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# Stability Analysis of Three-Species Syn Ecology Consisting of a Prey-Predator and a Super Predator with Limited and Unlimited Resources

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**Abstract:**

In this paper, we study on a three species ($S_1$, $S_2$, $S_3$) syn ecology consisting of a prey, predator and super predator with limited and unlimited resources. Predator ($S_2$) surviving on the prey ($S_1$) and super predator ($S_3$) surviving on the predator ($S_2$). The mathematical model equations of the system constitute a set of three first order non-linear simultaneous differential equations. All the possible five equilibrium points of the model are identified. The system would be stable, if all the characteristic roots are negative, in case they are real and have negative real parts, in case they are complex. Further, the criteria for global stability of linearized equations are discussed employing a properly constructed Liapunov's function.

**Keywords:** Asymptotically stable, Characteristic equation, Equilibrium State, Stable, Unstable.

**2010 Mathematics Subject Classification:** 92D25, 92D40

## 1. Introduction

Theoretical Ecology is the study of the interactions between organisms and their environment. The organisms include animals and plants, the environment includes the surroundings of animals. So ecology relates to the study of living beings (animals and plants) in relation to their habits and habitats. This discipline of knowledge is a branch of evolutionary biology purported to explain how or to what extent the living beings are regulated in nature. Allied to the problem of population regulation is the problem of species distribution- commensalism, prey-predator, competition and so on. Significant researches in the area of theoretical ecology have been discussed by Gillman [3] and by Kot [4]. Several ecologists and mathematicians contributed to the growth of this area of knowledge. Mathematical ecology can be broadly divided into two main sub-divisions, Autecology and

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Synecology, which are described in the treatises of Anna Sher [1], Arumugam [2] and Sharma [21].

Mathematical Modeling plays a key role in providing insight into the mutual relationships (positive, negative) between the interacting species. Several authors Ma [6], Moghadas [7], Murray [8] and Sze-Bi Hsu [23] were introduced the general concepts of Modeling in Biological Science. Srinivas [22] studied the competitive ecosystem of two species and three species with limited and unlimited resources. Later, Narayan [9] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Further, Kumar [5] studied some mathematical models of ecological commensalism. The present author Prasad [10-20] investigated continuous and discrete models on two, three and four species syn-ecosystems.

Predation is the interaction where one species gets benefits at the expense of the other. In this interaction between two organisms one organism captures bio-mass from another. This is called as predator. In this interaction one organism eats away the another with its closeness of association.

Some real-life examples of the model are given in the following Table.1

**Table.1**

| Sl. No. | $S_1$ | $S_2$ | $S_3$ |
|---|---|---|---|
| 1 | Butterfly | Frog | Eagle |
| 2 | Grasshopper | Rat | Wolf |
| 3 | Frog | Python | Eagle |

## 2. Notation

$N_i(t)$ : The population strength of $S_i$ at time $t$, $i = 1,2,3$

$t$ : Time instant

$a_i$ : Natural growth rate of $S_i$, $i = 1,2,3$

$a_{11}$ : Self inhibition coefficients of $S_1$

$a_{23}, a_{32}$ : Interaction coefficients of $S_2$ due to $S_3$

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$a_{12}, a_{21}$ : Interaction coefficients of $S_1$ due to $S_2$

$k_1 = \frac{a_1}{a_{11}}$ : Carrying capacity of $S_1$

Further the variables $N_1, N_2, N_3$ are non-negative and the model parameters $a_1, a_2, a_3, a_{11}$, $a_{12}, a_{21}, a_{23}, a_{32}$ are assumed to be non-negative constants.

### 3. Basic Equations

The model equations for syn ecosystem is given by the following system of first order non-linear ordinary differential equations.

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^{\,2} - a_{12} N_1 N_2 \qquad (1)$$

$$\frac{dN_2}{dt} = a_2 N_2 + a_{21} N_1 N_2 - a_{23} N_2 N_3 \qquad (2)$$

$$\frac{dN_3}{dt} = a_3 N_3 + a_{32} N_2 N_3 \qquad (3)$$

### 4. Equilibrium States

The system under investigation has five equilibrium states given by $\frac{dN_i}{dt} = 0, i = 1, 2, 3$

$$E_1 : \overline{N_1} = 0,\ \overline{N_2} = 0,\ \overline{N_3} = 0;\ E_2 : \overline{N_1} = k_1,\ \overline{N_2} = 0,\ \overline{N_3} = 0$$

$$E_3 : \overline{N_1} = 0,\ \overline{N_2} = \frac{-a_3}{a_{32}},\ \overline{N_3} = \frac{a_2}{a_{23}};\ E_4 : \overline{N_1} = \frac{-a_2}{a_{21}},\ \overline{N_2} = \theta_2,\ \overline{N_3} = 0$$

$$\text{Where } \theta_2 = \frac{a_1 a_{21} + a_{11} a_2}{a_{12} a_{21}} > 0 \qquad (4)$$

$$E_5 : \overline{N_1} = \theta_1,\ \overline{N_2} = \frac{-a_3}{a_{32}},\ \overline{N_3} = \alpha_3$$

$$\text{Where } \theta_1 = \frac{a_1 a_{32} + a_{12} a_3}{a_{11} a_{32}} > 0,\ \alpha_3 = \frac{a_2 + \theta_1 a_{21}}{a_{23}} > 0 \qquad (5)$$

### 5. Stability of Equilibrium States

Let $N = (N_1, N_2, N_3) = \overline{N} + U$ $\qquad (6)$

Where $U = (u_1, u_2, u_3)^T$ is a small perturbation over the equlibrium state $\overline{N} = \left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right)$.

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The equations (1), (2), (3) are quasi-linearized to obtain the equation for the perturbed state as,

$$\frac{dU}{dt} = AU \tag{7}$$

Where $A=\begin{pmatrix} a_1 - 2a_{11}\overline{N_1} - a_{12}\overline{N_2} & -a_{12}\overline{N_1} & 0 \\ a_{21}\overline{N_2} & a_2 + a_{21}\overline{N_1} - a_{23}\overline{N_3} & -a_{23}\overline{N_2} \\ 0 & a_{32}\overline{N_3} & a_3 + a_{32}\overline{N_2} \end{pmatrix}$ (8)

The Characteristic equation is $|A - \lambda I| = 0$ (9)

The Equilibrium state is stable, if all the roots of equation (9) are negative.

***5.1 Stability of*** $E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$

In this case, we have $A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ (10)

The characteristic equation is $(\lambda - a_1)(\lambda - a_2)(\lambda - a_3) = 0$ (11)

The Characteristic roots are $a_1, a_2, a_3$. Since all the three roots are positive values.

Hence the fully washed out state is **unstable** and the solutions of the equations (7) are

$$u_i = u_{i0} e^{a_i . t}; i = 1,2,3 \tag{12}$$

where $u_{10}, u_{20}, u_{30}$ are the initial values of $u_1, u_2, u_3$ respectively.

**5.2 Equilibrium state** $E_2 : \overline{N_1} = k_1,\ \overline{N_2} = 0,\ \overline{N_3} = 0$

In this state, we have $A = \begin{pmatrix} -a_1 & -a_{12}k_1 & 0 \\ 0 & a_2 + k_1 a_{21} & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ (13)

The characteristic equation is $(\lambda + a_1)\left[\lambda - (a_2 + k_1 a_{21})\right](\lambda - a_3) = 0$ (14)

$-a_1,\ a_2 + k_1 a_{21}$ and $a_3$ are the characteristic roots of (14). Since two of these three roots are positive values, hence the state is **unstable** and the equations (7) yield the solutions.

$$u_1 = (u_{10} - A_{10}) e^{-a_1 t} + A_{10} e^{(a_2 + a_{21}k_1)t};\ u_2 = u_{20} e^{(a_2 + a_{21}k_1)t};\ u_3 = u_{30} e^{a_3 t} \tag{15}$$

where $A_{10} = \dfrac{-k_1 a_{12} u_{20}}{a_1 + a_2 + a_{21}k_1}$ (16)

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**5.3 Equilibrium state** $E_3 : \overline{N_1} = 0,\ \overline{N_2} = \frac{-a_3}{a_{32}},\ \overline{N_3} = \frac{a_2}{a_{23}}$

In this state, we have $A = \begin{pmatrix} \delta_1 & 0 & 0 \\ \frac{-a_{21}a_3}{a_{32}} & 0 & \frac{a_{23}a_3}{a_{32}} \\ 0 & \frac{a_2 a_{32}}{a_{23}} & 0 \end{pmatrix}$ (17)

where $\delta_1 = a_1 + \frac{a_{12}a_3}{a_{32}} > 0$ (18)

The characteristic equation is $(\lambda - \delta_1)(\lambda^2 - a_2 a_3) = 0$ (19)

The Characteristic roots of (19) are $\delta_1,\ \sqrt{a_2 a_3},\ -\sqrt{a_2 a_3}$

Since, two of these three roots are positive, hence the state is **unstable** and the equations (7) yield the solutions.

$$u_1 = u_{10}e^{\delta_1 t};\ u_2 = \left(u_{20} - A_{20}\right)\cosh\sqrt{a_2 a_3}t + \left(B_3 u_{30} - A_{30}\right)\sinh\sqrt{a_2 a_3}t + A_{20}e^{\delta_1 t} \quad (20)$$

$$u_3 = \left(\frac{u_{20} - A_{20}}{B_3}\right)\sinh\sqrt{a_2 a_3}t + \left(\frac{B_3 u_{30} - A_{30}}{B_3}\right)\cosh\sqrt{a_2 a_3}t + Be^{\delta_1 t} \quad (21)$$

where

$$A_{20} = \frac{a_{21}a_3 u_{10}\delta_1}{a_{32}\left(a_2 a_3 - \delta_1^2\right)},\ B_3 = \frac{a_{23}a_3}{a_{32}\sqrt{a_2 a_3}} > 0,\ B = \frac{a_{32}A_{20}\delta_1 + a_{21}a_3 u_{10}}{a_{23}a_3} > 0,\ A_{30} = BB_3 \quad (22)$$

**5.4 Equilibrium state** $E_4 : \overline{N_1} = \frac{-a_2}{a_{21}},\ \overline{N_2} = \theta_2,\ \overline{N_3} = 0$

In this state, we get $A = \begin{pmatrix} \frac{a_{11}a_2}{a_{21}} & \frac{a_{12}a_2}{a_{21}} & 0 \\ a_{12}\theta_2 & 0 & -a_{23}\theta_2 \\ 0 & 0 & a_3 + a_{32}\theta_2 \end{pmatrix}$ (23)

The characteristic equation is $\left[\lambda - \left(a_3 + a_{32}\theta_2\right)\right]\left[\lambda^2 - \frac{a_{11}a_2}{a_{21}}\lambda - a_{12}a_2\theta_2\right] = 0$ (24)

Since, one root $a_3 + a_{32}\theta_2$ is positive, hence the state is **unstable**.

Let $\lambda_1,\ \lambda_2$ be the zeros of the quadratic polynomial on the above equation (24).

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The solutions are given by

$$u_1 = \left[\frac{\overline{A_2}\left(\overline{C}-u_{10}\right)-\left(\overline{D}-u_{20}\right)}{\overline{A_1}-\overline{A_2}}\right]e^{\lambda_1 t} + \left[\frac{\left(\overline{D}-u_{20}\right)-\overline{A_1}\left(\overline{C}-u_{10}\right)}{\overline{A_1}-\overline{A_2}}\right]e^{\lambda_2 t} + \overline{C}e^{(a_3+a_{32}\theta_2)t} \qquad (25)$$

$$u_2 = \left[\frac{\overline{A_2}\left(\overline{C}-u_{10}\right)-\left(\overline{D}-u_{20}\right)}{\overline{A_1}-\overline{A_2}}\right]\overline{A_1}e^{\lambda_1 t} + \left[\frac{\left(\overline{D}-u_{20}\right)-\overline{A_1}\left(\overline{C}-u_{10}\right)}{\overline{A_1}-\overline{A_2}}\right]\overline{A_2}e^{\lambda_2 t} + \overline{D}e^{(a_3+a_{32}\theta_2)t} \qquad (26)$$

$$u_3 = u_{30}e^{(a_3+a_{32}\theta_2)t} \qquad (27)$$

where

$$\left.\begin{array}{l} \overline{A_1} = \dfrac{a_{21}\lambda_1}{a_{12}a_2} - \dfrac{a_{11}}{a_{12}},\ \ \overline{A_2} = \dfrac{a_{21}\lambda_2}{a_{12}a_2} - \dfrac{a_{11}}{a_{12}},\ \ \overline{B} = \dfrac{a_{12}a_{23}a_2\theta_2 u_{30}}{a_{21}} > 0, \\ \overline{C} = \dfrac{\overline{B}}{a_{12}a_2\theta_2 + \dfrac{a_{11}a_2}{a_{21}}\left(a_3+a_{32}\theta_2\right) - \left(a_3+a_{32}\theta_2\right)^2},\ \ \overline{D} = \dfrac{a_{21}\left(a_3+a_{32}\theta_2\right)\overline{C}}{a_{12}a_2} - \dfrac{a_{11}\overline{C}}{a_{12}} \end{array}\right\} \qquad (28)$$

with $a_{12}a_2\theta_2 + \dfrac{a_{11}a_2}{a_{21}}\left(a_3+a_{32}\theta_2\right) \neq \left(a_3+a_{32}\theta_2\right)^2$

**5.5 Equilibrium state** $E_5 : \overline{N_1} = \theta_1,\ \overline{N_2} = \dfrac{-a_3}{a_{32}},\ \overline{N_3} = \alpha_3$

In this state, the matrix $A = \begin{pmatrix} -a_{11}\theta_1 & -a_{12}\theta_1 & 0 \\ \dfrac{-a_{21}a_3}{a_{32}} & 0 & \dfrac{a_{23}a_3}{a_{32}} \\ 0 & a_{32}\alpha_3 & 0 \end{pmatrix} \qquad (29)$

The characteristic equation is $\lambda^3 + a_{11}\theta_1\lambda^2 - \left[a_{23}a_3\alpha_3 + \dfrac{a_{12}a_{21}a_3\theta_1}{a_{32}}\right]\lambda - a_{11}a_{23}a_3\theta_1\alpha_3 = 0 \quad (30)$

Let $\lambda_1, \lambda_2, \lambda_3$ be the roots of above equation.

If $\lambda_1, \lambda_2, \lambda_3$ noted to be negative, hence the state is **stable** and the equations (7) yield the solutions.

$$u_1 = \frac{\overline{\alpha_1}}{\overline{\overline{\alpha}}}e^{\lambda_1 t} + \frac{\overline{\alpha_2}}{\overline{\overline{\alpha}}}e^{\lambda_2 t} + \left(u_{10} - \frac{\overline{\alpha_3}}{\overline{\overline{\alpha}}}\right)e^{\lambda_3 t} \qquad (31)$$

$$u_2 = \frac{X_1\overline{\alpha_1}}{\overline{\overline{\alpha}}}e^{\lambda_1 t} + \frac{X_2\overline{\alpha_2}}{\overline{\overline{\alpha}}}e^{\lambda_2 t} + X_3\left(u_{10} - \frac{\overline{\alpha_3}}{\overline{\overline{\alpha}}}\right)e^{\lambda_3 t} \qquad (32)$$

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$$u_3 = \frac{Y_1\overline{\alpha_1}}{\overline{\alpha}} e^{\lambda_1 t} + \frac{Y_2\overline{\alpha_2}}{\overline{\alpha}} e^{\lambda_2 t} + Y_3\left(u_{10} - \frac{\overline{\alpha_3}}{\overline{\alpha}}\right) e^{\lambda_3 t} \tag{33}$$

where

$$\left.\begin{array}{l}
X_1 = \frac{\lambda_1}{a_{12}\theta_1} - \frac{a_{11}}{a_{12}},\ X_2 = \frac{\lambda_2}{a_{12}\theta_1} - \frac{a_{11}}{a_{12}},\ X_3 = \frac{\lambda_3}{a_{12}\theta_1} - \frac{a_{11}}{a_{12}}, \\
\mathrm{Y}_1 = -\left(\frac{a_{21}}{a_{23}} + \frac{a_{32}\lambda_1 X_1}{a_{23}a_3}\right),\ \mathrm{Y}_2 = -\left(\frac{a_{21}}{a_{23}} + \frac{a_{32}\lambda_2 X_2}{a_{23}a_3}\right),\ \mathrm{Y}_3 = -\left(\frac{a_{21}}{a_{23}} + \frac{a_{32}\lambda_3 X_3}{a_{23}a_3}\right) \\
\overline{\alpha} = (X_3 - X_1)(Y_3 - Y_2) - (X_3 - X_2)(Y_3 - Y_1), \\
\overline{\alpha_1} = (X_3 - X_2)(u_{30} - Y_3 u_{10}) - (Y_3 - Y_2)(u_{20} - Y_3 u_{10}), \\
\overline{\alpha_2} = (Y_3 - Y_1)(u_{20} - X_3 u_{10}) - (X_3 - X_1)(u_{30} - Y_3 u_{10}), \\
\overline{\alpha_3} = \overline{\alpha_1} + \overline{\alpha_2}
\end{array}\right\} \tag{34}$$

## 6. Liapunov's Function for Global Stability

In section 5 we discussed the local stability of all five equilibrium states. From which only one state $E_5\left(\bar{N}_1, \bar{N}_2, \bar{N}_3\right)$ is **stable** and rest of them **unstable**. We now examine the global stability of dynamical system (1), (2) and (3) at this state by suitable Liapunov's function.

**Theorem:** *The equilibrium state* $E_5\left(\bar{N}_1, \bar{N}_2, \bar{N}_3\right)$ *is globally asymptotically stable.*

**Proof:** Let us consider the following Liapunov's function

$$L\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right) = N_1 - \overline{N_1} - \overline{N_1}\ln\left(\frac{N_1}{\overline{N_1}}\right) + l_1\left[N_2 - \overline{N_2} - \overline{N_2}\ln\left(\frac{N_2}{\overline{N_2}}\right)\right] + l_2\left[N_3 - \overline{N_3} - \overline{N_3}\ln\left(\frac{N_3}{\overline{N_3}}\right)\right]$$

where $l_1$ and $l_2$ are suitable constants to be determined as in the subsequent steps.

Now, the time derivative of L, along with solutions of (1), (2) and (3) can be written as

$$\frac{dL}{dt} = \left[\frac{N_1 - \overline{N_1}}{N_1}\right]\frac{dN_1}{dt} + l_1\left[\frac{N_2 - \overline{N_2}}{N_2}\right]\frac{dN_2}{dt} + l_2\left[\frac{N_3 - \overline{N_3}}{N_3}\right]\frac{dN_3}{dt}$$

$$\frac{dL}{dt} = \left[N_1 - \overline{N_1}\right](a_1 - a_{11}N_1 - a_{12}N_2) + l_1\left[N_2 - \overline{N_2}\right](a_2 + a_{21}N_1 - a_{23}N_3) + l_2\left[N_3 - \overline{N_3}\right](a_3 + a_{32}N_2)$$

$$\frac{dL}{dt} = -a_{11}\left(N_1 - \overline{N_1}\right)^2 + (l_1 a_{21} - a_{12})\left(N_1 - \overline{N_1}\right)\left(N_2 - \overline{N_2}\right) + (l_2 a_{32} - l_1 a_{23})\left(N_2 - \overline{N_2}\right)\left(N_3 - \overline{N_3}\right)$$

The positive constants $l_1$ and $l_2$ as so chosen that, the coefficients of $\left(N_1 - \bar{N}_2\right)\left(N_2 - \bar{N}_3\right)$ and $\left(N_2 - \bar{N}_2\right)\left(N_3 - \bar{N}_3\right)$ vanish.

Then we have $l_1 = \frac{a_{12}}{a_{21}} > 0$ , $l_2 = \frac{a_{23}a_{12}}{a_{21}a_{32}} > 0$ and, with this choice of the constants $l_1$ and $l_2$

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$\frac{dL}{dt} < 0$ (Negative definite)

Hence, the steady state is **globally asymptotically stable.**

## 6. Conclusion

This paper deals with an investigation on the stability on three species syn ecology consisting of a prey - predator and a super predator with limited resources for the prey. The model equations constitute a set of three first order non-linear coupled differential equations. All possible equilibrium states of the model are identified and criteria for their stability is discussed. It is observed that, in all states, only the equilibrium state $E_5$ is **stable.** Further, the global stability of the system is established with the aid of suitably constructed Liapunov's function.

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