NearlyProjectiveSemimodules

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ABSTRACT
Other researchers have previously proposed and studied the concept of a nearly projective module. Inthis paper, the above concept will be analyzed for semimodules, along with related ideas. A semimodule Pis considered to beroughly projective if ∀surjective-homα:A→B, where A,B are any two semimodules, and ∀ homβ: P→B, ∃γ: P→A homs.t παγ=πβ where π:B→B/J(B) is the natural map.

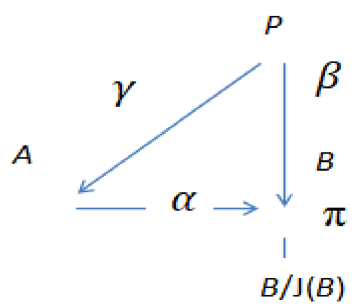
Keywords:studied,semimodule,concept

1. INTRODUCTION

Naoum and Al-Mothafar [2,9] proposed the notion of a nearly projective module as a generalization of a projective module relative to Jacobson radical. A module P is nearly projective if ∀ epimf:M→N and homg:P→N, ∃a homh:P→M such that (fh-g)(a)∈J(N), ∀a∈P.M and N are arbitrary modules, and J(N) is the Jacobsonradical of the module N. It is clear that an equivalent condition to the condition "(h-g)(a)∈J(N),for each a∈P" is πfh=πg where π is the natural map of N onto N/J(N). This fact helps to define nearly projective semimodule. Characterizations of this concept are given with some properties. Some results relating nearly projective with projective semimodules(presented in a previous paper[6,8]) are concluded. Conditions for nearly projective semimodule to beprojective are proved. Finally, a nearly-quasi- projectivesemimodule is defined, and some properties are proved. It is proved that for the class of semimodules over a semiring in which the direct sum of any two nearly-quasi-projectivesemimodules is nearly-quasi-projective, in such class any nearly-quasi-projectiveis nearly projective.
Risasemiringwithidentityinwhatfollows,andthesemimodulesareunitaryleftR-semimodules.

2. NearlyProjectivesemimodules

Definition2.1. A semimodule P is said to be nearly projective (N-projective) if ∀ surjective homα:A→B, where A,B are any two semimodules and ∀ homβ: P→B, ∃ a homeγ: P→A such that παγ=πβ where π:B→B/J(B) is the natural map.



Definition2.2.[5].AnR-semimoduleAisArtinianifanynon-emptysetof S-subsemimodules of A have minimal membersconcerning set inclusion.
Definition2.3. [6] An epimα:P→M is called a projective cover of M↔P is projective, and α is a small epim.
Definition2.4.[5].AleftR-semimoduleNisretractedofaleftR-semimoduleM↔ ∀asurjectiveR- homθ:M→N and an R-hom
δ:N→Msatisfyingtheconditionthatθδ=1_N.
Remark2.5.
1- Everyprojectivesemimoduleisnearlyprojective.

2- If a semimoduleP hasno maximalsubsemimodule,e.g.,$J(P)=P,\beta(P)=\beta(J(P))\subseteq J(B)$, hence$\pi\beta=0$,that is,Pis nearly projective. Thus, an almost projective semimodule may not be projective.

Lemma2.6.If$\alpha:F\to P$isasurjectivehomofsemimoduleand$\theta\in End(F)\exists \ker\alpha\subseteq \ker\theta$,then$\exists\alpha':P\to F$ such that $\alpha'\alpha=\theta$.

Proof:Since$\alpha:F\to P$issurjective,thenPisisomorphictoF/kerα,so$\exists\delta:P\to F/\ker\alpha$,whichisan isomorphism.Define$\beta:F/\ker\alpha\to F/\ker\theta$as$\beta(x/\ker\alpha)=x/\ker\theta$whereiswelldefined,anddefine $\sigma:F/\ker\theta\to \theta(F)\leq F$ as$\sigma(x/\ker\theta)=\theta(x)\leq F$.

Now $\alpha'=\sigma\beta\delta:P\to F$,then$\alpha'\alpha=\sigma\beta\delta\alpha=\theta$.Thus$\alpha'\alpha=\theta$.[$\theta\in End(F)$].

Lemma2.7.If$g:P\to B$isahom,then$\exists g':P/J(P)\to B/J(B)$suchthat$g'\pi_1=\pi_2 g$where$\pi_1:P\to P/J(P)$and $\pi_2:B\to B/J(B)$ are the natural maps.

Proof: Assum π_1,π_2betwo natural maps as$\pi_1:x\mapsto x/J(P)$,$\pi_2:y\mapsto y/J(B)$and g is a hom of semimodules byassumption.

Define $g':P/J(P)\to B/J(B)$ as $g'(x/J(P))=g(x)/J(B)$, since $g(J(P))\subseteq J(B)$, then g'Is well-defined. $g'\pi_1(x)=g'(x/J(P))=g(x)/J(B)=\pi_2 g(x)$,so$g'\pi_1=\pi_2 g$.

Theorem2.8. AssumePbeR-semimodule:

1. IfPisnearlyprojective,thenforeachexactsequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\ \alpha\ } P \longrightarrow 0$$

SuchthatFisfreeandK=kerα,$\exists$ahome$\theta\in End(F)$satisfies:

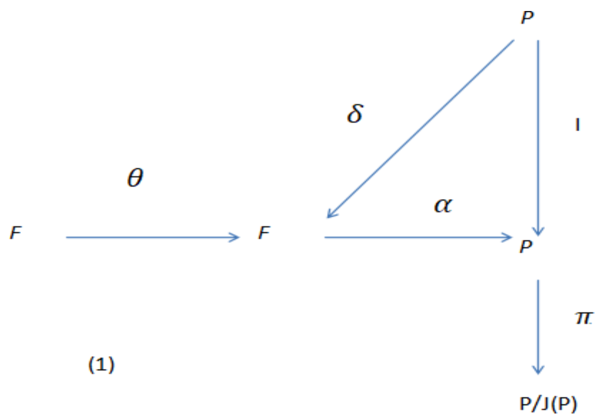
a. $\pi\alpha\theta=\pi\alpha$where$\pi:P\to P/J(P)$isthenaturalmap.

b. $\ker\alpha\subseteq\ker\theta$.

2-Ifthereisanexactsequenceasabovewithk-regular$\alpha$and$\theta\in End(F)$satisfying(a)and(b)of(1), then P is N-projective.

Proof(1):AssumethatPisnearlyprojective,inthediagram(1),$\exists\delta:P\to F$hom,suchthat $\pi\alpha\delta=\pi I=\pi$.....(1),take $\theta=\delta\alpha$.....(2).

Then$\pi\alpha\theta=\pi\alpha\delta\alpha=\pi\alpha$.Forevery $a\in\ker\alpha$impliesa$\in\ker\delta\alpha=\ker\theta$,thus $\ker\alpha\subseteq\ker\theta$.



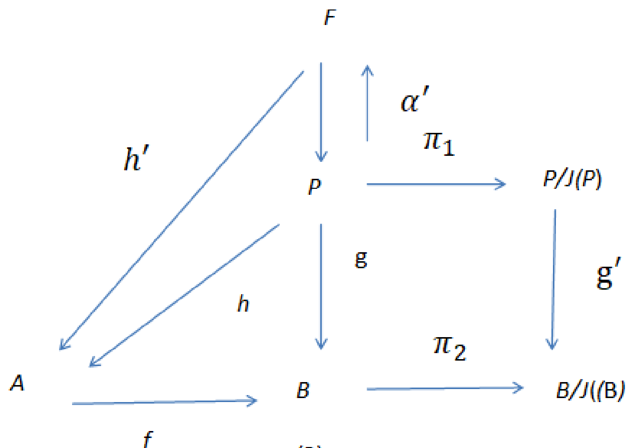
(2)Byassumption$\exists\theta\in End(F)$satisfying(a)and(b).

Define$\alpha':P\to F$by$\alpha'(a)=\theta(x_a)$where$\{x_a\}$isabasisforFand$\{\alpha(x_a)=a\}$isageneratingsetofP; if$\alpha(x_a)=\alpha(y_a)=a$, then $x_a+k=y_a+k'$ for some $k,k'\in\ker\alpha$(Since$\alpha:F\to P$ is k-regular). So,$\theta(x_a)=\theta(y_a)$, that is,α'Is well-defined.

Since F is a projectiveR-semimodule,$\exists h':F\to A$ such that $fh'=g\alpha$. Define$h=h'\alpha'$.

In diagram (2):

$\pi_1:P\to P/J(P)$ and $\pi_2:B\to B/J(B)$ are the natural maps where $g'\pi_1=\pi_2 g$ by [Lemma1.3.]$\pi_2 fh(a)=\pi_2 fh'\alpha'(a)=\pi_2 fh'\theta(x_a)=\pi_2\theta(x_a)=g'\pi_1\alpha\ \theta(x_a)=g'\pi_1\alpha(x_a) = g'\pi_1(a)=\pi_2 g(a)$,thatis, $\pi_2 fh=\pi_2 g$.SoP is N-projective.



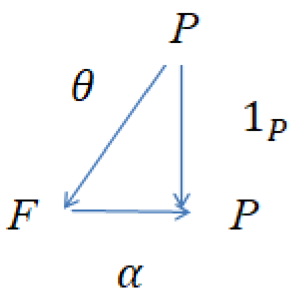
(2)

Definition.2.9.[5]. AleftR-semimoduleNisaretractofR-semimoduleMifandonlyif∃asurjectiveR- homθ:M→N and an R-homδ:N→Msatisfying the condition that θδ is the identity map on N.

Lemma2.10. Ifθδisanisomorphismwhere then there exist $\beta:N\to N$ such that $\theta\delta\beta=1_N$.
Lemma2.11. IfPisaretractofafreesemimodule,thenPisprojective.[Golan]
Proposition2.12.AssumePisanN-projectiveR-semimodule.IfJ(P)issmallinP,Pisprojective[8]. Proof: Since P is N-projective, then by [Theorem 1.4.]for each exact sequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\alpha} P \longrightarrow 0$$ such t

F is free R-semimodule and K=ker α∃a homθ ∈ End(F) satisfying $\pi\alpha\theta = \pi\alpha$ where $\pi:P\to P/J(P)$ is the natural map and since α surjective P=α(F), π(α(F))=π(α(θ(F))) implies P=α(F)=α(θ(F))+J(P) but J(P) is small, so α(F)= α(θ(F)), that is,Pis projective.
Corollary2.13.AssumePis a Hopfian N-projective R-semimodule; if J(P) is small,P is projective.Where Risa semiring and MR-semimodule. If every surjectiveR-endomorphism of M is an isomorphism, one calls M Hopfian.
Proof: In the diagram,P is a Hopfian N-projective,andF is a free semimodule with α is surjective then by (Theorem 1.4.) $\alpha(\theta(P)=1_P(P)=P$, that is,αθ is surjective.Since P Hopfianand αθ∈ End(P), then αθis an isomorphismthat is P is a retract of F. By Lemma.1.5.P is projective.



Lemma2.14.IfPisafinitelygeneratedprojectivesemimodule,J(P)issmallinP. Proof: Similar to the case's evidence in modules, see [7.p.159.].
Corollary2.15.AssumePisafinitelygeneratedR-semimodule;PisprojectiveonlyifPisN-projective. Proof: Since Pis projective, then P is N-projective.
Conversely, SinceP is finitely generated, it is Hopfian and J(P) is small in P. By [Lemma1.5],P is projective.
Corollary2.16.AssumePis an N-projective R-semimodule with J(P) is small in P; if P is a multiplication semimodule, P is projective.

Proof: Since Pis anN-projectiveR-semimoduleand J(P) is small in P, if weprove P Hopfian, thenit will be projective by[Lemma 1.7.], now assumefP→P be a surjective map, then f(P)=P, assum K=ker f, then K=JP for some J≤R, then f(K)=J f(P)=JP=K=0, so f is one-one, that is P is Hopfian. Hence,P is projective.

Theorem2.17.ForanyR-semimoduleP,thefollowingstatementsareequivalent.

1- PisanN-projectiveR-semimodule.

2- Foreveryfamily $\{a_i{:}i\in I\}$ofgeneratorsof P overR,∃afamily $\{f_i{:}i\in I\}, f_i\in P^*=Hom(P,R)$with

   a- Foralla∈P, $f_i(a)\neq 0$onlyforfinitelymanyi∈I.

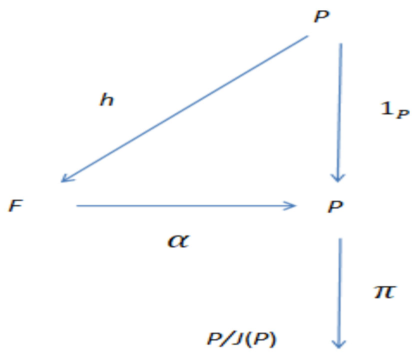
   b- Foralla∈P,$\pi(\sum f_i(a)a_i)=\pi(a)$where$\pi{:}P\to P/J(P)$isthesurjective map.

Proof: Assume that P is N-projective, and assumeFto be afree R-semimodule over P, assum$\{x_i{:}i\in I\}$ be abasis for F, and assum $\alpha{:}F\to P$ be surjective map $\alpha(x_i)=a_i$For all i.

Definehomes$\alpha_i{:}F\to R$asfollows$\alpha_i(\sum r_jx_j)=r_i$ForallI∈I.

Itisclearthatα_iItiswell-defined,andifweput$r_j=0$inthecasethattheindexdoesnotappearin$\sum r_jx_j$ thenforallxinF,$x=\sum r_jx_j$and $\alpha_i(x)\neq 0$foronlyfinitelymanyi∈I.Moreover,$x=\sum \alpha_j(x)x_j$…….(*).

Now, since P is N-projective, then by definition [N-projective], ∃a homh:P→F such that $\pi(\alpha(h(a)))=\pi(a)$, for all an in P (in the diagram) put $f_i=\alpha_i h$, i∈I, then $f_i\in P^*$and for all a∈P, $f_i(a)=\alpha_i h(a)\neq 0$ for only finitely manyi∈I,furthermoreforalla∈P,$a=\sum r_ia_i=\sum r_i\alpha\ (x_i)$.Thus$\pi(\sum f_i(a)a_i)=\ \pi(a)$implies $\pi(\sum(\alpha_i h)(a)\alpha(x_i)=\pi(a)$, so $\pi(\alpha \sum \alpha_i h(a)\ x_i)=\pi(a)$.



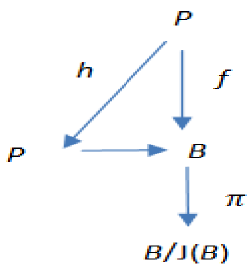
For the converse,assume $\{a_i{:}i\in I\}$ be a set of generators for P. Assum $\{x_i{:}i\in I\}$ be a basis for F such that $\alpha(x_i)=a_i$. Define a map $\theta{:}F\to F$:assumenext show that kerα=kerθ. Assum x∈F such that α(x)=0, then $\alpha(x)=0=\sum f_i(0)a_i$+t.But $f_i(0)=0$ for all I; thus, it is clear that θ(x)=0. It follows from theorem (2.8.) that P is an N-projective R-semimodule.

3. Nearly-Quasi-ProjectiveSemimodules

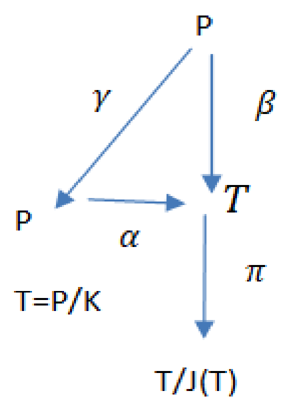
Definition3.1. AnR-semimodulePiscallednearly-quasi-projective (NQ-projective), if for every R-semimodule B and surjective homg:P→B andanyhomf:P→B∃R-hom h:P→Psuchthatπgh=πf,whereπisthenaturalsurjectivemap of B onto B/J(B).

Remark3.1.Everynearlyprojectivesemimoduleisnearlyquasi-projective.

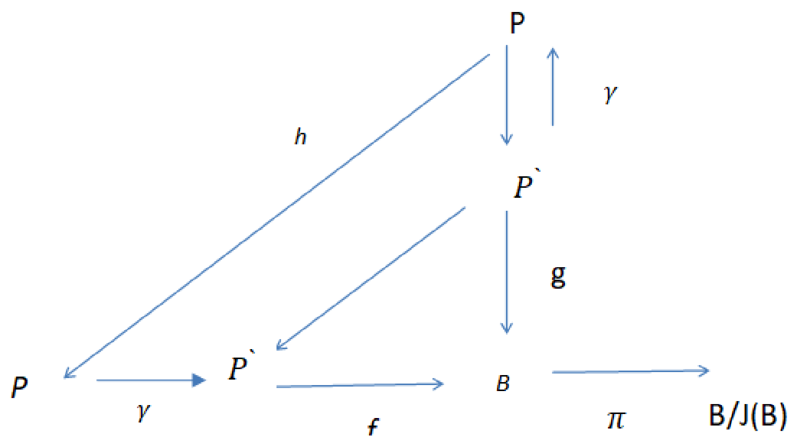
Remark3.2.TheretractionoftheN-quasiprojectiveistheN-quasiprojective.



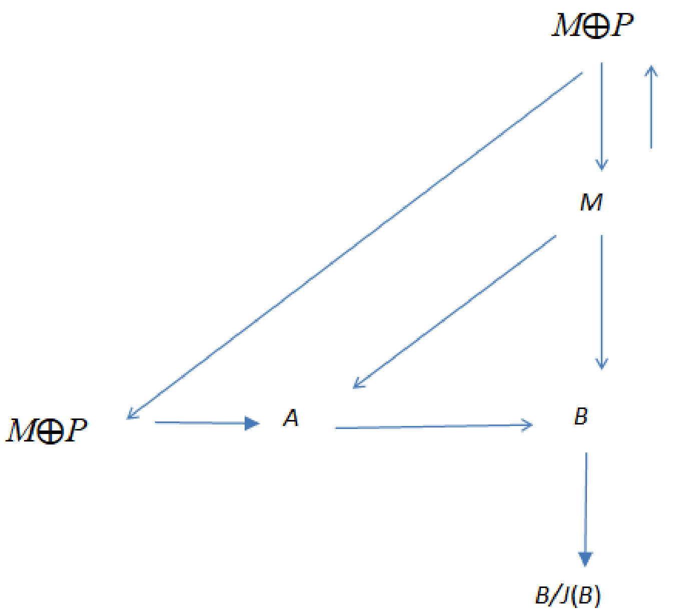
Proposition3.4.AnR-semimodulePisnearlyquasi-projectiveifandonlyifforanyK≤P;β:P→P/K, ∃γ:P→Psuchthatπαγ=πβ.
Proof: Inthediagram,assumePtobequasi-projectiveandK≤P;
αsurjectivehom and β any home, then there
existγS-homsuchthatπαγ=πβ.



Proposition3.5.AnyR-semimoduleretractofanyN-quasiprojective is
N-quasi projective.
Proof: Consider the diagram P is N-quasi projective P`Is a semimodule such that ∃γ:P→ P`and γ`:P` → P with γγ`=$1_{P`}$, then P`is N-quasi projective such that πfγh=πgγ, assum h`= γhγ`so πfγ`=πfγhγ`=πgγγ`=πg, then P is N-projective.



Proposition 3.6. Assum R bea semiringsuchthatany R-semimodule has a projective cover,andthe direct sum of any two N-quasi-projective R-semimodule is N-quasi-projective. Then any N-quasi-projectiveR-semimodule is N-projective R-semimodule.
Proof. AssumeMistheN-quasi-projectiveR-semimoduleandPistheprojectivecoverofM. We
must prove that M is N-projective.In the diagram:
M⊕PisN-quasi-projective,then πgαh`=πfα.
Assumh=αh`i,thenπgh=πgαh`i=πfαi=πf,thenMisN-projective.



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