

MultiSumIdentitiesInvolvingq-BasicSeries

PriyaWadhwa¹,A.K.Shrivastav^{2*},HarwinderKaur³

^{1,3}Department of Mathematics, RIMT University, FGS, Punjab, India 147301

²Department of Mathematics, MMEC, Maharishi Markandeshwar (Deemed to be University), Mullana-Ambala, Haryana, India 133207

Email: priyapuri093@gmail.com¹, aakkaasshhkumar8888@gmail.com², harwinderkaur808@gmail.com³

Received: 18.04.2024

Revised: 24.05.2024

Accepted: 26.05.2024

ABSTRACT

The q-basic infinite products can be used to express certain multi-summation identities of Rogers-Ramanujan. This work attempts to prove certain results concerning the ratio of q-basic infinite products and multi summation expressions utilizing well-known m- dissections of the power series.

Keywords: Bailey pair’s, bailey lemma, m-dissection, RR-type identities.

1. INTRODUCTION

Both applied mathematicians and pure mathematicians from earlier ages have been drawn to continued fractions. Numerous conclusions from earlier centuries have been produced in terms of continuous fraction. It’s additionally a tool that serves as a conduit between applied and pure mathematicians. Thus, the continuous fraction has also become more appealing to mathematicians nowadays.

Ramanujan’s second notebook [5] contains numerous continuing fraction identities, as he was a trailblazer in the field of continued fraction theory. The techniques employed by the renowned Indian mathematician Srinivasa Ramanujan to arrive at numerous intriguing findings are still a mystery. Several authors, including Andrews, have addressed and developed this portion of Ramanujan’s work. Simultaneously Number theory, orthogonal polynomials, affine root systems, Lie algebras and groups, physics, and other sciences all heavily rely on q-Series, often known as basic hypergeometric series.

Among the helpful resources for studying special functions is the inequality method; see [17] for a list of numerous works discussing q-integral and inequalities.

The m-dissection of the power series;

$$P = \sum_{n=0}^{\infty} a_n q^n$$

is the representation of P as

$$P = P_0 + P_1 + \dots + P_{m-1}, \text{ where } P_k = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$$

From last 3 decades, a number of researchers [7], [16] & [20] have given the 2 dissection, 4 dissection and 5 dissections of the continued fraction. Denis et al. [19] gave equivalent continued fraction representations for ratios of infinite products.

$$\begin{aligned} S(q) &= \frac{(q^3, q^5; q^8)_{\infty}}{(q, q^7; q^8)_{\infty}} \\ &= 1 + \frac{q + q^2 q^4 q^3 + q^5 q^8 q^5 + q^{10}}{1 + 1 + \frac{q + q^2 q + q^4}{1 + 1 + \frac{q + q^2 q + q^4}{1 + 1 + \frac{q + q^2 q + q^4}{1 + 1 + \dots}}} \frac{q^{12}}{1 + \dots} \\ &= 1 + \frac{q + q^2 q + q^4}{1 - q + 1 - q + 1 - q + \dots} \frac{q + q^6}{q^2} \\ &= 1 + q + \frac{q^2}{1 + q^3 + 1 + q^5 + 1 + q^7 + \dots} \frac{q^5}{q^4} \end{aligned}$$

One well-known and often applied method in the theory of partitions is the Bailey chain. It enlightens from W. N. Bailey’s formula [21] that the Rogers-Ramanujan identities might be obtained from the simple observation that if $\{\alpha_0, \alpha_1, \dots\}$ and $\{\delta_0, \delta_1, \dots\}$ are sequences that satisfy;

$$\beta_k = \sum_{r=0}^k \alpha_r u_{k-r} v_{k+r}$$

and

$$\gamma_k = \sum_{r=k}^{\infty} \delta_r u_{r-k} v_{r+k}$$

Then

$$\sum_{k=0}^{\infty} \alpha_k \gamma_k = \sum_{k=0}^{\infty} \beta_k \delta_k$$

which stipulated consistent convergence of all infinite sums.

His concept served as the basis for L.J. Slater's list of 130 identities that fit the Rogers-Ramanujan type [13, 14]. Over the past twenty years or more, several authors have explored intriguing and practical applications of Bailey's identity (see for e.g., [1-4]; see also [9-12], the very recent work).

Here an attempt has been made to obtain certain results involving the multi summation expressions and ratio of infinite products by using well known m dissections of the power series.

2. Notation

Suppose that $|q| < 1$, where q is non-zero complex number, this condition ensures that all the infinite products that we will converge. We will use the notation as follows;

$$(z; q)_{\infty} = \prod_{n=0}^{\infty} (1 - zq^n),$$

(2.1)

$$[z; q]_{\infty} = (z; q)_{\infty} (z^{-1}q; q)_{\infty}; \text{ (for } z \neq 0 \text{) and often we write,}$$

(2.2)

$$[z_1, z_2, \dots, z_n; q]_{\infty} = [z_1; q]_{\infty} [z_2; q]_{\infty} \dots [z_n; q]_{\infty}.$$

(2.3)

And also the following fact can be easily verified;

$$[z^{-1}; q]_{\infty} = -z^{-1} [z; q]_{\infty} = [zq; q]_{\infty},$$

(2.4)

$$[z, zq; q^2]_{\infty} = [z; q]_{\infty},$$

(2.5)

$$[z, -z; q]_{\infty} = [z^2; q^2]_{\infty},$$

(2.6)

$$[z^{-1}q; q]_{\infty} = [z; q]_{\infty},$$

(2.7)

$$[-1; q]_{\infty} [q; q^2]_{\infty} = 2.$$

(2.8)

We have the following general relations;

Suppose $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n \in C \setminus \{0\}$ satisfy

i) $a_i \neq q^n a_j$ for $i \neq j$ and $n \in \mathbb{Z}$,

ii) $a_1, a_2, \dots, a_n = b_1, b_2, \dots, b_n$, Then

$$\sum_{i=1}^n \frac{\prod_{j=1}^n [a^{-1} b_j; q]_{\infty}}{\prod_{j=1, j \neq i}^n [a_i^{-1} a_j; q]_{\infty}} = 0$$

(2.9)

This theorem appears without proof as given by Slater [4] and with proof as given by Lewis [8]. Also, we have the following well known Rogers-Ramanujan identity as;

$$\sum_{s_1, s_2, s_3 \geq 0} \frac{q^{9s_1^2 + 3s_2^2 + s_3^2} (q^3; q^3)_{2s_1} (q; q)_{3s_2 - s_3}}{(q^9; q^9)_{2s_1} (q^9; q^9)_{s_1 - s_2} (q^3; q^3)_{2s_2} (q^3; q^3)_{s_2 - s_3} (q; q)_{s_3}} = \frac{(q^{29}, q^{14}, q^{15}; q^{29})_{\infty}}{(q^9; q^9)_{\infty}}$$

(2.10)

$$\sum_{s_1, s_2, s_3 \geq 0} \frac{q^{9s_1^2 + 3s_2^2 + s_3^2} (q^3; q^3)_{2s_1} (q; q)_{3s_2 - s_3}}{(q^9; q^9)_{2s_1} (q^9; q^9)_{s_1 - s_2} (q^3; q^3)_{2s_2} (q^3; q^3)_{s_2 - s_3} (q; q)_{s_3}} = \frac{(q^{29}, q^{14}, q^{15}; q^{29})_{\infty}}{(q^9; q^9)_{\infty}}$$

(2.11)

[D.Bressoud, 1; p452]

3. Main Result

$$\begin{aligned}
 & \sum_{s_1, s_2, s_3 \geq 0} \frac{q^{9s^2+3s^2+s^2} (q; q)_{s_1-s_2}^3 (q; q)_{3s_2-s_3}}{(q^9; q^9)_{2s_1} (q^9; q^9)_{s_1-s_2} (q^3; q^3)_{2s_2} (q^3; q^3)_{s_2-s_3} (q; q)_{s_3}} \\
 &= \frac{(q^5, q^{24}, q^{29}; q^{29})_\infty}{(q^9; q^9)_\infty} \frac{q^9 (q^9, q^{20}, q^{38}, q^{49}; q^{58})_\infty (q^{28}, q^{88}; q^{116})_\infty}{(q^{29}, q^{29}; q^{58})_\infty (q^{10}, q^{18}, q^{58}, q^{58}, q^{98}, q^{106}; q^{116})_\infty} \\
 & \quad + \frac{q^9 (q^9, q^{20}, q^{38}, q^{49}; q^{58})_\infty (q^{30}, q^{86}; q^{116})_\infty}{(q^{29}, q^{29}; q^{58})_\infty (q^{18}, q^{48}, q^{58}, q^{58}, q^{68}, q^{98}; q^{116})_\infty} \\
 & \quad + \frac{q^9 (q^9, q^{20}, q^{38}, q^{49}; q^{58})_\infty (q^{30}, q^{86}; q^{116})_\infty}{(q^{29}, q^{29}; q^{58})_\infty (q^{18}, q^{48}, q^{58}, q^{58}, q^{68}, q^{98}; q^{116})_\infty} \\
 & \quad + \frac{q^2 (q^3, q^4, q^{22}, q^{36}, q^{54}, q^{55}; q^{58})_\infty (q^{32}, q^{34}, q^{84}, q^{86}; q^{116})_\infty}{(q^{16}, q^{17}, q^{41}, q^{42}; q^{58})_\infty (q^8, q^{10}, q^{48}, q^{58}, q^{58}, q^{68}, q^{106}, q^{108}; q^{116})_\infty} \\
 \end{aligned} \tag{3.1}$$

$$\begin{aligned}
 & \frac{(q^5, q^{24}, q^{29}; q^{29})_\infty}{(q^9; q^9)_\infty} = \frac{(q^9, q^{20}, q^{38}, q^{49}; q^{58})_\infty (q^{48}, q^{68}; q^{116})_\infty}{(q^{29}, q^{29}; q^{58})_\infty (q^{18}, q^{30}, q^{58}, q^{58}, q^{86}, q^{98}; q^{116})_\infty} \\
 & \quad + \frac{q^9 (q^9, q^{20}, q^{38}, q^{49}; q^{58})_\infty (q^{10}, q^{106}; q^{116})_\infty}{(q^{29}, q^{29}; q^{58})_\infty (q^{18}, q^{30}, q^{58}, q^{58}, q^{86}, q^{98}; q^{116})_\infty} \\
 & \quad + \frac{q^2 (q^3, q^4, q^{22}, q^{36}, q^{54}, q^{55}; q^{58})_\infty (q^{32}, q^{34}, q^{82}, q^{84}; q^{116})_\infty}{(q^{16}, q^{17}, q^{41}, q^{42}; q^{58})_\infty (q^8, q^{28}, q^{30}, q^{58}, q^{58}, q^{86}, q^{88}, q^{108}; q^{116})_\infty} \\
 & \quad + \frac{q^2 (q^4, q^{12}, q^{13}, q^{45}, q^{46}, q^{54}, q^{58})_\infty (q^{14}, q^{52}, q^{64}, q^{102}; q^{116})_\infty}{(q^7, q^{26}, q^{32}, q^{51}; q^{58})_\infty (q^8, q^{28}, q^{30}, q^{58}, q^{58}, q^{86}, q^{88}, q^{108}; q^{116})_\infty} \\
 & \times \sum_{s_1, s_2, s_3 \geq 0} \frac{q^{9s^2+3s^2+s^2} (q; q)_{s_1-s_2}^3 (q; q)_{3s_2-s_3}}{(q^9; q^9)_{2s_1} (q^9; q^9)_{s_1-s_2} (q^3; q^3)_{2s_2} (q^3; q^3)_{s_2-s_3} (q; q)_{s_3}} \\
 \end{aligned} \tag{3.2}$$

4. Proofs

On considering

$$\begin{aligned}
 & \frac{(q^{14}, q^{15}; q^{29})_\infty}{(q^5, q^{24}; q^{29})_\infty} \\
 & \text{which can be written as} \\
 \end{aligned}$$

$$(q) = \frac{[q^{14}, q^{43}; q^{58}]_\infty}{[q^5, q^{34}; q^{58}]_\infty}$$

by using (2.2).

Now, setting $(a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4) = (1, -1, q^5, q^{34}; q^{14}, q^{43}, -q^{-9}, q^{-9})$ and taking q^{58} for q in (2.9);

$$\begin{aligned}
 & \frac{[q^{14}, q^{43}, -q^{-9}, q^{-9}; q^{58}]_\infty}{[q^9, q^{38}, -q^{-14}, q^{-14}; q^{58}]_\infty} + \frac{[-q^{14}, -q^{43}, -q^{-9}, q^{-9}; q^{58}]_\infty}{[q^9, q^{-20}, -q^{-43}, q^{-43}; q^{58}]_\infty} \\
 & + \frac{[-1, q^5, q^{34}; q^{58}]_\infty}{[q^{-5}, -q^{-5}, q^{29}; q^{58}]_\infty} + \frac{[-1, -q^5, -q^{34}; q^{58}]_\infty}{[q^{-34}, -q^{-34}, q^{-29}; q^{58}]_\infty} = 0 \\
 \end{aligned} \tag{4.1}$$

(4.1)

By using (2.6) and (2.8) in (4.1)

$$\begin{aligned}
 & \frac{[q^{14}, q^{43}; q^{58}]_\infty}{[q^5, q^{34}; q^{58}]_\infty} + \frac{[-q^{14}, -q^{43}; q^{58}]_\infty}{[-q^5, -q^{34}; q^{58}]_\infty} \\
 &= \frac{2}{[q^{18}, q^{58}; q^{116}]_\infty (-q^{-18})} \frac{[q^9, q^{38}; q^{58}]_\infty [q^{-28}; q^{116}]_\infty}{[q^{29}, q^{58}]_\infty [q^{-10}; q^{116}]_\infty} - \frac{[q^9, q^{-20}; q^{58}]_\infty [q^{-86}; q^{116}]_\infty}{[q^{-29}, q^{58}]_\infty [q^{-68}; q^{116}]_\infty} \\
 \end{aligned}$$

By applying (2.4),

$$\begin{aligned}
 & \frac{[q^{14}, q^{43}; q^{58}]_\infty}{[q^5, q^{34}; q^{58}]_\infty} + \frac{[-q^{14}, -q^{43}; q^{58}]_\infty}{[-q^5, -q^{34}; q^{58}]_\infty} = \frac{2}{[q^{18}, q^{58}; q^{116}]_\infty (-q^{-18})} \\
 & \times \left[-\frac{[q^9, q^{38}; q^{58}]_\infty [q^{28}; q^{116}]_\infty}{[q^{29}; q^{58}]_\infty [q^{10}; q^{116}]_\infty} (q^{-18}) - \frac{[q^9, q^{-20}; q^{58}]_\infty [q^{86}; q^{116}]_\infty}{[q^{29}; q^{58}]_\infty [q^{68}; q^{116}]_\infty} (q^{-9}) \right] \\
 \end{aligned}$$

Or,

$$\frac{[q^{14}, q^{43}; q^{58}]_\infty}{[q^5, q^{34}; q^{58}]_\infty} + \frac{[-q^{14}, -q^{43}; q^{58}]_\infty}{[-q^5, -q^{34}; q^{58}]_\infty}$$

Thus,

$$C(q)+C(q)=\frac{2}{[q^{18},q^{58};q^{116}]_{\infty}}\left[\frac{[q^9,q^{38};q^{58}]_{\infty}[q^{28};q^{116}]_{\infty}}{[q^{29};q^{58}]_{\infty}[q^{10};q^{116}]_{\infty}}+\frac{q^9[q^9,q^{20};q^{58}]_{\infty}[q^{86};q^{116}]_{\infty}}{[q^{29};q^{58}]_{\infty}[q^{68};q^{116}]_{\infty}}\right]$$

$$C(q)+C(q)=\frac{2[q^9,q^{38};q^{58}]_{\infty}[q^{28};q^{116}]_{\infty}}{[q^{29};q^{58}]_{\infty}[q^{10},q^{18},q^{58};q^{116}]_{\infty}}+\frac{2q^9[q^9,q^{20};q^{58}]_{\infty}[q^{86};q^{116}]_{\infty}}{[q^{29};q^{58}]_{\infty}[q^{18},q^{58},q^{68};q^{116}]_{\infty}} \tag{4.2}$$

$$= \frac{\alpha(q)}{2} [C(q) + C'(q)]$$

$$= \frac{[q^9,q^{38};q^{58}]_{\infty}[q^{28};q^{116}]_{\infty}}{[q^{29};q^{58}]_{\infty}[q^{10},q^{18},q^{58};q^{116}]_{\infty}} + \frac{q^9[q^9,q^{20};q^{58}]_{\infty}[q^{86};q^{116}]_{\infty}}{[q^{29};q^{58}]_{\infty}[q^{18},q^{58},q^{68};q^{116}]_{\infty}}$$

(4.3)

Again, setting $(a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4) = (1, -1, q^{56}, -q^{60}; q^{44}, q^{43}, -q^5, -q^{24})$ and taking q^{58} for q in (2.9),

$$\frac{[q^{44}, q^{43}, -q^5, -q^{24}; q^{58}]_{\infty}}{[-1, q^{56}, -q^{60}; q^{58}]_{\infty}} + \frac{[-q^{44}, -q^{43}, q^5, q^{24}; q^{58}]_{\infty}}{[-1, -q^{56}, q^{60}; q^{58}]_{\infty}}$$

$$+ \frac{[q^{-12}, q^{-13}, -q^{-51}, -q^{-32}; q^{58}]_{\infty}}{[q^{-56}, -q^{-56}, -q^4; q^{58}]_{\infty}} + \frac{[-q^{-16}, -q^{-17}, q^{-55}, q^{-36}; q^{58}]_{\infty}}{[q^{-60}, -q^{-60}, -q^{-4}; q^{58}]_{\infty}} = 0$$

By applying (2.4) and (2.7),

$$\frac{q^2[[q^{14}, q^{43}, -q^5, -q^{34}; q^{58}]_{\infty} - [-q^{14}, -q^{43}, q^5, q^{34}; q^{58}]_{\infty}]}{q^4[q^{12}, q^{13}, -q^{32}, -q^{51}; q^{58}]_{\infty}} - \frac{[-1, q^2, -q^2; q^{58}]_{\infty}}{[q^{60}, -q^{60}, -q^4; q^{58}]_{\infty}} = 0$$

By using (2.6)

$$= \frac{2[q^4; q^{116}]_{\infty}}{q^2[q^{58}; q^{116}]_{\infty}} \left[\frac{[[q^{14}, q^{43}, -q^5, -q^{34}; q^{58}]_{\infty} - [-q^{14}, -q^{43}, q^5, q^{34}; q^{58}]_{\infty}]}{q^4[q^{12}, q^{13}, -q^{32}, -q^{51}; q^{58}]_{\infty}} + \frac{[-q^{16}, -q^{17}, q^{36}, q^{55}; q^{58}]_{\infty}}{q^5[q^{60}, -q^{60}, -q^4; q^{58}]_{\infty}} \right]$$

Or,

$$= \left[\frac{[[q^{14}, q^{43}, -q^5, -q^{34}; q^{58}]_{\infty} - [-q^{14}, -q^{43}, q^5, q^{34}; q^{58}]_{\infty}]}{[q^7, q^{26}; q^{58}]_{\infty}[q^8, q^{58}; q^{116}]_{\infty}} - \frac{2q^2[q^3, q^4, q^{22}; q^{58}]_{\infty}[q^{32}, q^{34}; q^{116}]_{\infty}}{q^{116}[q^{16}, q^{17}; q^{58}]_{\infty}[q^8, q^{58}; q^{116}]_{\infty}} \right]$$

Dividing by $[-q^5, q^5, q^{34}, -q^{34}; q^{58}]_{\infty}$ and using (2.6),

$$= \frac{1}{[q^{10}, q^{68}; q^{116}]_{\infty}} \left[\frac{2q^2[q^4, q^{12}, q^{13}; q^{58}]_{\infty}[q^{14}, q^{52}; q^{116}]_{\infty}[q^7]_{\infty}}{q^{26}; q^{58}]_{\infty}[q^8, q^{58}; q^{116}]_{\infty}} - \frac{2q^2[q^3, q^4, q^{22}; q^{58}]_{\infty}[q^{32}, q^{34}; q^{116}]_{\infty}}{q^{116}[q^{16}, q^{17}; q^{58}]_{\infty}[q^8, q^{58}; q^{116}]_{\infty}} \right]$$

Or,

$$= \left[\frac{[q^{14}, q^{43}; q^{58}]_{\infty}}{[q^5, q^{34}; q^{58}]_{\infty}} - \frac{[-q^{14}, -q^{43}; q^{58}]_{\infty}}{[-q^5, -q^{34}; q^{58}]_{\infty}} \right]$$

$$= \left[\frac{2q^2[q^4, q^{12}, q^{13}; q^{58}]_{\infty}[q^{14}, q^{52}; q^{116}]_{\infty}}{[q^7, q^{26}; q^{58}]_{\infty}[q^8, q^{10}, q^{58}, q^{68}; q^{116}]_{\infty}} - \frac{2q^2[q^3, q^4, q^{22}; q^{58}]_{\infty}[q^{32}, q^{34}; q^{116}]_{\infty}}{[q^{16}, q^{17}; q^{58}]_{\infty}[q^8, q^{10}, q^{58}, q^{68}; q^{116}]_{\infty}} \right]$$

Thus,

$$C(q)-C(q)=\left[\frac{2q^2[q^4,q^{12},q^{13};q^{58}]_{\infty}[q^{14},q^{52};q^{116}]_{\infty}}{[q^7,q^{26};q^{58}]_{\infty}[q^8,q^{10},q^{58},q^{68};q^{116}]_{\infty}}-\frac{2q^2[q^3,q^4,q^{22};q^{58}]_{\infty}[q^{32},q^{34};q^{116}]_{\infty}}{[q^{16},q^{17};q^{58}]_{\infty}[q^8,q^{10},q^{58},q^{68};q^{116}]_{\infty}}\right],$$

(4.4)

$$\beta_1(q)=C(q)-C(q)=\left[\frac{q^2[q^4,q^{12},q^{13};q^{58}]_{\infty}[q^{14},q^{52};q^{116}]_{\infty}}{[q^7,q^{26};q^{58}]_{\infty}[q^8,q^{10},q^{58},q^{68};q^{116}]_{\infty}}-\frac{q^2[q^3,q^4,q^{22};q^{58}]_{\infty}[q^{32},q^{34};q^{116}]_{\infty}}{[q^{16},q^{17};q^{58}]_{\infty}[q^8,q^{10},q^{58},q^{68};q^{116}]_{\infty}}\right]$$

(4.5)

By adding (4.3) and (4.5) as

$$C(q)=\alpha_1(q)+\beta_1(q),$$

$$C(q)=\frac{[q^9,q^{38};q^{58}]_{\infty}[q^{28};q^{116}]_{\infty}}{[q^{29};q^{58}]_{\infty}[q^{10},q^{18},q^{58};q^{116}]_{\infty}}+\frac{q^9[q^9,q^{20};q^{58}]_{\infty}[q^{86};q^{116}]_{\infty}}{[q^{29};q^{58}]_{\infty}[q^{18},q^{58},q^{68};q^{116}]_{\infty}}$$

$$+\frac{q^2[q^4,q^{12},q^{13};q^{58}]_{\infty}[q^{14},q^{52};q^{116}]_{\infty}}{[q^7,q^{26};q^{58}]_{\infty}[q^8,q^{10},q^{58},q^{68};q^{116}]_{\infty}}$$

$$(4.6) \quad \frac{q^2[q^3, q^4, q^{22}; q^{58}]_\infty [q^{32}, q^{34}; q^{116}]_\infty}{[q^{16}, q^{17}; q^{58}]_\infty [q^8, q^{10}, q^{58}, q^{68}; q^{116}]_\infty}$$

By applying (2.2) in (4.6),

$$(4.7) \quad C(q) = \frac{(q^9, q^{20}, q^{38}, q^{49}; q^{58})_\infty (q^{28}, q^{88}; q^{116})_\infty}{(q^{29}, q^{29}; q^{58})_\infty (q^{10}, q^{18}, q^{58}, q^{58}, q^{98}, q^{106}; q^{116})_\infty} \\ + \frac{q^9 (q^9, q^{20}, q^{38}, q^{49}; q^{58})_\infty (q^{30}, q^{86}; q^{116})_\infty}{(q^{29}, q^{29}; q^{58})_\infty (q^{18}, q^{48}, q^{58}, q^{58}, q^{68}, q^{98}; q^{116})_\infty} \\ + \frac{q^2 (q^4, q^{12}, q^{13}, q^{45}, q^{46}, q^{54}; q^{58})_\infty (q^{14}, q^{52}, q^{64}, q^{102}; q^{116})_\infty}{(q^7, q^{26}, q^{32}, q^{51}; q^{58})_\infty (q^8, q^{10}, q^{48}, q^{58}, q^{68}, q^{106}, q^{108}; q^{116})_\infty} \\ - \frac{q^2 (q^3, q^4, q^{22}, q^{36}, q^{54}, q^{55}; q^{58})_\infty (q^{32}, q^{34}, q^{84}, q^{86}; q^{116})_\infty}{(q^{16}, q^{17}, q^{41}, q^{42}; q^{58})_\infty (q^8, q^{10}, q^{48}, q^{58}, q^{68}, q^{106}, q^{108}; q^{116})_\infty}$$

(4.7)

By taking known identity (2.10)

$$\sum_{s_1, s_2, s_3 \geq 0} \frac{q^{9s^2+3s^2+s^2} (q^3; q^3)_{s_1-s_2} (q; q)_{3s_2-s_3}}{(q^9; q^9)_{2s_1} (q^9; q^9)_{s_1-s_2} (q^3; q^3)_{2s_2} (q^3; q^3)_{s-2s_3} (q; q)_{s_3}} = \frac{(q^{29}, q^{14}, q^{15}; q^{29})_\infty}{(q^9; q^9)_\infty}$$

or,

$$\sum_{s_1, s_2, s_3 \geq 0} \frac{q^{9s^2+3s^2+s^2} (q^3; q^3)_{s_1-s_2} (q; q)_{3s_2-s_3}}{(q^9; q^9)_{2s_1} (q^9; q^9)_{s-2s_2} (q^3; q^3)_{2s_2} (q^3; q^3)_{s-2s_3} (q; q)_{s_3}} = \frac{(q^5, q^{24}, q^{29}; q^{29})_\infty (q^{14}, q^{15}; q^{29})_\infty}{(q^9; q^9)_\infty (q^5, q^{24}; q^{29})_\infty}$$

or,

$$\sum_{s_1, s_2, s_3 \geq 0} \frac{q^{9s^2+3s^2+s^2} (q^3; q^3)_{s_1-s_2} (q; q)_{3s_2-s_3}}{(q^9; q^9)_{2s_1} (q^9; q^9)_{s_1-s_2} (q^3; q^3)_{2s_2} (q^3; q^3)_{s-2s_3} (q; q)_{s_3}} = \frac{(q^5, q^{24}, q^{29}; q^{29})_\infty}{(q^9; q^9)_\infty} C(q),$$

(4.8)

where

$$C(q) = \frac{(q^{14}, q^{15}, q^{29})_\infty}{(q^5, q^{24}, q^{29})_\infty}$$

Now, by putting the value of C(q) from (4.7) in (4.8),

$$\sum_{s_1, s_2, s_3 \geq 0} \frac{q^{9s^2+3s^2+s^2} (q^3; q^3)_{s_1-s_2} (q; q)_{3s_2-s_3}}{(q^9; q^9)_{2s_1} (q^9; q^9)_{s_1-s_2} (q^3; q^3)_{2s_2} (q^3; q^3)_{s_2-s_3} (q; q)_{s_3}} \\ = \frac{(q^5, q^{24}, q^{29}; q^{29})_\infty (q^9, q^{20}, q^{38}, q^{49}; q^{58})_\infty (q^{28}, q^{88}; q^{116})_\infty}{(q^9; q^9)_\infty [(q^{29}, q^{29}; q^{58})_\infty (q^{10}, q^{18}, q^{58}, q^{58}, q^{98}, q^{106}; q^{116})_\infty \\ + (q^9, q^{20}, q^{38}, q^{49}; q^{58})_\infty (q^{30}, q^{86}; q^{116})_\infty \\ + (q^{29}, q^{29}; q^{58})_\infty (q^{18}, q^{48}, q^{58}, q^{58}, q^{68}, q^{98}; q^{116})_\infty \\ + q^2 (q^4, q^{12}, q^{13}, q^{45}, q^{46}, q^{54}; q^{58})_\infty (q^{14}, q^{52}, q^{64}, q^{102}; q^{116})_\infty \\ + (q^7, q^{26}, q^{32}, q^{51}; q^{58})_\infty (q^8, q^{10}, q^{48}, q^{58}, q^{68}, q^{106}, q^{108}; q^{116})_\infty \\ + q^2 (q^3, q^4, q^{22}, q^{36}, q^{54}, q^{55}; q^{58})_\infty (q^{32}, q^{34}, q^{84}, q^{86}; q^{116})_\infty]} \\ - (q^{16}, q^{17}, q^{41}, q^{42}; q^{58})_\infty (q^8, q^{10}, q^{48}, q^{58}, q^{68}, q^{106}, q^{108}; q^{116})_\infty]$$

(4.9)

This is the main result (3.1)

Now, considering (4.2)

$$C(q) + C'(q) = \frac{2[q^9, q^{38}; q^{58}]_\infty [q^{28}, q^{116}]_\infty}{[q^{29}; q^{58}]_\infty [q^{10}, q^{18}, q^{58}; q^{116}]_\infty} + \frac{2q^9 [q^9, q^{20}; q^{58}]_\infty [q^{86}; q^{116}]_\infty}{[q^{29}; q^{58}]_\infty [q^{18}, q^{58}, q^{68}; q^{116}]_\infty}$$

Multiplying on both sides of (4.2) by

$$\frac{[q^{10}, q^{68}; q^{116}]_\infty}{q^{28}, q^{86}; q^{116}]_\infty}$$

The following obtained as;

$$\frac{[-q^5, -q^{34}; q^{58}]_\infty [q^5, q^{34}; q^{58}]_\infty}{[-q^{14}, -q^{43}; q^{58}]_\infty [q^{14}, q^{43}; q^{58}]_\infty} + \frac{2[q^9, q^{38}; q^{58}]_\infty [q^{68}; q^{116}]_\infty}{[q^{29}; q^{58}]_\infty [q^{18}, q^{58}, q^{86}; q^{116}]_\infty} + \frac{2q^9 [q^9, q^{20}; q^{58}]_\infty [q^{10}; q^{116}]_\infty}{[q^{29}; q^{58}]_\infty [q^{18}, q^{28}, q^{58}; q^{116}]_\infty}$$

or,

$$C(q)^{-1} + C'(q)^{-1} = \frac{2[q^9, q^{38}; q^{58}]_\infty [q^{68}; q^{116}]_\infty}{[q^{29}; q^{58}]_\infty [q^{18}, q^{58}, q^{86}; q^{116}]_\infty} + \frac{2q^9 [q^9, q^{20}; q^{58}]_\infty [q^{10}; q^{116}]_\infty}{[q^{29}; q^{58}]_\infty [q^{18}, q^{28}, q^{58}; q^{116}]_\infty}$$

(4.10)

Next, on arranging (4.10) as;

$$\alpha_2(q) = -[C'(q)^{-1} + C(q)^{-1}] = 2 \frac{[q^9, q^{38}; q^{58}]_\infty [q^{68}; q^{116}]_\infty}{[q^{29}; q^{58}]_\infty [q^{18}, q^{58}, q^{86}; q^{116}]_\infty} + \frac{q^9 [q^9, q^{20}; q^{58}]_\infty [q^{10}; q^{116}]_\infty}{[q^{29}; q^{58}]_\infty [q^{18}, q^{28}, q^{58}; q^{116}]_\infty}$$

(4.11)

Again, multiplying (4.4) by,

$$\frac{[q^{10}, q^{68}; q^{116}]_\infty [q^{28}, q^{86}; q^{116}]_\infty}{q}$$

get the following as;

$$C(q)^{-1} - C'(q)^{-1} = \left[\frac{2q^2 [q^3, q^4, q^{22}; q^{58}]_\infty [q^{32}, q^{34}; q^{116}]_\infty}{[q^{16}, q^{17}; q^{58}]_\infty [q^8, q^{28}, q^{58}, q^{86}; q^{116}]_\infty} - \frac{2q^2 [q^4, q^{12}, q^{13}; q^{58}]_\infty [q^{14}, q^{52}; q^{116}]_\infty}{[q^7, q^{26}; q^{58}]_\infty [q^8, q^{28}, q^{58}, q^{86}; q^{116}]_\infty} \right]$$

(4.12)

And, on arranging (4.12) as;

$$\beta_2(q) = \frac{[C(q)^{-1} - C'(q)^{-1}]}{2} = \frac{q^2 [q^3, q^4, q^{22}; q^{58}]_\infty [q^{32}, q^{34}; q^{116}]_\infty}{[q^{16}, q^{17}; q^{58}]_\infty [q^8, q^{28}, q^{58}, q^{86}; q^{116}]_\infty} - \frac{q^2 [q^4, q^{12}, q^{13}; q^{58}]_\infty [q^{14}, q^{52}; q^{116}]_\infty}{[q^7, q^{26}; q^{58}]_\infty [q^8, q^{28}, q^{58}, q^{86}; q^{116}]_\infty}$$

(4.13)

By adding (4.11) and (4.13),

$$C(q)^{-1} = \alpha_2(q) + \beta_2(q)$$

Then

$$C(q)^{-1} = \frac{[q^9, q^{38}; q^{58}]_\infty [q^{68}; q^{116}]_\infty}{[q^{29}; q^{58}]_\infty [q^{18}, q^{58}, q^{86}; q^{116}]_\infty} + \frac{q^9 [q^9, q^{20}; q^{58}]_\infty [q^{10}; q^{116}]_\infty}{[q^{29}; q^{58}]_\infty [q^{18}, q^{28}, q^{58}; q^{116}]_\infty} + \frac{q^2 [q^3, q^4, q^{22}; q^{58}]_\infty [q^{32}, q^{34}; q^{116}]_\infty}{[q^{16}, q^{17}; q^{58}]_\infty [q^8, q^{28}, q^{58}, q^{86}; q^{116}]_\infty} - \frac{q^2 [q^4, q^{12}, q^{13}; q^{58}]_\infty [q^{14}, q^{52}; q^{116}]_\infty}{[q^7, q^{26}; q^{58}]_\infty [q^8, q^{28}, q^{58}, q^{86}; q^{116}]_\infty}$$

(4.14)

By applying (2.2) in (4.14),

$$C(q)^{-1} = \frac{(q^9, q^{20}, q^{38}, q^{49}; q^{58})_\infty (q^{48}, q^{68}; q^{116})_\infty}{(q^{29}, q^{29}; q^{58})_\infty (q^{18}, q^{30}, q^{58}, q^{58}, q^{86}, q^{98}; q^{116})_\infty} + \frac{q^9 (q^9, q^{20}, q^{38}, q^{49}; q^{58})_\infty (q^{10}, q^{106}; q^{116})_\infty}{(q^{29}, q^{29}; q^{58})_\infty (q^{18}, q^{30}, q^{58}, q^{58}, q^{86}, q^{98}; q^{116})_\infty} + \frac{q^2 (q^3, q^4, q^{22}, q^{36}, q^{54}, q^{55}; q^{58})_\infty (q^{32}, q^{34}, q^{82}, q^{84}; q^{116})_\infty}{(q^{16}, q^{17}, q^{41}, q^{42}; q^{58})_\infty (q^8, q^{28}, q^{30}, q^{58}, q^{58}, q^{86}, q^{88}, q^{108}; q^{116})_\infty} - \frac{q^2 (q^4, q^{12}, q^{13}, q^{45}, q^{46}, q^{54}; q^{58})_\infty (q^{14}, q^{52}, q^{64}, q^{102}; q^{116})_\infty}{(q^7, q^{26}, q^{32}, q^{51}; q^{58})_\infty (q^8, q^{28}, q^{30}, q^{58}, q^{58}, q^{86}, q^{88}, q^{108}; q^{116})_\infty}$$

(4.15)

Now, from (4.8)

$$C(q)^{-1} \sum_{s_1, s_2, s_3 \geq 0} \frac{q^{9s_2 + 3s_1^2 + s_2^2} (q^3; q^3)_{s_1 - s_2} (q; q)_{3s_2 - s_3}}{(q^9; q^9)_{2s_1} (q^9; q^9)_{s_1 - s_2} (q^3; q^3)_{2s_2} (q^3; q^3)_{s - 2s_3} (q; q)_{s_3}} = \frac{(q^5, q^{24}, q^{29}; q^{29})_\infty}{(q^9; q^9)_\infty}$$

(4.16)

By using (4.15) in (4.16)

$$\frac{(q^5, q^{24}, q^{29}; q^{29})_\infty}{(q^9; q^9)_\infty} = \frac{(q^9, q^{20}, q^{38}, q^{49}; q^{58})_\infty (q^{48}, q^{68}; q^{116})_\infty}{(q^{29}, q^{29}; q^{58})_\infty (q^{18}, q^{30}, q^{58}, q^{58}, q^{86}, q^{98}; q^{116})_\infty} + \frac{q^9 (q^9, q^{20}, q^{38}, q^{49}; q^{58})_\infty (q^{10}, q^{106}; q^{116})_\infty}{(q^{29}, q^{29}; q^{58})_\infty (q^{18}, q^{30}, q^{58}, q^{58}, q^{86}, q^{98}; q^{116})_\infty} + \frac{q^2 (q^3, q^4, q^{22}, q^{36}, q^{54}, q^{55}; q^{58})_\infty (q^{32}, q^{34}, q^{82}, q^{84}; q^{116})_\infty}{(q^{16}, q^{17}, q^{41}, q^{42}; q^{58})_\infty (q^8, q^{28}, q^{30}, q^{58}, q^{58}, q^{86}, q^{88}, q^{108}; q^{116})_\infty} - \frac{q^2 (q^4, q^{12}, q^{13}, q^{45}, q^{46}, q^{54}; q^{58})_\infty (q^{14}, q^{52}, q^{64}, q^{102}; q^{116})_\infty}{(q^7, q^{26}, q^{32}, q^{51}; q^{58})_\infty (q^8, q^{28}, q^{30}, q^{58}, q^{58}, q^{86}, q^{88}, q^{108}; q^{116})_\infty} \times \sum_{s_1, s_2, s_3 \geq 0} \frac{(q^9; q^9)_{2s_1} (q^9; q^9)_{s_1 - s_2} (q^3; q^3)_{2s_2} (q^3; q^3)_{s - 2s_3} (q; q)_{s_3}}{q^{9s_2 + 3s_1^2 + s_2^2} (q; q)_{3s_2 - s_3}}$$

(4.17)

This is main result (3.2)

5. CONCLUSION

This paper outlines the concept of m -dissections to establish our results involving q -basic infinite products, which expressed in terms of RR identities. These expansions will shape the subject matter of our ensuing communications.

REFERENCES

- [1] A. K. Shrivastav, N. A. Herbert, H. Kaur, Certain Rogers-Ramanujan type Multi Sum Identities and Ratio of Infinite Products, *South East Asian Journal of Mathematics and Mathematical Sciences*, 15(2), (2019), 85-96.
- [2] A. K. Shrivastav, N. A. Herbert, On Certain Results Involving Bilateral Basic Hypergeometric Functions and Ratio of Infinite Products, *International J. of Scientific and Engineering Research*, 8(9), (2017), 1598-1612.
- [3] A. K. Shrivastav, N. A. Herbert, On Transformation Formulae for Basic Hypergeometric Functions by using WP Bailey's Pairs, *International Journal of Mathematics Trends and Technology*, 57(2), (2018), 122-127.
- [4] A. K. Shrivastav, N. A. Herbert, On Bailey's Transform and Transformation Formulae for Basic Hypergeometric Functions, *Journal of Computer and Mathematical Sciences*, 9(7), (2018), 880-885.
- [5] B. C. Berndt, P. T. Joshi, B. M. Wilson, Chapter 2 of Ramanujan's second notebook. *Glasgow Mathematical Journal*, 22(2), (1981), 199-216.
- [6] D. Bressoud, M. E. H. Ismail, and D. Stanton, Change of Base in Bailey Pairs, *The Ramanujan Journal*, 4, (2000), 435-453.
- [7] G. E. Andrews, Multiple series Rogers-Ramanujan type identities, *Pac. J. Math.* 114, (1984), 267-283.
- [8] G. E. Andrews, Ramanujan's lost notebook III-The Rogers-Ramanujan's Continued Fraction, *Adv. Math.* 41 (1981), 186-208.
- [9] Kacheru, G. (2018). Blockchain Technology: Architecture, Applications, and Challenges. *Turkish Journal of Computer and Mathematics Education (TURCOMAT)*, 9(3), 1433-1438.
- [10] Harwinder Kaur, and A. K. Shrivastav, On transformations formulae and certain transformations, *Journal of Physics: Conference Series (Iraqi Academics Syndicate International Conference for Pure and Applied Sciences)*, (2021), 1- 7.
- [11] Kacheru, G. "AI-powered test automation frameworks: choosing the right tools." *International journal of artificial intelligence & machine learning (IJAIML)* 3.02 (2024): 110.
- [12] Kuljot Kaur, Priya Wadhwa, Harwinder Kaur, and A. K. Shrivastav, On identities for basic hypergeometric series through WP-bailey's pairs, *Journal of Critical Review*, 7(19), (2020), 3669- 3674.
- [13] L. J. Slater, A new proof of Roger's transformations of infinite series, *Proc. London Math. Soc. (2)* 53, (1951), 460-475.
- [14] L. J. Slater, Further identities of the Rogers-Ramanujan type, *Proc. London Math. Soc. (2)* 54, (1952), 147-167.
- [15] L. J. Slater, *Generalized Hypergeometric Function*, Cambridge Univ. Press, Cambridge, (1996).
- [16] M. D. Hirschhorn, on the expansion of Ramanujan's Continued Fraction, *The Ramanujan's Journal*, 2 (1998), 521-527.
- [17] Priya Wadhwa, A. K. Shrivastav, An Aspect of Basic $(q-)$ Integral for q -Basic Series, *Advances in Nonlinear Variational Inequalities*, 26(2), (2023), 41-45.
- [18] R. P. Lewis, On the Rank of Partitions Modulo 9, *Bull. London Math. Soc.*, 23(1991), 417-421.
- [19] R. Y. Denis, S. N. Singh and N. A. Herbert, On certain Bilateral q -series and Ramanujan's Continued Fraction, *South East Asia J. Math. and Math Sc.*, 3(1), (2004), 49-61.
- [20] Richard Lewis and Zhi-Guo Liu, A Conjecture of Hirschhorn on the 4-dissection of Ramanujan's Continued Fraction, *The Ramanujan's Journal*, 4 (2001), 347-352.
- [21] W. N. Bailey, Identities of the Rogers-Ramanujan type, *Proc. London Math. Soc. (2)* 50, (1949), 1-10.