediately arises:

"What value of d,(A) is solvable in integers x and y, given $c\neq 1$?" A key scenario for this topic is $c = -1$, or the negative Pell equation $x^2-dy^2=-1$ (A.1)($x, y \in Z$)

It is required that $d \equiv 1$ or 2 (mod 4) and that all odd prime divisors of d are of the type Congruent to1 modulo 4 in order for (A.1)to be solvable in integers. These prerequisites, however, are not enough for a solution to exist. When d=2,5,10,13,17,26,29,37,41,50,..., such that (A.1)can be solved in integers x and y

A few number theorists have dedicated decades to defining the requirements for the solvability of (A.1). According to Newman, if $d = \prod_{i=1}^{r} p_i$ if pi's are primes congruent to 1 modulo 4, and r = 2 or r is odd. and satisfy $\left(\frac{p_i}{p_j}\right) = -1$ for all $1 \leq i; j \leq r$ with $i \neq j$, then (A.1) is solvable in integers.

However, Mollin found a relationship between the equation $x^2 - dy^2 = 1$ and the negative Pell equation (A.1). He demonstrated that (A.1) may be solved in integers x and y if and only if $x_0 \equiv -1(mod\ 2d)$ is satisfied by the basic solution (x_0; y_0) of $x^2 - dy^2 = 1$. Despite the significance of these findings, their approaches are constrained if d is less than 1. For instance, there are several situations when the fundamental answers are rather huge. Therefore, extensive computations are required to verify these criteria. Reputable solvability standards for negative Pell equations have recently been developed. Two methods are available: one involves calculating the period of the simple continuous fraction of $\sqrt{d}$, and the other involves verifying if a primitive Pythagorean triple exists for d.

Method of analysis:

The negative Pell equation representing hyperbola under consideration is

$$\boldsymbol{y^2 = 23x^2 - 11} \qquad \textbf{(1)}$$

whose smallest positive integer solution is $x_0 = 2, y_0 = 9$

To obtain the other solutions of (1),consider the pell equation $\boldsymbol{y^2 = 23x^2 + 1}$

whose general solution is given by

$$\boldsymbol{\breve{y}_n = \frac{1}{2} f_n \;;\; \breve{x}_n = \frac{1}{2\sqrt{23}} g_n}$$

$$\boldsymbol{f_n = (24 + 5\sqrt{23})^{n+1} + (24 - 5\sqrt{23})^{n+1}} \qquad \textbf{(2)}$$

$$\boldsymbol{g_n = (24 + 5\sqrt{23})^{n+1} - (24 - 5\sqrt{23})^{n+1}}, \textbf{ n=0,1,2,3.......} \qquad \textbf{(3)}$$

Applying Brahmagupta lemma between (x_0, y_0) *and* (x_n, y_n) the other integer solutions of (1) are given by

$$\boldsymbol{46x_{n+1} = 9\sqrt{23}g_n + 46f_n} \qquad \textbf{(4)}$$

$$\boldsymbol{2y_{n+1} = 2\sqrt{23}g_n + 9f_n} \qquad \textbf{(5)}$$

Some numerical examples of x& y satisfying (1) are given in the table below

n	x_n	y_n
0	**2**	**9**
1	**93**	**446**
2	**4462**	**21399**
3	**214083**	**1026706**
4	**10271522**	**49260489**

From the above table, we observe some interesting relations among the solutions which are presented below:

1) The recurrence relations satisfied by the solutions of (1) are given by

$$x_{n+2} - 48\, x_{n+1} + \ x_n = 0 \quad (6)$$
$$y_{n+2} - 48\, y_{n+1} + \ y_n = 0 \quad (7)$$

2) $x_{n+2} - x_n = 10\, y_{n+1}$
3) $y_{\ n+2} - y_n = 230\, x_{n+1}$
4) $x_{n+2} - 24x_{n+1} = 5y_{n+1}$
5) $24x_{n+2} - x_{n+1} = 5y_{n+2}$
6) $1151x_{n+2} - 24x_{n+1} = 5y_{n+3}$
7) $1151x_{n+1=}x_{n+3} - 240y_{n+1}$
8) $2645x_{n+2=}552y_{n+2} - 23y_{n+1}$
9) $2645x_{n+2=}23y_{n+3} - 552y_{n+2}$
10) $x_{n+2} = 24x_{n+3} - 5y_{n+3}$
11) $1150x_{n+2} = 5y_{n+3} - 5y_{n+1}$
12) $24x_{n+2} = x_{n+3} - 5y_{n+2}$
13) $1151x_{n+2} = 24x_{n+3} - 5y_{n+1}$
14) $2645x_{n+1} = 23y_{n+2} - 552y_{n+1}$
15) $1151\ {x_{n+2}}^2 = 24x_{n+3}x_{n+2} - 5y_{n+1}x_{n+2}$
16) $1151\ {x_{n+1}}^2 = x_{n+3}x_{n+1} - 240y_{n+1}x_{n+1}$
17) $23x_{n+1}y_{n+1} = 552y_{n+2}x_{n+1} - 2645x_{n+1}x_{n+2}$
18) $552x_{n+2}y_{n+1} = 23y_{n+2}x_{n+2} - 2645x_{n+1}x_{n+2}$
19) $\frac{92x_{2n+2} - 18y_{2n+2}}{11} + 2\ is\ perfect\ square$

Proof: Eliminating g_n between (4) and (5), we get

$$92x_{n+1} - 18y_{n+1} = 11f_n \quad (8)$$

Similarly, Eliminating f_n between (4) and (5), we get

$$4\sqrt{23}y_{n+1} - 18\sqrt{23}x_{n+1} = 11g_n \quad (9)$$

Replacing n by 2n+1 in (8), It is seen that

$$92x_{2n+2} - 18y_{2n+2} = 11\left[{f_n}^2 - 2\right]$$

Therefore,

$$\frac{92x_{2n+2} - 18y_{2n+2} + 22}{11} \text{ is a perfect square}$$

Similarly, each of the following expressions is a perfect square

i. $\frac{21399x_{2n+3} - 446x_{2n+4} + 99}{131}$
ii. $\frac{446x_{2n+2} - 9x_{2n+3} + 99}{131}$
iii. $\frac{12052x_{2n+2} - 9y_{2n+3} + 100746}{1441}$
iv. $\frac{92x_{2n+3} - 892y_{2n+2} + 89716}{1441}$
v. $\frac{46x_{2n+3} - 9y_{2n+3-} - 36828}{15851}$

20) $\frac{92x_{3n+3}-18y_{3n+3}+1584}{11}$ is a cubical integer

Proof:

Replacing n by 3n+2 in (8), it is seen that

$\frac{92x_{3n+3}-18y_{3n+3}+1584}{11}$ is a cubical integer

REMARKABLE OBSERVATIONS:

It is seen that ${f_n}^2 - {g_n}^2 = 4$ (10)

Define ($X= 92x_{n+1}-18y_{n+1}$, $Y = 4\sqrt{23}y_{n+1} - 18\sqrt{23}x_{n+1}$)

Therefore $f_n = \frac{X}{11}, g_n = \frac{Y}{11}$

Substituting the above values of (f_n , g_n) in (10), we have

$X^2 - Y^2 = 484$ which represents a hyperbola.

Similarly, employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the table 1 below

TABLE: 1

S.No	Hyperbola	(X, Y)
1)	$X^2 - Y^2 = 484$	$(92x_{n+1} - 18y_{n+1}, \frac{92y_{n+1}-414x_{n+1}}{\sqrt{23}})$
2)	$X^2 - Y^2 = 6969600$	$(21399x_{n+1} - 9x_{n+3}, \frac{46x_{n+3}-102626x_{n+1}}{\sqrt{23}})$
3)	$X^2 - Y^2 = 60710000$	$(12052x_{n+2} - 892y_{n+2}, \frac{12052y_{n+2}-20516x_{n+2}}{\sqrt{23}})$
4)	$X^2 - Y^2 = 116013312$	$(92x_{n+2} - 892y_{n+1}, \frac{552y_{n+1}-48x_{n+2}}{\sqrt{23}})$
5)	$X^2 - Y^2 = 641203684$	$(205252x_{n+1} - 18y_{n+3}, \frac{92y_{n+3}-984354x_{n+1}}{\sqrt{23}})$

II Define $(X = 92\,x_{2n+2} - 18y_{2n+2} + 22\ , Y = 4\sqrt{23}y_{n+1} - 18\sqrt{23}x_{n+1})$

Therefore $f_n{}^2 = \frac{X}{11}\ , g_n{}^2 = \frac{Y}{11}$

Substituting the above values of $(f_n{}^2\ , g_n{}^2)$ in (10), we have $Y^2 = 11X - 484$
which represents a parabola.

Similarly, Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the table 2 below

TABLE: 2

S.No	PARABOLA	(X,Y)
1)	$Y^2 = 11X - 484$	$\left(103x_{2n+2} - 18y_{2n+2} + 22,\ \frac{96y_{n+1} - 414x_{n+1}}{\sqrt{23}}\right)$
2)	$Y^2 = 63360X - 6969600$	$\left(21399x_{n+1} - 9x_{n+3},\ \frac{46x_{n+3} - 102626x_{n+1}}{\sqrt{23}}\right)$
3)	$Y^2 = 552X - 13156$	$\left(92x_{n+1} - 18y_{n+1},\ \frac{92y_{n+1} - 414x_{n+1}}{\sqrt{23}}\right)$
4)	$Y^2 = 12672X - 116013312$	$\left(92x_{n+2} - 892y_{n+1},\ \frac{552y_{n+1} - 48x_{n+2}}{\sqrt{23}}\right)$

III Let p,q ; p>q>0 be the generators of the Pythagorean triangle T(α,β, γ) , where α = 2pq,

β= p^2-q^2, γ= p^2+q^2, p>q>0 Let A,P represent the area and perimeter of T respectively, where

A = pq$(p^2 - q^2)$ P=2p(p+q)

Note that γ- β= $2q^2$, γ –α =$(p - q)^2$

2(γ –α)=23(γ- β)-22 (11)

Gives $(p - q)^2 = 23q^2 - 11$ (12)

Comparing (12) with (1), we have $p = x_{n+1} + y_{n+1}, q = x_{n+1}$ (13)

Thus, the Pythagorean triangle T with generators p, q given by (13) is such that

2α – 23β + 21γ = 22

In a similar manner, the other relations for the Pythagorean triangle T are presented below.

a) $25\beta - 23\gamma - \frac{8A}{P} = -22$
b) $2\alpha - \frac{4A}{P} + \beta = (2x_{n+1} + y_{n+1})^2$
c) $\gamma - \frac{4A}{P} - \alpha + \beta = 2{y_{n+1}}^2$
d) $\frac{2A}{P} = x_{n+1}y_{n+1}$

Conclusion

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative Pell equation $y^2=23x^2-11$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of negative Pell equations and determine their integer solutions along with suitable properties.

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