

## PRIVATE STOCHASTIC OPTIMIZATION WITH LARGE WORST-CASE LIPSCHITZ PARAMETER

ANDREW LOWY AND MEISAM RAZAVIYAYN

University of Wisconsin-Madison  
*e-mail address:* [alowy@wisc.edu](mailto:alowy@wisc.edu)  
*URL:* <https://sites.google.com/view/andrewlowy>

University of Southern California  
*e-mail address:* [razaviya@usc.edu](mailto:razaviya@usc.edu)  
*URL:* <https://sites.usc.edu/razaviyayn/>

**ABSTRACT.** We study differentially private (DP) stochastic optimization (SO) with loss functions whose worst-case Lipschitz parameter over all data points may be extremely large or infinite. To date, the vast majority of work on DP SO assumes that the loss is uniformly Lipschitz continuous over data (i.e. stochastic gradients are uniformly bounded over all data points). While this assumption is convenient, it often leads to pessimistic excess risk bounds. In many practical problems, the worst-case (uniform) Lipschitz parameter of the loss over all data points may be extremely large due to outliers and/or heavy-tailed data. In such cases, the error bounds for DP SO, which scale with the worst-case Lipschitz parameter of the loss, are vacuous. To address these limitations, this work provides improved excess risk bounds that do not depend on the uniform Lipschitz parameter of the loss. Building on a recent line of work (Wang et al., 2020; Kamath et al., 2022), we assume that stochastic gradients have bounded  $k$ -th order *moments* for some  $k \geq 2$ . Compared with works on uniformly Lipschitz DP SO, our excess risk scales with the  $k$ -th moment bound instead of the uniform Lipschitz parameter of the loss, allowing for significantly faster rates in the presence of outliers and/or heavy-tailed data.

For *smooth* convex and strongly convex loss functions, we provide *linear-time* algorithms with state-of-the-art excess risk. We complement our excess risk upper bounds with novel lower bounds. In certain parameter regimes, our linear-time excess risk bounds are *minimax optimal*. Second, we provide the first algorithm to handle *non-smooth* convex loss functions. To do so, we develop novel algorithmic and stability-based proof techniques, which we believe will be useful for future work in obtaining optimal excess risk. Finally, our work is the first to address *non-convex* non-uniformly Lipschitz loss functions satisfying the *Proximal-PL inequality*; this covers some practical machine learning models. Our Proximal-PL algorithm has *near-optimal* excess risk.

---

*Key words and phrases:* differential privacy, private optimization, stochastic optimization, heavy-tailed data, uniform Lipschitz continuity, convex optimization, non-convex optimization, non-smooth optimization, stability, generalization.

\* An initial version of this work appeared in The 34th International Conference on Algorithmic Learning Theory, and in TPDP 2023.

Most of this work was done while AL was at University of Southern California.

## 1. INTRODUCTION

As the use of machine learning (ML) models in industry and society has grown dramatically in recent years, so too have concerns about the privacy of personal data that is used in training such models. It is well-documented that ML models may leak training data, e.g., via model inversion attacks and membership-inference attacks (Fredrikson et al., 2015; Shokri et al., 2017; Faizullahoy and Korolova, 2018; Nasr et al., 2019; Carlini et al., 2021). *Differential privacy* (DP) (Dwork et al., 2006) is a rigorous notion of data privacy, and a plethora of work has been devoted to differentially private machine learning and optimization (Chaudhuri and Monteleoni, 2008; Duchi et al., 2013; Bassily et al., 2014; Ullman, 2015; Wang et al., 2017; Bassily et al., 2019; Feldman et al., 2020a; Lowy and Razaviyayn, 2023; Cheu et al., 2021; Asi et al., 2021b). Of particular importance is the fundamental problem of DP *stochastic (convex) optimization* (S(C)O): given  $n$  i.i.d. samples  $X = (x_1, \dots, x_n) \in \mathcal{X}^n$  from an unknown distribution  $\mathcal{D}$ , we aim to privately solve

$$\min_{w \in \mathcal{W}} \{F(w) := \mathbb{E}_{x \sim \mathcal{D}}[f(w, x)]\}, \quad (1.1)$$

where  $f : \mathcal{W} \times \mathcal{X} \rightarrow \mathbb{R}$  is the loss function and  $\mathcal{W} \subset \mathbb{R}^d$  is the parameter domain. Since finding the exact solution to (1.1) is not generally possible, we measure the quality of the obtained solution via *excess risk* (a.k.a. excess population loss): The excess risk of a (randomized) algorithm  $\mathcal{A}$  for solving Eq. (1.1) is defined as  $\mathbb{E}F(\mathcal{A}(X)) - \min_{w \in \mathcal{W}} F(w)$ , where the expectation is taken over both the random draw of the data  $X$  and the algorithm  $\mathcal{A}$ .

A large body of literature is devoted to characterizing the optimal achievable differentially private excess risk of Eq. (1.1) when the function  $f(\cdot, x)$  is uniformly  $L_f$ -Lipschitz for all  $x \in \mathcal{X}$ —see e.g., (Bassily et al., 2019; Feldman et al., 2020a; Asi et al., 2021b; Bassily et al., 2021b; Lowy and Razaviyayn, 2023). In these works, the gradient of  $f$  is assumed to be uniformly bounded with  $\sup_{w \in \mathcal{W}, x \in \mathcal{X}} \|\nabla_w f(w, x)\| \leq L_f$ , and excess risk bounds scale with  $L_f$ . While this assumption is convenient for bounding the *sensitivity* (Dwork et al., 2006) of the steps of the algorithm, it is often unrealistic in practice or leads to pessimistic excess risk bounds. In many practical applications, data contains outliers, is unbounded or heavy-tailed (see e.g. (Crovella et al., 1998; Markovich, 2008; Woolson and Clarke, 2011) and references therein for such applications). Consequently,  $L_f$  may be prohibitively large. For example, even the linear regression loss  $f(w, x) = \frac{1}{2}(\langle w, x^{(1)} \rangle - x^{(2)})^2$  with compact  $\mathcal{W}$  and data from  $\mathcal{X} = \mathcal{X}^{(1)} \times \mathcal{X}^{(2)}$ , leads to  $L_f \geq \text{diameter}(\mathcal{X}^{(1)})^2$ , which could be huge or even infinite. Similar observations can be made for other useful ML models such as deep neural nets (Lei and Ying, 2021), and the situation becomes even grimmer in the presence of heavy-tailed data. In these cases, existing excess risk bounds, which scale with  $L_f$ , becomes vacuous.

While  $L_f$  can be very large in practice (due to outliers), the  $k$ -th *moment* of the stochastic gradients is often reasonably small for some  $k \geq 2$  (see, e.g., Example 1). This is because the  $k$ -th moment  $\tilde{r}_k := \mathbb{E}[\sup_{w \in \mathcal{W}} \|\nabla_w f(w, x)\|_2^k]^{1/k}$  depends on the *average* behavior of the stochastic gradients, while  $L_f$  depends on the *worst-case* behavior over all data points. Motivated by this observation and building on the prior results (Wang et al., 2020; Kamath et al., 2022), this work makes progress towards answering the following fundamental open questions:

- Question I: What are the minimax optimal rates for (strongly) convex DP SO?
- Question II: What utility guarantees are achievable for non-convex DP SO?

Prior works have sought to address the first question above:<sup>1</sup> The work of Wang et al. (2020) provided the first excess risk upper bounds for *smooth* DP (strongly) convex SO. Kamath et al. (2022) gave improved, yet suboptimal, upper bounds for *smooth* (strongly) convex  $f(\cdot, x)$ , and lower bounds for (strongly) convex SO. In this work, we provide both new lower bounds and new algorithms with improved excess risk for (strongly) convex losses. In certain practical parameter regimes, our excess risk bounds for smooth  $F$  are *minimax optimal*, giving a partial answer to Question I. We also provide a novel algorithm for *non-smooth* convex  $F$ . Regarding Question II, we give the *first algorithm for DP SO with non-convex loss* functions satisfying the Proximal Polyak-Lojasiewicz (PL) condition (Polyak, 1963; Karimi et al., 2016). We provide a summary of our results for the case  $k = 2$  in Figure 1, and a thorough discussion of related work in Appendix A.

**1.1. Preliminaries.** Let  $\|\cdot\|$  be the  $\ell_2$  norm. Let  $\mathcal{W}$  be a convex, compact set of  $\ell_2$  diameter  $D$ . Function  $g : \mathcal{W} \rightarrow \mathbb{R}$  is  $\mu$ -strongly convex if  $g(\alpha w + (1 - \alpha)w') \leq \alpha g(w) + (1 - \alpha)g(w') - \frac{\alpha(1-\alpha)\mu}{2}\|w - w'\|^2$  for all  $\alpha \in [0, 1]$  and all  $w, w' \in \mathcal{W}$ . If  $\mu = 0$ , we say  $g$  is *convex*. For convex  $f(\cdot, x)$ , denote any *subgradient* of  $f(w, x)$  w.r.t.  $w$  by  $\nabla f(w, x) \in \partial_w f(w, x)$ : i.e.  $f(w', x) \geq f(w, x) + \langle \nabla f(w, x), w' - w \rangle$  for all  $w' \in \mathcal{W}$ . Function  $g$  is  $\beta$ -smooth if it is differentiable and its derivative  $\nabla g$  is  $\beta$ -Lipschitz. For  $\beta$ -smooth,  $\mu$ -strongly convex  $g$ , denote its *condition number* by  $\kappa = \beta/\mu$ . For functions  $a$  and  $b$  of input parameters, write  $a \lesssim b$  if there is an absolute constant  $A$  such that  $a \leq Ab$  for all feasible values of input parameters. Write  $a = \tilde{\mathcal{O}}(b)$  if  $a \lesssim \ell b$  for a logarithmic function  $\ell$  of input parameters. We assume that the stochastic gradient distributions have bounded  $k$ -th moment for some  $k \geq 2$ :

**Assumption 1.** *There exists  $k \geq 2$  and  $\tilde{r}^{(k)} > 0$  such that  $\mathbb{E} [\sup_{w \in \mathcal{W}} \|\nabla f(w, x)\|_2^k] \leq \tilde{r}^{(k)}$  for all  $\nabla f(w, x_i) \in \partial_w f(w, x_i)$ . Denote  $\tilde{r}_k := (\tilde{r}^{(k)})^{1/k}$ .*

Clearly,  $\tilde{r}_k \leq L_f = \sup_{\{\nabla f(w, x) \in \partial_w f(w, x)\}} \sup_{w, x} \|\nabla f(w, x)\|$ , but this inequality is often very loose:

**Example 1.** *For linear regression on a unit ball  $\mathcal{W}$  with 1-dimensional data  $x^{(1)}, x^{(2)} \in [-10^{10}, 10^{10}]$  having Truncated Normal distributions and  $\text{Var}(x^{(1)}) = \text{Var}(x^{(2)}) \leq 1$ , we have  $L_f \geq 10^{20}$ . On the other hand,  $\tilde{r}_k$  is much smaller than  $L_f$  for  $k < \infty$ : e.g.,  $\tilde{r}_2 \leq 5$ ,  $\tilde{r}_4 \leq 8$ , and  $\tilde{r}_8 \leq 14$ .*

**Differential Privacy:** *Differential privacy* (Dwork et al., 2006) ensures that no adversary—even one with enormous resources—can infer much more about any person who contributes training data than if that person’s data were absent. If two data sets  $X$  and  $X'$  differ in a single entry (i.e.  $d_{\text{hamming}}(X, X') = 1$ ), then we say that  $X$  and  $X'$  are *adjacent*.

**Definition 1** (Differential Privacy). *Let  $\varepsilon \geq 0$ ,  $\delta \in [0, 1]$ . A randomized algorithm  $\mathcal{A} : \mathcal{X}^n \rightarrow \mathcal{W}$  is  $(\varepsilon, \delta)$ -differentially private (DP) if for all pairs of adjacent data sets  $X, X' \in \mathcal{X}^n$  and all measurable subsets  $S \subseteq \mathcal{W}$ , we have  $\mathbb{P}(\mathcal{A}(X) \in S) \leq e^\varepsilon \mathbb{P}(\mathcal{A}(X') \in S) + \delta$ .*

In this work, we focus on *zero-concentrated differential privacy* (Bun and Steinke, 2016):

<sup>1</sup>(Wang et al., 2020; Kamath et al., 2022) consider a slightly different problem class than the class  $\tilde{r}_k$ , which we consider: see Appendix B. However, our results imply improved rates for the problem class considered in (Wang et al., 2020; Kamath et al., 2022), by Appendix B and the precise versions of our linear-time upper bounds given in Appendix F.

**Definition 2** (Zero-Concentrated Differential Privacy (zCDP)). *A randomized algorithm  $\mathcal{A} : \mathcal{X}^n \rightarrow \mathcal{W}$  satisfies  $\rho$ -zero-concentrated differential privacy ( $\rho$ -zCDP) if for all pairs of adjacent data sets  $X, X' \in \mathcal{X}^n$  and all  $\alpha \in (1, \infty)$ , we have  $D_\alpha(\mathcal{A}(X) \parallel \mathcal{A}(X')) \leq \rho\alpha$ , where  $D_\alpha(\mathcal{A}(X) \parallel \mathcal{A}(X'))$  is the  $\alpha$ -Rényi divergence<sup>2</sup> between the distributions of  $\mathcal{A}(X)$  and  $\mathcal{A}(X')$ .*

zCDP is weaker than  $(\varepsilon, 0)$ -DP, but stronger than  $(\varepsilon, \delta)$ -DP ( $\delta > 0$ ) in the following sense:

**Proposition 3.** (*Bun and Steinke, 2016, Proposition 1.3*) *If  $\mathcal{A}$  is  $\rho$ -zCDP, then  $\mathcal{A}$  is  $(\rho + 2\sqrt{\rho \log(1/\delta)}, \delta)$  for any  $\delta > 0$ .*

Thus, if  $\varepsilon \leq \sqrt{\log(1/\delta)}$ , then any  $\frac{\varepsilon^2}{2}$ -zCDP algorithm is  $(2\varepsilon\sqrt{\log(1/\delta)}, \delta)$ -DP. Appendix D contains more background on differential privacy.

**1.2. Contributions and Related Work.** We discuss our contributions in the context of related work. See Figure 1 for a summary of our results when  $k = 2$ , and Appendix A for a more thorough discussion of related work.

**Improved Risk for Smooth Convex Losses in Linear Time (Section 4):** For convex,  $\beta$ -smooth  $F$ , we provide an accelerated DP algorithm (Algorithm 2), building on the work of (Ghadimi and Lan, 2012).<sup>3</sup> Our algorithm is *linear time* and attains excess risk that improves over the previous state-of-the-art (*not linear time*) algorithm (Kamath et al., 2022, Theorem 5.4) in practical parameter regimes (e.g.  $d \gtrsim n^{1/6}$ ). The excess risk of our algorithm is *minimax optimal* in certain cases: e.g.,  $d \gtrsim (\varepsilon n)^{2/3}$  or “sufficiently smooth”  $F$  (see Remark 10).

For  $\mu$ -strongly convex,  $\beta$ -smooth losses, acceleration results in excessive bias accumulation, so we propose a simple noisy clipped SGD. Our algorithm builds on (Kamath et al., 2022), but uses a lower-bias clipping mechanism from (Barber and Duchi, 2014) and a new, tighter analysis. We attain excess risk that is near-optimal up to a  $\tilde{\mathcal{O}}((\beta/\mu)^{(k-1)/k})$  factor: see Theorem 11. Our bound strictly improves over the best previous bound of (Kamath et al., 2022).

We complement our excess risk upper bounds with information-theoretic *lower bounds*. Our lower bounds refine (to describe the dependence on  $\tilde{r}_k, D, \mu$ ), extend (to  $k \gg 1$ ), and tighten (for  $\mu = 0$ ) the lower bounds of (Kamath et al., 2022).

Special cases of our main results for smooth (strongly) convex losses are stated below:

**Theorem 4** (Informal/special cases, see Theorems 7 to 11). *Let  $F$  be convex and  $\beta$ -smooth and assume the constraint set  $\mathcal{W}$  has  $\ell_2$ -diameter  $D$ . Grant Assumption 1. If  $\beta \lesssim \frac{\tilde{r}_k}{D} \left(\frac{\sqrt{d}}{\varepsilon n}\right)^{\frac{k-1}{k}} + \frac{\tilde{r}_2}{D} \frac{1}{\sqrt{n}}$ , then there is a linear-time  $\frac{\varepsilon^2}{2}$ -zCDP algorithm  $\mathcal{A}$  such that  $\mathbb{E}F(\mathcal{A}(X)) - F^* = \tilde{\mathcal{O}}\left(D \left(\frac{\tilde{r}_2}{\sqrt{n}} + \tilde{r}_k \left(\frac{\sqrt{d}}{\varepsilon n}\right)^{(k-1)/k}\right)\right)$ . If  $F$  is  $\mu$ -strongly convex and  $\beta/\mu = \mathcal{O}(1)$ , then there is a linear-time  $\frac{\varepsilon^2}{2}$ -zCDP algorithm  $\mathcal{A}'$  such that  $\mathbb{E}F(\mathcal{A}'(X)) - F^* =$*

<sup>2</sup>For distributions  $P$  and  $Q$  with probability density/mass functions  $p$  and  $q$ ,  $D_\alpha(P \parallel Q) := \frac{1}{\alpha-1} \ln \left( \int p(x)^\alpha q(x)^{1-\alpha} dx \right)$  (Rényi, 1961, Eq. 3.3).

<sup>3</sup>In contrast to (Wang et al., 2020; Kamath et al., 2022), we do not require  $f(\cdot, x)$  to be  $\beta_f$ -smooth for all  $x$ .

$\tilde{\mathcal{O}}\left(\frac{1}{\mu}\left(\frac{\tilde{r}_2^2}{n} + \tilde{r}_k^2\left(\frac{\sqrt{d}}{\varepsilon n}\right)^{(2k-2)/k}\right)\right)$ . Moreover, the above excess risk bounds are minimax optimal up to logarithmic factors (in the stated parameter regimes).

As  $k \rightarrow \infty$ ,  $\tilde{r}_k \rightarrow L_f$  and Theorem 4 recovers the known rates for uniformly  $L_f$ -Lipschitz DP SCO (Bassily et al., 2019; Feldman et al., 2020a). However, when  $k < \infty$  and  $\tilde{r}_{2k} \ll L_f$ , the excess risk bounds in Theorem 4 may be much smaller than the uniformly Lipschitz excess risk bounds, which increase with  $L_f$ .

The works (Wang et al., 2020; Kamath et al., 2022) make a slightly different assumption than Assumption 1: they instead assume that the  $k$ -th order central moment of each coordinate  $\nabla_j f(w, x)$  is bounded by  $\gamma_k^{1/k}$  for all  $j \in [d], w \in \mathcal{W}$ . We also provide excess risk bounds for the class of problems satisfying the coordinate-wise moment assumption of (Wang et al., 2020; Kamath et al., 2022) and having *subexponential* stochastic subgradients: see Appendix G.4.

The previous state-of-the-art convex upper bound was  $\Omega\left(\tilde{r}_k D \sqrt{\frac{d}{n}}\right)$  (Kamath et al., 2022, Theorems 5.2 and 5.4).<sup>4</sup> Their result also required  $f(\cdot, x)$  to be  $\beta_f$ -smooth for all  $x \in \mathcal{X}$  with  $\beta_f \leq 10$ , which can be restrictive with outlier data: e.g. this implies that  $f(\cdot, x)$  is uniformly  $L_f$ -Lipschitz with  $L_f \leq \beta_f D \leq 10D$  if  $\nabla f(w^*(x), x) = 0$  for some  $w^*(x) \in \mathcal{W}$ . By comparison, our bounds hold even for  $f$  with  $L_f, \beta_f \gg 1$  or  $\beta_f = L_f = \infty$ .

Technically, a key bottleneck in the prior works Kamath et al. (2022); Wang et al. (2020) is the reliance on *uniform convergence* to bound the gradient estimation error simultaneously for all  $w \in \mathcal{W}$ . This approach leads to poor dependence on the dimension  $d$ . By contrast, our linear-time algorithm avoids uniform convergence arguments and instead uses disjoint batches of data in each iteration to maintain the independence between iterates  $w_t$  and  $w_{t+1}$ . This results in improved dependence on  $d$  in our bounds compared to the prior works. Additionally, we incorporate *acceleration* Ghadimi and Lan (2012) to speed up convergence. To prove our upper bound, we give the first analysis of accelerated SGD with biased stochastic gradients.

Our linear-time  $\mu$ -strongly convex bound also improves over the best previous upper bound of (Kamath et al., 2022, Theorem 5.6), which required uniform  $\beta_f$ -smoothness of  $f(\cdot, x)$ . In fact, (Kamath et al., 2022, Theorem 5.6) was incorrect, as we explain in Appendix C.<sup>5</sup> After communicating with the authors of (Kamath et al., 2022), they updated and corrected the result and proof in the arXiv version of their paper. The arXiv version of (Kamath et al., 2022, Theorem 5.6)—which we derive in Appendix C for completeness—is suboptimal by a factor of  $\tilde{\Omega}((\beta_f/\mu)^3)$ . In practice, the worst-case condition number  $\beta_f/\mu$  can be very large, especially in the presence of outliers or heavy-tailed data. Our excess risk bound removes this dependence on  $\beta_f/\mu$  and is minimax optimal for constant  $\beta/\mu$ .

**First Algorithm for Non-Smooth (Strongly) Convex Losses (Section 5):** Prior works (Wang et al., 2020; Kamath et al., 2022) required a strict uniform smoothness assumption in order to privately estimate the gradient of the loss function using Holland (2019). Additionally, their uniform convergence argument required a smoothness assumption.

<sup>4</sup>We write the bound in (Kamath et al., 2022, Theorem 5.4) in terms of Assumption 1, replacing their  $\gamma_k^{1/k} d$  by  $\tilde{r}_k \sqrt{d}$ .

<sup>5</sup>In short, the mistake is that Jensen’s inequality is used in the wrong direction to claim that the  $T$ -th iterate of their algorithm  $w_T$  satisfies  $\mathbb{E}[\|w_T - w^*\|^2] \leq (\mathbb{E}\|w_T - w^*\|)^2$ , which is false.

Function Class	Lower bound	Linear-Time Upper bound	Prior state-of-the-art
<b>Smooth Convex</b>	$\zeta := \sqrt{d} \left( \frac{1}{n} + \frac{\sqrt{d}}{\epsilon n} \right)^{1/2}$ (Theorem 7)	$\zeta + \left[ \zeta \left( \frac{n}{d} \right)^{1/16} \wedge \frac{d^{2/3}}{(\epsilon n)^{4/9}} \right]$ (Optimal if $d \gtrsim (\epsilon n)^{2/3}$ ) (Theorem 9)	(Kamath et al., 2022, Thm. 5.4) ( <b>not linear time</b> ): $\min \left\{ \frac{d}{\sqrt{\epsilon n}}, \zeta + \frac{d^{5/4}}{\sqrt{n}} \right\}$ if $f$ is <b>1-uniformly smooth</b> & $\nabla F(w^*) = 0$
<b>Smooth Strongly Convex</b>	$d \left( \frac{1}{n} + \frac{\sqrt{d}}{\epsilon n} \right)$ (Theorem 8)	$d \left( \frac{1}{n} + \frac{\sqrt{d\kappa}}{\epsilon n} \right)$ (Optimal if $\kappa \lesssim 1$ ) (Theorem 11)	$d \left( \frac{\kappa_f^3}{n} + \frac{\kappa_f^{3/2} \sqrt{d}}{\epsilon n} \right)$ if $f$ is <b>uniformly smooth</b> & $\nabla F(w^*) = 0$ (Kamath et al., 2022, Thm. 5.6) + <b>Appendix C</b>
<b>Proximal PL</b>		$d\kappa \left( \frac{1}{n} + \frac{\sqrt{d}}{\epsilon n} \right)$ (Optimal if $\kappa \lesssim 1$ ) (Theorem 19)	None

Figure 1: Smooth excess risk for  $k = 2$ ,  $\tilde{r}_2 = \sqrt{d}$ ; we omit logarithms.  $\kappa = \beta/\mu$  is the condition number of  $F$ ;  $\kappa_f = \beta_f/\mu$  is the worst-case condition number of  $f(\cdot, x)$ . See Theorems 12 and 18 for non-smooth upper bounds.

In this work, to handle non-smooth loss functions, we develop a novel algorithm and analysis that circumvents the bottlenecks of the previous works: To privately estimate the gradient of the loss function, we borrow the method of Barber and Duchi (2014), which does not require uniform smoothness. Additionally, to prove generalization of our algorithm, we prove and use novel *stability* results for non-smooth loss functions instead of the uniform convergence arguments of Wang et al. (2020); Kamath et al. (2022). We summarize our main results for non-smooth loss functions below:

**Theorem 5** (Informal, see Theorem 12, Theorem 18). *Let  $f(\cdot, x)$  be convex. Grant Assumption 1. Then, there is a polynomial-time  $\frac{\epsilon^2}{2}$ -zCDP algorithm  $\mathcal{A}$  such that  $\mathbb{E}F(\mathcal{A}(X)) - F^* = \tilde{\mathcal{O}} \left( \tilde{r}_{2k} D \left( \frac{1}{n} + \left( \frac{\sqrt{d}}{\epsilon n} \right)^{(k-1)/k} \right) \right)$ . If  $f(\cdot, x)$  is  $\mu$ -strongly convex, then  $\mathbb{E}F(\mathcal{A}(X)) - F^* = \tilde{\mathcal{O}} \left( \frac{\tilde{r}_{2k}^2}{\mu} \left( \frac{1}{n} + \left( \frac{\sqrt{d}}{\epsilon n} \right)^{(2k-2)/k} \right) \right)$ .*

The above upper bounds nearly resemble the lower bounds in Theorems 7 and 8 except that  $\tilde{r}_{2k}$  appears in place of  $\tilde{r}_k$ . We leave it as an open problem to obtain tight bounds with  $\tilde{r}_{2k} \mapsto \tilde{r}_k$ . We believe that this can be accomplished by building on our Algorithm 4 and our novel proof techniques.

Our Algorithm 4 combines the iterative localization technique of (Feldman et al., 2020a; Asi et al., 2021b) with a noisy *clipped* subgradient method. With clipped (hence *biased*) stochastic subgradients and non-Lipschitz/non-smooth  $f(\cdot, x)$ , the excess risk analysis of our algorithm is harder than in the uniformly Lipschitz setting. Instead of the uniform convergence analysis used in (Wang et al., 2020; Kamath et al., 2022), we derive new results about the *stability* (Kearns and Ron, 1999; Bousquet and Elisseeff, 2002) and generalization error of learning with loss functions that are not uniformly Lipschitz or differentiable; prior results (e.g. (Shalev-Shwartz et al., 2009; Lei and Ying, 2020)) were limited to  $\beta_f$ -smooth and/or  $L_f$ -Lipschitz  $f(\cdot, x)$ . Specifically, we show the following for *non-Lipschitz/non-smooth*  $f(\cdot, x)$ : a) *On-average model stability* (Lei and Ying, 2020) implies generalization (Proposition 16); and b) regularized empirical risk minimization is on-average model stable,

hence it generalizes (Proposition 17). We combine these results with an empirical error bound for *biased*, noisy subgradient method to bound the excess risk of our algorithm (Theorem 12). We obtain our strongly convex bound (Theorem 18) by a reduction to the convex case, ala (Hazan and Kale, 2014; Feldman et al., 2020a).

**First Algorithm for Non-Convex (Proximal-PL) Losses (Section 6):** We consider losses satisfying the *Proximal Polyak-Lojasiewicz (PPL) inequality* (Polyak, 1963; Karimi et al., 2016) (Definition 19), an extension of the classical PL inequality to the proximal setting. This covers important models like (some) neural nets, linear/logistic regression, and LASSO (Karimi et al., 2016; Lei and Ying, 2021). We propose a DP proximal clipped SGD to attain near-optimal excess risk that almost matches the *strongly convex* rate: see Theorem 20.

We also provide (in Appendix I) the first *shuffle differentially private (SDP)* (Cheu et al., 2019) algorithms for SO with large worst-case Lipschitz parameter. Our SDP algorithms achieve the same risk bounds as their zCDP counterparts *without requiring a trusted curator*.

## 2. PRIVATE HEAVY-TAILED MEAN ESTIMATION BUILDING BLOCKS

In each iteration of our SO algorithms, we need a way to privately estimate the mean  $\nabla F(w_t) = \mathbb{E}_{x \sim \mathcal{D}}[\nabla f(w_t, x)]$ . If  $f(\cdot, x)$  is uniformly Lipschitz, then one can simply draw a random sample  $x^t$  from  $X$  and add noise to the stochastic gradient  $\nabla f(w_t, x^t)$  to obtain a DP estimator of  $\nabla F(w_t)$ : the  $\ell_2$ -sensitivity of the stochastic gradients is bounded by  $\sup_{x, x' \in \mathcal{X}} \|\nabla f(w_t, x) - \nabla f(w_t, x')\| \leq 2L_f$ , so the Gaussian mechanism guarantees DP (by Proposition 23). However, in the setting that we consider,  $L_f$  (and hence the sensitivity) may be huge, leading the privacy noise variance to also be huge. Thus, we *clip* the stochastic gradients (to force the sensitivity to be bounded) before adding noise. Specifically, we invoke Algorithm 1 on a batch of  $s$  stochastic gradients at each iteration of our algorithms. In Algorithm 1,  $\Pi_C(z) := \operatorname{argmin}_{y \in B_2(0, C)} \|y - z\|^2$  denotes the projection onto the centered  $\ell_2$  ball of radius  $C$  in  $\mathbb{R}^d$ . Lemma 6 bounds the bias and variance of Algorithm 1.

---

**Algorithm 1:** `MeanOracle1`( $\{z_i\}_{i=1}^s; s; C; \frac{\varepsilon^2}{2}$ ) (Barber and Duchi, 2014)

---

- 1: **Input:**  $Z = \{z_i\}_{i=1}^s$ ,  $C > 0$ ,  $\varepsilon > 0$ . Set  $\sigma^2 = \frac{4C^2}{s^2\varepsilon^2}$  for  $\frac{\varepsilon^2}{2}$ -zCDP.
  - 2: Draw  $N \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_d)$  and compute  $\tilde{\nu} := \frac{1}{s} \sum_{i=1}^s \Pi_C(z_i) + N$ .
  - 3: **Output:**  $\tilde{\nu}$ .
- 

**Lemma 6** ((Barber and Duchi, 2014)). *Let  $\{z_i\}_{i=1}^s \sim \mathcal{D}^s$  be  $\mathbb{R}^d$ -valued random vectors with  $\mathbb{E}z_i = \nu$  and  $\mathbb{E}\|z_i\|^k \leq r^{(k)}$  for some  $k \geq 2$ . Denote the noiseless average of clipped samples by  $\hat{\nu} := \frac{1}{s} \sum_{i=1}^s \Pi_C(z_i)$  and  $\tilde{\nu} := \hat{\nu} + N$ . Then,  $\|\mathbb{E}\tilde{\nu} - \nu\| = \|\mathbb{E}\hat{\nu} - \nu\| \leq \mathbb{E}\|\hat{\nu} - \nu\| \leq \frac{r^{(k)}}{(k-1)C^{k-1}}$ , and  $\mathbb{E}\|\tilde{\nu} - \mathbb{E}\tilde{\nu}\|^2 = \mathbb{E}\|\tilde{\nu} - \mathbb{E}\hat{\nu}\|^2 \leq d\sigma^2 + \frac{r^{(2)}}{s}$ .*

### 3. EXCESS RISK LOWER BOUNDS FOR SMOOTH (STRONGLY) CONVEX LOSSES

In this section, we provide excess risk lower bounds for the class of DP SCO problems satisfying Assumption 1. These lower bounds will be useful for addressing *Question I* of the introduction and for calibrating the performance of our algorithms in later sections.

The work of (Kamath et al., 2022) proved lower bounds for  $D = \mu = 1$ ,  $\tilde{r}_k = \sqrt{d}$ , and  $k = \mathcal{O}(1)$ .<sup>6</sup> Our contribution in this subsection is: refining these lower bounds to display the correct dependence on  $\tilde{r}_k, D, \mu$ ; tightening the convex lower bound (Kamath et al., 2022, Theorem 6.4) in the regime  $d > n$ ; and extending (Kamath et al., 2022, Theorems 6.1 and 6.4) to  $k \gg 1$ .

Our first lower bound holds even for affine functions:

**Theorem 7** (Smooth Convex, Informal). *Let  $\rho \leq d$ . For any  $\rho$ -zCDP algorithm  $\mathcal{A}$ , there exist closed convex sets  $\mathcal{W}, \mathcal{X} \subset \mathbb{R}^d$  such that  $\|w - w'\| \leq 2D$  for all  $w, w' \in \mathcal{W}$ , a  $\beta_f$ -smooth, linear, convex (in  $w$ ) loss  $f : \mathcal{W} \times \mathcal{X} \rightarrow \mathbb{R}$ , and distribution  $\mathcal{D}$  on  $\mathcal{X}$  such that Assumption 1 holds and if  $X \sim \mathcal{D}^n$ , then*

$$\mathbb{E}F(\mathcal{A}(X)) - F^* = \Omega \left( D \left( \frac{\tilde{r}_2}{\sqrt{n}} + \tilde{r}_k \min \left\{ 1, \left( \frac{\sqrt{d}}{\sqrt{\rho n}} \right)^{\frac{k-1}{k}} \right\} \right) \right).$$

Remark 27 (in Appendix E) discusses parameter regimes in which Theorem 7 is strictly tighter than (Kamath et al., 2022, Theorem 6.4), as well as differences in our proof vs. theirs.

Next, we provide lower bounds for smooth, strongly convex loss functions:

**Theorem 8** (Smooth Strongly Convex, Informal). *Let  $\rho \leq d$ . For any  $\rho$ -zCDP algorithm  $\mathcal{A}$ , there exist compact convex sets  $\mathcal{W}, \mathcal{X} \subset \mathbb{R}^d$ , a  $\mu$ -smooth,  $\mu$ -strongly convex (in  $w$ ) loss  $f : \mathcal{W} \times \mathcal{X} \rightarrow \mathbb{R}$ , and distribution  $\mathcal{D}$  on  $\mathcal{X}$  such that: Assumption 1 holds, and if  $X \sim \mathcal{D}^n$ , then*

$$\mathbb{E}F(\mathcal{A}(X)) - F^* = \Omega \left( \frac{1}{\mu} \left( \frac{\tilde{r}_2^2}{n} + \tilde{r}_k^2 \min \left\{ 1, \left( \frac{\sqrt{d}}{\sqrt{\rho n}} \right)^{\frac{2k-2}{k}} \right\} \right) \right).$$

In the next section, we provide algorithms and excess risk upper bounds for DP SCO under Assumption 1.

### 4. LINEAR-TIME ALGORITHMS FOR SMOOTH (STRONGLY) CONVEX LOSSES

This section gives linear-time algorithms for smooth loss functions. We begin by providing our results for convex loss functions.

**4.1. Noisy Clipped Accelerated SGD for Smooth Convex Losses.** Algorithm 2 is a one-pass accelerated algorithm, which builds on (non-private) AC-SA of (Ghadimi and Lan, 2012); its privacy and excess risk guarantees are given in Theorem 9.

<sup>6</sup>The lower bounds asserted in (Kamath et al., 2022) only hold for constant  $k = \mathcal{O}(1)$  since the moments of the Gaussian distribution that they construct grow exponentially/factorially with  $k$ .

---

**Algorithm 2:** Noisy Clipped Accelerated SGD (AC-SA) for Heavy-Tailed DP SCO
 

---

- 1: **Input:** Data  $X \in \mathcal{X}^n$ , iteration number  $T \leq n$ , stepsize parameters  $\{\eta_t\}_{t \in [T]}$ ,  $\{\alpha_t\}_{t \in [T]}$  with  $\alpha_1 = 1, \alpha_t \in (0, 1) \forall t \geq 2$ .
  - 2: Initialize  $w_0^{ag} = w_0 \in \mathcal{W}$  and  $t = 1$ .
  - 3: **for**  $t \in [T]$  **do**
  - 4:    $w_t^{md} := (1 - \alpha_t)w_{t-1}^{ag} + \alpha_t w_{t-1}$ .
  - 5:   Draw new batch  $\mathcal{B}_t$  (without replacement) of  $n/T$  samples from  $X$ .
  - 6:    $\tilde{\nabla} F_t(w_t^{md}) := \text{MeanOracle1}(\{\nabla f(w_t^{md}, x)\}_{x \in \mathcal{B}_t}; \frac{n}{T}; \frac{\varepsilon^2}{2})$
  - 7:    $w_t := \operatorname{argmin}_{w \in \mathcal{W}} \left\{ \alpha_t \langle \tilde{\nabla} F_t(w_t^{md}), w \rangle + \frac{\eta_t}{2} \|w_{t-1} - w\|^2 \right\}$ .
  - 8:    $w_t^{ag} := \alpha_t w_t + (1 - \alpha_t)w_{t-1}^{ag}$ .
  - 9: **end for**
  - 10: **Output:**  $w_T^{ag}$ .
- 

**Theorem 9** (Informal). *Grant Assumption 1. Let  $F$  be convex and  $\beta$ -smooth. Then, there are parameters such that Algorithm 2 is  $\frac{\varepsilon^2}{2}$ -zCDP and:*

$$\mathbb{E}F(w_T^{ag}) - F^* \lesssim \frac{\tilde{r}_2 D}{\sqrt{n}} + \tilde{r}_k D \left[ \left( \frac{\sqrt{d}}{\varepsilon n} \right)^{\frac{k-1}{k}} + \min \left\{ \left( \left( \frac{\beta D}{\tilde{r}_k} \right)^{1/4} \frac{\sqrt{d}}{\varepsilon n} \right)^{\frac{4(k-1)}{5k-1}}, \left( \frac{\beta D}{\tilde{r}_k} \right)^{\frac{k-1}{4k}} \left( \frac{\sqrt{d}}{\varepsilon n} \right)^{\frac{k-1}{k}} n^{\frac{k-1}{8k}} \right\} \right] \quad (4.1)$$

Moreover, Algorithm 2 uses at most  $n$  gradient evaluations.

The key ingredient used to prove Eq. (4.1) is a novel convergence guarantee for AC-SA with *biased*, noisy stochastic gradients: see Proposition 30 in Appendix F.1. Combining Proposition 30 with Lemma 6 and a careful choice of stepsizes, clip threshold, and  $T$  yields Theorem 9.

Comparing Theorem 9 to the lower bound in Theorem 7, one sees that our excess risk bounds are *minimax optimal* in certain parameter regimes, giving a partial answer to Question I.

**Remark 10** (Optimal rate for “sufficiently smooth” functions). *Note that the upper bound Eq. (4.1) scales with the smoothness parameter  $\beta$ . Thus, for sufficiently small  $\beta$ —namely  $\beta \lesssim \frac{\tilde{r}_k}{D} \left( \frac{\sqrt{d}}{\varepsilon n} \right)^{\frac{k-1}{k}} + \frac{\tilde{r}_2}{D} \frac{1}{\sqrt{n}}$ —the optimal rates are attained (c.f. the lower bound in Theorem 7).*

For example, fix  $k = 2$  and consider linear regression with  $x \sim \mathcal{N}_d(0, \mathbf{I}_d)$  and  $\varepsilon = D = 1$ , and suppose the true labels  $y = \|x\| + \zeta$  for some random  $\zeta$  independent of  $x$ . Then,  $\beta = 1$  and  $\tilde{r}_2 \geq d$ , so that Eq. (4.1) is optimal whenever  $d \geq n^{2/5}$ .

As another example, for affine functions (which were not addressed in (Wang et al., 2020; Kamath et al., 2022) since these works assume  $\nabla F(w^*) = 0$ ),  $\beta = 0$  and Algorithm 2 is optimal.

Having discussed the dependence on  $\beta$ , let us focus on understanding how the bound in Theorem 9 scales with  $n, d$  and  $\varepsilon$ . Thus, let us fix  $\beta = D = 1$  and  $\tilde{r}_k = \sqrt{d}$  for simplicity. If  $k = 2$ , then the bound in Eq. (4.1) is  $\mathcal{O}\left(\sqrt{\frac{d}{n}} + \frac{d^{3/4}}{\sqrt{\varepsilon n}} + \min\left\{\frac{d^{2/3}}{(\varepsilon n)^{4/9}}, \frac{d^{3/4}}{\sqrt{\varepsilon n}} \left(\frac{n}{d}\right)^{1/16}\right\}\right)$ , whereas the lower bound in Theorem 7 (part 2) is  $\Omega\left(\sqrt{\frac{d}{n}} + \frac{d^{3/4}}{\sqrt{\varepsilon n}}\right)$ . Therefore, the bound in Eq. (4.1) is *optimal* if  $d \gtrsim n$  or  $d^{3/2} \gtrsim \varepsilon n$ . For general  $n, d, \varepsilon$ , Eq. (4.1) is suboptimal by a multiplicative factor of  $\min\left\{\left(\frac{n}{d}\right)^{1/16}, \left(\frac{\varepsilon n}{d^{3/2}}\right)^{1/18}\right\}$ . By comparison, the previous state-of-the-art (*not linear-time*) bound for  $\varepsilon \approx 1$  was  $\mathcal{O}\left(\frac{d}{\sqrt{n}}\right)$  (Kamath et al., 2022, Theorem 5.4). Our bound Eq. (4.1) improves over (Kamath et al., 2022, Theorem 5.4) if  $d \gtrsim n^{1/6}$ , which is typical in practical ML applications. As  $k \rightarrow \infty$ , Eq. (4.1) becomes  $\mathcal{O}\left(\sqrt{\frac{d}{n}} + \left(\frac{d}{n}\right)^{4/5}\right)$  for  $\varepsilon \approx 1$ , which is strictly better than the bound in (Kamath et al., 2022, Theorem 5.4).

**4.2. Noisy Clipped SGD for Strongly Convex Losses.** Our algorithm for strongly convex losses (Algorithm 6 in Appendix F.2) is a simple one-pass noisy clipped SGD. Compared to the algorithm of (Kamath et al., 2022), our approach differs in the choice of `MeanOracle`, step size, and iterate averaging weights, and in our *analysis*.

**Theorem 11 (Informal).** *Grant Assumption 1. Let  $F$  be  $\mu$ -strongly convex,  $\beta$ -smooth with  $\frac{\beta}{\mu} \leq n/\ln(n)$ . Then, there are algorithmic parameters such that Algorithm 6 is  $\frac{\varepsilon^2}{2}$ -zCDP and:*

$$\mathbb{E}F(\hat{w}_T) - F^* \lesssim \frac{1}{\mu} \left( \frac{\tilde{r}_2^2}{n} + \tilde{r}_k^2 \left( \frac{\sqrt{d\kappa \ln(n)}}{\varepsilon n} \right)^{\frac{2k-2}{k}} \right). \quad (4.2)$$

Moreover, Algorithm 6 uses at most  $n$  gradient evaluations.

The bound Eq. (4.2) is optimal up to a  $\tilde{\mathcal{O}}((\beta/\mu)^{(k-1)/k})$  factor and improves over the best previous bound in (Kamath et al., 2022, Theorem 5.6) by removing the dependence on  $\beta_f$  (which can be much larger than  $\beta$  in the presence of outliers): see Appendix C for details. In particular, for well-conditioned losses with  $\beta/\mu = \mathcal{O}(1)$ , Theorem 11 is optimal up to logarithms.

The proof of Theorem 11 (in Appendix F.2) relies on a novel convergence guarantee for projected SGD with biased noisy stochastic gradients: Proposition 32. Compared to results in (Asi et al., 2021a) for convex ERM and (Ajalloeian and Stich, 2020) for PL SO, Proposition 32 is tighter, which is needed to obtain near-optimal excess risk: we leverage smoothness and strong convexity. Our new analysis also avoids the issue in the proofs of (the ICML versions of) (Wang et al., 2020; Kamath et al., 2022).

## 5. ALGORITHM FOR NON-SMOOTH (STRONGLY) CONVEX LOSSES

In this section, we develop a novel algorithm for non-smooth convex loss functions. We present our result for convex losses in Section 5.1, and our result for strongly convex losses in Section 5.2. In Section 3, we provide lower bounds, which show that our upper bounds are tight (up to logarithms).

**5.1. Localized Noisy Clipped Subgradient Method for Convex Losses.** Our algorithm (Algorithm 4) uses iterative localization (Feldman et al., 2020a; Asi et al., 2021b) with clipping (in Algorithm 3) to handle stochastic subgradients with large norm.<sup>7</sup>

---

**Algorithm 3:** Noisy Clipped Regularized Subgradient Method for DP ERM

---

- 1: **Input:** Data  $X \in \mathcal{X}^n$ ,  $T \in \mathbb{N}$ , stepsize  $\eta$ , clip thresh.  $C$ , regularization  $\lambda \geq 0$ , constraint set  $\mathcal{W}$  and initialization  $w_0 \in \mathcal{W}$ .
  - 2: **for**  $t \in \{0, 1, \dots, T-1\}$  **do**
  - 3:    $\tilde{\nabla} F_t(w_t) := \text{MeanOracle1}(\{\nabla f(w_t, x_i)\}_{i=1}^n; n; C; \frac{\varepsilon^2}{2T})$  for subgradients  $\nabla f(w_t, x_i)$ .
  - 4:    $w_{t+1} = \Pi_{\mathcal{W}} \left[ w_t - \eta \left( \tilde{\nabla} F_t(w_t) + \lambda(w_t - w_0) \right) \right]$
  - 5: **end for**
  - 6: **Output:**  $w_T$ .
- 

---

**Algorithm 4:** Localized Noisy Clipped Subgradient Method for DP SCO

---

- 1: **Input:** Data  $X \in \mathcal{X}^n$ , stepsize  $\eta$ , clip thresh.  $\{C_i\}_{i=1}^{\log_2(n)}$ , iteration num.  $\{T_i\}_{i=1}^{\log_2(n)}$ .
  - 2: Initialize  $w_0 \in \mathcal{W}$ . Let  $l := \log_2(n)$  and  $p = 1 + 1/k$ .
  - 3: **for**  $i \in [l]$  **do**
  - 4:   Set  $n_i = 2^{-i}n$ ,  $\eta_i = 4^{-i}\eta$ ,  $\lambda_i = \frac{1}{\eta_i n_i^p}$ ,  $T_i = \tilde{\Theta} \left( \frac{1}{\lambda_i \eta_i} \right)$ , and  $D_i = \frac{4\tilde{r}_k(\sqrt{n}2^i)^{1/k}}{\lambda_i}$ .
  - 5:   Draw new batch  $\mathcal{B}_i$  of  $n_i = |\mathcal{B}_i|$  samples from  $X$  without replacement.
  - 6:   Let  $\hat{F}_i(w) := \frac{1}{n_i} \sum_{j \in \mathcal{B}_i} f(w, x_j) + \frac{\lambda_i}{2} \|w - w_{i-1}\|^2$ .
  - 7:   Use Algorithm 3 with initialization  $w_{i-1}$  to minimize  $\hat{F}_i$  over  $\mathcal{W}_i := \{w \in \mathcal{W} : \|w - w_{i-1}\| \leq D_i\}$ , for  $T_i$  iterations with clip threshold  $C_i$  and noise  $\sigma_i^2 = \frac{4C_i^2 T_i}{n_i^2 \varepsilon^2}$ . Let  $w_i$  be the output of Algorithm 3.
  - 8: **end for**
  - 9: **Output:**  $w_l$ .
- 

<sup>7</sup>We assume WLOG that  $n = 2^l$  for some  $l \in \mathbb{N}$ . If this is not the case, then throw out samples until it is; since the number of remaining samples is at least  $n/2$ , our bounds still hold up to a constant factor.

The main ideas of Algorithm 4 are:

- (1) *Clipping only the non-regularized component of the subgradient to control sensitivity and bias*: Notice that when we call Algorithm 3 in phase  $i$  of Algorithm 4, we only clip the subgradients of  $f(w, x_j)$  and not the regularized loss  $f(w, x_j) + \frac{\lambda}{2}\|w - w_{i-1}\|^2$ . Compared to clipping the full gradient of the regularized loss, our selective clipping approach significantly reduces the bias of our subgradient estimator. This is essential for obtaining our excess risk bound. Further, this reduction in bias comes at no cost to the variance of our subgradient estimator: the  $\ell_2$ -sensitivity of our estimator is unaffected by the regularization term.
- (2) *Solve regularized ERM subproblem with a stable DP algorithm*: We run a *multi-pass* zCDP solver on a *regularized* empirical loss: Multiple passes let us reduce the noise variance in phase  $i$  by a factor of  $T_i$  (via strong composition for zCDP) and get a more accurate solution to the ERM subproblem. Regularization makes the empirical loss strongly convex, which improves *on-average model stability* and hence generalization of the obtained solution (see Proposition 16 and 39).
- (3) *Localization* (Feldman et al., 2020a; Asi et al., 2021a) (i.e. iteratively “zooming in” on a solution): In early phases (small  $i$ ), when we are far away from the optimum  $w^*$ , we use more samples (larger  $n_i$ ) and large learning rate  $\eta_i$  to make progress quickly. As  $i$  increases,  $w_i$  is closer to  $w^*$ , so fewer samples and slower learning rate suffice. Since step size  $\eta_i$  shrinks (geometrically) faster than  $n_i$ , the effective variance of the privacy noise  $\eta_i^2 \sigma_i^2$  decreases as  $i$  increases. This prevents  $w_{i+1}$  from moving too far away from  $w_i$  (and hence from  $w^*$ ). We further enforce this localization behavior by increasing the regularization parameter  $\lambda_i$  and shrinking  $D_i$  over time. We choose  $D_i$  as small as possible subject to the constraint that  $\operatorname{argmin}_{w \in \mathcal{W}} \widehat{F}_i(w) \in \mathcal{W}_i$  with high probability. This constraint ensures that Algorithm 3 can find  $w_i$  with small expected excess risk.

Next, we provide privacy and excess risk guarantees for Algorithm 4:

**Theorem 12.** *Grant Assumption 1. Let  $\varepsilon \leq \sqrt{d}$  and let  $f(\cdot, x)$  be convex. Then, there are algorithmic parameters such that Algorithm 4 is  $\frac{\varepsilon^2}{2}$ -zCDP, and has excess risk*

$$\mathbb{E}F(w_l) - F^* \lesssim \tilde{r}_{2k} D \left( \frac{1}{\sqrt{n}} + \left( \frac{\sqrt{d \ln(n)}}{\varepsilon n} \right)^{\frac{k-1}{k}} \right).$$

Moreover, this excess risk is attained in  $\tilde{\mathcal{O}}(n^{2+1/k})$  subgradient evaluations.

**Remark 13** (Improved Computational Complexity for Approximate or Shuffle DP). *If one desires  $(\varepsilon, \delta)$ -DP or  $(\varepsilon, \delta)$ -SDP instead of zCDP, then the subgradient complexity of Algorithm 4 can be improved to  $\tilde{\mathcal{O}}\left(n^{\frac{3}{2}+1/k}\right)$ : see Appendix G.2.*

The excess risk bound in Theorem 12 nearly resembles the lower bound in Theorem 7, except that the upper bound scales with  $\tilde{r}_{2k}$  instead of  $\tilde{r}_k$ .<sup>8</sup> We conjecture that an appropriate modification of our algorithm and techniques can be used to get the optimal bound depending on  $\tilde{r}_k$ . We leave this as an interesting direction for future work.

A key feature of our bound is that it does not depend on  $L_f$  and holds even for heavy-tailed problems with  $L_f = \infty$ . By contrast, prior works (Wang et al., 2020; Kamath et al.,

<sup>8</sup>In fact, the term  $\tilde{r}_{2k}$  in Theorem 12 can be replaced by a smaller term, which is  $\mathcal{O}(r_{2k})$  as  $n \rightarrow \infty$  under mild assumptions. See Appendix G.2 and Appendix G.4.

2022) required uniform  $\beta_f$ -smoothness of  $f(\cdot, x)$ , which implies the restriction  $L_f \leq \beta_f D$  for loss functions that have a vanishing gradient at some point.<sup>9</sup>

We now sketch the proof of the privacy claim in Theorem 12:

*Sketch of the Proof of Theorem 12: Privacy.* Since the batches  $\{\mathcal{B}_i\}_{i=1}^l$  are disjoint, it suffices to show that  $w_i$  (produced by  $T_i$  iterations of Algorithm 3 in line 7 of Algorithm 4) is  $\frac{\epsilon^2}{2}$ -zCDP  $\forall i \in [l]$ . The  $\ell_2$  sensitivity of the clipped subgradient update is  $\Delta = \sup_{w, X \sim X'} \left\| \frac{1}{n_i} \sum_{j=1}^{n_i} \Pi_{C_i}(\nabla f(w, x_j)) - \Pi_{C_i}(\nabla f(w, x'_j)) \right\| \leq 2C_i/n_i$ . (Note that the regularization term does not contribute to sensitivity.) Thus, the privacy guarantees of the Gaussian mechanism (Proposition 23) and the composition theorem for zCDP (Lemma 24) imply that Algorithm 4 is  $\frac{\epsilon^2}{2}$ -zCDP.  $\square$

The proof of the excess risk bound in Theorem 12 consists of three main steps: i) We bound the empirical error of the noisy clipped subgradient subroutine (Lemma 14). ii) We prove that if an algorithm is *on-average model stable* (Definition 15), then it generalizes (Proposition 16). iii) We bound the on-average model stability of regularized ERM with non-smooth/non-Lipschitz  $f(\cdot, x)$  (Proposition 39), leading to an excess population loss bound for Algorithm 3 run on the regularized empirical objective (c.f. line 7 of Algorithm 4). By using these results with the proof technique of Feldman et al. (2020a), we can obtain Theorem 12.

First, we bound the empirical error of the step in line 7 of Algorithm 4, by extending the analysis of noisy subgradient method to *biased* subgradient oracles:

**Lemma 14.** Fix  $X \in \mathcal{X}^n$  and let  $\widehat{F}_\lambda(w) = \frac{1}{n} \sum_{i=1}^n f(w, x_i) + \frac{\lambda}{2} \|w - w_0\|^2$  for  $w_0 \in \mathcal{W}$ , where  $\mathcal{W}$  is a closed convex domain with diameter  $D$ . Assume  $f(\cdot, x)$  is convex and  $\widehat{r}_n(X)^{(k)} \geq \sup_{w \in \mathcal{W}} \left\{ \frac{1}{n} \sum_{i=1}^n \|\nabla f(w, x_i)\|^k \right\}$  for all  $\nabla f(w, x_i) \in \partial_w f(w, x_i)$ . Denote  $\widehat{r}_n(X) = [\widehat{r}_n(X)^{(k)}]^{1/k}$  and  $\widehat{w} = \operatorname{argmin}_{w \in \mathcal{W}} \widehat{F}_\lambda(w)$ . Let  $\eta \leq \frac{2}{\lambda}$ . Then, the output of Algorithm 3 satisfies

$$\mathbb{E} \|w_T - \widehat{w}\|^2 \leq \exp\left(-\frac{\lambda \eta T}{2}\right) \|w_0 - \widehat{w}\|^2 + \frac{8\eta}{\lambda} (\widehat{r}_n(X)^2 + \lambda^2 D^2 + d\sigma^2) + \frac{20}{\lambda^2} \left( \frac{\widehat{r}_n(X)^{(k)}}{(k-1)C^{k-1}} \right)^2,$$

where  $\sigma^2 = \frac{4C^2 T}{n^2 \epsilon^2}$ .

Detailed proofs for this subsection are deferred to Appendix G.2.

Our next goal is to bound the generalization error of regularized ERM with convex loss functions that are not differentiable or uniformly Lipschitz. We will use a stability argument to obtain such a bound. Recall the notion of *on-average model stability* (Lei and Ying, 2020):

**Definition 15.** Let  $X = (x_1, \dots, x_n)$  and  $X' = (x'_1, \dots, x'_n)$  be drawn independently from  $\mathcal{D}$ . For  $i \in [n]$ , let  $X^i := (x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)$ . We say randomized algorithm  $\mathcal{A}$  has *on-average model stability*  $\alpha$  (i.e.  $\mathcal{A}$  is  $\alpha$ -on-average model stable) if  $\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \|\mathcal{A}(X) - \mathcal{A}(X^i)\|^2 \right] \leq \alpha^2$ . The expectation is over the randomness of  $\mathcal{A}$  and the draws of  $X$  and  $X'$ .

<sup>9</sup>Additionally, Kamath et al. (2022) assumes  $\beta_f \leq 10$ .

On-average model stability is weaker than the notion of *uniform stability* (Bousquet and Elisseeff, 2002), which has been used in DP Lipschitz SCO (e.g. by (Bassily et al., 2019)); this is necessary for obtaining our learnability guarantees without uniform Lipschitz continuity.

The main result in Lei and Ying (2020) showed that on-average model stable algorithms generalize well if  $f(\cdot, x)$  is  $\beta_f$ -smooth for all  $x$ , which leads to a restriction on  $L_f$ . We show that neither smoothness nor Lipschitz continuity of  $f$  is needed to ensure generalizability:

**Proposition 16.** *Let  $f(\cdot, x)$  be convex for all  $x$ . Suppose  $\mathcal{A} : \mathcal{X}^n \rightarrow \mathcal{W}$  is  $\alpha$ -on-average model stable. Let  $\hat{F}_X(w) := \frac{1}{n} \sum_{i=1}^n f(w, x_i)$  be an empirical loss. Then for any  $\zeta > 0$ ,*

$$\mathbb{E}[F(\mathcal{A}(X)) - \hat{F}_X(\mathcal{A}(X))] \leq \frac{\tilde{r}^{(2)}}{2\zeta} + \frac{\zeta}{2}\alpha^2.$$

Proposition 39 (and its proof) in Appendix G.2 shows that regularized ERM is  $\alpha$ -on average model stable for  $\alpha \leq \frac{2\tilde{r}^2}{\lambda n}$ . The proof of Proposition 39 combines techniques from Lei and Ying (2020) with Proposition 16. By using Proposition 16 and Proposition 39, we can bound the generalization error and excess (population) risk of regularized ERM:

**Proposition 17.** *Let  $f(\cdot, x)$  be convex,  $w_{i-1}, y \in \mathcal{W}$ , and  $\hat{w}_i := \operatorname{argmin}_{w \in \mathcal{W}} \hat{F}_i(w)$ , where  $\hat{F}_i(w) := \frac{1}{n_i} \sum_{j \in \mathcal{B}_i} f(w, x_j) + \frac{\lambda_i}{2} \|w - w_{i-1}\|^2$  (c.f. line 6 of Algorithm 4). Then,*

$$\mathbb{E}[F(\hat{w}_i)] - F(y) \leq \frac{2\tilde{r}^{(2)}}{\lambda_i n_i} + \frac{\lambda_i}{2} \|y - w_{i-1}\|^2,$$

where the expectation is over both the random draws of  $X$  from  $\mathcal{D}$  and  $\mathcal{B}_i$  from  $X$ .

With the pieces developed above, we can now sketch the excess risk proof of Theorem 12:

*Sketch of the Proof of Theorem 12: Excess risk.* First, our choice of  $D_i$  ensures that  $\hat{w}_i \in \mathcal{W}_i$  for all  $i \in [l]$  with high probability  $\geq 1 - 1/\sqrt{n}$ , by Chebyshev's inequality. We will assume that this event occurs in the rest of the proof sketch: see the detailed proof in the Appendix for when this event breaks. Denote  $p = 1 + 1/k$ . Combining Lemma 14 with Lemma 6 and proper choices of  $\eta$  and  $T_i$ , we get:

$$\begin{aligned} \mathbb{E}\|w_i - \hat{w}_i\|^2 &\lesssim \eta_i^2 n_i^p (\tilde{r}_k^2 n^{1/k} 2^{2i/k} + d\sigma_i^2) + \frac{\eta_i^2 n_i^{2p} \tilde{r}^{(2k)}}{C_i^{2k-2}} \\ &\lesssim \left( \frac{\eta^2 n^p}{32^i} \left( \tilde{r}_k^2 n^{1/k} 2^{2i/k} + \frac{dC_i^2 T_i}{\varepsilon^2 n_i^2} + \frac{n^p \tilde{r}^{(2k)}}{C_i^{2k-2} 2^{pi}} \right) \right). \end{aligned} \quad (5.1)$$

Now, following the strategy used in the proof of (Feldman et al., 2020a, Theorem 4.4), we write  $\mathbb{E}F(w_l) - F(w^*) = \mathbb{E}[F(w_l) - F(\hat{w}_l)] + \sum_{i=1}^l \mathbb{E}[F(\hat{w}_i) - F(\hat{w}_{i-1})]$ , where  $\hat{w}_0 := w^*$ . Using Eq. (5.1) and  $\tilde{r}_k$ -Lipschitz continuity of  $F$  (which is implied by Assumption 1), we can bound  $\mathbb{E}[F(w_l) - F(\hat{w}_l)]$  for the right  $\eta$  and  $C_l$ . To bound the sum (second term), we use Proposition 17 to obtain

$$\begin{aligned} \sum_{i=1}^l \mathbb{E}[F(\hat{w}_i) - F(\hat{w}_{i-1})] &\lesssim r^2 \eta n^{p-1} + \frac{D^2}{\eta n^p} + \eta \tilde{r}_k^2 n^{1/k} + \eta \sum_{i=2}^l 4^{-i} n_i^p \tilde{r}_{2k}^2 \left( \frac{d \ln(n)}{\varepsilon^2 n_i^2} \right)^{\frac{k-1}{k}} \\ &\lesssim \eta \left[ r^2 n^{p-1} + \tilde{r}_k^2 n^{1/k} + \tilde{r}_{2k}^2 n^p \left( \frac{d \ln(n)}{\varepsilon^2 n^2} \right)^{\frac{k-1}{k}} \right] + \frac{D^2}{\eta n^p}, \end{aligned}$$

for the right choice of  $C_i$ . Then properly choosing  $\eta$  completes the excess risk proof.

**Computational complexity:** The choice  $T_i = \tilde{\mathcal{O}}(n_i^p) = \tilde{\mathcal{O}}(n_i^{1+1/k})$  implies that the number of subgradient evaluations is bounded by  $\sum_{i=1}^l n_i T_i = \tilde{\mathcal{O}}(n^{p+1}) = \tilde{\mathcal{O}}(n^{2+1/k})$ .  $\square$

**5.2. The Strongly Convex Case.** Following (Feldman et al., 2020a), we use a folklore reduction to the convex case (detailed in Appendix G.3) in order to obtain the following upper bound via Theorem 12:

**Theorem 18.** *Grant Assumption 1. Let  $\varepsilon \leq \sqrt{d}$  and let  $f(\cdot, x)$  be  $\mu$ -strongly convex. Then, there is a polynomial-time  $\frac{\varepsilon^2}{2}$ -zCDP algorithm  $\mathcal{A}$  based on Algorithm 4 with excess risk*

$$\mathbb{E}F(\mathcal{A}(X)) - F^* \lesssim \frac{\tilde{r}_{2k}^2}{\mu} \left( \frac{1}{n} + \left( \frac{\sqrt{d \ln(n)}}{\varepsilon n} \right)^{\frac{2k-2}{k}} \right).$$

## 6. ALGORITHM FOR NON-CONVEX PROXIMAL-PL LOSS FUNCTIONS

Assume:  $f(w, x) = f^0(w, x) + f^1(w)$ ;  $f^0(\cdot, x)$  is differentiable (maybe non-convex);  $f^1$  is proper, closed, and convex (maybe non-differentiable) for all  $x \in \mathcal{X}$ ; and  $F(w) = F^0(w) + f^1(w) = \mathbb{E}_{x \sim \mathcal{D}}[f^0(w, x)] + f^1(w)$  satisfies the *Proximal-PL* condition (Karimi et al., 2016):

**Definition 19** ( $\mu$ -PPL). *Let  $F(w) = F^0(w) + f^1(w)$  be bounded below;  $F^0$  is  $\beta$ -smooth and  $f^1$  is convex.  $F$  satisfies Proximal Polyak-Łojasiewicz inequality with parameter  $\mu > 0$  if*

$$\mu[F(w) - \inf_{w'} F(w')] \leq -\beta \min_y \left[ \langle \nabla F^0(w), y - w \rangle + \frac{\beta}{2} \|y - w\|^2 + f^1(y) - f^1(w) \right], \quad \forall w \in \mathbb{R}^d.$$

Definition 19 generalizes the classical PL condition ( $f^1 = 0$ ), allowing for constrained optimization or non-smooth regularizer depending on  $f^1$  (Polyak, 1963; Karimi et al., 2016).

Recall that the *proximal operator* of a convex function  $g$  is defined as  $\text{prox}_{\eta g}(z) := \text{argmin}_{y \in \mathbb{R}^d} (\eta g(y) + \frac{1}{2} \|y - z\|^2)$  for  $\eta > 0$ . We propose *Noisy Clipped Proximal SGD* (Algorithm 8 in Appendix H) for PPL losses. The algorithm runs as follows. For  $t \in [T]$ : first draw a new batch  $\mathcal{B}_t$  (without replacement) of  $n/T$  samples from  $X$ ; let  $\tilde{\nabla} F_t^0(w_t) := \text{MeanOracle1}(\{\nabla f^0(w_t, x)\}_{x \in \mathcal{B}_t}; \frac{n}{T}; \frac{\varepsilon^2}{2})$ ; then update  $w_{t+1} = \text{prox}_{\eta_t f^1}(w_t - \eta_t \tilde{\nabla} F_t^0(w_t))$ . Finally, return the last iterate,  $w_T$ . Thus, the algorithm is linear time. Furthermore:

**Theorem 20** (Informal). *Grant Assumption 1. Let  $F$  be  $\mu$ -PPL for  $\beta$ -smooth  $F^0$ , with  $\frac{\beta}{\mu} \leq n/\ln(n)$ . Then, there are parameters such that Algorithm 8 is  $\frac{\varepsilon^2}{2}$ -zCDP, and:*

$$\mathbb{E}F(w_T) - F^* \lesssim \frac{1}{\mu} \left( \tilde{r}_k^2 \left( \frac{\sqrt{d}}{\varepsilon n} (\beta/\mu) \ln(n) \right)^{\frac{2k-2}{k}} + \frac{\tilde{r}_2^2 (\beta/\mu) \ln(n)}{n} \right).$$

Moreover, Algorithm 8 uses at most  $n$  gradient evaluations.

The bound in Theorem 20 nearly matches the smooth *strongly convex* (hence PPL) lower bound in Theorem 8 up to  $\tilde{\mathcal{O}}((\beta/\mu)^{(2k-2)/2})$ , and is attained without convexity.

To prove Theorem 20, we derive a convergence guarantee for proximal SGD with generic biased, noisy stochastic gradients in terms of the bias and variance of the oracle (see Proposition 46). We then apply this guarantee for `MeanOracle1` (Algorithm 1) with carefully chosen stepsizes, clip threshold, and  $T$ , using Lemma 6. Proposition 46 generalizes (Ajalloeian and Stich, 2020, Theorem 6)—which covered the unconstrained classical PL problem—to the proximal setting. However, the proof of Proposition 46 is very different from the proof of (Ajalloeian and Stich, 2020, Theorem 6), since `prox` makes it hard to bound excess risk without convexity when the stochastic gradients are biased/noisy. Instead, our proof builds on the proof of (Lowy et al., 2023, Theorem 3.1), using techniques from the analysis of *objective perturbation* (Chaudhuri et al., 2011; Kifer et al., 2012). See Appendix H for details.

## 7. CONCLUDING REMARKS AND OPEN QUESTIONS

This paper was motivated by practical problems in which data contains outliers and potentially heavy tails, causing the worst-case Lipschitz parameter of the loss over all data points to be prohibitively large. In such cases, existing bounds for DP SO that scale with the worst-case Lipschitz parameter become vacuous. Thus, we operated under the more relaxed assumption of stochastic gradient distributions having bounded  $k$ -th moments. The  $k$ -th moment bound can be much smaller than the worst-case Lipschitz parameter in practice. For (strongly) convex loss functions, we established the optimal rates (up to logarithms) in certain parameter regimes, but not in every parameter regime. Thus, a natural problem for future work is to provide optimal risk bounds in all regimes. We believe that a promising approach will be to build on our algorithm and techniques in Section 5.1 in order to replace the  $\tilde{r}_{2k}$  term by  $\tilde{r}_k$  in our upper bounds. We also initiated the study of non-convex DP SO without uniform Lipschitz continuity, showing that the optimal strongly convex rates can nearly be attained without convexity, via the proximal-PL condition. We leave the treatment of general non-convex losses for future work.

## ACKNOWLEDGMENT

We thank John Duchi, Larry Goldstein, and Stas Minsker for very helpful conversations and pointers related to our lower bounds and the proof of Lemma 57. We also thank the authors of (Kamath et al., 2022) for clarifying some steps in the proof of their Theorem 4.1 and providing feedback on the first draft of this manuscript. We would also like to thank the USC-Meta Center for Research and Education in AI and Learning for supporting this research. AL is supported by NSF grant 2023239 and AFOSR MURI award FA9550-21-1-0084. Finally, we thank the anonymous ALT and TPDP reviewers for their insightful comments.

## REFERENCES

- M. Abadi, A. Chu, I. Goodfellow, H. B. McMahan, I. Mironov, K. Talwar, and L. Zhang. Deep learning with differential privacy. *Proceedings of the 2016 ACM SIGSAC Conference on Computer and Communications Security*, Oct 2016. <https://doi.org/10.1145/2976749.2978318>.
- J. Acharya, Z. Sun, and H. Zhang. Differentially private assouad, fano, and le cam. In *Algorithmic Learning Theory*, pages 48–78. PMLR, 2021. <https://proceedings.mlr.press/v132/acharya21a.html>.
- A. Ajalloeian and S. U. Stich. On the convergence of sgd with biased gradients. *arXiv preprint arXiv:2008.00051*, 2020. <https://arxiv.org/abs/2008.00051>.
- A. Andoni. COMS E6998-9: algorithmic techniques for massive data, 2015. [https://www.mit.edu/~andoni/F15\\_AlgoTechMassiveData/files/scribe2.pdf](https://www.mit.edu/~andoni/F15_AlgoTechMassiveData/files/scribe2.pdf).
- G. Andrew, O. Thakkar, B. McMahan, and S. Ramaswamy. Differentially private learning with adaptive clipping. *Advances in Neural Information Processing Systems*, 34, 2021. [https://papers.neurips.cc/paper\\_files/paper/2021/file/91cff01af640a24e7f9f7a5ab407889f-Paper.pdf](https://papers.neurips.cc/paper_files/paper/2021/file/91cff01af640a24e7f9f7a5ab407889f-Paper.pdf).
- R. Arora, R. Bassily, C. Guzmán, M. Menart, and E. Ullah. Differentially private generalized linear models revisited. *arXiv preprint arXiv:2205.03014*, 2022. <https://arxiv.org/pdf/2205.03014>.
- H. Asi, J. Duchi, A. Fallah, O. Javidsbakht, and K. Talwar. Private adaptive gradient methods for convex optimization. In *International Conference on Machine Learning*, pages 383–392. PMLR, 2021a. <https://proceedings.mlr.press/v139/asi21a/asi21a.pdf>.
- H. Asi, V. Feldman, T. Koren, and K. Talwar. Private stochastic convex optimization: Optimal rates in  $\ell_1$  geometry. In M. Meila and T. Zhang, editors, *Proceedings of the 38th International Conference on Machine Learning*, volume 139 of *Proceedings of Machine Learning Research*, pages 393–403. PMLR, 18–24 Jul 2021b. <https://proceedings.mlr.press/v139/asi21b.html>.
- R. F. Barber and J. C. Duchi. Privacy and statistical risk: Formalisms and minimax bounds. *arXiv preprint arXiv:1412.4451*, 2014. <https://arxiv.org/pdf/1412.4451>.
- R. Bassily, A. Smith, and A. Thakurta. Private empirical risk minimization: Efficient algorithms and tight error bounds. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*, pages 464–473. IEEE, 2014. <https://doi.org/10.1109/FOCS.2014.56>.
- R. Bassily, V. Feldman, K. Talwar, and A. Thakurta. Private stochastic convex optimization with optimal rates. In *Advances in Neural Information Processing Systems*, 2019. <https://openreview.net/pdf?id=BJLJWSSeLS>.
- R. Bassily, V. Feldman, C. Guzmán, and K. Talwar. Stability of stochastic gradient descent on nonsmooth convex losses. *Advances in Neural Information Processing Systems*, 33:4381–4391, 2020. <https://proceedings.neurips.cc/paper/2020/file/2e2c4bf7ceaa4712a72dd5ee136dc9a8-Paper.pdf>.
- R. Bassily, C. Guzmán, and M. Menart. Differentially private stochastic optimization: New results in convex and non-convex settings. *arXiv preprint arXiv:2107.05585*, 2021a. <https://arxiv.org/pdf/2107.05585>.
- R. Bassily, C. Guzmán, and A. Nandi. Non-euclidean differentially private stochastic convex optimization. In *Conference on Learning Theory*, pages 474–499. PMLR, 2021b. <https://proceedings.mlr.press/v134/bassily21a/bassily21a.pdf>.

- O. Bousquet and A. Elisseeff. Stability and generalization. *The Journal of Machine Learning Research*, 2:499–526, 2002. <https://www.jmlr.org/papers/volume2/bousquet02a/bousquet02a.pdf>.
- S. Bubeck. Convex optimization: Algorithms and complexity. *Foundations and Trends® in Machine Learning*, 8(3-4):231–357, 2015. <https://doi.org/10.1561/22000000050>.
- M. Bun and T. Steinke. Concentrated differential privacy: Simplifications, extensions, and lower bounds. In *Proceedings, Part I, of the 14th International Conference on Theory of Cryptography - Volume 9985*, page 635–658, Berlin, Heidelberg, 2016. Springer-Verlag. ISBN 9783662536407. [https://doi.org/10.1007/978-3-662-53641-4\\_24](https://doi.org/10.1007/978-3-662-53641-4_24).
- N. Carlini, F. Tramèr, E. Wallace, M. Jagielski, A. Herbert-Voss, K. Lee, A. Roberts, T. Brown, D. Song, U. Erlingsson, et al. Extracting training data from large language models. In *30th USENIX Security Symposium (USENIX Security 21)*, pages 2633–2650, 2021. <https://www.usenix.org/system/files/sec21-carlini-extracting.pdf>.
- K. Chaudhuri and C. Monteleoni. Privacy-preserving logistic regression. *Advances in neural information processing systems*, 21, 2008. [https://papers.nips.cc/paper\\_files/paper/2008/file/8065d07da4a77621450aa84fee5656d9-Paper.pdf](https://papers.nips.cc/paper_files/paper/2008/file/8065d07da4a77621450aa84fee5656d9-Paper.pdf).
- K. Chaudhuri, C. Monteleoni, and A. D. Sarwate. Differentially private empirical risk minimization. *Journal of Machine Learning Research*, 12(3), 2011. <https://jmlr.org/papers/volume12/chaudhuri11a/chaudhuri11a.pdf>.
- X. Chen, S. Z. Wu, and M. Hong. Understanding gradient clipping in private sgd: A geometric perspective. *Advances in Neural Information Processing Systems*, 33:13773–13782, 2020. <https://proceedings.neurips.cc/paper/2020/file/9ecff5455677b38d19f49ce658ef0608-Paper.pdf>.
- A. Cheu, A. Smith, J. Ullman, D. Zeber, and M. Zhilyaev. Distributed differential privacy via shuffling. In *Annual International Conference on the Theory and Applications of Cryptographic Techniques*, pages 375–403. Springer, 2019. [https://doi.org/10.1007/978-3-030-17653-2\\_13](https://doi.org/10.1007/978-3-030-17653-2_13).
- A. Cheu, M. Joseph, J. Mao, and B. Peng. Shuffle private stochastic convex optimization. *arXiv preprint arXiv:2106.09805*, 2021. <https://arxiv.org/abs/2106.09805>.
- M. E. Crovella, M. S. Taqqu, and A. Bestavros. Heavy-tailed probability distributions. *A Practical Guide to Heavy Tails Statistical Techniques and Applications*, 1998. <https://doi.org/10.5555/292595.292596>.
- R. Das, S. Kale, Z. Xu, T. Zhang, and S. Sanghavi. Beyond uniform lipschitz condition in differentially private optimization. *arXiv preprint arXiv:2206.10713*, 2022. <https://doi.org/10.5555/3618408.3618689>.
- J. Duchi. Lecture notes for statistics 311/electrical engineering 377. URL: <https://stanford.edu/class/stats311/lecture-notes.pdf>, 2021. <https://stanford.edu/class/stats311/lecture-notes.pdf>.
- J. C. Duchi, M. I. Jordan, and M. J. Wainwright. Local privacy and statistical minimax rates. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, pages 429–438, 2013. <https://doi.org/10.1109/FOCS.2013.53>.
- C. Dwork and A. Roth. *The Algorithmic Foundations of Differential Privacy*. 2014. <https://doi.org/10.1561/04000000042>.
- C. Dwork, F. McSherry, K. Nissim, and A. Smith. Calibrating noise to sensitivity in private data analysis. In *Theory of cryptography conference*, pages 265–284. Springer, 2006. [https://doi.org/10.1007/978-3-540-32732-5\\_32](https://doi.org/10.1007/978-3-540-32732-5_32).

- Ú. Erlingsson, V. Feldman, I. Mironov, A. Raghunathan, S. Song, K. Talwar, and A. Thakurta. Encode, shuffle, analyze privacy revisited: Formalizations and empirical evaluation. *arXiv preprint arXiv:2001.03618*, 2020. <https://arxiv.org/abs/2001.03618>.
- I. Faizullahoy and A. Korolova. Facebook’s advertising platform: New attack vectors and the need for interventions. *arXiv preprint arXiv:1803.10099*, 2018. <https://arxiv.org/pdf/1803.10099>.
- V. Feldman, T. Koren, and K. Talwar. Private stochastic convex optimization: optimal rates in linear time. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, pages 439–449, 2020a. <https://doi.org/10.1145/3357713.3384335>.
- V. Feldman, A. McMillan, and K. Talwar. Hiding among the clones: A simple and nearly optimal analysis of privacy amplification by shuffling, 2020b.
- M. Fredrikson, S. Jha, and T. Ristenpart. Model inversion attacks that exploit confidence information and basic countermeasures. In *Proceedings of the 22nd ACM SIGSAC Conference on Computer and Communications Security*, pages 1322–1333, 2015. <https://doi.org/10.1145/2810103.2813677>.
- S. Ghadimi and G. Lan. Optimal stochastic approximation algorithms for strongly convex stochastic composite optimization i: A generic algorithmic framework. *SIAM Journal on Optimization*, 22(4):1469–1492, 2012. <https://doi.org/10.1137/110848864>.
- E. Hazan and S. Kale. Beyond the regret minimization barrier: optimal algorithms for stochastic strongly-convex optimization. *The Journal of Machine Learning Research*, 15(1):2489–2512, 2014. <https://jmlr.org/papers/volume15/hazan14a/hazan14a.pdf>.
- M. J. Holland. Robust descent using smoothed multiplicative noise. In *The 22nd International Conference on Artificial Intelligence and Statistics*, pages 703–711. PMLR, 2019. <https://proceedings.mlr.press/v89/holland19a.html>.
- L. Hu, S. Ni, H. Xiao, and D. Wang. High dimensional differentially private stochastic optimization with heavy-tailed data. *arXiv preprint arXiv:2107.11136*, 2021. [10.1145/3517804.3524144](https://arxiv.org/abs/2107.11136).
- P. Kairouz, H. B. McMahan, B. Avent, A. Bellet, M. Bennis, A. N. Bhagoji, K. Bonawitz, Z. Charles, G. Cormode, R. Cummings, R. G. L. D’Oliveira, S. E. Rouayheb, D. Evans, J. Gardner, Z. Garrett, A. Gascón, B. Ghazi, P. B. Gibbons, M. Gruteser, Z. Harchaoui, C. He, L. He, Z. Huo, B. Hutchinson, J. Hsu, M. Jaggi, T. Javidi, G. Joshi, M. Khodak, J. Konečný, A. Korolova, F. Koushanfar, S. Koyejo, T. Lepoint, Y. Liu, P. Mittal, M. Mohri, R. Nock, A. Özgür, R. Pagh, M. Raykova, H. Qi, D. Ramage, R. Raskar, D. Song, W. Song, S. U. Stich, Z. Sun, A. T. Suresh, F. Tramèr, P. Vepakomma, J. Wang, L. Xiong, Z. Xu, Q. Yang, F. X. Yu, H. Yu, and S. Zhao. Advances and open problems in federated learning. *arXiv preprint:1912.04977*, 2019. <https://arxiv.org/abs/1912.04977>.
- G. Kamath, V. Singhal, and J. Ullman. Private mean estimation of heavy-tailed distributions. In J. Abernethy and S. Agarwal, editors, *Proceedings of Thirty Third Conference on Learning Theory*, volume 125 of *Proceedings of Machine Learning Research*, pages 2204–2235. PMLR, 09–12 Jul 2020. <https://proceedings.mlr.press/v125/kamath20a.html>.
- G. Kamath, X. Liu, and H. Zhang. Improved rates for differentially private stochastic convex optimization with heavy-tailed data. In K. Chaudhuri, S. Jegelka, L. Song, C. Szepesvari, G. Niu, and S. Sabato, editors, *Proceedings of the 39th International Conference on Machine Learning*, volume 162 of *Proceedings of Machine Learning Research*, pages 10633–10660. PMLR, 17–23 Jul 2022. <https://proceedings.mlr.press/v162/kamath22a.html>.

- H. Karimi, J. Nutini, and M. Schmidt. Linear convergence of gradient and proximal-gradient methods under the polyak-łojasiewicz condition. In *Joint European Conference on Machine Learning and Knowledge Discovery in Databases*, pages 795–811. Springer, 2016. [https://doi.org/10.1007/978-3-319-46128-1\\_50](https://doi.org/10.1007/978-3-319-46128-1_50).
- M. Kearns and D. Ron. Algorithmic stability and sanity-check bounds for leave-one-out cross-validation. *Neural computation*, 11(6):1427–1453, 1999. <https://doi.org/10.1162/089976699300016304>.
- D. Kifer, A. Smith, and A. Thakurta. Private convex empirical risk minimization and high-dimensional regression. In *Conference on Learning Theory*, pages 25–1. JMLR Workshop and Conference Proceedings, 2012. <https://proceedings.mlr.press/v23/kifer12/kifer12.pdf>.
- A. N. Kolmogorov and V. M. Tikhomirov.  $\varepsilon$ -entropy and  $\varepsilon$ -capacity of sets in function spaces. *Uspekhi Matematicheskikh Nauk*, 14(2):3–86, 1959.
- J. Kulkarni, Y. T. Lee, and D. Liu. Private non-smooth empirical risk minimization and stochastic convex optimization in subquadratic steps. *arXiv preprint arXiv:2103.15352*, 2021. <https://arxiv.org/pdf/2103.15352>.
- Y. Lei and Y. Ying. Fine-grained analysis of stability and generalization for stochastic gradient descent. In *International Conference on Machine Learning*, pages 5809–5819. PMLR, 2020. <https://proceedings.mlr.press/v119/lei20c/lei20c.pdf>.
- Y. Lei and Y. Ying. Sharper generalization bounds for learning with gradient-dominated objective functions. In *International Conference on Learning Representations*, 2021. <https://doi.org/10.48550/arXiv.2006.06522>.
- A. Lowy and M. Razaviyayn. Output perturbation for differentially private convex optimization with improved population loss bounds, runtimes and applications to private adversarial training. *arXiv preprint:2102.04704*, 2021. <https://arxiv.org/pdf/2102.04704>.
- A. Lowy and M. Razaviyayn. Private federated learning without a trusted server: Optimal algorithms for convex losses. In *The Eleventh International Conference on Learning Representations*, 2023. <https://openreview.net/forum?id=TVY6GoURrw>.
- A. Lowy, A. Ghafelebashi, and M. Razaviyayn. Private non-convex federated learning without a trusted server. In *International Conference on Artificial Intelligence and Statistics*, pages 5749–5786. PMLR, 2023. <https://proceedings.mlr.press/v206/lowy23a/lowy23a.pdf>.
- N. Markovich. *Nonparametric analysis of univariate heavy-tailed data: research and practice*. John Wiley & Sons, 2008. <https://doi.org/10.1002/9780470723609>.
- F. D. McSherry. Privacy integrated queries: an extensible platform for privacy-preserving data analysis. In *Proceedings of the 2009 ACM SIGMOD International Conference on Management of data*, pages 19–30, 2009. <https://doi.org/10.1145/1559845.1559850>.
- S. Minsker. U-statistics of growing order and sub-gaussian mean estimators with sharp constants. *arXiv preprint arXiv:2202.11842*, 2022. <https://arxiv.org/pdf/2202.11842>.
- M. Nasr, R. Shokri, and A. Houmansadr. Comprehensive privacy analysis of deep learning: Passive and active white-box inference attacks against centralized and federated learning. In *2019 IEEE symposium on security and privacy (SP)*, pages 739–753. IEEE, 2019. <https://doi.org/10.1109/SP.2019.00065>.
- B. T. Polyak. Gradient methods for the minimisation of functionals. *USSR Computational Mathematics and Mathematical Physics*, 3(4):864–878, 1963. [https://doi.org/10.1016/0041-5553\(63\)90382-3](https://doi.org/10.1016/0041-5553(63)90382-3).

- A. Rényi. On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, volume 4, pages 547–562. University of California Press, 1961. <https://projecteuclid.org/proceedings/berkeley-symposium-on-mathematical-statistics-and-probability/Proceedings-of-the-Fourth-Berkeley-Symposium-on-Mathematical-Statistics-and-Chapter/On-Measures-of-Entropy-and-Information/bsmsp/1200512181>.
- S. Shalev-Shwartz, O. Shamir, N. Srebro, and K. Sridharan. Stochastic convex optimization. 2(4):5, 2009. <https://www.cs.cornell.edu/~sridharan/convex.pdf>.
- R. Shokri, M. Stronati, C. Song, and V. Shmatikov. Membership inference attacks against machine learning models. In *2017 IEEE symposium on security and privacy (SP)*, pages 3–18. IEEE, 2017. <https://doi.org/10.1109/SP.2017.41>.
- S. Song, T. Steinke, O. Thakkar, and A. Thakurta. Evading the curse of dimensionality in unconstrained private glms. In *International Conference on Artificial Intelligence and Statistics*, pages 2638–2646. PMLR, 2021. <https://proceedings.mlr.press/v130/song21a/song21a.pdf>.
- S. U. Stich. Unified optimal analysis of the (stochastic) gradient method. *arXiv preprint:1907.04232*, 2019. <https://arxiv.org/pdf/1907.04232>.
- Y. Tao, Y. Wu, P. Zhao, and D. Wang. Optimal rates of (locally) differentially private heavy-tailed multi-armed bandits. *arXiv preprint arXiv:2106.02575*, 2021. <https://arxiv.org/pdf/2106.02575>.
- J. Ullman. Private multiplicative weights beyond linear queries. In *Proceedings of the 34th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems*, pages 303–312, 2015. <https://doi.org/10.1145/2745754.2745755>.
- V. N. Vapnik. An overview of statistical learning theory. *IEEE transactions on neural networks*, 10(5):988–999, 1999.
- R. Vershynin. *High-dimensional probability: An introduction with applications in data science*, volume 47. Cambridge university press, 2018.
- D. Wang and J. Xu. Differentially private  $\ell_1$ -norm linear regression with heavy-tailed data. *arXiv preprint arXiv:2201.03204*, 2022. <https://arxiv.org/pdf/2201.03204>.
- D. Wang, M. Ye, and J. Xu. Differentially private empirical risk minimization revisited: Faster and more general. In *Proc. 31st Annual Conference on Advances in Neural Information Processing Systems (NIPS 2017)*, 2017. [https://papers.nips.cc/paper\\_files/paper/2017/hash/f337d999d9ad116a7b4f3d409fcc6480-Abstract.html](https://papers.nips.cc/paper_files/paper/2017/hash/f337d999d9ad116a7b4f3d409fcc6480-Abstract.html).
- D. Wang, H. Xiao, S. Devadas, and J. Xu. On differentially private stochastic convex optimization with heavy-tailed data. In *International Conference on Machine Learning*, pages 10081–10091. PMLR, 2020. <https://doi.org/10.5555/3524938.3525873>.
- R. F. Woolson and W. R. Clarke. *Statistical methods for the analysis of biomedical data*. John Wiley & Sons, 2011. <https://doi.org/10.1002/9781118033050>.
- B. Yu. Assouad, fano, and le cam. In *Festschrift for Lucien Le Cam*, pages 423–435. Springer, 1997. [https://doi.org/10.1007/978-1-4612-1880-7\\_29](https://doi.org/10.1007/978-1-4612-1880-7_29).

## APPENDIX

## APPENDIX A. ADDITIONAL DISCUSSION OF RELATED WORK

**DP SCO Without Uniform Lipschitz Continuity:** The study of DP SCO without uniformly Lipschitz continuous loss functions was initiated by (Wang et al., 2020), who provided upper bounds for smooth convex/strongly convex loss. The work of (Kamath et al., 2022) provided lower bounds and improved, yet still suboptimal, upper bounds for the convex case. Both of the works (Wang et al., 2020; Kamath et al., 2022) require  $f$  to be  $\beta_f$ -smooth. It is also worth mentioning that (Wang et al., 2020; Kamath et al., 2022) restricted attention to losses satisfying  $\nabla F(w^*) = 0$  for  $w^* \in \mathcal{W}$ , i.e.  $\mathcal{W}$  is a compact set containing the *unconstrained* optimum  $w^* = \operatorname{argmin}_{w \in \mathbb{R}^d} F(w) \in \mathcal{W}$ . By comparison, we consider the more general *constrained* optimization problem  $\min_{w \in \mathcal{W}} F(w)$ , where  $\mathcal{W}$  need not contain the global unconstrained optimum.

Here we provide a brief discussion of the techniques used in (Wang et al., 2020; Kamath et al., 2022). The work of (Wang et al., 2020) used a full batch (clipped, noisy) gradient descent based algorithm, building on the heavy-tailed mean estimator of (Holland, 2019). They bounded the excess risk of their algorithm by using a uniform convergence (Vapnik, 1999) argument, resulting in a suboptimal dependence on the dimension  $d$ . The work of (Kamath et al., 2022) used essentially the same approach as (Wang et al., 2020), but obtained an improved rate with a more careful analysis.<sup>10</sup> However, as discussed, the bound in (Kamath et al., 2022) is  $\mathcal{O}\left(rD\sqrt{\frac{d}{n}}\right)$  when  $\varepsilon \approx 1$ , which is still suboptimal.<sup>11</sup>

More recently, DP optimization with outliers was studied in special cases of sparse learning (Hu et al., 2021), multi-arm bandits (Tao et al., 2021), and  $\ell_1$ -norm linear regression (Wang and Xu, 2022).

**DP ERM and DP GLMs without Uniform Lipschitz continuity:** The work of (Asi et al., 2021a) provides bounds for constrained DP *ERM* with arbitrary convex loss functions using a Noisy Clipped SGD algorithm that is similar to our Algorithm 6, except that their algorithm is multi-pass and ours is one-pass. In a concurrent work, (Das et al., 2022) considered DP *ERM* in the *unconstrained* setting with convex and non-convex loss functions. Their algorithm, noisy clipped SGD, is also similar to Algorithm 6 and the algorithm of (Asi et al., 2021a). The results in (Das et al., 2022) are not directly comparable to (Asi et al., 2021a) since (Das et al., 2022) consider the unconstrained setting while (Asi et al., 2021a) consider the constrained setting, but the rates in (Asi et al., 2021a) are faster. (Das et al., 2022) also analyzes the convergence of noisy clipped SGD with smooth non-convex loss functions.

The works of (Song et al., 2021; Arora et al., 2022) consider *generalized linear models (GLMs)*, a particular subclass of convex loss functions and provide empirical and population risk bounds for the *unconstrained* DP optimization problem. The unconstrained setting is not comparable to the constrained setting that we consider here: in the unconstrained case,

<sup>10</sup>Additionally, (Kamath et al., 2022, Theorem 5.2) provided a bound via noisy gradient descent with the clipping mechanism of (Kamath et al., 2020), but this bound is inferior (in the practical privacy regime  $\varepsilon \approx 1$ ) to their bound in (Kamath et al., 2022, Theorem 5.4) that used the estimator of (Holland, 2019).

<sup>11</sup>The bound in (Kamath et al., 2022, Theorem 5.4) for  $k = 2$  is stated in the notation of Assumption 3 and thus has an extra factor of  $\sqrt{d}$ , compared to the bound written here. We write their bound in terms of our Assumption 1, replacing their  $\gamma_k d$  term by  $r\sqrt{d}$ .

a dimension-independent upper bound is achievable, whereas our lower bounds (which apply to GLMs) imply that a dependence on the dimension  $d$  is necessary in the constrained case.

**Other works on gradient clipping:** The gradient clipping technique (and adaptive variants of it) has been studied empirically in works such as (Abadi et al., 2016; Chen et al., 2020; Andrew et al., 2021), to name a few. The work of (Chen et al., 2020) shows that gradient clipping can prevent SGD from converging, and describes the clipping bias with a disparity measure between the gradient distribution and a geometrically symmetric distribution.

**Optimization with biased gradient oracles:** The works (Ajalloeian and Stich, 2020; Asi et al., 2021a) analyze SGD with biased gradient oracles. Our work provides a tighter bound for smooth, strongly convex functions and analyzes accelerated SGD and proximal SGD with biased gradient oracles.

**DP SO with Uniformly Lipschitz loss functions:** In the absence of outlier data, there are a multitude of works studying uniformly Lipschitz DP SO, mostly in the convex/strongly convex case. We do not attempt to provide a comprehensive list of these here, but will name the most notable ones, which provide optimal or state-of-the-art utility guarantees. The first suboptimal bounds for DP SCO were provided in (Bassily et al., 2014). The work of (Bassily et al., 2019) established the optimal rate for non-strongly convex DP SCO, by bounding the uniform stability of Noisy DP SGD (without clipping). The strongly convex case was addressed by (Feldman et al., 2020a), who also provided optimal rates in linear times for sufficiently smooth, convex losses. Since then, other works have provided faster and simpler (optimal) algorithms for the non-smooth DP SCO problem (Bassily et al., 2020; Asi et al., 2021b; Kulkarni et al., 2021; Bassily et al., 2021a) and considered DP SCO with different geometries (Asi et al., 2021b; Bassily et al., 2021b). State-of-the-art rates for DP SO with the proximal PL condition are due to (Lowy et al., 2023).

## APPENDIX B. OTHER BOUNDED MOMENT CONDITIONS BESIDES ASSUMPTION 1

In this section, we give the alternate bounded moment assumption made in (Wang et al., 2020; Kamath et al., 2022) and a third bounded moment condition, and discuss the relationships between these assumptions. The notation presented here will be necessary in order to state the sharper versions of our linear-time excess risk bounds and the asymptotically optimal excess risk bounds under the coordinate-wise assumption of (Wang et al., 2020; Kamath et al., 2022) (which our Algorithm 4 also attains). First, we introduce a relaxation of Assumption 1:

**Assumption 2.** *There exists  $k \geq 2$  and  $r^{(k)} > 0$  such that  $\sup_{w \in \mathcal{W}} \mathbb{E} [\|\nabla f(w, x)\|_2^k] \leq r^{(k)}$ , for all subgradients  $\nabla f(w, x_i) \in \partial_w f(w, x_i)$ . Denote  $r_k := (r^{(k)})^{1/k}$ .*

Assumption 1 implies Assumption 2 for  $r \leq \tilde{r}$ . Next, we precisely state the coordinate-wise moment bound assumption that is used in (Wang et al., 2020; Kamath et al., 2022) for differentiable  $f$ :

**Assumption 3. (Used by (Wang et al., 2020; Kamath et al., 2022)<sup>12</sup>, but not in this work)** *There exists  $k \geq 2$  and  $\gamma_k > 0$  such that  $\sup_{w \in \mathcal{W}} \mathbb{E} |\langle \nabla f(w, x) -$*

<sup>12</sup>The work of (Kamath et al., 2022) assumes that  $L \lesssim \gamma_k^{1/k} = 1$ . On the other hand, (Wang et al., 2020) assumes that  $F$  is  $\beta$ -smooth and  $\nabla F(w^*) = 0$  for some  $w^* \in \mathcal{W}$ , which implies  $L \leq 2\beta D$ .

$|\langle \nabla F(w), e_j \rangle|^k \leq \gamma_k$ , for all  $j \in [d]$ , where  $e_j$  denotes the  $j$ -th standard basis vector in  $\mathbb{R}^d$ . Also,  $L \triangleq \sup_{w \in \mathcal{W}} \|\nabla F(w)\| \leq \sqrt{d}\gamma_k^{1/k}$ .

Lemma 21 allows us compare our results in Section 4 obtained under Assumption 2 to the results in (Wang et al., 2020; Kamath et al., 2022), which require Assumption 3.

**Lemma 21.** *Suppose Assumption 3 holds. Then, Assumption 2 holds for  $r_k \leq 4\sqrt{d}\gamma_k^{1/k}$ .*

*Proof.* We use the following inequality, which can easily be verified inductively, using Cauchy-Schwartz and Young's inequalities: for any vectors  $u, v \in \mathbb{R}^d$ , we have

$$\|u\|^k \leq 2^{k-1} \left( \|u - v\|^k + \|v\|^k \right). \quad (\text{B.1})$$

Therefore,

$$\begin{aligned} r^{(k)} &= \sup_{w \in \mathcal{W}} \mathbb{E} \|\nabla f(w, x)\|^k \\ &\leq 2^{k-1} \left( \sup_{w \in \mathcal{W}} \mathbb{E} \|\nabla f(w, x) - \nabla F(w)\|^k + L^k \right) \\ &= 2^{k-1} \left( \sup_{w \in \mathcal{W}} \mathbb{E} \left[ \left\{ \sum_{j=1}^d |\langle \nabla f(w, x) - \nabla F(w), e_j \rangle|^2 \right\}^{k/2} \right] + L^k \right) \\ &\leq (2L)^k + 2^k d^{k/2} \sup_{w \in \mathcal{W}} \mathbb{E} \left[ \frac{1}{d} \sum_{j=1}^d |\langle \nabla f(w, x) - \nabla F(w), e_j \rangle|^k \right], \end{aligned}$$

where we used convexity of the function  $\phi(y) = y^{k/2}$  for all  $y \geq 0, k \geq 2$  and Jensen's inequality in the last inequality. Now using linearity of expectation and Assumption 3 gives us

$$r^{(k)} \leq 2^k \left( L^k + d^{k/2} \gamma_k \right) \leq 2^{k+1} d^{k/2} \gamma_k,$$

since  $L^k \leq d^{k/2} \gamma_k$  by hypothesis.  $\square$

**Remark 22.** *Since Assumption 2 is implied by Assumption 3, the upper bounds that we obtain under Assumption 2 also hold (up to constants) if we grant Assumption 3 instead, with  $r_k \leftrightarrow \sqrt{d}\gamma_k^{1/k}$ . In particular, the upper bounds stated in Appendix F also imply improved upper bounds under Assumption 3. Also, in Appendix G.4, we will use Lemma 21 to show that our non-smooth excess risk bounds under Assumption 1 imply similar excess risk bounds under Assumption 3.*

### APPENDIX C. CORRECTING THE ERRORS IN THE STRONGLY CONVEX UPPER BOUNDS CLAIMED IN (KAMATH ET AL., 2022; WANG ET AL., 2020)

While the ICML 2022 paper (Kamath et al., 2022, Theorem 5.6) claimed an upper bound for smooth strongly convex losses that is tight up to a factor of  $\tilde{\mathcal{O}}(\kappa_f^2)$ —where  $\kappa_f = \beta_f/\mu$  is the uniform condition number of  $f(\cdot, x)$  over all  $x \in \mathcal{X}$ —we identify an issue with their proof that invalidates their result. A similar issue appears in the proof of (Wang et al., 2020, Theorems 5 and 7), which (Kamath et al., 2022) built upon. We then show how to salvage a

correct upper bound within the framework of (Kamath et al., 2022), albeit at the cost of an additional factor of  $\kappa_f$ .<sup>13</sup>

The proof of (Kamath et al., 2022, Theorem 5.6) relies on (Kamath et al., 2022, Theorem 3.2). The proof of (Kamath et al., 2022, Theorem 3.2), in turn, bounds  $\mathbb{E}\|w_T - w^*\| \leq \frac{(\lambda+L)(M+1)G}{\lambda L}$  in the notation of (Kamath et al., 2022), where  $L$  is the smoothness parameter,  $\lambda$  is the strong convexity parameter (so  $L \geq \lambda$ ), and  $M$  is the diameter of  $\mathcal{W}$ . Then, it is *incorrectly* deduced that  $\mathbb{E}[\|w_T - w^*\|^2] \leq \left(\frac{(\lambda+L)(M+1)G}{\lambda L}\right)^2$  (final line of the proof). Notice that  $\mathbb{E}[\|w_T - w^*\|^2]$  can be much larger than  $(\mathbb{E}\|w_T - w^*\|)^2$  in general: for example, if  $\|w_T - w^*\|$  has the Pareto distribution with shape parameter  $\alpha \in (1, 2]$  and scale parameter 1, then  $(\mathbb{E}\|w_T - w^*\|)^2 = \left(\frac{\alpha}{\alpha-1}\right)^2 \ll \mathbb{E}(\|w_T - w^*\|^2) = \infty$ .

As a first attempt to correct this issue, one could use Young's inequality to instead bound

$$\begin{aligned} \mathbb{E}[\|w_T - w^*\|^2] &\leq 2 \left(1 - \frac{2\lambda L}{(\lambda + L)^2}\right) \mathbb{E}[\|w_{T-1} - w^*\|^2] + \frac{2G^2}{(\lambda + L)^2} \\ &\leq \left[2 \left(1 - \frac{2\lambda L}{(\lambda + L)^2}\right)\right]^T \|w_0 - w^*\|^2 + \frac{2G^2}{(\lambda + L)^2} \sum_{t=0}^{T-1} \left[2 \left(1 - \frac{2\lambda L}{(\lambda + L)^2}\right)\right]^t, \end{aligned}$$

but the geometric series above diverges to  $+\infty$  as  $T \rightarrow \infty$ , since  $2 \left(1 - \frac{2\lambda L}{(\lambda + L)^2}\right) \geq 1 \iff (\lambda - L)^2 \geq 0$ .

Next, we show how to modify the proof of (Kamath et al., 2022, Theorem 5.6) in order to obtain a correct excess risk upper bound of

$$\tilde{\mathcal{O}} \left( \frac{\gamma_k^{2/k}}{\mu} d \left[ \frac{(\beta_f/\mu)^3}{n} + \left( \frac{\sqrt{d}(\beta_f/\mu)^3}{\varepsilon n} \right)^{(2k-2)/k} \right] \right) \quad (\text{C.1})$$

(in our notation). This correction was derived in collaboration with the authors of (Kamath et al., 2022), who have also updated the arXiv version of their paper accordingly. By waiting until the very of the proof of (Kamath et al., 2022, Theorem 3.2) to take expectation, we can derive

$$\|w_t - w^*\| \leq \left(1 - \frac{\lambda L}{(\lambda + L)^2}\right) \|w_{t-1} - w^*\| + \frac{\|\tilde{\nabla}F(w_{t-1}) - \nabla F(w_{t-1})\|}{\lambda + L} \quad (\text{C.2})$$

for all  $t$ , where we use their  $L = \beta_f$  and  $\lambda = \mu$  notation but our notation  $F$  and  $\tilde{\nabla}F$  for the population loss and its biased noisy gradient estimate (instead of their  $L_{\mathcal{D}}$  notation). By iterating Eq. (C.2), we can get

$$\begin{aligned} \|w_T - w^*\| &\leq \left(1 - \frac{2\lambda L}{(\lambda + L)^2}\right)^T \|w_0 - w^*\| \\ &\quad + \sum_{t=0}^{T-1} \left(1 - \frac{2\lambda L}{(\lambda + L)^2}\right)^t \left[ \frac{\|\tilde{\nabla}F(w_{T-t}) - \nabla F(w_{T-t})\|}{\lambda + L} \right]. \end{aligned}$$

Squaring both sides and using Cauchy-Schwartz, we get

<sup>13</sup>The corrected result of (Kamath et al., 2022), derived here, is also included in the latest arXiv version of their paper. We communicated with the authors of (Kamath et al., 2022) to obtain this correct version.

$$\begin{aligned} \|w_T - w^*\|^2 &\leq 2 \left(1 - \frac{2\lambda L}{(\lambda + L)^2}\right)^{2T} \|w_0 - w^*\|^2 \\ &\quad + T \sum_{t=0}^{T-1} \left(1 - \frac{2\lambda L}{(\lambda + L)^2}\right)^{2t} \left[ \frac{\|\tilde{\nabla} F(w_{T-t}) - \nabla F(w_{T-t})\|}{\lambda + L} \right]^2. \end{aligned}$$

Using  $L$ -smoothness of  $F$  and the assumption made in (Kamath et al., 2022) that  $\nabla F(w^*) = 0$ , and then taking expectation yields

$$\mathbb{E}F(w_T) - F^* \lesssim L \|w_0 - w^*\|^2 \left(1 - \frac{2\lambda L}{(\lambda + L)^2}\right)^{2T} + TG^2 \frac{L}{\lambda}, \quad (\text{C.3})$$

where  $G^2 \geq \mathbb{E} \left[ \|\tilde{\nabla} F(w_{T-t}) - \nabla F(w_{T-t})\|^2 \right]$  for all  $t$ . It is necessary and sufficient to choose  $T = \tilde{\Omega}(L/\lambda)$  to make the first term in the right-hand side of Eq. (C.3) less than the second term (up to logarithms). With this choice of  $T$ , we get

$$\mathbb{E}F(w_T) - F^* = \tilde{\mathcal{O}}(G^2 \kappa_f^2), \quad (\text{C.4})$$

where  $\kappa_f = L/\lambda$ . Next, we apply the bound on  $G^2$  for the `MeanOracle` that is used in (Kamath et al., 2022); this bound is stated in the version of (Kamath et al., 2022, Lemma B.5) that appears in the updated (November 1, 2022) arXiv version of their paper. The bound (for general  $\gamma_k$ ) is  $G^2 = \tilde{\mathcal{O}}\left(\gamma_k^{2/k} \left[\frac{Td}{n} + d \left(\frac{\sqrt{dT}^{3/2}}{\varepsilon n}\right)^{(2k-2)/k}\right]\right)$ . Plugging this bound on  $G^2$  into Eq. (C.4) yields Eq. (C.1).

#### APPENDIX D. MORE DIFFERENTIAL PRIVACY PRELIMINARIES

We collect some basic facts about DP algorithms that will be useful in the proofs of our results. Our algorithms use the *Gaussian mechanism* to achieve zCDP:

**Proposition 23.** (Bun and Steinke, 2016, Proposition 1.6) *Let  $q : \mathcal{X}^n \rightarrow \mathbb{R}$  be a query with  $\ell_2$ -sensitivity  $\Delta := \sup_{X \sim X'} \|q(X) - q(X')\|$ . Then the Gaussian mechanism, defined by  $\mathcal{M} : \mathcal{X}^n \rightarrow \mathbb{R}$ ,  $M(X) := q(X) + u$  for  $u \sim \mathcal{N}(0, \sigma^2)$ , is  $\rho$ -zCDP if  $\sigma^2 \geq \frac{\Delta^2}{2\rho}$ .*

The (adaptive) composition of zCDP algorithms is zCDP, with privacy parameters adding:

**Lemma 24.** (Bun and Steinke, 2016, Lemma 2.3) *Suppose  $\mathcal{A} : \mathcal{X}^n \rightarrow \mathcal{Y}$  satisfies  $\rho$ -zCDP and  $\mathcal{A}' : \mathcal{X}^n \times \mathcal{Y} \rightarrow \mathcal{Z}$  satisfies  $\rho'$ -zCDP (as a function of its first argument). Define the composition of  $\mathcal{A}$  and  $\mathcal{A}'$ ,  $\mathcal{A}'' : \mathcal{X}^n \rightarrow \mathcal{Z}$  by  $\mathcal{A}''(X) = \mathcal{A}'(X, \mathcal{A}(X))$ . Then  $\mathcal{A}''$  satisfies  $(\rho + \rho')$ -zCDP. In particular, the composition of  $T$   $\rho$ -zCDP mechanisms is a  $T\rho$ -zCDP mechanism.*

The definitions of DP and zCDP given above do not dictate *how* the algorithm  $\mathcal{A}$  operates. In particular, they allow  $\mathcal{A}$  to send sensitive data to a third party curator/analyst, who can then add noise to the data. However, in certain practical applications (e.g. federated learning (Kairouz et al., 2019)), there is no third party that can be trusted to handle sensitive user data. On the other hand, it is often more realistic to have a secure *shuffler* (a.k.a. mixnet): in each iteration of the algorithm, the shuffler receives encrypted noisy reports

(e.g. noisy stochastic gradients) from each user and applies a uniformly random permutation to the  $n$  reports, thereby anonymizing them (and amplifying privacy) (Cheu et al., 2019; Erlingsson et al., 2020; Feldman et al., 2020b). An algorithm is *shuffle private* if all of these “shuffled” reports are DP:

**Definition 25.** (*Shuffle Differential Privacy* (Cheu et al., 2019)) *A randomized algorithm is  $(\varepsilon, \delta)$ -shuffle DP (SDP) if the collection of reports output by the shuffler satisfies Definition 1.*

### APPENDIX E. DETAILS AND PROOFS FOR SECTION 3: LOWER BOUNDS

In this section, we prove the lower bounds stated in Section 3, and also provide tight lower bounds under Assumptions 2 and 3.

**Theorem 26** (Precise Statement of Theorem 7). *Let  $k \geq 2$ ,  $D, \gamma_k, r^{(k)}, \tilde{r}^{(k)} > 0$ ,  $\beta_f \geq 0$ ,  $d \geq 40, n > 7202$ , and  $\rho \leq d$ . Then, for any  $\rho$ -zCDP algorithm  $\mathcal{A}$ , there exist  $\mathcal{W}, \mathcal{X} \subset \mathbb{R}^d$  such that  $\|w - w'\| \leq 2D$  for all  $w, w' \in \mathcal{W}$ , a  $\beta_f$ -smooth, linear, convex (in  $w$ ) loss  $f : \mathcal{W} \times \mathcal{X} \rightarrow \mathbb{R}$ , and distributions  $\mathcal{D}$  and  $\mathcal{D}'$  on  $\mathcal{X}$  such that:*

1. *Assumption 1 holds and if  $X' \sim \mathcal{D}'^n$ , then*

$$\mathbb{E}F(\mathcal{A}(X')) - F^* = \Omega \left( D \left( \frac{\tilde{r}_2}{\sqrt{n}} + \tilde{r}_k \min \left\{ 1, \left( \frac{\sqrt{d}}{\sqrt{\rho n}} \right)^{\frac{k-1}{k}} \right\} \right) \right). \quad (\text{E.1})$$

2. *Assumption 2 holds and if  $X' \sim \mathcal{D}'^n$ , then*

$$\mathbb{E}F(\mathcal{A}(X')) - F^* = \Omega \left( D \left( \frac{r_2}{\sqrt{n}} + r_k \min \left\{ 1, \left( \frac{\sqrt{d}}{\sqrt{\rho n}} \right)^{\frac{k-1}{k}} \right\} \right) \right). \quad (\text{E.2})$$

3. *Assumption 3 holds and if  $X \sim \mathcal{D}^n$ , then*

$$\mathbb{E}F(\mathcal{A}(X)) - F^* = \Omega \left( D \left( \gamma_2^{1/2} \sqrt{\frac{d}{n}} + \gamma_k^{1/k} \sqrt{d} \min \left\{ 1, \left( \frac{\sqrt{d}}{\sqrt{\rho n}} \right)^{\frac{k-1}{k}} \right\} \right) \right).$$

*Proof.* We will prove part 3 first.

3. We begin by proving the result for  $\gamma_k = D = 1$ . In this case, it is proved in (Kamath et al., 2022) that

$$\mathbb{E}F(\mathcal{A}(X)) - F^* = \Omega \left( \sqrt{d} \min \left\{ 1, \left( \frac{\sqrt{d}}{\sqrt{\rho n}} \right)^{\frac{k-1}{k}} \right\} \right)$$

for  $f(w, x) = -\langle w, x \rangle$  with  $\mathcal{W} = B_2^d(0, 1)$  and  $\mathcal{X} = \{\pm 1\}^d$ , and a distribution satisfying Assumption 3 with  $\gamma_k = 1$ . Then  $f(\cdot, x)$  is linear, convex, and  $\beta_f$ -smooth for all  $\beta_f \geq 0$ . We prove the first (non-private) term in the lower bound. By the Gilbert-Varshamov bound (see e.g. (Acharya et al., 2021, Lemma 6)) and the assumption  $d \geq 40$ , there exists a set  $\mathcal{V} \subseteq \{\pm 1\}^d$  with  $|\mathcal{V}| \geq 2^{d/20}$ ,  $d_{\text{Ham}}(\nu, \nu') \geq \frac{d}{8}$  for all  $\nu, \nu' \in \mathcal{V}$ ,  $\nu \neq \nu'$ . For  $\nu \in \mathcal{V}$ , define the product distribution  $Q_\nu = (Q_{\nu_1}, \dots, Q_{\nu_d})$ , where for all  $j \in [d]$ ,

$$Q_{\nu_j} = \begin{cases} 1 & \text{with probability } \frac{1+\delta_{\nu_j}}{2} \\ -1 & \text{with probability } \frac{1-\delta_{\nu_j}}{2} \end{cases}$$

for  $\delta_{\nu_j} \in (0, 1)$  to be chosen later. Then  $\mathbb{E}Q_{\nu_j} := \mu_{\nu_j} = \delta_{\nu_j}$  and for any  $w \in \mathcal{W}$ ,  $x \sim Q_{\nu}$ , we have

$$\mathbb{E}|\langle \nabla f(w, x) - \nabla F(w), e_j \rangle|^2 = \mathbb{E}|\langle -x + \mathbb{E}x, e_j \rangle|^k \quad (\text{E.3})$$

$$= \mathbb{E}|x_j - \mu_{\nu_j}|^k \quad (\text{E.4})$$

$$\leq \mathbb{E}|x_j|^k \quad (\text{E.5})$$

$$\leq 1 \quad (\text{E.6})$$

for  $\delta_{\nu_j} \in (0, 1)$ . Thus, our unscaled hard instance/distribution for the non-private lower bound term (which will be chosen from among  $Q_{\nu}, \nu \in \mathcal{V}$ ) has  $k$ -th moment bounded by 1 for any  $k \in [2, \infty]$ . Now, let  $p := \sqrt{d/n}$  and  $\delta_{\nu_j} := \frac{p\nu_j}{\sqrt{d}}$ . Note that  $\mathbb{E}Q_{\nu} := \mu_{\nu} = \frac{p\nu}{\sqrt{d}}$  and  $w_{\nu} := \frac{\mu_{\nu}}{\|\mu_{\nu}\|} = \frac{\nu}{\|\nu\|}$ . Also,  $\|\mu_{\nu}\| = p := \|\mu\|$  for all  $\nu \in \mathcal{V}$ . Now, denoting  $F_{Q_{\nu}}(w) := \mathbb{E}_{x \sim Q_{\nu}} f(w, x)$ , we have for any  $w \in \mathcal{W}$  (possibly depending on  $X \sim Q_{\nu}^n$ ) that

$$\max_{\nu \in \mathcal{V}} \mathbb{E} \left[ F_{Q_{\nu}}(w) - \min_{w' \in \mathcal{W}} F_{Q_{\nu}}(w') \right] = \max_{\nu \in \mathcal{V}} \mathbb{E} \left[ \left\langle \frac{\mu_{\nu}}{\|\mu_{\nu}\|}, \mu_{\nu} \right\rangle - \langle w, \mu_{\nu} \rangle \right] \quad (\text{E.7})$$

$$= \max_{\nu \in \mathcal{V}} \mathbb{E} [\|\mu_{\nu}\| - \langle w, \mu_{\nu} \rangle] \quad (\text{E.8})$$

$$= \max_{\nu \in \mathcal{V}} \mathbb{E} (\|\mu_{\nu}\| [1 - \langle w, w_{\nu} \rangle]) \quad (\text{E.9})$$

$$\geq \max_{\nu \in \mathcal{V}} \mathbb{E} \left[ \frac{1}{2} \|\mu_{\nu}\| \|w - w_{\nu}\|^2 \right], \quad (\text{E.10})$$

since  $\|w\|, \|w_{\nu}\| \leq 1$ . Further, denoting  $\hat{w} := \operatorname{argmin}_{\nu \in \mathcal{V}} \|w_{\nu} - w\|$ , we have  $\|\hat{w} - w_{\nu}\|^2 \leq 4\|w_{\nu} - w\|^2$  for all  $\nu \in \mathcal{V}$  (via Young's inequality). Hence

$$\max_{\nu \in \mathcal{V}} \mathbb{E} \left[ F_{Q_{\nu}}(w) - \min_{w' \in \mathcal{W}} F_{Q_{\nu}}(w') \right] \geq \frac{\|\mu\|}{8} \max_{\nu \in \mathcal{V}} \mathbb{E} \|\hat{w} - w_{\nu}\|^2. \quad (\text{E.11})$$

Now we apply Fano's method (see e.g. (Yu, 1997, Lemma 3)) to lower bound  $\max_{\nu \in \mathcal{V}} \mathbb{E} \|\hat{w} - w_{\nu}\|^2$ . For all  $\nu \neq \nu'$ , we have  $\|w_{\nu} - w_{\nu'}\|^2 \geq \frac{\|\nu - \nu'\|^2}{\|\nu\|^2} \geq 1$  since  $d_{\text{Ham}}(\nu, \nu') \geq \frac{d}{2}$  and  $\nu \in \{\pm 1\}^d$  implies  $\|\nu - \nu'\|^2 \geq \frac{d}{2}$  and  $\|\nu\|^2 = d$ . Also, a straightforward computation shows that for any  $j \in [d]$  and  $\nu, \nu' \in \mathcal{V}$ ,

$$D_{KL}(Q_{\nu_j} \| Q_{\nu'_j}) \leq \frac{1 + \frac{p}{\sqrt{d}}}{2} \left[ \log \left( \frac{\sqrt{d} + p}{\sqrt{d}} \right) + \log \left( \frac{\sqrt{d}}{\sqrt{d} - p} \right) \right] \quad (\text{E.12})$$

$$\leq \log \left( \frac{1 + \frac{p}{\sqrt{d}}}{1 - \frac{p}{\sqrt{d}}} \right) \quad (\text{E.13})$$

$$\leq \frac{3p}{\sqrt{d}}, \quad (\text{E.14})$$

for our choice of  $p$ , provided  $\frac{p}{\sqrt{d}} = \frac{1}{\sqrt{n}} \in (0, \frac{1}{2})$ , which holds if  $n > 4$ . Hence by the chain rule for KL-divergence,

$$D_{KL}(Q_{\nu} \| Q_{\nu'}) \leq 3p\sqrt{d} = 3\frac{d}{\sqrt{n}}$$

for all  $\nu, \nu' \in \mathcal{V}$ . Thus, for any  $w \in \mathcal{W}$ , Fano's method yields

$$\max_{\nu \in \mathcal{V}} \mathbb{E} \|w - w_\nu\|^2 \geq \frac{1}{2} \left( 1 - \frac{3p\sqrt{d} + \log(2)}{(d/20)} \right) = \frac{1}{2} \left( 1 - \frac{60 \frac{d}{\sqrt{n}} - 20 \log(2)}{d} \right),$$

which is  $\Omega(1)$  for  $d \geq 40 > 20 \log(2)$  and  $n > 7202$ . Combining this with Eq. (E.11) and plugging in  $\|\mu\| = \sqrt{\frac{d}{n}}$  shows that

$$\mathbb{E} F_{Q_\nu}(\mathcal{A}(X)) - F_{Q_\nu}^* = \Omega \left( \sqrt{\frac{d}{n}} \right)$$

for some  $\nu \in \mathcal{V}$  (for any algorithm  $\mathcal{A}$ ), where  $X \sim Q_\nu^n$ .

Next, we scale our hard instance. For the non-private lower bound just proved, we scale the distribution  $Q_\nu \rightarrow \tilde{Q}_\nu = \gamma_k^{1/k} Q_\nu$ , which is supported on  $\tilde{\mathcal{X}} = \{\pm \gamma_k^{1/k}\}^d$ . Denote its mean by  $\mathbb{E} Q_\nu := \tilde{\mu}_\nu = \gamma_k^{1/k} \mu_\nu$ . Also we scale  $\mathcal{W} \rightarrow \tilde{\mathcal{W}} = D\mathcal{W} = B_2^d(0, D)$ . So our final (linear, convex, smooth) hard instance for the non-private lower bound is  $f: \tilde{\mathcal{W}} \times \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ ,  $f(\tilde{w}, \tilde{x}) = -\langle \tilde{w}, \tilde{x} \rangle$ ,  $\tilde{F}(\tilde{w}) := \mathbb{E}_{\tilde{x} \sim \tilde{Q}_\nu} f(\tilde{w}, \tilde{x})$ . Denote  $F(w) := \mathbb{E}_{x \sim Q_\nu} f(w, x)$ . Note that

$$\begin{aligned} \mathbb{E} |\langle \nabla f(\tilde{w}, \tilde{x}) - \nabla \tilde{F}(\tilde{w}), e_j \rangle|^k &= \mathbb{E} |-\tilde{x} + \mathbb{E} \tilde{x}, e_j|^k \\ &= \mathbb{E} |\tilde{x}_j - \tilde{\mu}_{\nu_j}|^k \\ &= \mathbb{E} |\gamma_k^{1/k} (x_j - \mu_{\nu_j})|^k \leq \gamma_k. \end{aligned}$$

Further, we have  $w^* := \operatorname{argmin}_{w \in \mathcal{W}} F(w) = \frac{\mu_\nu}{\|\mu_\nu\|}$  and  $\tilde{w}^* := \operatorname{argmin}_{\tilde{w} \in \tilde{\mathcal{W}}} \tilde{F}(\tilde{w}) = Dw^*$ . Therefore, for any  $w \in \mathcal{W}$ ,  $\tilde{w} = Dw \in \tilde{\mathcal{W}}$ , we have

$$\tilde{F}(\tilde{w}) - \tilde{F}(\tilde{w}^*) = -\langle \tilde{w}, \tilde{\mu}_\nu \rangle + \langle \tilde{w}^*, \tilde{\mu}_\nu \rangle \quad (\text{E.15})$$

$$= \langle D(w^* - w), \gamma_k^{1/k} \mu_\nu \rangle \quad (\text{E.16})$$

$$= D\gamma_k^{1/k} [F(w) - F(w^*)]. \quad (\text{E.17})$$

Thus,

$$\mathbb{E} \tilde{F}(\mathcal{A}(\tilde{X})) - \tilde{F}^* = \gamma_k^{1/k} D [\mathbb{E} F(\mathcal{A}(X)) - F^*] \geq \Omega \left( \gamma_2^{1/2} D \sqrt{\frac{d}{n}} \right),$$

by applying the non-private lower bound for the case  $D = \gamma_2 = 1$  (i.e. for the unscaled  $F$ ) and using monotonicity of moments. A similar scaling argument applied to the unscaled *private* lower bound instance of Kamath et al. (2022) completes the proof of part 3 of the theorem.

1. We will use nearly the same unscaled hard instances used to prove the private and non-private terms of the lower bound in part 3, but the scaling will differ. Starting with the *non-private* term, we scale the distribution  $Q_\nu \rightarrow \tilde{Q}_\nu = \frac{\tilde{r}_2}{\sqrt{d}} Q_\nu$  and  $\mathcal{X} \rightarrow \tilde{\mathcal{X}} = \frac{\tilde{r}_2}{\sqrt{d}} \mathcal{X}$ . Also, scale  $\mathcal{W} \rightarrow \tilde{\mathcal{W}} = D\mathcal{W} = B_2^d(0, D)$ . Let  $f(\tilde{w}, \tilde{x}) := -\langle \tilde{w}, \tilde{x} \rangle$ , which satisfies all the hypotheses of the theorem. Also,

$$\mathbb{E}_{\tilde{x} \sim \tilde{Q}_\nu} \left[ \sup_{\tilde{w}} \|\nabla f(\tilde{w}, \tilde{x})\|^k \right] = \left( \frac{\tilde{r}_k}{\sqrt{d}} \right)^k \mathbb{E}_{x \sim Q_\nu} \|x\|^k \leq \left( \frac{\tilde{r}_k}{\sqrt{d}} \right)^k d^{k/2} = \tilde{r}^{(k)}.$$

Now  $\tilde{w}^* = Dw^*$  as before and letting  $\tilde{F}(\cdot) := \mathbb{E}_{\tilde{x} \sim \tilde{Q}_\nu} f(\cdot, \tilde{x})$ , we have

$$\tilde{F}(\tilde{w}) - \tilde{F}(\tilde{w}^*) = \frac{\tilde{r}_k D}{\sqrt{d}} [F(w) - F^*]$$

for any  $\tilde{w} = Dw$ . Thus, applying the unscaled non-private lower bound established above yields the desired lower bound of  $\Omega\left(\frac{\tilde{r}_k D}{\sqrt{n}}\right)$  on the non-private excess risk of our scaled instance.

Next, we turn to the scaled *private* lower bound. The unscaled hard distribution  $Q'_\nu$  given by

$$Q'_\nu = \begin{cases} 0 & \text{with probability } 1 - p \\ p^{-1/k} \nu & \text{with probability } p \end{cases}$$

(with the same linear  $f$  and same  $\mathcal{W}$ ) provides the unscaled lower bound

$$\mathbb{E}F(\mathcal{A}(X)) - F^* = \Omega\left(\sqrt{d} \min\left\{1, \left(\frac{\sqrt{d}}{\sqrt{\rho n}}\right)^{\frac{k-1}{k}}\right\}\right),$$

by the proof of (Kamath et al., 2022, Theorem 6.4). We scale  $Q'_\nu \rightarrow \tilde{Q}'_\nu = \frac{\tilde{r}}{\sqrt{d}} Q'_\nu$ ,  $\mathcal{X} \rightarrow \tilde{\mathcal{X}} = \frac{\tilde{r}}{\sqrt{d}} \mathcal{X}$ , and  $\mathcal{W} \rightarrow \tilde{\mathcal{W}} = D\mathcal{W}$ . Then for any  $\tilde{w} \in \tilde{\mathcal{W}}$ ,

$$\mathbb{E}_{\tilde{x} \sim \tilde{Q}'_\nu} \left[ \sup_{\tilde{w}} \|\nabla f(\tilde{w}, \tilde{x})\|^k \right] = \left(\frac{\tilde{r}_k}{\sqrt{d}}\right)^k \mathbb{E}_{x \sim Q'_\nu} \|x\|^k = p \|p^{-1/k} \nu\|^k = \tilde{r}^{(k)}.$$

Moreover, excess risk scales by a factor of  $\frac{\tilde{r}_k D}{\sqrt{d}}$ . Thus, applying the unscaled lower bound completes the proof of part 1.

2. We use an identical construction to that used above in part 1 except that the scaling factor  $\tilde{r}_k$  gets replaced by  $r_k$ . It is easy to see that  $\mathbb{E} \left[ \sup_{w \in \mathcal{W}} \|\nabla f(w, x)\|^k \right] = \sup_{w \in \mathcal{W}} \mathbb{E} \left[ \|\nabla f(w, x)\|^k \right]$  for our construction, hence the result follows.  $\square$

**Remark 27.** *The main differences in our proof of part 3 of Theorem 7 from the proof of (Kamath et al., 2022, Theorem 6.4) (for  $\gamma_k = D = 1$ ) are: 1) we construct a Bernoulli product distribution (built on (Duchi, 2021, Example 7.7)) instead of a Gaussian, which establishes a lower bound that holds for all  $k \geq 2$  instead of just  $k = \mathcal{O}(1)$ ; and 2) we choose a different parameter value (larger  $p$  in the notation of the proof) in our application of Fano's method, which results in a tighter lower bound: the term  $\min\{1, \sqrt{d/n}\}$  in (Kamath et al., 2022, Theorem 6.4) gets replaced with  $\sqrt{d/n}$ .<sup>14</sup> Also, there exist parameter settings for which our lower bound is indeed strictly greater than the lower bound in (Kamath et al., 2022, Theorem 6.4): for instance, if  $d > n > d/\rho$  and  $k \rightarrow \infty$ , then our lower bound simplifies to  $\Omega(\sqrt{\frac{d}{n}})$ . On the other hand, the lower bound in (Kamath et al., 2022, Theorem 6.4) breaks as  $k \rightarrow \infty$  (since the  $k$ -th moment of their Gaussian goes to infinity); however, even if we extended to  $k \rightarrow \infty$  (e.g. by replacing their Gaussian with our Bernoulli distribution), then the resulting lower bound  $\Omega(1 + \frac{d}{\sqrt{\rho n}})$  would still be smaller than the one we prove above.<sup>15</sup>*

<sup>14</sup>Note that (Kamath et al., 2022, Theorem 6.4) writes  $\sqrt{d/n}$  for the first term. However, the proof (see Equation 16 in their paper) only establishes the bound  $\min\{1, \sqrt{d/n}\}$ .

<sup>15</sup>By Lemma 21, lower bounds under Assumption 3 imply lower bounds under Assumption 2 with  $\gamma_k^{1/k}$  replaced by  $r/\sqrt{d}$ . Nevertheless, we provide direct proofs under both assumptions for additional clarity.

**Theorem 28** (Precise Statement of Theorem 8). *Let  $k \geq 2$ ,  $\mu, \gamma_k, \tilde{r}_k, r_k > 0$ ,  $n \in \mathbb{N}$ ,  $d \geq 40$ , and  $\rho \leq d$ . Then, for any  $\rho$ -zCDP algorithm  $\mathcal{A}$ , there exist convex, compact sets  $\mathcal{W}, \mathcal{X} \subset \mathbb{R}^d$  of diameter  $D$ , a  $\mu$ -smooth,  $\mu$ -strongly convex (in  $w$ ) loss  $f : \mathcal{W} \times \mathcal{X} \rightarrow \mathbb{R}$ , and distributions  $\mathcal{D}$  and  $\mathcal{D}'$  on  $\mathcal{X}$  such that:*

1. *Assumption 1 holds with  $D \approx \frac{\tilde{r}_k}{\mu}$ , and if  $X' \sim \mathcal{D}'^n$ , then*

$$\mathbb{E}F(\mathcal{A}(X')) - F^* = \Omega \left( \frac{1}{\mu} \left( \frac{\tilde{r}_k^2}{n} + \tilde{r}_k^2 \min \left\{ 1, \left( \frac{\sqrt{d}}{\sqrt{\rho n}} \right)^{\frac{2k-2}{k}} \right\} \right) \right).$$

2. *Assumption 2 holds with  $D \approx \frac{r_k}{\mu}$ , and if  $X' \sim \mathcal{D}'^n$ , then*

$$\mathbb{E}F(\mathcal{A}(X')) - F^* = \Omega \left( \frac{1}{\mu} \left( \frac{r_k^2}{n} + r_k^2 \min \left\{ 1, \left( \frac{\sqrt{d}}{\sqrt{\rho n}} \right)^{\frac{2k-2}{k}} \right\} \right) \right).$$

3. *Assumption 3 holds,  $D \approx \frac{\gamma_k^{1/k} \sqrt{d}}{\mu}$ , and if  $X \sim \mathcal{D}^n$ , then*

$$\mathbb{E}F(\mathcal{A}(X)) - F^* = \Omega \left( \frac{1}{\mu} \left( \frac{\gamma_k^{2/k} d}{n} + \gamma_k^{2/k} d \min \left\{ 1, \left( \frac{\sqrt{d}}{\sqrt{\rho n}} \right)^{\frac{2k-2}{k}} \right\} \right) \right).$$

*Proof.* We will prove part 3 first. 3. We first consider  $\gamma_k = \mu = 1$  and then scale our hard instance. For  $f(w, x) := \frac{1}{2} \|w - x\|^2$ , (Kamath et al., 2022) construct a convex/compact domain  $\mathcal{W} \times \mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}^d$  and distribution  $\mathcal{D}$  on  $\mathcal{X}$  such that

$$\mathbb{E}F(\mathcal{A}(X)) - F^* = \Omega \left( d \min \left\{ 1, \left( \frac{\sqrt{d}}{\sqrt{\rho n}} \right)^{\frac{2k-2}{k}} \right\} \right)$$

for any  $k$  and any  $\rho$ -zCDP algorithm  $\mathcal{A} : \mathcal{X}^n \rightarrow \mathcal{W}$  if  $X \sim \mathcal{D}^n$ .<sup>16</sup> So, it remains to a) prove the first term ( $d/n$ ) in the lower bound, and then b) show that the scaled instance satisfies the exact hypotheses in the theorem and has excess loss that scales by a factor of  $\gamma_k^{2/k}/\mu$ . We start with task a). Observe that for  $f$  defined above and any distribution  $\mathcal{D}$  such that  $\mathbb{E}\mathcal{D} \in \mathcal{W}$ , we have

$$\mathbb{E}F(\mathcal{A}(X)) - F^* = \frac{1}{2} \mathbb{E} \|\mathcal{A}(X) - \mathbb{E}\mathcal{D}\|^2 \quad (\text{E.18})$$

(see (Kamath et al., 2022, Lemma 6.2)), and

$$\mathbb{E} |\langle \nabla f(w, x) - \nabla F(w), e_j \rangle|^k = \mathbb{E} |\langle x - \mathbb{E}x, e_j \rangle|^k.$$

Thus, it suffices to prove that  $\mathbb{E} \|\mathcal{A}(X) - \mathbb{E}\mathcal{D}\|^2 \gtrsim \frac{d}{n}$  for some  $\mathcal{D}$  such that  $\mathbb{E} |\langle x - \mathbb{E}x, e_j \rangle|^k \leq 1$ . This is a known result for products of Bernoulli distributions; nevertheless, we provide a detailed proof below. First consider the case  $d = 1$ . Then the proof follows along the lines of (Duchi, 2021, Example 7.7). Define the following pair of distributions on  $\{\pm 1\}$ :

$$P_0 := \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

<sup>16</sup>In fact,  $\mathcal{W}$  and  $\mathcal{X}$  can be chosen to be Euclidean balls of radius  $\sqrt{d} p^{-1/k}$  for  $p$  defined in the proof of (Kamath et al., 2022, Lemma 6.3), which ensures that  $\mathbb{E}\mathcal{D} \in \mathcal{W} = \mathcal{X}$ .

and

$$P_1 := \begin{cases} 1 & \text{with probability } \frac{1+\delta}{2} \\ -1 & \text{with probability } \frac{1-\delta}{2} \end{cases}$$

for  $\delta \in (0, 1)$  to be chosen later. Notice that if  $X$  is a random variable with distribution  $P_\nu$  ( $\nu \in \{0, 1\}$ ), then  $\mathbb{E}|X - \mu|^k \leq \mathbb{E}|X|^k \leq 1$ . Also,  $\mathbb{E}P_\nu = \delta\nu$  for  $\nu \in \{0, 1\}$  and  $|\mathbb{E}P_1 - \mathbb{E}P_0| = \delta$  (i.e. the two distributions are  $\delta$ -separated with respect to the metric  $\rho(a, b) = |a - b|$ ). Then by Le Cam's method (see e.g. (Duchi, 2021, Eq. 7.33) and take  $\Phi(\cdot) = (\cdot)^2$ ),

$$\max_{\nu \in \{0, 1\}} \mathbb{E}_{X \sim P_\nu^n} |\mathcal{A}(X) - \mathbb{E}P_\nu|^2 \geq \frac{\delta^2}{8} [1 - \|P_0^n - P_1^n\|_{TV}].$$

Now, by Pinsker's inequality and the chain rule for KL-divergence, we have

$$\|P_0^n - P_1^n\|_{TV}^2 \leq \frac{1}{2} D_{KL}(P_0^n \| P_1^n) = \frac{n}{2} D_{KL}(P_0 \| P_1) = \frac{n}{2} \log \left( \frac{1}{1 - \delta^2} \right).$$

Choosing  $\delta = \frac{1}{\sqrt{2n}} < \frac{1}{\sqrt{2}}$  implies  $\|P_0^n - P_1^n\|_{TV}^2 \leq n\delta^2 = \frac{1}{2}$ . Hence there exists a distribution  $\hat{\mathcal{D}} \in \{P_0, P_1\}$  on  $\mathbb{R}$  such that

$$\mathbb{E}_{X \sim \hat{\mathcal{D}}^n} |\mathcal{A}(X) - \mathbb{E}\hat{\mathcal{D}}|^2 \geq \frac{\delta^2}{8} \left[ 1 - \frac{1}{\sqrt{2}} \right] \geq \frac{1}{64n}$$

For general  $d \geq 1$ , we take the product distribution  $\mathcal{D} := \hat{\mathcal{D}}^d$  on  $\mathcal{X} = \{\pm 1\}^d$  and choose  $\mathcal{W} = B_2^d(0, \sqrt{d})$  to ensure  $\mathbb{E}\mathcal{D} \in \mathcal{W}$  (so that Eq. (E.18) holds). Clearly,  $\mathbb{E}|\langle \mathcal{D} - \mathbb{E}\mathcal{D}, e_j \rangle|^k \leq 1$  for all  $j \in [d]$ . Further, the mean squared error of any algorithm for estimating the mean of  $\mathcal{D}$  is

$$\mathbb{E}_{X \sim \mathcal{D}^n} \|\mathcal{A}(X) - \mathbb{E}\mathcal{D}\|^2 = \sum_{j=1}^d \mathbb{E}|\mathcal{A}(X)_j - \mathbb{E}\mathcal{D}_j|^2 \geq \frac{d}{64n}, \quad (\text{E.19})$$

by applying the  $d = 1$  result to each coordinate.

Next, we move to task b). For this, we re-scale each of our hard distributions (non-private given above, and private given in the proof of (Kamath et al., 2022, Lemma 6.3) and below in our proof of part 2 of the theorem—see Eq. (E.23)):  $\mathcal{D} \rightarrow \frac{\gamma_k^{1/k}}{\mu} \mathcal{D} = \tilde{\mathcal{D}}$ ,  $\mathcal{X} \rightarrow \frac{\gamma_k^{1/k}}{\mu} \mathcal{X} = \tilde{\mathcal{X}}$ ,  $\mathcal{W} \rightarrow \frac{\gamma_k^{1/k}}{\mu} \mathcal{W} = \tilde{\mathcal{W}}$  and  $f : \mathcal{W} \times \mathcal{X} \rightarrow \mu f = \tilde{f} : \tilde{\mathcal{W}} \times \tilde{\mathcal{X}}$ . Then  $\tilde{f}(\cdot, \tilde{x})$  is  $\mu$ -strongly convex and  $\mu$ -smooth for all  $\tilde{x} \in \tilde{\mathcal{X}}$  and

$$\begin{aligned} \mathbb{E}|\langle \nabla \tilde{f}(\tilde{w}, \tilde{x}) - \nabla \tilde{F}(\tilde{w}), e_j \rangle|^k &= \mu^k \mathbb{E}|\langle \tilde{x} - \mathbb{E}\tilde{x}, e_j \rangle|^k \\ &= \mu^k \mathbb{E} \left| \left( \frac{\gamma_k^{1/k}}{\mu} \right) \langle x - \mathbb{E}x, e_j \rangle \right|^k \\ &= \gamma_k \mathbb{E}|\langle x - \mathbb{E}x, e_j \rangle|^k \leq \gamma_k \end{aligned}$$

for any  $j \in [d]$ ,  $x \sim \mathcal{D}$ ,  $\tilde{x} \sim \tilde{\mathcal{D}}$ ,  $\tilde{w} \in \tilde{\mathcal{W}}$ . Thus, the scaled hard instance is in the required class of functions/distributions. Further, denote  $\tilde{F}(w) = \mathbb{E}\tilde{f}(w, x)$ ,  $\tilde{w}^* := \operatorname{argmin}_{\tilde{w} \in \tilde{\mathcal{W}}} \tilde{F}(\tilde{w}) =$

$\mathbb{E}\tilde{\mathcal{D}} = \frac{\gamma_k^{1/k}}{\mu} \mathbb{E}\mathcal{D}$ . Then, for any  $w \in \mathcal{W}$ ,  $\tilde{w} := \frac{\gamma_k^{1/k}}{\mu} w$ , we have:

$$\tilde{F}(\tilde{w}) - \tilde{F}(\tilde{w}^*) = \frac{\mu}{2} \mathbb{E} [\|\tilde{w} - \tilde{x}\|^2 - \|\tilde{w}^* - \tilde{x}\|^2] \quad (\text{E.20})$$

$$= \frac{\mu}{2} \left( \frac{\gamma_k^{2/k}}{\mu^2} \right) \mathbb{E} [\|w - x\|^2 - \|w^* - x\|^2] \quad (\text{E.21})$$

$$= \frac{\gamma_k^{2/k}}{\mu} [F(w) - F(w^*)]. \quad (\text{E.22})$$

In particular, for  $w := \mathcal{A}(X)$  and  $\tilde{w} := \frac{\gamma_k^{1/k}}{\mu} \mathcal{A}(X)$ , we get

$$\mathbb{E}_{\mathcal{A}, X \sim \mathcal{D}^n} \left[ \tilde{F} \left( \frac{\gamma_k^{1/k}}{\mu} \mathcal{A}(X) \right) - \tilde{F}^* \right] = \frac{\gamma_k^{2/k}}{\mu} \mathbb{E}_{\mathcal{A}, X \sim \mathcal{D}^n} [F(\mathcal{A}(X)) - F^*]$$

for any algorithm  $\mathcal{A} : \mathcal{X}^n \rightarrow \mathcal{W}$ . Writing  $\tilde{\mathcal{A}}(\tilde{X}) := \frac{\gamma_k^{1/k}}{\mu} \mathcal{A}(X)$  and  $\tilde{X} := \frac{\gamma_k^{1/k}}{\mu} X$  for  $X \in \mathcal{X}^n$ , we conclude

$$\mathbb{E}_{\tilde{\mathcal{A}}, \tilde{X} \sim \tilde{\mathcal{D}}^n} \left[ \tilde{F}(\tilde{\mathcal{A}}(\tilde{X})) - \tilde{F}^* \right] = \frac{\gamma_k^{2/k}}{\mu} \mathbb{E}_{\mathcal{A}, X \sim \mathcal{D}^n} [F(\mathcal{A}(X)) - F^*]$$

for any  $\tilde{\mathcal{A}} : \tilde{\mathcal{X}}^n \rightarrow \tilde{\mathcal{W}}$ . Therefore, an application of the unscaled lower bound

$$\mathbb{E}_{\mathcal{A}, X \sim \mathcal{D}^n} [F(\mathcal{A}(X)) - F^*] = \Omega \left( \frac{d}{n} + d \min \left\{ 1, \left( \frac{\sqrt{d}}{n\sqrt{\rho}} \right)^{\frac{2k-2}{k}} \right\} \right),$$

which follows by combining part 3a) above with (Kamath et al., 2022, Lemma 6.3), completes the proof of part 3.

1. We begin by proving the first (non-private) term in the lower bound: For our *unscaled* hard instance, we will take the same distribution  $\mathcal{D} = P_\nu^d$  (for some  $\nu \in \{0, 1\}$ ) on  $\mathcal{X} = \{\pm 1\}^d$  and quadratic  $f$  described above in part 1a with  $\mathcal{W} := B_2^d(0, \sqrt{d})$ . The choice of  $\mathcal{W}$  ensures  $\mathbb{E}\mathcal{D} \in \mathcal{W}$  so that Eq. (E.18) holds. Further,

$$\mathbb{E} \left[ \sup_{w \in \mathcal{W}} \|\nabla f(w, x)\|^k \right] = \mathbb{E} \left[ \sup_{w \in \mathcal{W}} \|w - x\|^k \right] \leq \mathbb{E}[\|3x\|^k] \leq (9d)^{k/2}.$$

Thus, if we scale  $f \rightarrow \tilde{f} = \mu f$ ,  $\mathcal{W} \rightarrow \tilde{\mathcal{W}} := \frac{\tilde{r}_k}{\mu\sqrt{9d}} \mathcal{W}$ ,  $\mathcal{X} \rightarrow \tilde{\mathcal{X}} := \frac{\tilde{r}_k}{\mu\sqrt{9d}} \mathcal{X}$  and  $\mathcal{D} \rightarrow \tilde{\mathcal{D}} = \frac{\tilde{r}_k}{\mu\sqrt{9d}} \mathcal{D}$ , then  $\tilde{f}(\cdot, \tilde{x})$  is  $\mu$ -strongly convex and  $\mu$ -smooth, and

$$\mathbb{E} \left[ \sup_{\tilde{w} \in \tilde{\mathcal{W}}} \|\nabla \tilde{f}(\tilde{w}, \tilde{x})\|^k \right] = \mathbb{E} \left[ \sup_{\tilde{w} \in \tilde{\mathcal{W}}} \|\tilde{w} - \tilde{x}\|^k \right] = \mu^k \tilde{r}^{(k)} \left( \frac{1}{\mu\sqrt{9d}} \right)^k \mathbb{E} \left[ \sup_{w \in \mathcal{W}} \|\nabla f(w, x)\|^k \right] \leq \tilde{r}^{(k)}.$$

Moreover, if  $\left( \frac{\tilde{r}_k}{3\mu\sqrt{d}} \right) \mathcal{A} = \tilde{\mathcal{A}} : \tilde{\mathcal{X}}^n \rightarrow \tilde{\mathcal{W}}$  is any algorithm and  $\tilde{X} \sim \tilde{\mathcal{D}}^n$ , then by Eq. (E.19) and Eq. (E.18), we have

$$\mathbb{E}\tilde{F}(\tilde{\mathcal{A}}(\tilde{X})) - \tilde{F}^* = \frac{\mu}{2} \mathbb{E} \|\tilde{\mathcal{A}}(\tilde{X}) - \mathbb{E}\tilde{\mathcal{D}}\|^2 = \frac{\mu}{2} \left( \frac{\tilde{r}_k}{\mu\sqrt{9d}} \right)^2 \mathbb{E} \|\mathcal{A}(X) - \mathbb{E}\mathcal{D}\|^2 \gtrsim \frac{\tilde{r}_k^2}{\mu n}.$$

Next, we prove the second (private) term in the lower bound. Let  $f$  be as defined above. For our unscaled hard distribution, we follow (Barber and Duchi, 2014; Kamath et al., 2022) and define a family of distributions  $\{Q_\nu\}_{\nu \in \mathcal{V}}$  on  $\mathbb{R}^d$ , where  $\mathcal{V} \subset \{\pm 1\}^d$  will be defined later. For any given  $\nu \in \mathcal{V}$ , we define the distribution  $Q_\nu$  as follows:  $X_\nu \sim Q_\nu$  iff

$$X_\nu = \begin{cases} 0 & \text{with probability } 1 - p \\ p^{-1/k} \nu & \text{with probability } p \end{cases} \quad (\text{E.23})$$

where  $p := \min\left(1, \frac{\sqrt{d}}{n\sqrt{\rho}}\right)$ . Now, we select a set  $\mathcal{V} \subset \{\pm 1\}^d$  such that  $|\mathcal{V}| \geq 2^{d/20}$  and  $d_{\text{Ham}}(\nu, \nu') \geq \frac{d}{8}$  for all  $\nu, \nu' \in \mathcal{V}, \nu \neq \nu'$ : such  $\mathcal{V}$  exists by standard Gilbert-Varshamov bound (see e.g. (Acharya et al., 2021, Lemma 6)). For any  $\nu \in \mathcal{V}$ , if  $x \sim Q_\nu$  and  $w \in \mathcal{W} := B_2^d(0, \sqrt{d}p^{-1/k})$ , then

$$\begin{aligned} \mathbb{E} \left[ \sup_{w \in \mathcal{W}} \|\nabla f(w, x)\|^k \right] &= \mathbb{E} \left[ \sup_{w \in \mathcal{W}} \|w - x\|^k \right] \\ &\leq \mathbb{E}[\|2x\|^k] = 2^k (p \|p^{-1/k} \nu\|^k) = 2^k \|\nu\|^k = 2^k d^{k/2}. \end{aligned}$$

Note also that our choice of  $\mathcal{W}$  and  $p \leq 1$  ensures that  $\mathbb{E}[Q_\nu] \in \mathcal{W}$ . Moreover, as in the proof of (Kamath et al., 2022, Lemma 6.3), zCDP Fano's inequality (see Kamath et al., 2022, Theorem 1.4) implies that for any  $\rho$ -zCDP algorithm  $\mathcal{A}$ ,

$$\sup_{\nu \in \mathcal{V}} \mathbb{E}_{X \sim Q_\nu^n, \mathcal{A}} \|\mathcal{A}(X) - \mathbb{E}Q_\nu\|^2 = \Omega \left( d \min \left\{ 1, \left( \frac{\sqrt{d}}{n\sqrt{\rho}} \right)^{\frac{2k-2}{k}} \right\} \right). \quad (\text{E.24})$$

Thus,

$$\mathbb{E}_{X \sim Q_\nu^n, \mathcal{A}} F(\mathcal{A}(X)) - F^* = \Omega \left( d \min \left\{ 1, \left( \frac{\sqrt{d}}{n\sqrt{\rho}} \right)^{\frac{2k-2}{k}} \right\} \right)$$

for some  $\nu \in \mathcal{V}$ , by Eq. (E.18). Now we scale our hard instance:  $f \rightarrow \tilde{f} = \mu f$ ,  $\mathcal{W} \rightarrow \tilde{\mathcal{W}} := \frac{\tilde{r}_k}{2\mu\sqrt{d}} \mathcal{W}$ ,  $\mathcal{X} \rightarrow \tilde{\mathcal{X}} := \frac{\tilde{r}_k}{2\mu\sqrt{d}} \mathcal{X}$  and  $\mathcal{D} \rightarrow \tilde{\mathcal{D}} = \frac{\tilde{r}_k}{2\mu\sqrt{d}} \mathcal{D}$ . Then  $\tilde{f}(\cdot, \tilde{x})$  is  $\mu$ -strongly convex and  $\mu$ -smooth, and

$$\mathbb{E} \left[ \sup_{\tilde{w} \in \tilde{\mathcal{W}}} \|\nabla \tilde{f}(\tilde{w}, \tilde{x})\|^k \right] = \mathbb{E} \left[ \sup_{\tilde{w} \in \tilde{\mathcal{W}}} \|\tilde{w} - \tilde{x}\|^k \right] = \mu^k \left( \frac{\tilde{r}_k}{2\mu\sqrt{d}} \right)^k \mathbb{E} \left[ \sup_{w \in \mathcal{W}} \|\nabla f(w, x)\|^k \right] \leq \tilde{r}^{(k)}.$$

Moreover, if  $\left(\frac{\tilde{r}_k}{2\mu\sqrt{d}}\right) \mathcal{A} = \tilde{\mathcal{A}} : \tilde{X}^n \rightarrow \tilde{\mathcal{W}}$  is any  $\rho$ -zCDP algorithm and  $\tilde{X} \sim \tilde{\mathcal{D}}^n$ , then

$$\begin{aligned} \mathbb{E} \tilde{F}(\tilde{\mathcal{A}}(\tilde{X})) - \tilde{F}^* &= \frac{\mu}{2} \mathbb{E} \|\tilde{\mathcal{A}}(\tilde{X}) - \mathbb{E} \tilde{\mathcal{D}}\|^2 \\ &= \frac{\mu}{2} \left( \frac{\tilde{r}_k}{2\mu\sqrt{d}} \right)^2 \mathbb{E} \|\mathcal{A}(X) - \mathbb{E} \mathcal{D}\|^2 \\ &\geq \frac{\tilde{r}_k^2}{16\mu d} \Omega \left( d \min \left\{ 1, \left( \frac{\sqrt{d}}{n\sqrt{\rho}} \right)^{\frac{2k-2}{k}} \right\} \right), \end{aligned}$$

by Eq. (E.24).

2. We use an identical construction to that used above in part 1 except that the scaling factor  $\tilde{r}_k$  gets replaced by  $r_k$ . It is easy to see that  $\mathbb{E} [\sup_{w \in \mathcal{W}} \|\nabla f(w, x)\|^k] \approx \sup_{w \in \mathcal{W}} \mathbb{E} [\|\nabla f(w, x)\|^k]$  for our construction, and the lower bound in part 2 follows just as it did in part 1. This completes the proof.  $\square$

**Remark 29.** *Note that the lower bound proofs construct bounded (hence subexponential) distributions and uniformly  $L_f$ -Lipschitz,  $\beta_f$ -smooth losses that easily satisfy the conditions in all of our upper bound theorems, including those in Appendix G.4.*

## APPENDIX F. DETAILS AND PROOFS FOR SECTION 4: LINEAR TIME ALGORITHMS FOR SMOOTH (STRONGLY) CONVEX LOSSES

### F.1. Noisy Clipped Accelerated SGD for Smooth Convex Losses (Section 4.1).

We first present Algorithm 5, which is a generalized version of Algorithm 2 that allows for any `MeanOracle`. This will be useful for our analysis. Proposition 30 provides excess risk

---

#### Algorithm 5: Generic Framework for DP Accelerated Stochastic Approximation (AC-SA)

---

- 1: **Input:** Data  $X \in \mathcal{X}^n$ , iteration number  $T \leq n$ , stepsize parameters  $\{\eta_t\}_{t \in [T]}, \{\alpha_t\}_{t \in [T]}$  with  $\alpha_1 = 1, \alpha_t \in (0, 1) \forall t \geq 2$ , DP `MeanOracle`.
  - 2: Initialize  $w_0^{ag} = w_0 \in \mathcal{W}$  and  $t = 1$ .
  - 3: **for**  $t \in [T]$  **do**
  - 4:    $w_t^{md} := (1 - \alpha_t)w_{t-1}^{ag} + \alpha_t w_{t-1}$ .
  - 5:   Draw new batch  $\mathcal{B}_t$  (without replacement) of  $n/T$  samples from  $X$ .
  - 6:    $\tilde{\nabla} F_t(w_t^{md}) := \text{MeanOracle}(\{\nabla f(w_t^{md}, x)\}_{x \in \mathcal{B}_t}; \frac{n}{T}; \frac{\epsilon^2}{2})$
  - 7:    $w_t := \operatorname{argmin}_{w \in \mathcal{W}} \left\{ \alpha_t \langle \tilde{\nabla} F_t(w_t^{md}), w \rangle + \frac{\eta_t}{2} \|w_{t-1} - w\|^2 \right\}$ .
  - 8:    $w_t^{ag} := \alpha_t w_t + (1 - \alpha_t)w_{t-1}^{ag}$ .
  - 9: **end for**
  - 10: **Output:**  $w_T^{ag}$ .
- 

guarantees for Algorithm 5 in terms of the bias and variance of the `MeanOracle`.

**Proposition 30.** *Consider Algorithm 5 run with a `MeanOracle` satisfying  $\tilde{\nabla} F_t(w_t^{md}) = \nabla F(w_t^{md}) + b_t + N_t$ , where  $\|b_t\| \leq B$  (with probability 1),  $\mathbb{E}N_t = 0$ ,  $\mathbb{E}\|N_t\|^2 \leq \Sigma^2$  for all  $t \in [T - 1]$ , and  $\{N_t\}_{t=1}^T$  are independent. Assume that  $F : \mathcal{W} \rightarrow \mathbb{R}$  is convex and  $\beta$ -smooth,  $F(w_0) - F^* \leq \Delta$ , and  $\|w_0 - w^*\| \leq D$ . Suppose parameters are chosen in Algorithm 5 so that for all  $t \in [T]$ ,  $\eta_t > \beta\alpha_t^2$  and  $\eta_t/\gamma_t = \eta_1/\gamma_1$ , where*

$$\gamma_t := \begin{cases} 1, & t = 1 \\ (1 - \alpha_t)\gamma_t, & t \geq 2. \end{cases}$$

Then,

$$\mathbb{E}F(w_T^{ag}) - F^* \leq \frac{\gamma_T \eta_1 D^2}{2} + \gamma_T \sum_{t=1}^T \left[ \frac{2\alpha_t^2(\Sigma^2 + B^2)}{\gamma_t(\eta_t - \beta\alpha_t^2)} + \frac{\alpha_t}{\gamma_t} BD \right].$$

In particular, choosing  $\alpha_t = \frac{2}{t+1}$  and  $\eta_t = \frac{4\eta}{t(t+1)}$ ,  $\forall t \geq 1$ , where  $\eta \geq 2\beta$  implies

$$\mathbb{E}F(w_T^{ag}) - F^* \leq \frac{4\eta D^2}{T(T+1)} + \frac{4(\Sigma^2 + B^2)(T+2)}{3\eta} + BD.$$

Further, setting  $\eta = \max\left\{2\beta, \frac{T^{3/2}\sqrt{\Sigma^2+B^2}}{D}\right\}$  implies

$$\mathbb{E}F(w_T^{ag}) - F^* \lesssim \frac{\beta D^2}{T^2} + \frac{D(\Sigma + B)}{\sqrt{T}} + BD. \quad (\text{F.1})$$

*Proof.* We begin by extending (Ghadimi and Lan, 2012, Proposition 4) to biased/noisy stochastic gradients. Fix any  $w_{t-1}, w_{t-1}^{ag} \in \mathcal{W}$ . By (Ghadimi and Lan, 2012, Lemma 3), we have

$$F(w_t^{ag}) \leq (1 - \alpha_t)F(w_{t-1}^{ag}) + \alpha_t[F(z) + \langle \nabla F(z), w_t - z \rangle] + \frac{\beta}{2}\|w_t^{ag} - z\|^2, \quad (\text{F.2})$$

for any  $z \in \mathcal{W}$ . Denote

$$\Upsilon_t(w) := \alpha_t \langle N_t + b_t, w - w_{t-1} \rangle + \frac{\alpha_t^2 \|N_t + b_t\|^2}{\eta_t - \beta \alpha_t^2}$$

and  $d_t := w_t^{ag} - w_t^{md} = \alpha_t(w_t - w_{t-1})$ . Then using Eq. (F.2) with  $z = w_t^{md}$ , we have

$$\begin{aligned} F(w_t^{ag}) &\leq (1 - \alpha_t)F(w_{t-1}^{ag}) + \alpha_t[F(w_t^{md}) + \langle \nabla F(w_t^{md}), w_t - w_t^{md} \rangle] + \frac{\beta}{2}\|d_t\|^2 \\ &= (1 - \alpha_t)F(w_{t-1}^{ag}) + \alpha_t[F(w_t^{md}) + \langle \nabla F(w_t^{md}), w_t - w_t^{md} \rangle] \\ &\quad + \frac{\eta_t}{2}\|w_{t-1} - w_t\|^2 - \frac{\eta_t - \beta \alpha_t^2}{2\alpha_t^2}\|d_t\|^2, \end{aligned} \quad (\text{F.3})$$

by the expression for  $d_t$ . Now we apply (Ghadimi and Lan, 2012, Lemma 2) with  $p(u) = \alpha_t[\langle \tilde{\nabla} F_t(w_t^{md}, u) \rangle]$ ,  $\mu_1 = 0$ ,  $\mu_2 = \eta_t$ ,  $\tilde{x} = w_t^{md}$ , and  $\tilde{y} = w_{t-1}$  to obtain (conditional on all randomness) for any  $w \in \mathcal{W}$ :

$$\begin{aligned} &\alpha_t[F(w_t^{md}) + \langle \tilde{\nabla} F_t(w_t^{md}), w_t - w_t^{md} \rangle] + \frac{\eta_t}{2}\|w_{t-1} - w_t\|^2 \\ &\leq \alpha_t[F(w_t^{md}) + \langle \nabla F(w_t^{md}), w - w_t^{md} \rangle] \\ &\quad + \alpha_t \langle N_t + b_t, w - w_t^{md} \rangle + \frac{\eta_t}{2}\|w_{t-1} - w\|^2 - \frac{\eta_t}{2}\|w_t - w\|^2. \end{aligned}$$

Next, we combine the above inequality with Eq. (F.3) to get

$$\begin{aligned} F(w_t^{ag}) &\leq (1 - \alpha_t)F(w_{t-1}^{ag}) + \alpha_t[F(w_t^{md}) + \langle \nabla F(w_t^{md}), w - w_t^{md} \rangle] \\ &\quad + \frac{\eta_t}{2} [\|w_{t-1} - w\|^2 - \|w_t - w\|^2] + \underbrace{-\frac{\eta_t - \beta \alpha_t^2}{2\alpha_t^2}\|d_t\|^2 + \alpha_t \langle N_t + b_t, w - w_t \rangle}_{U_t}, \end{aligned} \quad (\text{F.4})$$

for all  $w \in \mathcal{W}$ . By Cauchy-Schwartz, we can bound

$$\begin{aligned} U_t &\leq -\frac{\eta_t - \beta \alpha_t^2}{2\alpha_t^2}\|d_t\|^2 + \|N_t + b_t\|\|d_t\| + \alpha_t \langle N_t + b_t, w - w_{t-1} \rangle \\ &\leq \Upsilon_t(w), \end{aligned} \quad (\text{F.5})$$

where the last inequality follows from maximizing the concave quadratic function  $q(\|d_t\|) := -\left[\frac{\eta_t - \beta\alpha_t^2}{2\alpha_t^2}\right]\|d_t\|^2 + \|N_t + b_t\|\|d_t\|$  with respect to  $\|d_t\|$ . Plugging the bound Eq. (F.5) back into Eq. (F.4) shows that

$$\begin{aligned} F(w_t^{ag}) &\leq (1 - \alpha_t)F(w_{t-1}^{ag}) + \alpha_t[F(w_t^{md}) + \langle \nabla F(w_t^{md}), w - w_t^{md} \rangle] \\ &\quad + \frac{\eta_t}{2} [\|w_{t-1} - w\|^2 - \|w_t - w\|^2] + \Upsilon_t(w). \end{aligned} \quad (\text{F.6})$$

Then it can be shown (see (Ghadimi and Lan, 2012, Proposition 5)) that the assumptions on  $\eta_t$  and  $\alpha_t$  imply that

$$F(w_T^{ag}) - \gamma_T \sum_{t=1}^T \left[ \frac{\alpha_t}{\gamma_t} \left( F(w_t^{md}) + \langle \nabla F(w_t^{md}), w - w_t^{md} \rangle \right) \right] \leq \gamma_T \sum_{t=1}^T \frac{\eta_t}{2\gamma_t} [\|w_{t-1} - w\|^2 - \|w_t - w\|^2] \quad (\text{F.7})$$

$$+ \gamma_T \sum_{t=1}^T \frac{\Upsilon_t(w)}{\gamma_t}, \quad (\text{F.8})$$

for any  $w \in \mathcal{W}$  and any  $T \geq 1$ . Now,

$$\sum_{t=1}^T \frac{\alpha_t}{\gamma_t} = \frac{1}{\gamma_T}$$

by definition. Hence by convexity of  $F$ ,

$$\sum_{t=1}^T \left[ \frac{\alpha_t}{\gamma_t} \left( F(w_t^{md}) + \langle \nabla F(w_t^{md}), w - w_t^{md} \rangle \right) \right] \leq F(w), \quad \forall w \in \mathcal{W}.$$

Also, since  $\gamma_t/\eta_t = \gamma_1/\eta_1$  for all  $t \geq 1$ , we have

$$\gamma_T \sum_{t=1}^T \frac{\eta_t}{2\gamma_t} [\|w_{t-1} - w\|^2 - \|w_t - w\|^2] = \gamma_T \frac{\eta_1}{2\gamma_1} [\|w_0 - w\|^2 - \|w_T - w\|^2] \leq \gamma_T \eta_1 \frac{1}{2} \|w_0 - w\|^2,$$

since  $\gamma_1 = 1$ . Substituting the above bounds into Eq. (F.7), we get

$$F(w_T^{ag}) - F(w) \leq \gamma_T \eta_1 \frac{1}{2} \|w_0 - w\|^2 + \gamma_T \sum_{t=1}^T \frac{\Upsilon_t(w)}{\gamma_t}, \quad \forall w \in \mathcal{W}. \quad (\text{F.9})$$

Now, setting  $w = w^*$  and taking expectation yields

$$\mathbb{E}[F(w_T^{ag}) - F^*] \leq \frac{\gamma_T \eta_1 D^2}{2} + \gamma_T \sum_{t=1}^T \frac{\mathbb{E}\Upsilon_t(w^*)}{\gamma_t} \quad (\text{F.10})$$

$$\leq \frac{\gamma_T \eta_1 D^2}{2} + \gamma_T \sum_{t=1}^T \left[ \frac{1}{\gamma_t} \left( \alpha_t \mathbb{E}\langle b_t, w^* - w_{t-1} \rangle + \frac{2\alpha_t^2(\Sigma^2 + B^2)}{\eta_t - \beta\alpha_t^2} \right) \right] \quad (\text{F.11})$$

$$\leq \frac{\gamma_T \eta_1 D^2}{2} + \gamma_T \sum_{t=1}^T \left[ \frac{1}{\gamma_t} \left( \alpha_t B D + \frac{2\alpha_t^2(\Sigma^2 + B^2)}{\eta_t - \beta\alpha_t^2} \right) \right], \quad (\text{F.12})$$

where we used conditional independence of  $N_t$  and  $w^* - w_{t-1}$  given  $w_{t-1}$ , Young's inequality, Cauchy-Schwartz, and the definitions of  $B^2$  and  $\Sigma^2$ . This establishes the first claim of the

theorem. The second and third claims are simple corollaries, which can be verified as in (Ghadimi and Lan, 2012, Proposition 7) and the ensuing discussion.  $\square$

**Theorem 31** (Complete Version of Theorem 9). *Grant Assumption 2. Let  $\varepsilon > 0$  and assume  $F$  is convex and  $\beta$ -smooth. Then, there are algorithmic parameters such that Algorithm 2 is  $\frac{\varepsilon^2}{2}$ -zCDP. Further, if*

$$n \geq T := \left[ \min \left\{ \left( \frac{\beta D}{r_k} \right)^{2k/(5k-1)} \left( \frac{\varepsilon n}{\sqrt{d}} \right)^{(2k-2)/(5k-1)}, \sqrt{\frac{\beta D}{r_k}} n^{1/4} \right\} \right],$$

then,

$$\mathbb{E}F(w_T^{ag}) - F^* \lesssim \frac{r_2 D}{\sqrt{n}} + r_k D \left[ \left( \frac{\sqrt{d}}{\varepsilon n} \right)^{\frac{k-1}{k}} + \min \left\{ \left( \left( \frac{\beta D}{r_k} \right)^{1/4} \frac{\sqrt{d}}{\varepsilon n} \right)^{\frac{4(k-1)}{5k-1}}, \left( \frac{\beta D}{r_k} \right)^{\frac{k-1}{4k}} \left( \frac{\sqrt{d}}{\varepsilon n} \right)^{\frac{k-1}{k}} n^{\frac{k-1}{8k}} \right\} \right].$$

*Proof. Privacy:* Choose  $\sigma^2 = \frac{4C^2 T^2}{\varepsilon^2 n^2}$ . First, the collection of all  $\tilde{\nabla} F_t(w_t^{md})$ ,  $t \in [T]$  is  $\frac{\varepsilon^2}{2}$ -zCDP: since the batches of data drawn in each iteration are disjoint, it suffices (by parallel composition (McSherry, 2009)) to show that  $\tilde{\nabla} F_t(w_t^{md})$  is  $\frac{\varepsilon^2}{2}$ -zCDP for all  $t$ . Now, the  $\ell_2$  sensitivity of each clipped gradient update is bounded by  $\Delta = \sup_{w, X \sim X'} \left\| \frac{T}{n} \sum_{x \in \mathcal{B}_t} \Pi_C(\nabla f(w, x)) - \sum_{x' \in \mathcal{B}_t} \Pi_C(\nabla f(w, x')) \right\| = \sup_{w, x, x'} \left\| \frac{T}{n} \Pi_C(\nabla f(w, x)) - \Pi_C(\nabla f(w, x')) \right\| \leq \frac{2CT}{n}$ . Thus,  $\tilde{\nabla} F_t(w_t^{md})$  is  $\frac{\varepsilon^2}{2}$ -zCDP by Proposition 23. Second, the iterates  $w_t^{ag}$  are deterministic functions of  $\tilde{\nabla} F_t(w_t^{md})$ , so the post-processing property of differential privacy (Dwork and Roth, 2014; Bun and Steinke, 2016) ensures that Algorithm 2 is  $\frac{\varepsilon^2}{2}$ -zCDP.

**Excess risk:** Consider round  $t \in [T]$  of Algorithm 2, where Algorithm 1 is run on input data  $\{\nabla f(w_t, x_i^t)\}_{i=1}^{n/T}$ . Denote the bias of Algorithm 1 by  $b_t := \mathbb{E} \tilde{\nabla} F_t(w_t) - \nabla F(w_t)$ , where  $\tilde{\nabla} F_t(w_t) = \tilde{v}$  in the notation of Algorithm 1. Also let  $\hat{\nabla} F_t(w_t) := \hat{\mu}$  (in the notation of Lemma 6) and denote the noise by  $N_t = \tilde{\nabla} F_t(w_t) - \nabla F(w_t) - b_t = \tilde{\nabla} F_t(w_t) - \mathbb{E} \tilde{\nabla} F_t(w_t)$ . Then we have  $B := \sup_{t \in [T]} \|b_t\| \leq \frac{r^{(k)}}{(k-1)C^{k-1}}$  and  $\Sigma^2 := \sup_{t \in [T]} \mathbb{E}[\|N_t\|^2] \leq d\sigma^2 + \frac{r^{(2)}T}{n} \lesssim \frac{dC^2 T^2}{\varepsilon^2 n^2} + \frac{r^{(2)}T}{n}$ , by Lemma 6. Plugging these estimates for  $B$  and  $\Sigma^2$  into Proposition 30 and setting  $C = r_k \left( \frac{\varepsilon n}{\sqrt{dT}} \right)^{1/k}$ , we get

$$\begin{aligned} \mathbb{E}F(w_T^{ag}) - F^* &\lesssim \frac{\beta D^2}{T^2} + \frac{D(\Sigma + B)}{\sqrt{T}} + BD \\ &\lesssim \frac{\beta D^2}{T^2} + \frac{CD\sqrt{dT}}{\varepsilon n} + \frac{r_2 D}{\sqrt{n}} + \frac{r^{(k)}D}{C^{k-1}} \\ &\lesssim \frac{\beta D^2}{T^2} + D \left[ \frac{r_2}{\sqrt{n}} + r_k \left( \frac{\sqrt{dT}}{\varepsilon n} \right)^{(k-1)/k} \right]. \end{aligned} \quad (\text{F.13})$$

Now, our choice of  $T$  implies that  $\frac{\beta D^2}{T^2} \leq r_k D \left[ \frac{1}{\sqrt{n}} + \left( \frac{\sqrt{dT}}{\varepsilon n} \right)^{(k-1)/k} \right]$  and we get the result upon plugging in  $T$ .  $\square$

**Algorithm 6:** Noisy Clipped SGD for Heavy-Tailed DP SCO

- 
- 1: **Input:** Data  $X \in \mathcal{X}^n$ ,  $T \leq n$ , stepsizes  $\{\eta_t\}_{t=0}^T$ , averaging weights  $\{\zeta_t\}_{t=0}^T$ ,  $w_0 \in \mathcal{W}$ .
  - 2: **for**  $t \in \{0, 1, \dots, T\}$  **do**
  - 3:   Draw new batch  $\mathcal{B}_t$  (without replacement) of  $n/T$  samples from  $X$ .
  - 4:    $\tilde{\nabla}F_t(w_t) := \text{MeanOracle1}(\{\nabla f(w_t, x)\}_{x \in \mathcal{B}_t}; \frac{n}{T}; \frac{\varepsilon^2}{2})$
  - 5:    $w_{t+1} = \Pi_{\mathcal{W}} [w_t - \eta_t \tilde{\nabla}F_t(w_t)]$
  - 6: **end for**
  - 7: **Output:**  $\hat{w}_T := \frac{1}{Z_T} \sum_{t=0}^T \zeta_t w_{t+1}$ , where  $Z_T = \sum_{t=0}^T \zeta_t$ .
- 

**F.2. Noisy Clipped SGD for Strongly Convex Losses (Section 4.2).** We begin by presenting the pseudocode for our noisy clipped SGD in Algorithm 6.

Algorithm 7 is a generalized version of Algorithm 6 that allows for any `MeanOracle` and will be useful in our analysis. In Proposition 32, we provide the convergence guarantees

**Algorithm 7:** Generic Noisy Clipped SGD Framework for Heavy-Tailed SCO

- 
- 1: **Input:** Data  $X \in \mathcal{X}^n$ ,  $T \leq n$ , `MeanOracle`, stepsizes  $\{\eta_t\}_{t=0}^T$ , averaging weights  $\{\zeta_t\}_{t=0}^T$ .
  - 2: Initialize  $w_0 \in \mathcal{W}$ .
  - 3: **for**  $t \in \{0, 1, \dots, T\}$  **do**
  - 4:   Draw new batch  $\mathcal{B}_t$  (without replacement) of  $n/T$  samples from  $X$ .
  - 5:    $\tilde{\nabla}F_t(w_t) := \text{MeanOracle}(\{\nabla f(w_t, x)\}_{x \in \mathcal{B}_t}; \frac{n}{T}; \frac{\varepsilon^2}{2})$
  - 6:    $w_{t+1} = \Pi_{\mathcal{W}} [w_t - \eta_t \tilde{\nabla}F_t(w_t)]$
  - 7: **end for**
  - 8: **Output:**  $\hat{w}_T := \frac{1}{Z_T} \sum_{t=0}^T \zeta_t w_{t+1}$ , where  $Z_T = \sum_{t=0}^T \zeta_t$ .
- 

for Algorithm 7 in terms of the bias and variance of the `MeanOracle`.

**Proposition 32.** *Let  $F : \mathcal{W} \rightarrow \mathbb{R}$  be  $\mu$ -strongly convex and  $\beta$ -smooth with condition number  $\kappa := \frac{\beta}{\mu}$ . Let  $w_{t+1} := \Pi_{\mathcal{W}}[w_t - \eta_t \tilde{\nabla}F_t(w_t)]$ , where  $\tilde{\nabla}F_t(w_t) = \nabla F(w_t) + b_t + N_t$ , such that the bias and noise (which can depend on  $w_t$  and the samples drawn) satisfy  $\|b_t\| \leq B$  (with probability 1),  $\mathbb{E}N_t = 0$ ,  $\mathbb{E}\|N_t\|^2 \leq \Sigma^2$  for all  $t \in [T-1]$ , and that  $\{N_t\}_{t=1}^T$  are independent. Then, there exist stepsizes  $\{\eta_t\}_{t=1}^T$  and weights  $\{\zeta_t\}_{t=0}^T$  such that the average iterate  $\hat{w}_T := \frac{1}{\sum_{t=0}^T \zeta_t} \sum_{t=0}^T \zeta_t w_{t+1}$  satisfies*

$$\mathbb{E}F(\hat{w}_T) - F^* \leq 32\beta D^2 \exp\left(-\frac{T}{4\kappa}\right) + \frac{72\Sigma^2}{\mu T} + \frac{2B^2}{\mu}.$$

*Proof.* Define  $g(w_t) = -\frac{1}{\eta_t}(w_{t+1} - w_t)$ . Then

$$\begin{aligned} \mathbb{E}\|w_{t+1} - w^*\|^2 &= \mathbb{E}\|w_t - \eta_t g(w_t) - w^*\|^2 \\ &= \mathbb{E}\|w_t - w^*\|^2 - 2\eta_t \mathbb{E}\langle g(w_t), w_t - w^* \rangle + \eta_t^2 \mathbb{E}\|g(w_t)\|^2. \end{aligned} \quad (\text{F.14})$$

Now, conditional on all randomness, we use smoothness and strong convexity to write:

$$\begin{aligned}
F(w_{t+1}) - F(w^*) &= F(w_{t+1}) - F(w_t) + F(w_t) - F(w^*) \\
&\leq \langle F(w_t), w_{t+1} - w_t \rangle + \frac{\beta}{2} \|w_{t+1} - w_t\|^2 + \langle \nabla F(w_t), w_t - w^* \rangle - \frac{\mu}{2} \|w_t - w^*\|^2 \\
&= \langle \tilde{\nabla} F_t(w_t), w_{t+1} - w^* \rangle + \langle \nabla F(w_t) - \tilde{\nabla} F_t(w_t), w_{t+1} - w^* \rangle + \frac{\beta \eta_t^2}{2} \|g(w_t)\|^2 \\
&\quad - \frac{\mu}{2} \|w_t - w^*\|^2 \\
&\leq \langle g(w_t), w_{t+1} - w^* \rangle + \langle \nabla F(w_t) - \tilde{\nabla} F_t(w_t), w_{t+1} - w^* \rangle + \frac{\beta \eta_t^2}{2} \|g(w_t)\|^2 \\
&\quad - \frac{\mu}{2} \|w_t - w^*\|^2 \\
&= \langle g(w_t), w_{t+1} - w_t \rangle + \langle g(w_t), w_t - w^* \rangle - \langle b_t + N_t, w_{t+1} - w^* \rangle + \frac{\beta \eta_t^2}{2} \|g(w_t)\|^2 \\
&\quad - \frac{\mu}{2} \|w_t - w^*\|^2 \\
&= \langle g(w_t), w_t - w^* \rangle - \langle b_t + N_t, w_{t+1} - w^* \rangle + \left( \frac{\beta \eta_t^2}{2} - \eta_t \right) \|g(w_t)\|^2 - \frac{\mu}{2} \|w_t - w^*\|^2,
\end{aligned}$$

where we used the fact that  $\langle \Pi_{\mathcal{W}}(y) - x, \Pi_{\mathcal{W}}(y) - y \rangle \leq 0$  for all  $x \in \mathcal{W}, y \in \mathbb{R}^d$  (c.f. (Bubeck, 2015, Lemma 3.1)) to obtain the last inequality. Thus,

$$\begin{aligned}
-2\eta_t \mathbb{E} \langle g(w_t), w_t - w^* \rangle &\leq -2\eta_t \mathbb{E} [F(w_{t+1}) - F^*] \\
&\quad + 2\eta_t \mathbb{E} \left[ -\langle b_t + N_t, w_{t+1} - w^* \rangle + \left( \frac{\beta \eta_t^2}{2} - \eta_t \right) \|g(w_t)\|^2 - \frac{\mu}{2} \|w_t - w^*\|^2 \right].
\end{aligned}$$

Combining the above inequality with Eq. (F.14), we get

$$\begin{aligned}
\mathbb{E} \|w_{t+1} - w^*\|^2 &\leq (1 - \mu \eta_t) \mathbb{E} \|w_t - w^*\|^2 - 2\eta_t \mathbb{E} [F(w_{t+1}) - F^*] - 2\eta_t \mathbb{E} \langle b_t + N_t, w_{t+1} - w^* \rangle \\
&\quad + 2\eta_t \left( \frac{\eta_t^2 \beta}{2} - \eta_t \right) \mathbb{E} \|g(w_t)\|^2. \tag{F.15}
\end{aligned}$$

Next, consider

$$\begin{aligned}
|\mathbb{E} \langle b_t + N_t, w_{t+1} - w^* \rangle| &\leq |\mathbb{E} \langle b_t + N_t, w_{t+1} - w_t \rangle| + |\mathbb{E} \langle b_t + N_t, w_t - w^* \rangle| \\
&= |\mathbb{E} \langle b_t + N_t, w_{t+1} - w_t \rangle| + |\mathbb{E} \langle b_t, w_t - w^* \rangle| \\
&\leq |\mathbb{E} \langle b_t + N_t, w_{t+1} - w_t \rangle| + \frac{B^2}{\mu} + \frac{\mu}{4} \mathbb{E} \|w_t - w^*\|^2
\end{aligned}$$

by independence of  $N_t$  (which has zero mean) and  $w_t - w^*$ , and Young's inequality. Next, note that  $v := w_t - \eta_t(\nabla F(w_t) + b_t)$  is independent of  $N_t$ , so  $\mathbb{E}\langle N_t, \Pi_{\mathcal{W}}(v) \rangle = 0$ . Thus,

$$\begin{aligned} |\mathbb{E}\langle N_t, w_{t+1} - w_t \rangle| &= |\mathbb{E}\langle N_t, w_{t+1} \rangle| \\ &= |\mathbb{E}\langle N_t, \Pi_{\mathcal{W}}[w_t - \eta_t(\nabla F(w_t) + b_t) + N_t] \rangle| \\ &= |\mathbb{E}\langle N_t, \Pi_{\mathcal{W}}[v - \eta_t N_t] \rangle| \\ &= |\mathbb{E}\langle N_t, \Pi_{\mathcal{W}}[v] - \Pi_{\mathcal{W}}[v - \eta_t N_t] \rangle| \\ &\leq \mathbb{E}[\|N_t\| \|\Pi_{\mathcal{W}}[v] - \Pi_{\mathcal{W}}[v - \eta_t N_t]\|] \\ &\leq \mathbb{E}[\|N_t\| \|\eta_t N_t\|] \\ &\leq \eta_t \Sigma^2, \end{aligned}$$

by Cauchy-Schwartz and non-expansiveness of projection. Further,

$$\begin{aligned} |\mathbb{E}\langle b_t, w_{t+1} - w_t \rangle| &= |\mathbb{E}\langle b_t, -\eta_t g(w_t) \rangle| \\ &\leq \frac{B^2}{\mu} + \frac{\eta_t^2 \mu}{4} \mathbb{E}\|g(w_t)\|^2, \end{aligned}$$

by Young's inequality. Therefore,

$$-2\eta_t \mathbb{E}\langle b_t + N_t, w_{t+1} - w^* \rangle \leq 2\eta_t \left[ \frac{2B^2}{\mu} + \frac{\eta_t^2 \mu}{4} \mathbb{E}\|g(w_t)\|^2 + \eta_t \Sigma^2 + \frac{\mu}{4} \mathbb{E}\|w_t - w^*\|^2 \right].$$

Plugging this bound back into Eq. (F.15) and choosing  $\eta_t \leq \frac{1}{\beta} \leq \frac{1}{\mu}$  yields:

$$\begin{aligned} \mathbb{E}\|w_{t+1} - w^*\|^2 &\leq \left(1 - \frac{\mu\eta_t}{2}\right) \mathbb{E}\|w_t - w^*\|^2 - 2\eta_t \mathbb{E}[F(w_{t+1}) - F^*] + \frac{4\eta_t B^2}{\mu} + 2\eta_t^2 \Sigma^2 \\ &\quad + 2\eta_t \left( \frac{\eta_t^2 \beta}{2} - \eta_t + \frac{\eta_t^2 \mu}{4} \right) \mathbb{E}\|g(w_t)\|^2 \\ &\leq \left(1 - \frac{\mu\eta_t}{2}\right) \mathbb{E}\|w_t - w^*\|^2 - 2\eta_t \mathbb{E}[F(w_{t+1}) - F^*] + \frac{4\eta_t B^2}{\mu} + 2\eta_t^2 \Sigma^2. \end{aligned}$$

Next, we apply Lemma 33 (see below) with  $r_t := \mathbb{E}\|w_t - w^*\|^2$ ,  $s_t := \mathbb{E}F(w_{t+1}) - F^* - \frac{2B^2}{\mu}$ ,  $a := \frac{\mu}{2}$ ,  $b := 2$ ,  $c = 2\Sigma^2$ , and  $g = \beta$ . We may assume  $s_t \geq 0$  for all  $t$ : if this inequality breaks for some  $t$ , then simply return  $w_{t+1}$  instead of  $\hat{w}_T$  to obtain  $\mathbb{E}F(w_t) - F^* < \frac{2B^2}{\mu}$ . Thus,

$$\frac{1}{\gamma_T} \sum_{t=0}^T \gamma_t \mathbb{E}[F(w_{t+1}) - F^*] \leq \frac{1}{2} \left[ 32\beta D^2 \exp\left(\frac{-\mu T}{4\beta}\right) + \frac{144\Sigma^2}{\mu T} + \frac{2B^2}{\mu} \right]$$

Finally, Jensen's inequality yields the proposition.  $\square$

**Lemma 33.** (*Stich, 2019, Lemma 3*) *Let  $b > 0$ , let  $a, c \geq 0$ , and  $\{\eta_t\}_{t \geq 0}$  be non-negative step-sizes such that  $\eta_t \leq \frac{1}{g}$  for all  $t \geq 0$  for some parameter  $g \geq a$ . Let  $\{r_t\}_{t \geq 0}$  and  $\{s_t\}_{t \geq 0}$  be two non-negative sequences of real numbers which satisfy*

$$r_{t+1} \leq (1 - a\eta_t)r_t - b\eta_t s_t + c\eta_t^2$$

*for all  $t \geq 0$ . Then there exist particular choices of step-sizes  $\eta_t \leq \frac{1}{g}$  and averaging weights  $\zeta_t \geq 0$  such that*

$$\frac{b}{\gamma_T} \sum_{t=0}^T s_t \zeta_t + ar_{T+1} \leq 32gr_0 \exp\left(\frac{-aT}{2g}\right) + \frac{36c}{aT},$$

where  $\gamma_T := \sum_{t=0}^T \gamma_t$ .

We are now prepared to prove Theorem 11.

**Theorem 34** (Precise statement of Theorem 11). *Grant Assumption 2. Let  $\varepsilon > 0$ , and assume  $F$  is  $\mu$ -strongly convex and  $\beta$ -smooth with  $\kappa = \frac{\beta}{\mu} \leq n/\ln(n)$ . Then, there are parameters such that Algorithm 6 is  $\frac{\varepsilon^2}{2}$ -zCDP, and*

$$\mathbb{E}F(\hat{w}_T) - F^* \lesssim \frac{1}{\mu} \left( \frac{r_2^2}{n} + r_k^2 \left( \frac{\sqrt{d\kappa \ln(n)}}{\varepsilon n} \right)^{\frac{2k-2}{k}} \right). \quad (\text{F.16})$$

*Proof. Privacy:* Choose  $\sigma^2 = \frac{4C^2T^2}{\varepsilon^2n^2}$ . Since the batches of data drawn in each iteration are disjoint, it suffices (by parallel composition (McSherry, 2009)) to show that  $\tilde{\nabla}F_t(w_t)$  is  $\frac{\varepsilon^2}{2}$ -zCDP for all  $t$ . Now, the  $\ell_2$  sensitivity of each clipped gradient update is bounded by  $\Delta = \sup_{w, X \sim X'} \left\| \frac{T}{n} \sum_{x \in \mathcal{B}_t} \Pi_C(\nabla f(w, x)) - \sum_{x' \in \mathcal{B}'_t} \Pi_C(\nabla f(w, x')) \right\| = \sup_{w, x, x'} \left\| \frac{T}{n} \Pi_C(\nabla f(w, x)) - \Pi_C(\nabla f(w, x')) \right\| \leq \frac{2CT}{n}$ . Hence Proposition 23 implies that the algorithm is  $\frac{\varepsilon^2}{2}$ -zCDP.

*Excess risk:* For any iteration  $t \in [T]$ , denote the bias of Algorithm 1 by  $b_t := \mathbb{E}\tilde{\nabla}F_t(w_t) - \nabla F(w_t)$ , where  $\tilde{\nabla}F_t(w_t) = \tilde{v}$  in the notation of Algorithm 1. Also let  $\hat{\nabla}F_t(w_t) := \hat{v}$  (in the notation of Lemma 6) and denote the noise by  $N_t = \hat{\nabla}F_t(w_t) - \nabla F(w_t) - b_t = \hat{\nabla}F_t(w_t) - \mathbb{E}\tilde{\nabla}F_t(w_t)$ . Then we have  $B := \sup_{t \in [T]} \|b_t\| \leq \frac{r^{(k)}}{(k-1)C^{k-1}}$  and  $\Sigma^2 := \sup_{t \in [T]} \mathbb{E}[\|N_t\|^2] \leq d\sigma^2 + \frac{r^{(2)}T}{n} \lesssim \frac{dC^2T^2}{\varepsilon^2n^2} + \frac{r^{(2)}T}{n}$ , by Lemma 6. Plugging these bias and variance estimates into Proposition 32, we get

$$\mathbb{E}F(\hat{w}_T) - F^* \lesssim \beta D^2 \exp\left(-\frac{T}{4\kappa}\right) + \frac{1}{\mu T} \left( \frac{dC^2T^2}{\varepsilon^2n^2} + \frac{r^{(2)}T}{n} \right) + \frac{(r^{(k)})^2}{C^{2k-2}\mu}.$$

Choosing  $C = r_k \left( \frac{\varepsilon^2 n^2}{dT} \right)^{1/2k}$  implies

$$\mathbb{E}F(\hat{w}_T) - F^* \lesssim \beta D^2 \exp\left(-\frac{T}{4\kappa}\right) + \frac{1}{\mu} \left( \frac{r^{(2)}}{n} + r_k^2 \left( \frac{dT}{\varepsilon^2 n^2} \right)^{(k-1)/k} \right).$$

Finally, choosing  $T = \left\lceil 4\kappa \ln \left( \frac{\mu\beta D^2}{r_k^2} \left( n + \left( \frac{\varepsilon^2 n^2}{d} \right)^{(k-1)/k} \right) \right) \right\rceil \lesssim \kappa \ln(n)$  yields the result.  $\square$

## APPENDIX G. DETAILS AND PROOFS FOR SECTION 5: ALGORITHM FOR NON-SMOOTH (STRONGLY) CONVEX LOSSES

In order to precisely state (sharper forms of) Theorems 12 and 18, we will need to introduce some notation.

**G.1. Notation.** For a batch of data  $X \in \mathcal{X}^m$ , we define the  $k$ -th *empirical moment* of  $f(w, \cdot)$  by

$$\widehat{r}_m(X)^{(k)} = \sup_{w \in \mathcal{W}} \sup_{\{\nabla f(w, x_i) \in \partial_w f(w, x_i)\}} \frac{1}{m} \sum_{i=1}^m \|\nabla f(w, x_i)\|^k,$$

where the supremum is also over all subgradients  $\nabla f(w, x_i) \in \partial_w f(w, x_i)$  in case  $f$  is not differentiable. For  $X \sim \mathcal{D}^m$ , we denote the  $k$ -th *expected empirical moment* by

$$\widetilde{e}_m^{(k)} := \mathbb{E}[\widehat{r}_m(X)^{(k)}]$$

and let

$$\widetilde{r}_{k,m} := (\widetilde{e}_m^{(k)})^{1/k}.$$

Note that  $\widetilde{r}_{k,1} = \widetilde{r}_k$ . Our excess risk upper bounds will depend on a weighted average of the expected empirical moments for different batch sizes  $m \in \{1, 2, 4, 8, \dots, n\}$ , with more weight being given to  $\widetilde{r}_m$  for large  $m$  (which are smaller, by Lemma 35 below): for  $n = 2^l$ , define

$$\widetilde{R}_{k,n} := \sqrt[l]{\sum_{i=1}^l 2^{-i} \widetilde{r}_{k,n_i}^2},$$

where  $n_i = 2^{-i}n$ .

**Lemma 35.** *Under Assumptions 1 and 2, we have:  $\widetilde{r}^{(k)} = \widetilde{e}_1^{(k)} \geq \widetilde{e}_2^{(k)} \geq \widetilde{e}_4^{(k)} \geq \widetilde{e}_8^{(k)} \geq \dots \geq r^{(k)}$ . In particular,  $\widetilde{R}_{k,n} \leq \widetilde{r}_k$ .*

*Proof.* Let  $l \in \mathbb{N}$ ,  $n = 2^l$  and consider

$$\begin{aligned} \widehat{r}_n(X)^{(k)} &= \frac{1}{n} \sup_w \left( \sum_{i=1}^{n/2} \|\nabla f(w, x_i)\|^k + \sum_{i=n/2+1}^n \|\nabla f(w, x_i)\|^k \right) \\ &\leq \frac{1}{n} \left( \sup_w \sum_{i=1}^{n/2} \|\nabla f(w, x_i)\|^k + \sup_w \sum_{i=n/2+1}^n \|\nabla f(w, x_i)\|^k \right). \end{aligned}$$

Taking expectations over the random draw of  $X \sim \mathcal{D}^n$  yields  $\widetilde{e}_n^{(k)} \leq \widetilde{e}_{n/2}^{(k)}$ . Thus,  $\widetilde{R}_{k,n} \leq \widetilde{r}_k$  by the definition of  $\widetilde{R}_n$ .  $\square$

**G.2. Localized Noisy Clipped Subgradient Method (Section 5.1).** We begin by proving the technical ingredients that will be used in the proof of Theorem 12. First, we will prove a variant of Lemma 6 that bounds the bias and variance of the subgradient estimator in Algorithm 3.

**Lemma 36.** *Let  $\widehat{F}_\lambda(w) = \frac{1}{n} \sum_{i=1}^n f(w, x_i) + \frac{\lambda}{2} \|w - w_0\|^2$  be a regularized empirical loss on a closed convex domain  $\mathcal{W}$  with  $\ell_2$ -diameter  $D$ . Let  $\widetilde{\nabla} F_\lambda(w_t) = \nabla \widehat{F}_\lambda(w_t) + b_t + N_t = \frac{1}{n} \sum_{i=1}^n \Pi_C(\nabla f(w, x_i)) + \lambda(w - w_0) + N_t$  be the biased, noisy subgradients of the regularized empirical loss in Algorithm 3, with  $N_t \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_d)$  and  $b_t = \frac{1}{n} \sum_{i=1}^n \Pi_C(\nabla f(w_t, x_i)) -$*

$\frac{1}{n} \sum_{i=1}^n \nabla f(w_t, x_i)$ . Assume  $\hat{r}_n(X)^{(k)} \geq \sup_{w \in \mathcal{W}} \left\{ \frac{1}{n} \sum_{i=1}^n \|\nabla f(w, x_i)\|^k \right\}$  for all  $\nabla f(w, x_i) \in \partial_w f(w, x_i)$ . Then, for any  $T \geq 1$ , we have:

$$\hat{B} := \sup_{t \in [T]} \|b_t\| \leq \frac{\hat{r}_n(X)^{(k)}}{(k-1)C^{k-1}}$$

and

$$\hat{\Sigma}^2 := \sup_{t \in [T]} \mathbb{E} \|N_t\|^2 = d\sigma^2.$$

*Proof.* Fix any  $t$ . We have

$$\begin{aligned} \|b_t\| &= \left\| \frac{1}{n} \sum_{i=1}^n \Pi_C(\nabla f(w_t, x_i)) - \frac{1}{n} \sum_{i=1}^n \nabla f(w_t, x_i) \right\| \\ &\leq \frac{1}{(k-1)C^{k-1}} \left[ \frac{1}{n} \sum_{i=1}^n \|\nabla f(w_t, x_i)\|^k \right], \end{aligned} \quad (\text{G.1})$$

by Lemma 6 applied with  $\mathcal{D}$  as the empirical distribution on  $X$ , and  $z_i$  in Lemma 6 corresponding to  $\nabla f(w_t, x_i)$  in Eq. (G.1). Taking supremum over  $t$  of both sides of Eq. (G.1) and recalling the definition of  $\hat{r}_n(X)^{(k)}$  proves the bias bound. The noise variance bound is immediate from the distribution of  $N_t$ .  $\square$

Using Lemma 36, we can obtain the following convergence guarantee for Algorithm 3:

**Lemma 37** (Re-statement of Lemma 14). *Fix  $X \in \mathcal{X}^n$  and let  $\hat{F}_\lambda(w) = \frac{1}{n} \sum_{i=1}^n f(w, x_i) + \frac{\lambda}{2} \|w - w_0\|^2$  for  $w_0 \in \mathcal{W}$ , where  $\mathcal{W}$  is a closed convex domain with diameter  $D$ . Assume  $f(\cdot, x)$  is convex and  $\hat{r}_n(X)^{(k)} \geq \sup_{w \in \mathcal{W}} \left\{ \frac{1}{n} \sum_{i=1}^n \|\nabla f(w, x_i)\|^k \right\}$  for all  $\nabla f(w, x_i) \in \partial_w f(w, x_i)$ . Denote  $\hat{r}_n(X) = [\hat{r}_n(X)^{(k)}]^{1/k}$  and  $\hat{w} = \operatorname{argmin}_{w \in \mathcal{W}} \hat{F}_\lambda(w)$ . Let  $\eta \leq \frac{2}{\lambda}$ . Then, the output of Algorithm 3 satisfies*

$$\mathbb{E} \|w_T - \hat{w}\|^2 \leq \exp\left(-\frac{\lambda\eta T}{2}\right) \|w_0 - \hat{w}\|^2 + \frac{8\eta}{\lambda} (\hat{r}_n(X)^2 + \lambda^2 D^2 + d\sigma^2) + \frac{20}{\lambda^2} \left( \frac{\hat{r}_n(X)^{(k)}}{(k-1)C^{k-1}} \right)^2,$$

where  $\sigma^2 = \frac{4C^2T}{n^2\varepsilon^2}$ .

*Proof.* We use the notation of Lemma 36 and write  $\tilde{\nabla} F_\lambda(w_t) = \nabla \hat{F}_\lambda(w_t) + b_t + N_t = \frac{1}{n} \sum_{i=1}^n \Pi_C(\nabla f(w_t, x_i)) + \lambda(w_t - w_0) + N_t$  as the biased, noisy subgradients of the regularized empirical loss in Algorithm 3, with  $N_t \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_d)$  and  $b_t = \frac{1}{n} \sum_{i=1}^n \Pi_C(\nabla f(w_t, x_i)) - \frac{1}{n} \sum_{i=1}^n \nabla f(w_t, x_i)$ . Denote  $y_{t+1} = w_t - \eta \tilde{\nabla} F_\lambda(w_t)$ , so that  $w_{t+1} = \Pi_{\mathcal{W}}(y_{t+1})$ . For now,

condition on the randomness of the algorithm (noise). By strong convexity, we have

$$\begin{aligned}
\widehat{F}_\lambda(w_t) - \widehat{F}_\lambda(\hat{w}) &\leq \langle \nabla \widehat{F}_\lambda(w_t), w_t - \hat{w} \rangle - \frac{\lambda}{2} \|w_t - \hat{w}\|^2 \\
&= \langle \widetilde{\nabla} F_\lambda(w_t), w_t - \hat{w} \rangle - \frac{\lambda}{2} \|w_t - \hat{w}\|^2 + \langle \nabla \widehat{F}_\lambda(w_t) - \widetilde{\nabla} F_\lambda(w_t), w_t - \hat{w} \rangle \\
&= \frac{1}{2\eta} (\|w_t - \hat{w}\|^2 + \|w_t - y_{t+1}\|^2 - \|y_{t+1} - \hat{w}\|^2) - \frac{\lambda}{2} \|w_t - \hat{w}\|^2 \\
&\quad + \langle \nabla \widehat{F}_\lambda(w_t) - \widetilde{\nabla} F_\lambda(w_t), w_t - \hat{w} \rangle \\
&= \frac{1}{2\eta} (\|w_t - \hat{w}\|^2(1 - \lambda\eta) - \|y_{t+1} - \hat{w}\|^2) + \frac{\eta}{2} \|\widetilde{\nabla} F_\lambda(w_t)\|^2 \\
&\quad + \langle \nabla \widehat{F}_\lambda(w_t) - \widetilde{\nabla} F_\lambda(w_t), w_t - \hat{w} \rangle \\
&\leq \frac{1}{2\eta} (\|w_t - \hat{w}\|^2(1 - \lambda\eta) - \|y_{t+1} - \hat{w}\|^2) + \frac{\eta}{2} \|\widetilde{\nabla} F_\lambda(w_t)\|^2 - \langle b_t + N_t, w_t - \hat{w} \rangle,
\end{aligned}$$

where we used non-expansiveness of projection and the definition of  $\widetilde{\nabla} F_\lambda(w_t)$  in the last line. Now, re-arranging this inequality and taking expectation, we get

$$\begin{aligned}
\mathbb{E}[\|w_{t+1} - \hat{w}\|^2] &\leq -2\eta \mathbb{E}[\widehat{F}_\lambda(w_t) - \widehat{F}_\lambda(\hat{w})] + \mathbb{E}\|w_t - \hat{w}\|^2(1 - \lambda\eta) + \eta^2 \mathbb{E}\|\widetilde{\nabla} F_\lambda(w_t)\|^2 \\
&\quad - 2\eta \mathbb{E}\langle b_t + N_t, w_t - \hat{w} \rangle \\
&\leq \mathbb{E}\|w_t - \hat{w}\|^2(1 - \lambda\eta) + \eta^2 \mathbb{E}\|\widetilde{\nabla} F_\lambda(w_t)\|^2 - 2\eta \mathbb{E}\langle b_t, w_t - \hat{w} \rangle,
\end{aligned}$$

by optimality of  $\hat{w}$  and the assumption that the noise  $N_t$  is independent of  $w_t - \hat{w}$  and zero mean. Also,

$$\begin{aligned}
\mathbb{E}\|\widetilde{\nabla} F_\lambda(w_t)\|^2 &\leq 2 \left( \mathbb{E}\|\nabla \widehat{F}_\lambda(w_t)\|^2 + \|b_t\|^2 + \mathbb{E}\|N_t\|^2 \right) \\
&\leq 2 \left( 2\widehat{r}_n(X)^2 + 2\lambda^2 D^2 + \widehat{B}^2 + \widehat{\Sigma}^2 \right),
\end{aligned}$$

where  $\widehat{B} := \sup_{t \in [T]} \|b_t\| \leq \frac{\widehat{r}_n(X)^{(k)}}{(k-1)C^{k-1}}$  and  $\widehat{\Sigma}^2 := \sup_{t \in [T]} \mathbb{E}\|N_t\|^2 = d\sigma^2$ . by Lemma 36. We also used Young's and Jensen's inequalities and the fact that  $\mathbb{E}N_t = 0$ . Further,

$$|\mathbb{E}\langle b_t, w_t - \hat{w} \rangle| \leq \frac{\widehat{B}^2}{\lambda} + \frac{\lambda}{4} \mathbb{E}\|w_t - \hat{w}\|^2,$$

by Young's inequality. Combining these pieces yields

$$\mathbb{E}\|w_{t+1} - \hat{w}\|^2 \leq \left(1 - \frac{\lambda\eta}{2}\right) \mathbb{E}\|w_t - \hat{w}\|^2 + 4\eta^2 \left(\widehat{r}_n(X)^2 + \lambda^2 D^2 + \widehat{B}^2 + \widehat{\Sigma}^2\right) + \frac{2\eta\widehat{B}^2}{\lambda}. \quad (\text{G.2})$$

Iterating Eq. (G.2) gives us

$$\begin{aligned}
\mathbb{E}\|w_T - \hat{w}\|^2 &\leq \left(1 - \frac{\lambda\eta}{2}\right)^T \|w_0 - \hat{w}\|^2 + \left[4\eta^2 \left(\hat{r}_n(X)^2 + \lambda^2 D^2 + \hat{B}^2 + \hat{\Sigma}^2\right) + \frac{2\eta\hat{B}^2}{\lambda}\right] \sum_{t=0}^{T-1} \left(1 - \frac{\lambda\eta}{2}\right)^t \\
&\leq \left(1 - \frac{\lambda\eta}{2}\right)^T \|w_0 - \hat{w}\|^2 + \left[4\eta^2 \left(\hat{r}_n(X)^2 + \lambda^2 D^2 + \hat{B}^2 + \hat{\Sigma}^2\right) + \frac{2\eta\hat{B}^2}{\lambda}\right] \left(\frac{2}{\lambda\eta}\right) \\
&= \left(1 - \frac{\lambda\eta}{2}\right)^T \|w_0 - \hat{w}\|^2 + \frac{8\eta}{\lambda} \left(\hat{r}_n(X)^2 + \lambda^2 D^2 + \hat{B}^2 + \hat{\Sigma}^2\right) + \frac{4\hat{B}^2}{\lambda^2} \\
&\leq \exp\left(-\frac{\lambda\eta T}{2}\right) \|w_0 - \hat{w}\|^2 + \frac{8\eta}{\lambda} \left(\hat{r}_n(X)^2 + \lambda^2 D^2 + \hat{B}^2 + \hat{\Sigma}^2\right) + \frac{4\hat{B}^2}{\lambda^2} \\
&\leq \exp\left(-\frac{\lambda\eta T}{2}\right) \|w_0 - \hat{w}\|^2 + \frac{8\eta}{\lambda} \left(\hat{r}_n(X)^2 + \lambda^2 D^2 + \hat{\Sigma}^2\right) + \frac{20\hat{B}^2}{\lambda^2},
\end{aligned}$$

since  $\eta \leq \frac{2}{\lambda}$ . Plugging in the bounds on  $\hat{B}$  and  $\hat{\Sigma}$  from Lemma 36 completes the proof.  $\square$

**Proposition 38** (Precise statement of Proposition 16). *Let  $f(\cdot, x)$  be convex for all  $x$  and grant Assumption 2 for  $k = 2$ . Suppose  $\mathcal{A} : \mathcal{X}^n \rightarrow \mathcal{W}$  is  $\alpha$ -on-average model stable. Then for any  $\zeta > 0$ , we have*

$$\mathbb{E}[F(\mathcal{A}(X)) - \hat{F}_X(\mathcal{A}(X))] \leq \frac{r^{(2)}}{2\zeta} + \frac{\zeta}{2}\alpha^2.$$

*Proof.* Let  $X, X', X^i$  be constructed as in Definition 15. We may write  $\mathbb{E}[F(\mathcal{A}(X)) - \hat{F}_X(\mathcal{A}(X))] = \mathbb{E}[\frac{1}{n} \sum_{i=1}^n f(\mathcal{A}(X^i), x_i) - f(\mathcal{A}(X), x_i)]$ , by symmetry and independence of  $x_i$  and  $\mathcal{A}(X^i)$  (c.f. (Lei and Ying, 2020, Equation B.2)). Then by convexity, we have

$$\begin{aligned}
\mathbb{E}[F(\mathcal{A}(X)) - \hat{F}_X(\mathcal{A}(X))] &\leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\langle \mathcal{A}(X^i) - \mathcal{A}(X), \nabla f(\mathcal{A}(X^i), x_i) \rangle] \\
&\leq \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \frac{\zeta}{2} \|\mathcal{A}(X^i) - \mathcal{A}(X)\|^2 + \frac{1}{2\zeta} \|\nabla f(\mathcal{A}(X^i), x_i)\|^2 \right].
\end{aligned}$$

Now, since  $\mathcal{A}(X^i)$  is independent of  $x_i$ , we have:

$$\begin{aligned}
\mathbb{E}\|\nabla f(\mathcal{A}(X^i), x_i)\|^2 &= \mathbb{E}[\mathbb{E}[\|\nabla f(\mathcal{A}(X^i), x_i)\|^2 | \mathcal{A}(X^i)]] \\
&\leq \sup_{w \in \mathcal{W}} \mathbb{E}[\|\nabla f(\mathcal{A}(X^i), x_i)\|^2 | \mathcal{A}(X^i) = w] \\
&= \sup_{w \in \mathcal{W}} \mathbb{E}[\|\nabla f(w, x_i)\|^2] \\
&\leq r^{(2)}.
\end{aligned}$$

Combining the above inequalities and recalling Definition 15 yields the result.  $\square$

To prove our excess risk bound for regularized ERM (i.e. Proposition 17), we require the following bound on the generalization error of ERM with strongly convex loss:

**Proposition 39.** *Let  $f(\cdot, x)$  be  $\lambda$ -strongly convex, and grant Assumption 2. Let  $\mathcal{A}(X) := \operatorname{argmin}_{w \in \mathcal{W}} \widehat{F}_X(w)$  be the ERM algorithm. Then,*

$$\mathbb{E}[F(\mathcal{A}(X)) - \widehat{F}_X(\mathcal{A}(X))] \leq \frac{2r^{(2)}}{\lambda n}.$$

*Proof.* We first bound the stability of ERM and then use Proposition 16 to get a bound on the generalization error. The beginning of the proof is similar to the proof of (Lei and Ying, 2020, Proposition D.6): Let  $X, X', X^i$  be constructed as in Definition 15. By strong convexity of  $\widehat{F}_{X^i}$  and optimality of  $\mathcal{A}(X^i)$ , we have

$$\frac{\lambda}{2} \|\mathcal{A}(X) - \mathcal{A}(X^i)\|^2 \leq \widehat{F}_{X^i}(\mathcal{A}(X)) - \widehat{F}_{X^i}(\mathcal{A}(X^i)),$$

which implies

$$\frac{1}{n} \sum_{i=1}^n \|\mathcal{A}(X) - \mathcal{A}(X^i)\|^2 \leq \frac{2}{\lambda n} \sum_{i=1}^n [\widehat{F}_{X^i}(\mathcal{A}(X)) - \widehat{F}_{X^i}(\mathcal{A}(X^i))]. \quad (\text{G.3})$$

Now, for any  $w \in \mathcal{W}$ ,

$$\begin{aligned} n \sum_{i=1}^n \widehat{F}_{X^i}(w) &= \sum_{i=1}^n [f(w, x'_i) + \sum_{j \neq i} f(w, x_j)] \\ &= (n-1)n\widehat{F}_X(w) + n\widehat{F}_{X'}(w). \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{n} \mathbb{E} \left[ \sum_{i=1}^n \widehat{F}_{X^i}(\mathcal{A}(X)) \right] &= \left( \frac{n-1}{n} \right) \mathbb{E} \widehat{F}_X(\mathcal{A}(X)) + \frac{1}{n} \mathbb{E} \widehat{F}_{X'}(\mathcal{A}(X)) \\ &= \left( \frac{n-1}{n} \right) \frac{1}{n} \mathbb{E} \left[ \sum_{i=1}^n \widehat{F}_{X^i}(\mathcal{A}(X^i)) \right] + \frac{1}{n} \mathbb{E} F(\mathcal{A}(X)), \end{aligned}$$

by symmetry and independence of  $\mathcal{A}(X)$  and  $X'$ . Re-arranging the above equality and using symmetry yields

$$\frac{1}{n} \mathbb{E} \left[ \sum_{i=1}^n \widehat{F}_{X^i}(\mathcal{A}(X)) - \widehat{F}_{X^i}(\mathcal{A}(X^i)) \right] = \frac{1}{n} \mathbb{E} [F(\mathcal{A}(X)) - \widehat{F}_X(\mathcal{A}(X))]. \quad (\text{G.4})$$

Combining Eq. (G.3) with Eq. (G.4) shows that ERM is  $\alpha$ -on-average model stable for

$$\alpha^2 = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \|\mathcal{A}(X) - \mathcal{A}(X^i)\|^2 \right] \leq \frac{2}{\lambda n} \mathbb{E} [F(\mathcal{A}(X)) - \widehat{F}_X(\mathcal{A}(X))]. \quad (\text{G.5})$$

The rest of the proof is where we depart from the analysis of (Lei and Ying, 2020) (which required smoothness of  $f(\cdot, x)$ ): Bounding the right-hand side of Eq. (G.5) by Proposition 16 yields

$$\alpha^2 \leq \frac{2}{\lambda n} \left( \frac{r^{(2)}}{2\zeta} + \frac{\zeta}{2} \alpha^2 \right)$$

for any  $\zeta > 0$ . Choosing  $\zeta = \frac{\lambda n}{2}$ , we obtain

$$\frac{\alpha^2}{2} \leq \frac{r^{(2)}}{\lambda n \zeta} = \frac{2r^{(2)}}{\lambda^2 n^2},$$

and  $\alpha^2 \leq \frac{4r^{(2)}}{\lambda^2 n^2}$ . Applying Proposition 16 again yields (for any  $\zeta' > 0$ )

$$\begin{aligned} \mathbb{E}[F(\mathcal{A}(X)) - \widehat{F}_X(\mathcal{A}(X))] &\leq \frac{r^{(2)}}{2\zeta'} + \frac{\zeta'}{2} \left( \frac{4r^{(2)}}{\lambda^2 n^2} \right) \\ &\leq \frac{2r^{(2)}}{\lambda n}, \end{aligned}$$

by the choice  $\zeta' = \frac{\lambda n}{2}$ .  $\square$

**Proposition 40** (Precise statement of Proposition 17). *Let  $f(\cdot, x)$  be convex,  $w_{i-1}, y \in \mathcal{W}$ , and  $\hat{w}_i := \operatorname{argmin}_{w \in \mathcal{W}} \widehat{F}_i(w)$ , where  $\widehat{F}_i(w) := \frac{1}{n_i} \sum_{j \in \mathcal{B}_i} f(w, x_j) + \frac{\lambda_i}{2} \|w - w_{i-1}\|^2$  (c.f. line 6 of Algorithm 4). Then,*

$$\mathbb{E}[F(\hat{w}_i)] - F(y) \leq \frac{2r^{(2)}}{\lambda_i n_i} + \frac{\lambda_i}{2} \|y - w_{i-1}\|^2,$$

where the expectation is over both the random draws of  $X$  from  $\mathcal{D}$  and  $\mathcal{B}_i$  from  $X$ .

*Proof.* Denote the regularized population loss by  $G_i(w) := \mathbb{E}[\widehat{F}_i(w)] = F(w) + \frac{\lambda_i}{2} \|w - w_{i-1}\|^2$ . By Proposition 39, we have

$$\mathbb{E}[G_i(\hat{w}_i) - \widehat{F}_i(\hat{w}_i)] \leq \frac{2r^{(2)}}{\lambda_i n_i}.$$

Thus,

$$\begin{aligned} \frac{\lambda_i}{2} \mathbb{E} \|\hat{w}_i - w_{i-1}\|^2 + \mathbb{E} F(\hat{w}_i) &= \mathbb{E} G_i(\hat{w}_i) \\ &\leq \frac{2r^{(2)}}{\lambda_i n_i} + \mathbb{E}[\widehat{F}_i(\hat{w}_i)] \\ &\leq \frac{2r^{(2)}}{\lambda_i n_i} + \frac{\lambda_i}{2} \|y - w_{i-1}\|^2 + F(y), \end{aligned} \tag{G.6}$$

since  $\mathbb{E}[\widehat{F}_i(\hat{w}_i)] = \mathbb{E}[\min_{w \in \mathcal{W}} \widehat{F}_i(w)] \leq \min_{w \in \mathcal{W}} \mathbb{E}[\widehat{F}_i(w)] = \min_{w \in \mathcal{W}} G_i(w) \leq \frac{\lambda_i}{2} \|y - w_{i-1}\|^2 + F(y)$ . Subtracting  $F(y)$  from both sides of Eq. (G.6) completes the proof.  $\square$

We are ready to state and prove the precise form of Theorem 12, using the notation of Appendix G.1:

**Theorem 41** (Precise statement of Theorem 12). *Grant Assumption 1. Let  $f(\cdot, x)$  be convex and let  $\varepsilon \leq \sqrt{d}$ . Then, there are algorithmic parameters such that Algorithm 4 is  $\frac{\varepsilon^2}{2}$ -zCDP, and has excess risk*

$$\mathbb{E}F(w_l) - F^* \lesssim \widetilde{R}_{2k,n} D \left( \frac{1}{\sqrt{n}} + \left( \frac{\sqrt{d \ln(n)}}{\varepsilon n} \right)^{\frac{k-1}{k}} \right).$$

Moreover, this excess risk is attained in  $\widetilde{\mathcal{O}}(n^{2+1/k})$  subgradient evaluations.

*Proof.* We choose  $\sigma_i^2 = \frac{4C_i^2 T_i}{n_i^2 \varepsilon^2}$  for  $C_i$  and  $T_i$  to be determined exactly later. Note that for  $\lambda_i$  and  $\eta_i$  defined in Algorithm 4, we have  $\eta_i \leq \frac{2}{\lambda_i}$  for all  $i \in [l]$ .

**Privacy:** Since the batches  $\{\mathcal{B}_i\}_{i=1}^l$  are disjoint, it suffices (by parallel composition (McSherry, 2009)) to show that  $w_i$  (produced by  $T_i$  iterations of Algorithm 3 in line 7 of Algorithm 4)

is  $\frac{\epsilon^2}{2}$ -zCDP for all  $i \in [l]$ . With clip threshold  $C_i$  and batch size  $n_i$ , the  $\ell_2$  sensitivity of the clipped subgradient update is bounded by  $\Delta = \sup_{w, X \sim X'} \frac{1}{n_i} \left\| \sum_{j=1}^{n_i} \Pi_{C_i}(\nabla f(w, x_j)) - \Pi_{C_i}(\nabla f(w, x'_j)) \right\| = \frac{1}{n_i} \sup_{w, x, x'} \left\| \Pi_{C_i}(\nabla f(w, x)) - \Pi_{C_i}(\nabla f(w, x')) \right\| \leq \frac{2C_i}{n_i}$ . (Note that the terms arising from regularization cancel out.) Thus, by Proposition 23, conditional on the previous updates  $w_{1:i}$ , the  $(i+1)$ -st update in line 5 of Algorithm 3 satisfies  $\frac{\epsilon^2}{2T_i}$ -zCDP. Hence, Lemma 24 implies that  $w_i$  (in line 7 of Algorithm 4) is  $\frac{\epsilon^2}{2}$ -zCDP.

**Excess risk:** Recall Chebyshev's inequality:

$$\mathbb{P}\left(Y \geq t\mathbb{E}[Y^k]^{1/k}\right) \leq \frac{1}{t^k}$$

for any non-negative random variable  $Y$ ,  $t \geq 0$ , and  $k \geq 2$ . This implies

$$\mathbb{P}(f(\cdot, x) \text{ is not } s\text{-Lipschitz on } \mathcal{W}) = \mathbb{P}\left(\sup_{w \in \mathcal{W}} \|\nabla f(w, x)\| > s\right) \leq \frac{\tilde{r}_k^k}{s^k}$$

for any  $s \geq 0$ . Thus, for  $i \in [l]$ , the following event holds with probability at least  $1 - \frac{1}{t_i^k}$  (over the random draw of  $\mathcal{B}_i$ ):

$$\begin{aligned} \widehat{F}_i(\hat{w}_i) &= \frac{1}{n_i} \sum_{j \in \mathcal{B}_i} f(\hat{w}_i, x_j) + \frac{\lambda_i}{2} \|\hat{w}_i - w_{i-1}\|^2 \leq \widehat{F}_i(w_{i-1}) = \frac{1}{n_i} \sum_{j \in \mathcal{B}_i} f(w_{i-1}, x_j) \\ &\implies \frac{\lambda_i}{2} \|\hat{w}_i - w_{i-1}\|^2 \leq t_i \tilde{r}_k \|\hat{w}_i - w_{i-1}\| \\ &\implies \|\hat{w}_i - w_{i-1}\| \leq \frac{2t_i \tilde{r}_k}{\lambda_i}, \end{aligned}$$

by definition of  $\hat{w}_i$  and (with high probability)  $s := t_i \tilde{r}_k$ -Lipschitz continuity of  $\widehat{F}_i(\cdot)$ . Thus, for  $i \in [l]$ , if we choose  $D_i \geq \frac{2\tilde{r}_k t_i}{\lambda_i}$ , then  $\hat{w}_i \in \mathcal{W}_i$  with probability at least  $1 - \frac{1}{t_i^k}$ . By a union bound, it suffices to choose  $t_i = (2^i \sqrt{n})^{1/k}$  to ensure

$$\mathbb{P}(\exists i \in [l], \hat{w}_i \notin \mathcal{W}_i) \leq \sum_{i=1}^{\log_2(n)} \frac{1}{t_i^k} \leq \frac{1}{\sqrt{n}}.$$

Therefore,  $D_i = \frac{4\tilde{r}_k 2^{i/k} \sqrt{n}^{1/k}}{\lambda_i}$  ensures that  $\hat{w}_i \in \mathcal{W}_i$  for all  $i \in [l]$  with probability at least  $1 - \frac{1}{\sqrt{n}}$ . Conditional on the event  $\hat{w}_i \in \mathcal{W}_i$  for all  $i \in [l]$ , we have by Lemma 14:

$$\mathbb{E}\|w_i - \hat{w}_i\|^2 \leq \exp\left(-\frac{\lambda_i \eta_i T_i}{2}\right) \|w_{i-1} - \hat{w}_i\|^2 + \frac{8\eta_i}{\lambda_i} \left(\widehat{r}_{n_i}(\mathcal{B}_i)^{(2)} + \lambda_i^2 D_i^2 + d\sigma_i^2\right) + \frac{20}{\lambda_i^2} \left(\frac{\widehat{r}_{n_i}(\mathcal{B}_i)^{(k)}}{(k-1)C_i^{k-1}}\right)^2,$$

conditional on  $w_{i-1}$  and the draws of  $X \sim \mathcal{D}^n$  and  $\mathcal{B}_i \sim X^{n_i}$ . Denote  $p := 1 + 1/k$ . Taking expectation over the random sampling yields

$$\mathbb{E}\|w_i - \hat{w}_i\|^2 \leq \exp\left(-\frac{\lambda_i \eta_i T_i}{2}\right) \|w_{i-1} - \hat{w}_i\|^2 + \frac{8\eta_i}{\lambda_i} \left(\widehat{e}_{n_i}^{(2)} + \lambda_i^2 D_i^2 + d\sigma_i^2\right) + \frac{20}{\lambda_i^2} \frac{\widehat{e}_{n_i}^{(2k)}}{C_i^{2k-2}(k-1)^2},$$

where  $d\sigma_i^2 \leq \frac{4dC_i^2T_i}{n_i^2\varepsilon^2}$ . Choosing  $T_i = \frac{1}{\lambda_i\eta_i} \ln\left(\frac{D^2\lambda_i}{d\sigma_i^2\eta_i}\right) \lesssim n_i^p \ln(n)$  and  $\eta$  to be determined later (polynomial in  $n$ ), we get

$$\begin{aligned} \mathbb{E}\|w_i - \hat{w}_i\|^2 &\lesssim \frac{\eta_i}{\lambda_i} \left( \left( \tilde{r}_k 2^{i/k} \sqrt{n}^{1/k} \right)^2 + d\sigma_i^2 \right) + \frac{\tilde{e}_{n_i}^{(2k)}}{\lambda_i^2 C_i^{2k-2}} \\ &\lesssim \eta_i^2 n_i^p \left( \left( \tilde{r}_k 2^{i/k} \sqrt{n}^{1/k} \right)^2 + d\sigma_i^2 \right) + \frac{\eta_i^2 n_i^{2p} \tilde{e}_{n_i}^{(2k)}}{C_i^{2k-2}} \\ &\lesssim \left( \frac{\eta^2 n^p}{16^i 2^{ip}} \left( \left( \tilde{r}_k 2^{i/k} \sqrt{n}^{1/k} \right)^2 + \frac{dC_i^2 T_i}{\varepsilon^2 n_i^2} + \frac{n^p \tilde{e}_{n_i}^{(2k)}}{C_i^{2k-2} 2^{ip}} \right) \right). \end{aligned} \quad (\text{G.7})$$

Note that under Assumption 1,  $F$  is  $L$ -Lipshitz, where  $L = \sup_{w \in \mathcal{W}} \|\nabla F(w)\| \leq \tilde{r}_k$  by Jensen's inequality. Now, following the strategy used in the proof of (Feldman et al., 2020a, Theorem 4.4), we write

$$\mathbb{E}F(w_l) - F(w^*) = \mathbb{E}[F(w_l) - F(\hat{w}_l)] + \sum_{i=1}^l \mathbb{E}[F(\hat{w}_i) - F(\hat{w}_{i-1})],$$

where  $\hat{w}_0 := w^*$ . Using Eq. (G.7), the first term can be bounded as follows:

$$\begin{aligned} \mathbb{E}[F(w_l) - F(\hat{w}_l)] &\leq L \sqrt{\mathbb{E}\|w_l - \hat{w}_l\|^2} \\ &\lesssim L \sqrt{\eta_l^2 \left( \left( \tilde{r}_k 2^{l/k} \sqrt{n}^{1/k} \right)^2 + \frac{C_l^2 d}{\varepsilon^2} + \frac{\tilde{e}_{n_l}^{(2k)}}{C_l^{2k-2}} \right)} \\ &\lesssim L \left[ \frac{\eta}{n^2} \left( \tilde{r}_k 2^{l/k} \sqrt{n}^{1/k} + \frac{\sqrt{d} C_l}{\varepsilon} + \frac{\tilde{r}_{2k}^k}{C_l^{k-1}} \right) \right] \\ &\lesssim L \left[ \frac{\eta}{n^2} \left( \tilde{r}_k n^{3/2k} + \tilde{r}_{2k} \left( \frac{\sqrt{d}}{\varepsilon} \right)^{(k-1)/k} \right) \right] \end{aligned}$$

if we choose  $C_l = \tilde{r}_{2k} \left( \frac{\varepsilon}{\sqrt{d}} \right)^{1/k}$ . Therefore,

$$\mathbb{E}[F(w_l) - F(\hat{w}_l)] \lesssim \tilde{R}_{2k,n} D \left( \frac{1}{\sqrt{n}} + \left( \frac{\sqrt{d \ln(n)}}{\varepsilon n} \right)^{\frac{k-1}{k}} \right), \quad (\text{G.8})$$

if we choose

$$\eta \lesssim \frac{\tilde{R}_{2k,n} D n^2}{L} \min \left( \frac{1}{\tilde{r}_k n^{3/2k}}, \frac{1}{\tilde{r}_{2k}} \left( \frac{\varepsilon}{\sqrt{d}} \right)^{(k-1)/k} \right) \left( \frac{1}{\sqrt{n}} + \left( \frac{\sqrt{d \ln(n)}}{\varepsilon n} \right)^{\frac{k-1}{k}} \right) =: \eta_A.$$

Next, Proposition 17 implies

$$\mathbb{E}[F(\hat{w}_i) - F(\hat{w}_{i-1})] \leq \frac{2r^2}{\lambda_i n_i} + \frac{\lambda_i}{2} \mathbb{E}\|\hat{w}_{i-1} - w_{i-1}\|^2$$

for all  $i \in [l]$ . Hence

$$\begin{aligned} \sum_{i=1}^l \mathbb{E}[F(\hat{w}_i) - F(\hat{w}_{i-1})] &\lesssim \frac{r^2}{\lambda_1 n_1} + \lambda_1 D^2 + \sum_{i=2}^l \left[ \frac{r^2}{\lambda_i n_i} + \lambda_i \eta_i^2 \left( n_i^p \left( \left( 2^{i/k} \sqrt{n}^{1/k} \tilde{r}_k \right)^2 + d\sigma_i^2 \right) + \frac{n_i^{2p} \tilde{e}_{n_i}^{(2k)}}{C_i^{2k-2}} \right) \right] \\ &\lesssim r^2 \eta n^{p-1} + \frac{D^2}{\eta n^p} + \sum_{i=2}^l r^2 \eta_i n_i^{p-1} + \sum_{i=2}^l \frac{\eta_i}{n_i^p} \left( n_i^p \left( \left( 2^{i/k} \sqrt{n}^{1/k} \tilde{r}_k \right)^2 + \frac{dC_i^2 T_i}{\varepsilon^2 n_i^2} \right) + \frac{n_i^{2p} \tilde{e}_{n_i}^{(2k)}}{C_i^{2k-2}} \right) \\ &\lesssim r^2 \eta n^{p-1} + \frac{D^2}{\eta n^p} + \sum_{i=2}^l \eta_i \left( \left( 2^{i/k} \sqrt{n}^{1/k} \tilde{r}_k \right)^2 + \frac{dC_i^2 n_i^p \ln(n)}{\varepsilon^2 n_i^2} + \frac{n_i^p \tilde{e}_{n_i}^{(2k)}}{C_i^{2k-2}} \right). \end{aligned}$$

Choosing  $C_i = \tilde{r}_{2k, n_i} \left( \frac{\varepsilon n_i}{\sqrt{d \ln(n)}} \right)^{1/k}$  approximately equalizes the two terms above involving  $C_i$  and we get

$$\begin{aligned} \sum_{i=1}^l \mathbb{E}[F(\hat{w}_i) - F(\hat{w}_{i-1})] &\lesssim r^2 \eta n^{p-1} + \frac{D^2}{\eta n^p} + \eta \tilde{r}_k^2 n^{1/k} + \eta \sum_{i=2}^l 4^{-i} n_i^p \tilde{r}_{2k, n_i}^2 \left( \frac{d \ln(n)}{\varepsilon^2 n_i^2} \right)^{\frac{k-1}{k}} \\ &\lesssim \eta \left[ r^2 n^{p-1} + \tilde{r}_k^2 n^{1/k} + \tilde{R}_{2k, n}^2 n^p \left( \frac{d \ln(n)}{\varepsilon^2 n^2} \right)^{\frac{k-1}{k}} \right] + \frac{D^2}{\eta n^p}. \end{aligned}$$

Now, choosing

$$\eta = \min \left( \eta_A, \frac{D}{n^{p/2}} \min \left\{ \frac{1}{rn^{(p-1)/2}}, \frac{1}{\tilde{r}_k n^{1/2k}}, \frac{1}{\tilde{R}_{2k, n} n^{p/2}} \left( \frac{\varepsilon n}{\sqrt{d \ln(n)}} \right)^{(k-1)/k} \right\} \right)$$

yields

$$\begin{aligned} \sum_{i=1}^l \mathbb{E}[F(\hat{w}_i) - F(\hat{w}_{i-1})] &\lesssim \tilde{R}_{2k, n} D \left( \frac{1}{\sqrt{n}} + \left( \frac{\sqrt{d \ln(n)}}{\varepsilon n} \right)^{\frac{k-1}{k}} \right) + \frac{\tilde{r}_k D n^{1/2k}}{n^{p/2}} + \frac{D^2}{\eta_A n^p} \\ &\lesssim \tilde{R}_{2k, n} D \left( \frac{1}{\sqrt{n}} + \left( \frac{\sqrt{d \ln(n)}}{\varepsilon n} \right)^{\frac{k-1}{k}} \right). \end{aligned}$$

Combining the above pieces, we obtain the desired excess risk bound conditional on the high-probability event that  $\hat{w}_i \in \mathcal{W}_i$  for all  $i \in [l]$ . Finally, by the law of total expectation, we have (unconditionally)

$$\begin{aligned} \mathbb{E}[F(w_l) - F(w^*)] &\lesssim \tilde{R}_{2k, n} D \left( \frac{1}{\sqrt{n}} + \left( \frac{\sqrt{d \ln(n)}}{\varepsilon n} \right)^{\frac{k-1}{k}} \right) + \frac{LD}{\sqrt{n}} \\ &\lesssim \tilde{R}_{2k, n} D \left( \frac{1}{\sqrt{n}} + \left( \frac{\sqrt{d \ln(n)}}{\varepsilon n} \right)^{\frac{k-1}{k}} \right). \end{aligned}$$

**Subgradient complexity:** Our choice of  $T_i = \tilde{\Theta} \left( \frac{1}{\lambda_i \eta_i} \right) \lesssim n_i^p \ln(n)$  implies that Algorithm 4 uses  $\sum_{i=1}^l n_i T_i \lesssim \ln(n) n^{p+1} = \ln(n) n^{2+1/k}$  subgradient evaluations.  $\square$

**Remark 42** (Details of Remark 13). *If one desires  $(\varepsilon, \delta)$ -DP or  $(\varepsilon, \delta)$ -SDP instead of zCDP, then the gradient complexity of Algorithm 4 can be improved to  $\mathcal{O}(n^{p+\frac{1}{2}} \ln(n)) = \mathcal{O}(n^{\frac{3}{2}+1/k} \ln(n))$  by using Clipped Noisy Stochastic Subgradient Method instead of Algorithm 3 as the subroutine in line 7 of Algorithm 4. Choosing batch sizes  $m_i \approx \sqrt{n_i} < n_i$  in this subroutine (and increasing  $\sigma_i^2$  by a factor of  $\mathcal{O}(\log(1/\delta))$ ) ensures  $(\varepsilon, \delta)$ -DP by (Abadi et al., 2016, Theorem 1) via privacy amplification by subsampling. The same excess risk bounds hold for any minibatch size  $m_i \in [n_i]$ , as the proof of Theorem 12 shows.*

**G.3. The Strongly Convex Case (Section 5.2).** Our algorithm is an instantiation of the meta-algorithm described in (Feldman et al., 2020a): Initialize  $w_0 \in \mathcal{W}$ . For  $j \in [M] := \lceil \log_2(\log_2(n)) \rceil$ , let  $N_j = 2^{j-2}n/\log_2(n)$ ,  $\mathcal{C}_j = \left\{ \sum_{h<j} N_h + 1, \dots, \sum_{h \leq j} N_h \right\}$ , and let  $w_j$  be the output of Algorithm 4 run with input data  $X_j = (x_s)_{s \in \mathcal{C}_j}$  initialized at  $w_{j-1}$ . Output  $w_M$ . Assume without loss of generality that  $N_j = 2^p$  for some  $p \in \mathbb{N}$ . Then, with the notation of Appendix G.1, we have the following guarantees:

**Theorem 43** (Precise Statement of Theorem 18). *Grant Assumption 1. Let  $\varepsilon \leq \sqrt{d}$  and  $f(\cdot, x)$  be  $\mu$ -strongly convex. Then, there is a polynomial-time  $\frac{\varepsilon^2}{2}$ -zCDP algorithm  $\mathcal{A}$  based on Algorithm 4 with excess risk*

$$\mathbb{E}F(\mathcal{A}(X)) - F^* \lesssim \frac{\tilde{R}_{2k, n/4}^2}{\mu} \left( \frac{1}{n} + \left( \frac{\sqrt{d \ln(n)}}{\varepsilon n} \right)^{\frac{2k-2}{k}} \right).$$

*Proof. Privacy:* Since the batches  $X_j$  used in each phase of the algorithm are disjoint and Algorithm 4 is  $\frac{\varepsilon^2}{2}$ -zCDP, privacy of the algorithm follows from parallel composition of DP (McSherry, 2009).

**Excess risk:** Note that  $N_j$  samples are used in phase  $j$  of the algorithm. For  $j \geq 0$ , let  $D_j^2 = \mathbb{E}[\|w_j - w^*\|^2]$  and  $\Delta_j = \mathbb{E}[F(w_j) - F^*]$ . By strong convexity, we have  $D_j^2 \leq \frac{2\Delta_j}{\mu}$ . Also,

$$\begin{aligned} \Delta_{j+1} &\leq a\tilde{R}_{2k, N_j} D_j \left( \frac{1}{\sqrt{N_j}} + \left( \frac{\sqrt{d \ln(N_j)}}{\varepsilon N_j} \right)^{\frac{k-1}{k}} \right) \\ &\leq a\tilde{R}_{2k, N_j} \sqrt{\frac{2\Delta_j}{\mu}} \left( \frac{1}{\sqrt{N_j}} + \left( \frac{\sqrt{d \ln(N_j)}}{\varepsilon N_j} \right)^{\frac{k-1}{k}} \right) \end{aligned} \quad (\text{G.9})$$

for an absolute constant  $a \geq 1$ , by Theorem 12. Denote  $E_j = \left[ a\tilde{R}_{2k, N_j} \sqrt{\frac{2}{\mu}} \left( \frac{1}{\sqrt{N_j}} + \left( \frac{\sqrt{d \ln(N_j)}}{\varepsilon N_j} \right)^{\frac{k-1}{k}} \right) \right]^2$ .

Then since  $N_j = 2N_{j+1}$ , we have

$$\begin{aligned} \frac{E_j}{E_{j+1}} &\leq 4 \left( \frac{\tilde{R}_{2k, N_j}}{\tilde{R}_{2k, N_{j+1}}} \right)^2 \\ &\leq 8, \end{aligned} \quad (\text{G.10})$$

where the second inequality holds because for any  $m = 2^q$ , we have:

$$\tilde{R}_{2k,m/2}^2 = \sum_{i=1}^{\log_2(m)-1} 2^{-i} \tilde{r}_{2k,2^{-(i+1)m}}^2 = \sum_{i=2}^{\log_2(m)} 2^{-(i-1)} \tilde{r}_{2k,2^{-im}}^2 = 2 \sum_{i=2}^{\log_2(m)} 2^{-i} \tilde{r}_{2k,2^{-im}}^2 \leq 2 \tilde{R}_{2k,m}^2.$$

Now, Eq. (G.10) implies that Eq. (G.9) can be re-arranged as

$$\frac{\Delta_{j+1}}{64E_{j+1}} \leq \sqrt{\frac{\Delta_j}{64E_j}} \leq \left(\frac{\Delta_0}{64E_0}\right)^{1/2^{j+1}}. \quad (\text{G.11})$$

Further, if  $M \geq \log \log \left(\frac{\Delta_0}{E_0}\right)$ , then

$$\frac{\Delta_M}{64E_M} \leq \left(\frac{\Delta_0}{64E_0}\right)^{1/2^M} \leq \left(\frac{\Delta_0}{64E_0}\right)^{1/\log(\Delta_0/E_0)} \leq 2^A \left(\frac{1}{64}\right)^{1/\log(\Delta_0/E_0)} \leq 2^A,$$

for an absolute constant  $A > 0$ , since  $\Delta_0 \leq \frac{2L^2}{\mu}$  and  $E_0 \geq \frac{2L^2}{\mu n}$  implies  $\Delta_0/E_0 = \frac{n}{a^2} \leq n$  and  $\frac{1}{\log(\Delta_0/E_0)} = \frac{1}{\log(n)-2\log(a)} \leq \frac{A}{\log(n)}$  for some  $A > 0$ , so that  $\left(\frac{\Delta_0}{E_0}\right)^{1/\log(\Delta_0/E_0)} \leq n^{A/\log(n)} \leq 2^A$ . Therefore,

$$\Delta_M \leq 2^A 64E_M = \mathcal{O} \left( \frac{\tilde{R}_{2k,n/4}^2}{\mu} \left( \frac{1}{n} + \left( \frac{\sqrt{d \ln(n)}}{\varepsilon n} \right)^{\frac{2k-2}{k}} \right) \right),$$

since  $N_M = n/4$ . □

**G.4. Asymptotic Upper Bounds Under Assumptions 2 and 3.** We first recall the notion of *subexponential* distribution:

**Definition 44** (Subexponential Distribution). *A random variable  $Y$  is subexponential if there is an absolute constant  $s > 0$  such that  $\mathbb{P}(|Y| \geq t) \leq 2 \exp(-\frac{t}{s})$  for all  $t \geq 0$ . For subexponential  $Y$ , we define  $\|Y\|_{\psi_1} := \inf \{s > 0 : \mathbb{P}(|Y| \geq t) \leq 2 \exp(-\frac{t}{s}) \forall t \geq 0\}$ .*

Essentially all (heavy-tailed) distributions that arise in practice are subexponential (?).

Now, we establish asymptotic upper bounds for a broad subclass of the problem class considered in (Wang et al., 2020; Kamath et al., 2022): namely, *subexponential* stochastic subgradient distributions satisfying Assumption 2 or Assumption 3. In Theorem 45 below (which uses the notation of Appendix G.1), we give upper bounds under Assumption 2:

**Theorem 45.** *Let  $f(\cdot, x)$  be convex. Assume  $\tilde{r}_{2k} < \infty$  and  $Y_i = \|\nabla f(w, x_i)\|^{2k}$  is subexponential with  $E_n \geq \max_{i \in [n]} (\|Y_i\|_{\psi_1}) \forall w \in \mathcal{W}, \nabla f(w, x_i) \in \partial_w f(w, x_i)$ . Assume that for sufficiently large  $n$ , we have  $\sup_{w,x} \|\nabla f(w, x)\|^{2k} \leq n^q r^{(2k)}$  for some  $q \geq 1$  and  $\max \left( \frac{E_n}{r^{(2k)}}, \frac{E_n^2}{(r^{(2k)})^2} \right) \ln \left( \frac{3nD\beta}{4r_{2k}} \right) \leq \frac{n}{dq}$ , where  $\|\nabla f(w, x) - \nabla f(w', x)\| \leq \beta \|w - w'\|$  for all  $w, w' \in \mathcal{W}, x \in \mathcal{X}$ , and subgradients  $\nabla f(w, x) \in \partial_w f(w, x)$ . Then,  $\lim_{n \rightarrow \infty} \tilde{R}_{2k,n} \leq 4r_{2k}$ . Further, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ , the output of Algorithm 4 satisfies*

$$\mathbb{E}F(w_l) - F^* = \mathcal{O} \left( r_{2k} D \left( \frac{1}{\sqrt{n}} + \left( \frac{\sqrt{d \ln(n)}}{\varepsilon n} \right)^{\frac{k-1}{k}} \right) \right).$$

If  $f(\cdot, x)$  is  $\mu$ -strongly convex, then the output of algorithm  $\mathcal{A}$  (in Section 5.2) satisfies

$$\mathbb{E}F(\mathcal{A}(X)) - F^* = \mathcal{O} \left( \frac{r_{2k}^2}{\mu} \left( \frac{1}{n} + \left( \frac{\sqrt{d \ln(n)}}{\varepsilon n} \right)^{\frac{2k-2}{k}} \right) \right).$$

While a bound on  $\sup_{w,x} \|\nabla f(w, x)\|$  is needed in Theorem 45, it can grow as fast as any polynomial in  $n$  and only needs to hold for sufficiently large  $n$ . As  $n \rightarrow \infty$ , this assumption is usually satisfied. Likewise, Theorem 45 depends only logarithmically on the Lipschitz parameter of the subgradients  $\beta$ , so the result still holds up to constant factors if, say,  $\beta \leq n^p(r/D)$  as  $n \rightarrow \infty$  for some  $p \geq 1$ . Crucially, our excess risk bounds do not depend on  $L_f$  or  $\beta$ .

Asymptotic upper bounds for Assumption 3 are an immediate consequence of Lemma 21 combined with Theorem 45. Namely, under Assumption 3, the upper bounds in Theorem 45 hold with  $r_k$  replaced by  $\sqrt{d}\gamma_k^{1/k}$  (by Lemma 21).

*Proof of Theorem 45. Step One:* There exists  $N \in \mathbb{N}$  such that  $\tilde{r}_{2k,n}^2 \leq 16r_{2k}^2$  for all  $n \geq N$ . We will first use a covering argument to show that  $\hat{r}_n(X)^{(2k)}$  is upper bounded by  $2^{2k+1}r^{(2k)}$  with high probability. For any  $\alpha > 0$ , we may choose an  $\alpha$ -net with  $N_\alpha \leq \left(\frac{3D}{2\alpha}\right)^d$  balls centered around points in  $\mathcal{W}_\alpha = \{w_1, w_2, \dots, w_{N_\alpha}\} \subset \mathcal{W}$  such that for any  $w \in \mathcal{W}$  there exists  $i \in [N_\alpha]$  with  $\|w - w_i\| \leq \alpha$  (see e.g. (Kolmogorov and Tikhomirov, 1959) for the existence of such  $\mathcal{W}_\alpha$ ). For  $w \in \mathcal{W}$ , let  $\tilde{w}$  denote the element of  $\mathcal{W}_\alpha$  that is closest to  $w$ , so that  $\|w - \tilde{w}\| \leq \alpha$ . Now, for any  $X \in \mathcal{X}^n$ , we have

$$\begin{aligned} \hat{r}_n(X)^{(2k)} &= \sup_w \left\{ \frac{1}{n} \sum_{i=1}^n \|\nabla f(w, x_i) - \nabla f(\tilde{w}, x_i) + \nabla f(\tilde{w}, x_i)\|^{2k} \right\} \\ &\leq 2^{2k} \sup_w \left\{ \frac{1}{n} \sum_{i=1}^n \|\nabla f(w, x_i) - \nabla f(\tilde{w}, x_i)\|^{2k} + \|\nabla f(\tilde{w}, x_i)\|^{2k} \right\} \\ &\leq 2^{2k} \left[ \beta^{2k} \alpha^{2k} + \frac{1}{n} \max_{j \in [N_\alpha]} \sum_{i=1}^n \|\nabla f(w_j, x_i)\|^{2k} \right], \end{aligned}$$

where we used Cauchy-Schwartz and Young's inequality for the first inequality, and the assumption of  $\beta$ -Lipschitz subgradients plus the definition of  $\mathcal{W}_\alpha$  for the second inequality. Further,

$$\begin{aligned} \mathbb{P} \left( \frac{2^{2k}}{n} \max_{j \in [N_\alpha]} \sum_{i=1}^n \|\nabla f(w_j, x_i)\|^{2k} \geq 2^{2k+1}r^{(2k)} \right) &\leq N_\alpha \max_{j \in [N_\alpha]} \mathbb{P} \left( \sum_{i=1}^n \|\nabla f(w_j, x_i)\|^{2k} \geq 2^{2k+1}r^{(2k)} \right) \\ &\leq N_\alpha \exp \left( -n \min \left( \frac{r^{(2k)}}{E_n}, \frac{(r^{(2k)})^2}{E_n^2} \right) \right), \end{aligned}$$

s by a union bound and Bernstein's inequality (see e.g. (Vershynin, 2018, Corollary 2.8.3)). Choosing  $\alpha = \frac{2r_{2k}}{\beta}$  ensures that  $\mathbb{P}(2^{2k}\beta^{2k}\alpha^{2k} > 2^{2k+1}r^{(2k)}) = 0$  and hence (by union bound)

$$\begin{aligned} \mathbb{P}\left(\widehat{r}_n(X)^{(2k)} \geq 2^{2k+1}r^{(2k)}\right) &\leq N_\alpha \exp\left(-n \min\left(\frac{r^{(2k)}}{E_n}, \frac{(r^{(2k)})^2}{E_n^2}\right)\right) \\ &\leq \left(\frac{3D\beta}{4r_{2k}}\right)^d \exp\left(-n \min\left(\frac{r^{(2k)}}{E_n}, \frac{(r^{(2k)})^2}{E_n^2}\right)\right) \\ &\leq \frac{1}{n^q}, \end{aligned}$$

by the assumption on  $n$ . Next, we use this concentration inequality to derive a bound on  $\widetilde{e}_n^{(2k)}$ :

$$\begin{aligned} \widetilde{e}_n^{(2k)} &= \mathbb{E}\left[\widehat{r}_n(X)^{(2k)}\right] \leq \mathbb{E}\left[\widehat{r}_n(X)^{(2k)} \mid \widehat{r}_n(X)^{(2k)} \geq 2^{2k+1}r^{(2k)}\right] \frac{1}{n^q} + 2^{2k+1}r^{(2k)} \\ &\leq \frac{\sup_{w,x} \|\nabla f(w,x)\|^{2k}}{n^q} + 2^{2k+1}r^{(2k)} \\ &\leq (1 + 2^{2k+1})r^{(2k)}, \end{aligned}$$

for sufficiently large  $n$ . Thus,  $\widetilde{r}_{2k,n}^2 \leq 16r_{2k}^2$  for all sufficiently large  $n$ . This establishes Step One.

**Step Two:**  $\lim_{n \rightarrow \infty} \widetilde{R}_{2k,n} \leq 4r_{2k}$ .

For all  $n = 2^l, l, i \in \mathbb{N}$ , define  $h_n(i) = 2^{-i}\widetilde{r}_{2k,2^{-i}n}^2 \mathbb{1}_{\{i \in [\log_2(n)]\}}$ . Note that  $0 \leq h_n(i) \leq g(i) := 2^{-i}\widetilde{r}_{2k}^2$  for all  $n, i$ , and that  $\sum_{i=1}^{\infty} g(i) = \widetilde{r}_{2k}^2 < \infty$  (i.e.  $g$  is integrable with respect to the counting measure). Furthermore, the limit  $\lim_{n \rightarrow \infty} h_n(i) = 2^{-i} \lim_{n \rightarrow \infty} \widetilde{r}_{2k,2^{-i}n}^2$  exists since Lemma 35 implies that the sequence  $\{\widetilde{r}_{2k,2^{-i}n}^2\}_{n=1}^{\infty}$  is monotonic and bounded for every  $i \in \mathbb{N}$ . Thus, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \widetilde{R}_{2k,n}^2 &= \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} h_n(i) \\ &= \sum_{i=1}^{\infty} \lim_{n \rightarrow \infty} h_n(i) \\ &\leq \sum_{i=1}^{\infty} 2^{-i} \lim_{n \rightarrow \infty} \widetilde{r}_{2k,2^{-i}n}^2 \\ &\leq 16 \sum_{i=1}^{\infty} 2^{-i} r_{2k}^2 \\ &= 16r_{2k}^2, \end{aligned}$$

where the last inequality follows from **Step One**. Therefore,  $\lim_{n \rightarrow \infty} \widetilde{R}_{2k,n} \leq 4r_{2k}$ . By Theorem 12 and Theorem 18, this also implies the last two claims in Theorem 45.  $\square$

APPENDIX H. DETAILS AND PROOFS FOR SECTION 6: NON-CONVEX PROXIMAL PL  
LOSSES

Pseudocode for our algorithm for PPL losses is given in Algorithm 8. Algorithm 9 is a

---

**Algorithm 8:** Noisy Clipped Proximal SGD for Heavy-Tailed DP SO

---

- 1: **Input:** Data  $X \in \mathcal{X}^n$ ,  $T \leq n$ , stepsizes  $\{\eta_t\}_{t=0}^{T-1}$ .
  - 2: Initialize  $w_0 \in \mathcal{W}$ .
  - 3: **for**  $t \in \{0, 1, \dots, T-1\}$  **do**
  - 4:   Draw new batch  $\mathcal{B}_t$  (without replacement) of  $n/T$  samples from  $X$ .
  - 5:    $\tilde{\nabla}F_t^0(w_t) := \text{MeanOracle1}(\{\nabla f^0(w_t, x)\}_{x \in \mathcal{B}_t}; \frac{n}{T}; \frac{\varepsilon^2}{2})$
  - 6:    $w_{t+1} = \text{prox}_{\eta_t f^1}(w_t - \eta_t \tilde{\nabla}F_t^0(w_t))$
  - 7: **end for**
  - 8: **Output:**  $w_T$ .
- 

generalization of Algorithm 8 which allows for arbitrary `MeanOracle`. This will be useful for our analysis. Proposition 46 provides a convergence guarantee for Algorithm 9 in terms of

---

**Algorithm 9:** Generic Noisy Proximal SGD Framework for Heavy-Tailed DP SO

---

- 1: **Input:** Data  $X \in \mathcal{X}^n$ ,  $T \leq n$ , `MeanOracle` (and truncation/minibatch parameters), privacy parameter  $\rho = \varepsilon^2/2$ , stepsizes  $\{\eta_t\}_{t=0}^{T-1}$ .
  - 2: Initialize  $w_0 \in \mathcal{W}$ .
  - 3: **for**  $t \in \{0, 1, \dots, T-1\}$  **do**
  - 4:   Draw new batch  $\mathcal{B}_t$  (without replacement) of  $n/T$  samples from  $X$ .
  - 5:    $\tilde{\nabla}F_t^0(w_t) := \text{MeanOracle}(\{\nabla f^0(w_t, x)\}_{x \in \mathcal{B}_t}; \frac{n}{T}; \frac{\varepsilon^2}{2})$
  - 6:    $w_{t+1} = \text{prox}_{\eta_t f^1}(w_t - \eta_t \tilde{\nabla}F_t^0(w_t))$
  - 7: **end for**
  - 8: **Output:**  $w_T$ .
- 

the bias and variance of the `MeanOracle`.

**Proposition 46.** *Consider Algorithm 9 with biased, noisy stochastic gradients:  $\tilde{\nabla}F_t^0(w_t) = \nabla F^0(w_t) + b_t + N_t$ , and stepsize  $\eta = \frac{1}{2\beta}$ . Assume that the bias and noise (which can depend on  $w_t$  and the samples drawn) satisfy  $\|b_t\| \leq B$  (with probability 1),  $\mathbb{E}N_t = 0$ ,  $\mathbb{E}\|N_t\|^2 \leq \Sigma^2$  for all  $t \in [T-1]$ , and that  $\{N_t\}_{t=1}^T$  are independent. Assume further that  $F$  is  $\mu$ -Proximal-PL,  $F^0$  is  $\beta$ -smooth, and  $F(w_0) - F^* \leq \Delta$ . Then,*

$$\mathbb{E}F(w_T) - F^* \leq \left(1 - \frac{\mu}{2\beta}\right)^T \Delta + \frac{4(B^2 + \Sigma^2)}{\mu}.$$

*Proof.* Our proof extends the ideas in (Lowy et al., 2023) to generic *biased* and noisy gradients without using Lipschitz continuity of  $f$ . By  $\beta$ -smoothness, for any  $r \in [T-1]$ , we

have

$$\begin{aligned}
\mathbb{E}F(w_{r+1}) &= \mathbb{E}[F^0(w_{r+1}) + f^1(w_r) + f^1(w_{r+1}) - f^1(w_r)] \\
&\leq \mathbb{E} \left\{ F(w_r) + \left[ \langle \tilde{\nabla} F_r^0(w_r), w_{r+1} - w_r \rangle + \frac{\beta}{2} \|w_{r+1} - w_r\|^2 + f^1(w_{r+1}) - f^1(w_r) \right] \right\} \\
&\quad + \mathbb{E} \langle \nabla F^0(w_r) - \tilde{\nabla} F_r^0(w_r), w_{r+1} - w_r \rangle \\
&= \mathbb{E}F(w_r) + \mathbb{E} \left[ \langle \nabla F^0(w_r), w_{r+1} - w_r \rangle + \frac{\beta}{2} \|w_{r+1} - w_r\|^2 + f^1(w_{r+1}) - f^1(w_r) \right. \\
&\quad \left. + \langle b_r + N_r, w_{r+1} - w_r \rangle \right] - \mathbb{E} \langle b_r + N_r, w_{r+1} - w_r \rangle \\
&\leq \mathbb{E}F(w_r) + \mathbb{E} \left[ \langle \nabla F^0(w_r), w_{r+1} - w_r \rangle + \beta \|w_{r+1} - w_r\|^2 + f^1(w_{r+1}) - f^1(w_r) \right. \\
&\quad \left. + \langle b_r + N_r, w_{r+1} - w_r \rangle \right] + \frac{B^2 + \Sigma^2}{\beta}, \tag{H.1}
\end{aligned}$$

where we used Young's inequality to bound

$$-\mathbb{E} \langle b_r + N_r, w_{r+1} - w_r \rangle \leq \frac{B^2 + \Sigma^2}{\beta} + \frac{\beta}{2} \|w_{r+1} - w_r\|^2. \tag{H.2}$$

Next, we will bound the quantity

$$\mathbb{E} \left[ \langle \nabla F^0(w_r), w_{r+1} - w_r \rangle + \beta \|w_{r+1} - w_r\|^2 + f^1(w_{r+1}) - f^1(w_r) + \langle b_r + N_r, w_{r+1} - w_r \rangle \right].$$

Denote  $H_r^{\text{priv}}(y) := \langle \nabla F^0(w_r), y - w_r \rangle + \beta \|y - w_r\|^2 + f^1(y) - f^1(w_r) + \langle b_r + N_r, y - w_r \rangle$  and  $H_r(y) := \langle \nabla F^0(w_r), y - w_r \rangle + \beta \|y - w_r\|^2 + f^1(y) - f^1(w_r)$ . Note that  $H_r$  and  $H_r^{\text{priv}}$  are  $2\beta$ -strongly convex. Denote the minimizers of these two functions by  $y_*$  and  $y_*^{\text{priv}}$  respectively. Now, conditional on  $w_r$  and  $N_r + b_r$ , we claim that

$$H_r(y_*^{\text{priv}}) - H_r(y_*) \leq \frac{\|N_r + b_r\|^2}{2\beta}. \tag{H.3}$$

To prove Eq. (H.3), we will need the following lemma:

**Lemma 47.** (*Lowy and Razaviyayn, 2021, Lemma B.2*) *Let  $H(y), h(y)$  be convex functions on some convex closed set  $\mathcal{Y} \subseteq \mathbb{R}^d$  and suppose that  $H$  is  $2\beta$ -strongly convex. Assume further that  $h$  is  $L_h$ -Lipschitz. Define  $y_1 = \arg \min_{y \in \mathcal{Y}} H(y)$  and  $y_2 = \arg \min_{y \in \mathcal{Y}} [H(y) + h(y)]$ . Then  $\|y_1 - y_2\|_2 \leq \frac{L_h}{2\beta}$ .*

We apply Theorem 47 with  $H(y) := H_r(y)$ ,  $h(y) := \langle N_r + b_r, y \rangle$ ,  $L_h = \|N_r + b_r\|$ ,  $y_1 = y_*$ , and  $y_2 = y_*^{\text{priv}}$  to get

$$\|y_* - y_*^{\text{priv}}\| \leq \frac{\|N_r + b_r\|}{2\beta}.$$

On the other hand,

$$H_r^{\text{priv}}(y_*^{\text{priv}}) = H_r(y_*^{\text{priv}}) + \langle N_r + b_r, y_*^{\text{priv}} \rangle \leq H_r^{\text{priv}}(y_*) = H_r(y_*) + \langle N_r + b_r, y_* \rangle.$$

Combining these two inequalities yields

$$\begin{aligned} H_r(y_*^{\text{priv}}) - H_r(y_*) &\leq \langle N_r + b_r, y_* - y_*^{\text{priv}} \rangle \\ &\leq \|N_r + b_r\| \|y_* - y_*^{\text{priv}}\| \\ &\leq \frac{\|N_r + b_r\|^2}{2\beta}, \end{aligned} \tag{H.4}$$

as claimed. Also, note that  $w_{r+1} = y_*^{\text{priv}}$ . Hence

$$\begin{aligned} &\mathbb{E} [\langle \nabla F^0(w_r), w_{r+1} - w_r \rangle + \beta \|w_{r+1} - w_r\|^2 + f^1(w_{r+1}) - f^1(w_r) + \langle b_r + N_r, w_{r+1} - w_r \rangle] \\ &= \mathbb{E} \left[ \min_{y \in \mathbb{R}^d} H_r^{\text{priv}}(y) \right] \end{aligned}$$

satisfies

$$\mathbb{E} \left[ \min_{y \in \mathbb{R}^d} H_r^{\text{priv}}(y) \right] \leq \mathbb{E} \left[ \min_y \{ \langle \nabla F^0(w_r), y - w_r \rangle + \beta \|y - w_r\|^2 + f^1(y) - f^1(w_r) \} \right] + \frac{\Sigma^2 + B^2}{\beta} \tag{H.5}$$

$$\leq -\frac{\mu}{2\beta} \mathbb{E} [F(w_r) - F^*] + \frac{\Sigma^2 + B^2}{\beta}, \tag{H.6}$$

where we used the assumptions that  $F$  is  $\mu$ -PPL and  $F^0$  is  $2\beta$ -smooth in the last inequality. Plugging the above bounds back into Eq. (H.1), we obtain

$$\mathbb{E} F(w_{r+1}) \leq \mathbb{E} F(w_r) - \frac{\mu}{2\beta} [F(w_r) - F^*] + \frac{2(\Sigma^2 + B^2)}{\beta}, \tag{H.7}$$

whence

$$\mathbb{E} [F(w_{r+1}) - F^*] \leq \mathbb{E} [F(w_r) - F^*] \left(1 - \frac{\mu}{2\beta}\right) + \frac{2(\Sigma^2 + B^2)}{\beta}. \tag{H.8}$$

Using Eq. (H.8) recursively and summing the geometric series, we get

$$\mathbb{E} [F(w_T) - F^*] \leq \Delta \left(1 - \frac{\mu}{2\beta}\right)^T + \frac{4(\Sigma^2 + B^2)}{\mu}. \tag{H.9}$$

□

**Theorem 48** (Precise statement of Theorem 20). *Grant Assumption 2. Let  $\varepsilon > 0$  and assume  $F(w) = F^0(w) + f^1(w)$  is  $\mu$ -PPL for  $\beta$ -smooth  $F^0$ , with  $\kappa = \frac{\beta}{\mu} \leq n/\ln(n)$ . Then, there are parameters such that Algorithm 8 is  $\frac{\varepsilon^2}{2}$ -zCDP, and*

$$\mathbb{E} F(w_T) - F^* \lesssim \frac{r_k^2}{\mu} \left( \left( \frac{\sqrt{d}}{\varepsilon n} \kappa \ln(n) \right)^{\frac{2k-2}{k}} + \frac{\kappa \ln(n)}{n} \right).$$

*Proof.* We choose  $\sigma^2 = \frac{4C^2T^2}{\varepsilon^2n^2}$ .

**Privacy:** By parallel composition (since each sample is used only once) and the post-processing property of DP (since the iterates are deterministic functions of the output of `MeanOracle1`), it suffices to show that  $\tilde{\nabla} F_t(w_t)$  is  $\frac{\varepsilon^2}{2}$ -zCDP for all  $t \geq 0$ . By our choice of  $\sigma^2$  and Proposition 23,  $\tilde{\nabla} F_t(w_t)$  is  $\frac{\varepsilon^2}{2}$ -zCDP, since its sensitivity is bounded by  $\sup_{X \sim X', w} \frac{T}{n} \left\| \sum_{x \in \mathcal{B}_t} \Pi_C[\nabla f^0(w, x)] - \sum_{x' \in \mathcal{B}_t} \Pi_C[\nabla f^0(w, x')] \right\| \leq \frac{T}{n} \sup_{x, x', w} \|\Pi_C[\nabla f^0(w, x)] - \Pi_C[\nabla f^0(w, x')]\| \leq \frac{2CT}{n}$ .

**Excess risk:** For any iteration  $t \in [T]$ , denote the bias of `MeanOracle1` (Algorithm 1) by  $b_t := \mathbb{E}\tilde{\nabla}F_t(w_t) - \nabla F(w_t)$ , where  $\tilde{\nabla}F_t(w_t) = \tilde{\nu}$  in the notation of Algorithm 1. Also let  $\widehat{\nabla}F_t(w_t) := \hat{\nu}$  (in the notation of Lemma 6) and denote the noise by  $N_t = \tilde{\nabla}F_t(w_t) - \nabla F(w_t) - b_t = \tilde{\nabla}F_t(w_t) - \mathbb{E}\tilde{\nabla}F_t(w_t)$ . Then we have  $B := \sup_{t \in [T]} \|b_t\| \leq \frac{r^{(k)}}{(k-1)C^{k-1}}$  and  $\Sigma^2 := \sup_{t \in [T]} \mathbb{E}[\|N_t\|^2] \leq d\sigma^2 + \frac{r^{(2)}T}{n} \leq \frac{4dC^2T^2}{\varepsilon^2n^2} + \frac{r_2^2T}{n}$ , by Lemma 6. Plugging these bounds on  $B^2$  and  $\Sigma^2$  into Proposition 46, and choosing  $T = 2 \left\lceil \kappa \ln \left( \frac{\Delta\mu}{B^2 + \Sigma^2} \right) \right\rceil \lesssim \kappa \ln(n)$  where  $\Delta \geq F(w_0) - F^*$ , we have:

$$\mathbb{E}F(w_T) - F^* \leq \frac{5(B^2 + \Sigma^2)}{\mu} \leq \frac{5}{\mu} \left( \frac{2r^{(2)}T}{n} + \frac{2(r^{(k)})^2}{(k-1)^2C^{2k-2}} + \frac{2dC^2T^2}{\varepsilon^2n^2} \right),$$

for any  $C > 0$ . Choosing  $C = r_k \left( \frac{\varepsilon^2n^2}{dT^2} \right)^{1/2k}$  makes the last two terms in the above display equal, and we get

$$\mathbb{E}F(w_T) - F^* \lesssim \frac{1}{\mu} \left( \left( \frac{r_k^2\sqrt{d}}{\varepsilon n} \kappa \ln(n) \right)^{\frac{2k-2}{k}} + \frac{r_2^2\kappa \ln(n)}{n} \right),$$

as desired. □

## APPENDIX I. SHUFFLE DIFFERENTIALLY PRIVATE ALGORITHMS

In the next two subsections, we present two SDP algorithms for DP heavy-tailed mean estimation. The first is an SDP version of Algorithm 1 and the second is an SDP version of the coordinate-wise protocol of (Kamath et al., 2020, 2022). Both of our algorithms offer the same utility guarantees as their zCDP counterparts (up to logarithms). In particular, this implies that the upper bounds obtained in the main body of this paper can also be attained via SDP protocols that do not require individuals to trust any third party curator with their sensitive data (assuming the existence of a secure shuffler).

**I.1.  $\ell_2$  Clip Shuffle Private Mean Estimator.** For heavy-tailed SO problems satisfying Assumption 2, we propose using the SDP mean estimation protocol described in Algorithm 10. Algorithm 10 relies on the shuffle private vector summation protocol of (Cheu et al., 2021), which is given in Algorithm 11. The useful properties of Algorithm 11 are contained in Lemma 49.

---

**Algorithm 10:**  $\ell_2$  Clip ShuffleMeanOracle1( $\{x_i\}_{i=1}^s; C; (\varepsilon, \delta)$ )

---

- 1: **Input:**  $X = \{x_i\}_{i=1}^s$ ,  $x_i = (x_{i,1}, \dots, x_{i,d}) \in \mathbb{R}^d$ ,  $C > 0$ ,  $(\varepsilon, \delta) \in (\mathbb{R}_+ \times (0, 1/2))$ .
  - 2: **for**  $i \in [s]$  **do**
  - 3:    $z_i := \Pi_C(x_i)$ .
  - 4: **end for**
  - 5:  $\tilde{\nu} := \mathcal{P}_{\text{vec}}(\{z_i\}_{i=1}^s; s; C; (\varepsilon, \delta))$ .
  - 6: **Output:**  $\tilde{\nu}$ .
-

---

**Algorithm 11:**  $\mathcal{P}_{\text{vec}}$ , a shuffle private protocol for vector summation

---

- 1: **Input:** database of  $d$ -dimensional vectors  $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_s)$  with maximum norm bounded by  $C > 0$ ; privacy parameters  $(\varepsilon, \delta)$ .
  - 2: **procedure:** Local Randomizer  $\mathcal{R}_{\text{vec}}(\mathbf{x}_i)$
  - 3:   **for**  $j \in [d]$  **do**
  - 4:     Shift component to enforce non-negativity:  $\mathbf{w}_{i,j} \leftarrow \mathbf{x}_{i,j} + C$
  - 5:      $\mathbf{m}_j \leftarrow \mathcal{R}_{1D}(\mathbf{w}_{i,j})$
  - 6:   **end for**
  - 7:   Output labeled messages  $\{(j, \mathbf{m}_j)\}_{j \in [d]}$
  - 8: **end procedure**
  - 9: **procedure:** Analyzer  $\mathcal{A}_{\text{vec}}(\mathbf{y})$
  - 10:   **for**  $j \in [d]$  **do**
  - 11:     Run analyzer on coordinate  $j$ 's messages  $z_j \leftarrow \mathcal{A}_{1D}(\mathbf{y}_j)$
  - 12:     Re-center:  $o_j \leftarrow z_j - L$
  - 13:   **end for**
  - 14:   Output the vector of estimates  $\mathbf{o} = (o_1, \dots, o_d)$
  - 15: **end procedure**
- 

**Lemma 49.** (*Cheu et al., 2021, Theorem 3.2*) Let  $\varepsilon \leq 15, \delta \in (0, 1/2)$ ,  $d, s \in \mathbb{N}$  and  $C > 0$ . There are choices of parameters  $b, g, p$  for  $\mathcal{P}_{1D}$  such that for an input data set  $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_s)$  of vectors with maximum norm  $\|\mathbf{x}_i\| \leq C$ , the following holds:

- 1) Algorithm 11 is  $(\varepsilon, \delta)$ -SDP.
- 2)  $\mathcal{P}_{\text{vec}}(\mathbf{X})$  is an unbiased estimate of  $\sum_{i=1}^s \mathbf{x}_i$  with bounded variance

$$\mathbb{E} \left[ \left\| \mathcal{P}_{\text{vec}}(\mathbf{X}) - \sum_{i=1}^s \mathbf{x}_i \right\|^2 \right] = \mathcal{O} \left( \frac{dC^2}{\varepsilon^2} \ln^2(d/\delta) \right).$$

By the post-processing property of DP, we immediately obtain:

**Lemma 50** (Privacy, Bias, and Variance of Algorithm 10). Let  $\{z_i\}_{i=1}^s \sim \mathcal{D}^s$  have mean  $\mathbb{E}z_i = \nu$  and  $\mathbb{E}\|z_i\|^k \leq r^{(k)}$  for  $k \geq 2$ . Denote the noiseless average of clipped samples in Algorithm 10 by  $\hat{\nu} := \frac{1}{n} \sum_{i=1}^s \Pi_C(z_i)$ . Then, there exist algorithmic parameters such that Algorithm 10 is  $(\varepsilon, \delta)$ -SDP and such that the following bias and variance bounds hold:

$$\|\mathbb{E}\tilde{\nu} - \nu\| = \|\mathbb{E}\hat{\nu} - \nu\| \leq \mathbb{E}\|\hat{\nu} - \nu\| \leq \frac{r^{(k)}}{(k-1)C^{k-1}}, \quad (\text{I.1})$$

and

$$\mathbb{E}\|\tilde{\nu} - \mathbb{E}\tilde{\nu}\|^2 = \mathbb{E}\|\hat{\nu} - \mathbb{E}\hat{\nu}\|^2 = \mathcal{O} \left( \frac{dC^2 \ln^2(d/\delta)}{\varepsilon^2 s^2} + \frac{r^2}{s} \right). \quad (\text{I.2})$$

*Proof. Privacy:* The privacy claim is immediate from Lemma 49 and the post-processing property of DP (Dwork and Roth, 2014, Proposition 2.1).

**Bias:** The bias bound follows as in Lemma 6, since  $\mathcal{P}_{\text{vec}}$  is an unbiased estimator (by Lemma 49).

**Variance:** We have

$$\begin{aligned}
\mathbb{E}\|\tilde{\nu} - \mathbb{E}\tilde{\nu}\|^2 &= \mathbb{E}\|\tilde{\nu} - \mathbb{E}\hat{\nu}\|^2 \\
&= \mathbb{E}\|\tilde{\nu} - \hat{\nu}\|^2 + \mathbb{E}\|\hat{\nu} - \mathbb{E}\hat{\nu}\|^2 \\
&\lesssim \frac{dC^2 \ln^2(d/\delta)}{\varepsilon^2 s^2} + \frac{1}{s} \mathbb{E}\|\Pi_C(z_1) - \mathbb{E}\Pi_C(z_1)\|^2 \\
&\leq \frac{dC^2 \ln^2(d/\delta)}{\varepsilon^2 s^2} + \frac{1}{s} \mathbb{E}\|z_1 - \mathbb{E}z_1\|^2 \\
&\leq \frac{dC^2 \ln^2(d/\delta)}{\varepsilon^2 s^2} + \frac{r^{(2)}}{s},
\end{aligned}$$

where we used that the samples  $\{z_i\}_{i=1}^s$  are i.i.d., the variance bound in Lemma 49, and (Barber and Duchi, 2014, Lemma 4), which states that  $\mathbb{E}\|\Pi_C(X) - \mathbb{E}\Pi_C(X)\|^2 \leq \mathbb{E}\|X - \mathbb{E}X\|^2$  for any random vector  $X$ .  $\square$

**Remark 51.** Comparing Lemma 50 to Lemma 6, we see that the bias and variance of the two *MeanOracles* are the same up to logarithmic factors. Therefore, replacing Algorithm 1 by Algorithm 10 in our stochastic optimization algorithms yields SDP algorithms with excess risk that matches the bounds provided in this paper (via Algorithm 1) up to logarithmic factors.

**I.2. Coordinate-wise Shuffle Private Mean Estimation Oracle.** For SO problems satisfying Assumption 3, we propose Algorithm 13 as a shuffle private mean estimation oracle. Algorithm 13 is a shuffle private variation of Algorithm 12, which was employed by (Kamath et al., 2020, 2022).

---

**Algorithm 12:** Coordinate-wise Private `MeanOracle2`( $\{x_i\}_{i=1}^s; s; \tau; \frac{\varepsilon^2}{2}; m$ ) (Kamath et al., 2020, 2022)

---

- 1: **Input:**  $X = \{x_i\}_{i=1}^s$ ,  $x_i = (x_{i,1}, \dots, x_{i,d}) \in \mathbb{R}^d$ ,  $\varepsilon > 0, \tau > 0, m \in [s]$  such that  $m$  divides  $s$ .
  - 2: **for**  $j \in [d]$  **do**
  - 3:   Partition  $j$ -th coordinates of data into  $m$  disjoint groups of size  $s/m$ .
  - 4:   **for**  $i \in [m]$  **do**
  - 5:     Clip data in  $i$ -th group:  $Z_j^i := \left\{ \Pi_{[-\tau, \tau]}(x_{(i-1)\frac{s}{m}+1, j}), \dots, \Pi_{[-\tau, \tau]}(x_{i\frac{s}{m}, j}) \right\}$ .
  - 6:     Compute average of  $Z_j^i$ :  $\hat{\nu}_j^i := \frac{m}{s} \sum_{z \in Z_j^i} z$ .
  - 7:   **end for**
  - 8:   Compute median of group means:  $\hat{\nu}_j := \text{median}(\hat{\nu}_j^1, \dots, \hat{\nu}_j^m)$ .
  - 9:   Draw  $u \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_d)$ , with  $\sigma^2 = \frac{4\tau^2 m^2 d}{s^2 \varepsilon^2}$ .
  - 10: **end for**
  - 11: **Output:**  $\tilde{\nu} = (\hat{\nu}_1, \dots, \hat{\nu}_d) + u$ .
- 

The bias/variance and privacy properties of Algorithm 12 are summarized in Lemma 52.

**Lemma 52** (Privacy, Bias, and Variance of Algorithm 12, (Kamath et al., 2020, 2022)). *Let  $\{x_i\}_{i=1}^s \sim \mathcal{D}^s$  have mean  $\mathbb{E}x_i = \nu$ ,  $\|\nu\| \leq L$ , and  $\mathbb{E}|x_{i,j} - \nu_j|^k \leq \gamma_k$  for some  $k \geq 2$ . Denote*

by  $\hat{\nu}$  the output of the non-private algorithm that is identical to Algorithm 12 except without the added Gaussian noise. Then, Algorithm 12 with  $\sigma^2 = \frac{72\tau^2 m^2 d}{\varepsilon^2 s^2}$  is  $\frac{\varepsilon^2}{2}$ -zCDP. Further, if  $\tau \geq 2L$ , then there is  $m = \tilde{\mathcal{O}}(1) \in [s]$  such that:

$$\|\mathbb{E}\tilde{\nu} - \nu\| = \|\mathbb{E}\hat{\nu} - \mu\| \leq \mathbb{E}\|\hat{\nu} - \nu\| = \tilde{\mathcal{O}}\left(\sqrt{d}\left(\frac{\gamma_k^{1/k}}{\sqrt{s}} + \left(\frac{2}{\tau}\right)^{k-1} \gamma_k\right)\right) =: B, \quad (\text{I.3})$$

and

$$\mathbb{E}\|N\|^2 = \mathbb{E}\|\tilde{\nu} - \mathbb{E}\tilde{\nu}\|^2 = \mathbb{E}\|\tilde{\nu} - \mathbb{E}\hat{\nu}\|^2 = \tilde{\mathcal{O}}(B^2 + d\sigma^2). \quad (\text{I.4})$$

Algorithm	13:	Coordinate-wise	Shuffle	Private
ShuffleMeanOracle2	$(\{x_i\}_{i=1}^s; s; \tau; \varepsilon; \delta; m)$			
1:	<b>Input:</b> $X = \{x_i\}_{i=1}^s$ , $x_i = (x_{i,1}, \dots, x_{i,d}) \in \mathbb{R}^d$ , $\varepsilon > 0$ , $\delta \in (0, 1)$ , $m \in [s]$ .			
2:	<b>for</b> $j \in [d]$ <b>do</b>			
3:	$\varepsilon_j := \frac{\varepsilon}{4\sqrt{2d \ln(1/\delta_j)}}$ , $\delta_j := \frac{\delta}{2d}$ .			
4:	Partition $j$ -th coordinates of data into $m$ disjoint groups of size $s/m$ .			
5:	<b>for</b> $i \in [m]$ <b>do</b>			
6:	Clip $j$ -th coordinate of data in $i$ -th group: $Z_j^i := \left\{ \Pi_{[-\tau, \tau]}(x_{(i-1)\frac{s}{m}+1, j}), \dots, \Pi_{[-\tau, \tau]}(x_{i\frac{s}{m}, j}) \right\}$ .			
7:	Shift to enforce non-negativity: $Z_j^i \leftarrow Z_j^i + (\tau, \dots, \tau)$ .			
8:	Compute noisy average of $s/m$ scalars in $Z_j^i$ : $\tilde{\nu}_j^i := \frac{m}{s} \mathcal{P}_{1D}(Z_j^i; \frac{s}{m}; 2\tau; \varepsilon_j, \delta_j)$ .			
9:	Re-center: $\tilde{\nu}_j^i \leftarrow \tilde{\nu}_j^i - \tau$ .			
10:	<b>end for</b>			
11:	Compute median of noisy means: $\tilde{\nu}_j := \text{median}(\tilde{\nu}_j^1, \dots, \tilde{\nu}_j^m)$ .			
12:	<b>end for</b>			
13:	<b>Output:</b> $\tilde{\nu} = (\tilde{\nu}_1, \dots, \tilde{\nu}_d)$ .			

The  $\mathcal{P}_{1D}$  subroutine used in Algorithm 13 is an SDP protocol for summing scalars that we borrow from (Cheu et al., 2021). It is outlined in Algorithm 14. Algorithm 14 decomposes into a local randomizer  $\mathcal{R}$  that individuals execute on their own data, and an analyzer component  $\mathcal{A}$  that the shuffler executes.  $\mathcal{S}(\mathbf{y})$  denotes the shuffled vector  $\mathbf{y}$ : i.e. the vector whose components are random permutations of the components of  $\mathbf{y}$ . We describe the privacy guarantee, bias, and variance of Algorithm 13 in Proposition 53.

**Proposition 53** (Privacy, Bias, and Variance of Algorithm 13). *Let  $\{x_i\}_{i=1}^s \sim \mathcal{D}^s$  have mean  $\mathbb{E}x_i = \nu$ ,  $\|\nu\| \leq L$ , and  $\mathbb{E}\|x_{i,j} - \nu_j\|^k \leq \gamma_k$  for some  $k \geq 2$ . Let  $\varepsilon \leq 8 \ln(2d/\delta)$ ,  $\delta \in (0, 1/2)$ , and choose  $\tau \geq 2L$ . Then, there exist choices of parameters  $(g, b, p, m)$  such that Algorithm 8 is  $(\varepsilon, \delta)$ -SDP, and has bias and variance bounded as follows:*

$$\|\mathbb{E}\tilde{\nu} - \nu\| = \tilde{\mathcal{O}}\left(\sqrt{d}\left(\frac{\gamma_k^{1/k}}{\sqrt{s}} + \left(\frac{2}{\tau}\right)^{k-1} \gamma_k\right)\right) \quad (\text{I.5})$$

and

$$\mathbb{E}\|N\|^2 = \mathbb{E}\|\tilde{\nu} - \mathbb{E}\tilde{\nu}\|^2 = \tilde{\mathcal{O}}\left(\frac{\tau^2 d \ln^2(d/\delta)}{s^2 \varepsilon^2} + d\left(\frac{1}{s} \gamma_k^{2/k} + \gamma_k^2 \left(\frac{2}{\tau}\right)^{2k-2}\right)\right). \quad (\text{I.6})$$

---

**Algorithm 14:**  $\mathcal{P}_{1D}$ , a shuffle private protocol for summing scalars (Cheu et al., 2021)

---

- 1: **Input:** Scalars  $Z = (z_1, \dots, z_s) \in [0, \tau]^s$ ; design parameters  $g, b \in \mathbb{N}; p \in (0, \frac{1}{2})$ .
  - 2: **procedure: Local Randomizer**  $\mathcal{R}_{1D}(z_i)$
  - 3:   **for**  $i \in [s]$  **do**
  - 4:      $z_i \leftarrow \lfloor z_i g / \tau \rfloor$ .
  - 5:     Sample rounding value  $\eta_1 \sim \mathbf{Ber}(z_i g / \tau - z_i)$ .
  - 6:     Set  $\hat{z}_i \leftarrow z_i + \eta_1$ .
  - 7:     Sample privacy noise value  $\eta_2 \sim \mathbf{Bin}(b, p)$ .
  - 8:     Report  $y_i = (y_{i,1}, \dots, y_{i,g+b}) \in \{0, 1\}^{g+b}$  containing  $\hat{z}_i + \eta_2$  copies of 1 and  $g + b - (\hat{z}_i + \eta_2)$  copies of 0.
  - 9:   **end for**
  - 10: **end procedure**
  - 11: **procedure: Shuffler**  $\mathcal{S}(\mathbf{y})$
  - 12:   Shuffler receives  $\mathbf{y} := (y_1, \dots, y_s)$ , draws a uniformly random permutation  $\pi$  of  $[g + b] \times [s]$ , and sends  $\mathcal{S}(\mathbf{y}) := (y_{\pi(1,1)}, \dots, y_{\pi(s,g+b)})$  to analyzer.
  - 13: **end procedure**
  - 14: **procedure: Analyzer**  $\mathcal{A}_{1D}(\mathcal{S}(\mathbf{y}))$
  - 15:   **Output:**  $\frac{\tau}{g} [(\sum_{i=1}^s \sum_{l=1}^{g+b} y_{\pi(i,l)}) - pbs]$ .
- 

**Remark 54.** Comparing Proposition 53 with Lemma 52, we see that the bias and variance of the two *MeanOracles* Algorithm 13 and Algorithm 12 are the same up to logarithmic factors. Therefore, using Algorithm 13 as *MeanOracle* for stochastic optimization results in the same excess risk bounds as one would get by using Algorithm 12, up to logarithmic factors.

The proof of Proposition 53 will require Lemma 56, which is due to (Cheu et al., 2021). First, we need the following notation:

**Definition 55** ( $\delta$ -Approximate Max Divergence). For random variables  $X$  and  $Y$ , define

$$D_\infty^\delta(X||Y) = \sup_{S \subseteq \text{supp}(X): \mathbb{P}(X \in S) \geq \delta} \ln \left[ \frac{\mathbb{P}(X \in S) - \delta}{\mathbb{P}(Y \in S)} \right].$$

An important fact is that a randomized algorithm  $\mathcal{A}$  is  $(\varepsilon, \delta)$ -DP if and only if  $D_\infty^\delta(\mathcal{A}(X)||\mathcal{A}(X')) \leq \varepsilon$  for all adjacent data sets  $X \sim X'$  (Dwork and Roth, 2014).

**Lemma 56.** (Cheu et al., 2021, Lemma 3.1) Let  $s \in \mathbb{N}, \varepsilon \leq 15$ . Let  $g \geq \tau\sqrt{s}$ ,  $b > \frac{180g^2 \ln(2/\delta)}{\varepsilon^2 s}$  and  $p = \frac{90g^2 \ln(2/\delta)}{b\varepsilon^2 s}$ . Then for any adjacent scalar databases  $Z, Z' \in [0, \tau]^r$  differing on user  $u$  ( $z_u \neq z'_u$ ), we have:

- a)  $D_\infty^\delta(\mathcal{S} \circ \mathcal{R}_{1D}^r(Z)||\mathcal{S} \circ \mathcal{R}_{1D}^r(Z')) \leq \varepsilon \left( \frac{2}{g} + \frac{|z_u - z'_u|}{\tau} \right)$ .
- b) *Unbiasedness:*  $\mathbb{E}[\mathcal{P}_{1D}(Z)] = \sum_{i=1}^s z_i$ .
- c) *Variance bound:*  $\mathbb{E}[(\mathcal{P}_{1D}(Z) - \sum_{i=1}^s z_i)^2] = \mathcal{O} \left( \frac{\tau^2}{\varepsilon^2} \ln(1/\delta) \right)$ .

To prove the utility guarantees in Proposition 53, we begin by providing (Lemma 57) high probability bounds on the bias and noise induced by the *non-private* version of Algorithm 13,

in which  $\mathcal{P}_{1D}$  in line 8 is replaced by the sum of  $z \in Z_j^i$ . Lemma 57 is a refinement of (Kamath et al., 2022, Theorem 4.1), with correct scaling for arbitrary  $\gamma_k > 0$  and exact constants:

**Lemma 57.** *Let  $\zeta \in (0, 1)$  and  $X \sim \mathcal{D}$  be a random  $d$ -dimensional vector with mean  $\nu$  such that  $\|\nu\| \leq L$  and  $\mathbb{E}|\langle X - \nu, e_j \rangle|^k \leq \gamma_k$  for some  $k \geq 2$  for all  $j \in [d]$ . Consider the non-private version of Algorithm 13 run with  $m = \lceil 20 \log(4d/\zeta) \rceil$  and  $\tau \geq 2L$ , where  $\mathcal{P}_{1D}$  in line 8 is replaced by the sum of  $z \in Z_j^i$ . Denote the output of this algorithm by  $\hat{\nu}$ . Then with probability at least  $1 - \zeta$ , we have*

$$\|\hat{\nu} - \nu\| \leq 10\sqrt{d} \left( \sqrt{\frac{m}{s}} \gamma_k^{1/k} + \gamma_k \left( \frac{2}{\tau} \right)^{k-1} \right).$$

*Proof.* Denote  $X = (x_1, \dots, x_d)$ ,  $\nu = (\nu_1, \dots, \nu_d)$ , and  $z_j := \Pi_{[-\tau, \tau]}(x_j)$  for  $j \in [d]$ . By Lemma 58 (stated and proved below),

$$|\mathbb{E}z_j - \nu_j| \leq 10\gamma_k \left( \frac{2}{\tau} \right)^{k-1}. \quad (\text{I.7})$$

Now an application of (Minsker, 2022, Lemma 3) with  $\rho(t) := \begin{cases} \frac{t^2}{2} & \text{if } t \in [-\tau, \tau] \\ \tau t & \text{if } t > \tau \\ -\tau t & \text{if } t < -\tau \end{cases}$  shows

that  $\mathbb{E}|z_j - \mathbb{E}z_j|^k \leq \gamma_k$ . Hence by Lemma 59 (stated and proved below),

$$\begin{aligned} \mathbb{P} \left( \left| \hat{\nu}_j^i - \mathbb{E}z_j \right| \leq 10\sqrt{\frac{m}{s}} \gamma_k^{1/k} \right) &= \mathbb{P} \left( \left| \frac{m}{s} \sum_{z \in Z_j^i} z - \mathbb{E}z_j \right| \leq 10\sqrt{\frac{m}{s}} \gamma_k^{1/k} \right) \\ &\geq 0.99, \quad \forall i \in [m], j \in [d]. \end{aligned} \quad (\text{I.8})$$

Next, using the ‘‘median trick’’ (via Chernoff/Hoeffding bound, see e.g. (Andoni, 2015)), we get

$$\mathbb{P} \left( \left| \hat{\nu}_j - \mathbb{E}z_j \right| \leq 10\sqrt{\frac{m}{s}} \gamma_k^{1/k} \right) \leq 2e^{-m/20} \leq \frac{\zeta}{2d}, \quad (\text{I.9})$$

where the last inequality follows from our choice of  $m$ . Now, by union bound, we have for any  $a > 0$  that

$$\mathbb{P}(\|\hat{\nu} - \nu\| \geq \sqrt{da}) \leq \sum_{j=1}^d \mathbb{P}(|\hat{\nu}_j - \nu_j| \geq a) \quad (\text{I.10})$$

$$\leq \sum_{j=1}^d [\mathbb{P}(|\hat{\nu}_j - \mathbb{E}z_j| \geq a/2) + \mathbb{P}(|\mathbb{E}z_j - \nu_j| \geq a/2)]. \quad (\text{I.11})$$

Plugging in  $a = 2 \left[ 10\gamma_k^{1/k} \sqrt{\frac{m}{s}} + 10\gamma_k \left( \frac{2}{\tau} \right)^{k-1} \right]$  implies

$$\mathbb{P} \left( \|\hat{\nu} - \nu\| \geq \sqrt{d} \left( 10\gamma_k^{1/k} \sqrt{\frac{m}{s}} + 10\gamma_k \left( \frac{2}{\tau} \right)^{k-1} \right) \right) \leq d \left( \frac{\zeta}{2d} + 0 \right) = \frac{\zeta}{2}, \quad (\text{I.12})$$

by Eq. (I.7) and Eq. (I.9).  $\square$

Below we give the lemmas that we used in the proof of Lemma 57. The following is a refinement of (Kamath et al., 2022, Lemma B.1) with the proper scaling in  $\gamma_k$ :

**Lemma 58.** *Let  $x \sim \mathcal{D}$  be a random variable with mean  $\nu$  and  $\mathbb{E}|x - \nu|^k \leq \gamma_k$  for some  $k \geq 2$ . Let  $z = \Pi_{[-\tau, \tau]}(x)$  for  $\tau \geq 2|\nu|$ . Then,*

$$|\nu - \mathbb{E}z| \leq 10\gamma_k \left(\frac{2}{\tau}\right)^{k-1}.$$

*Proof.* We begin by recalling the following form of Chebyshev's inequality:

$$\mathbb{P}(|x - \nu| \geq c) \leq \frac{\mathbb{E}|x - \nu|^k}{c^k} \leq \frac{\gamma_k}{c^k} \quad (\text{I.13})$$

for any  $c > 0$ , via Markov's inequality. By symmetry, we may assume without loss of generality that  $\nu \leq 0$ . Note that

$$\nu - \mathbb{E}z = \mathbb{E} \left[ x \mathbb{1}_{x < -\tau} + x \mathbb{1}_{x > \tau} + x \mathbb{1}_{x \in [-\tau, \tau]} - (-\tau \mathbb{1}_{x < -\tau} + \tau \mathbb{1}_{x > \tau} + x \mathbb{1}_{x \in [-\tau, \tau]}) \right] \quad (\text{I.14})$$

$$= \mathbb{E} \left[ (x + \tau) \mathbb{1}_{x < -\tau} + (x - \tau) \mathbb{1}_{x > \tau} \right]. \quad (\text{I.15})$$

So,

$$|\nu - \mathbb{E}z| \leq \underbrace{\mathbb{E}(x + \tau) \mathbb{1}_{x < -\tau}}_{\text{a)}} + \underbrace{\mathbb{E}(x - \tau) \mathbb{1}_{x > \tau}}_{\text{b)}}, \quad (\text{I.16})$$

by the triangle inequality. Now,

$$\begin{aligned} \text{a)} &= |\mathbb{E}[x - \nu - (-\tau - \nu) \mathbb{1}_{x < -\tau}]| \\ &\leq \mathbb{E}[|x - \nu| \mathbb{1}_{x < -\tau}] + |-\tau - \nu| \mathbb{E} \mathbb{1}_{x < -\tau} \\ &\leq \left( \mathbb{E}|x - \nu|^k \right)^{1/k} (\mathbb{P}(x < -\tau))^{(k-1)/k} + |-\tau - \nu| \mathbb{P}(x < -\tau), \end{aligned} \quad (\text{I.17})$$

by Holder's inequality. Also, since  $-\frac{\tau}{2} \leq \nu \leq 0$ , we have

$$\mathbb{P}(x < -\tau) \leq \mathbb{P}\left(x < \nu - \frac{\tau}{2}\right) \leq \mathbb{P}\left(|x - \nu| > \frac{\tau}{2}\right) \leq \frac{\gamma_k 2^k}{\tau^k}, \quad (\text{I.18})$$

via Eq. (I.13). Plugging Eq. (I.18) into Eq. (I.17) and using the bounded moment assumption, we get

$$\text{a)} \leq \gamma_k^{1/k} \left(\frac{\gamma_k 2^k}{\tau^k}\right)^{(k-1)/k} + (\tau + |\nu|) \frac{\gamma_k 2^k}{\tau^k} \quad (\text{I.19})$$

$$\leq \gamma_k \left(\frac{2}{\tau}\right)^{k-1} + 4\gamma_k \left(\frac{2}{\tau}\right)^{k-1} \quad (\text{I.20})$$

$$= 5\gamma_k \left(\frac{2}{\tau}\right)^{k-1}. \quad (\text{I.21})$$

Likewise, a symmetric argument shows that  $\text{b)} \leq 5\gamma_k \left(\frac{2}{\tau}\right)^{k-1}$ . Hence the lemma follows.  $\square$

Below is a more precise and general version of (Kamath et al., 2022, Lemma A.2) (scaling with  $\gamma_k$ ):

**Lemma 59.** *Let  $\mathcal{D}$  be a distribution over  $\mathbb{R}$  with mean  $\nu$  and  $\mathbb{E}|\mathcal{D} - \nu|^k \leq \gamma_k$  for some  $k \geq 2$ . Let  $x_1, \dots, x_n$  be i.i.d. samples from  $\mathcal{D}$ . Then, with probability at least 0.99,*

$$\left| \frac{1}{n} \sum_{i=1}^n x_i - \nu \right| \leq \frac{10\gamma_k^{1/k}}{\sqrt{n}}.$$

*Proof.* First, by Jensen's inequality, we have

$$\mathbb{E}[(x - \nu)^2] \leq \mathbb{E}[|x - \nu|^k]^{2/k} \leq \gamma_k^{2/k}. \quad (\text{I.22})$$

Hence,

$$\mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^n x_i - \nu \right)^2 \right] = \frac{1}{n^2} \mathbb{E} \left[ \sum_{i=1}^n (x_i - \nu)^2 \right] \quad (\text{I.23})$$

$$\leq \frac{\gamma_k^{2/k}}{n}, \quad (\text{I.24})$$

where we used the assumption that  $\{x_i\}_{i=1}^n$  are i.i.d. and Eq. (I.22). Thus, by Chebyshev's inequality,

$$\mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^n x_i - \nu \right| \geq \frac{10\gamma_k^{1/k}}{\sqrt{n}} \right) \leq \frac{1}{100}.$$

□

Now, we provide the proof of Proposition 53:

*Proof of Proposition 53. Privacy:* Let  $X$  and  $X'$  be adjacent data sets in  $\mathcal{X}^s$ . Assume without loss of generality that  $x_1 \neq x'_1$  and  $x_l = x'_l$  for  $l > 1$ . By the post-processing property of DP and the fact that each sample is only processed once (due to disjoint batches) during the algorithm, it suffices to fix  $i \in [m]$  and show that the composition of all  $d$  invocations of  $\mathcal{P}_{1D}$  (line 8) for  $j \in [d]$  is  $(\varepsilon, \delta)$ -DP. Assume without loss of generality that  $Z_j := Z_j^1$  and  $Z'_j := (Z_j^1)'$  contain the truncated  $z_{1,j} := \Pi_{[0,2\tau]}(x_{1,j})$  and  $z'_{1,j} := \Pi_{[0,2\tau]}(x'_{1,j})$ , respectively. Then, by the first part of Lemma 56, there are choices of  $b$  and  $p$  such that

$$D_{\infty}^{\delta_j}(\mathcal{S} \circ \mathcal{R}_{1D}^r(Z_j) \| \mathcal{S} \circ \mathcal{R}_{1D}^r(Z'_j)) \leq \varepsilon_j \left( \frac{2}{g} + \frac{|z_{1,j} - z'_{1,j}|}{2\tau} \right) \leq 2\varepsilon_j,$$

for all  $j$  by the triangle inequality, provided  $g \geq \max\{2, 2\tau\sqrt{s}\}$ . Hence each invocation of  $\mathcal{P}_{1D}$  is  $(2\varepsilon_j, \delta_j)$ -DP. Thus, by the advanced composition theorem (Dwork and Roth, 2014,

Theorem 3.20), Algorithm 13 is  $(\varepsilon', \delta)$ -SDP, where

$$\varepsilon' = \sum_{j=1}^d (2\varepsilon_j) (e^{2\varepsilon_j} - 1) + 2\sqrt{2 \sum_{j=1}^d \varepsilon_j^2 \ln(1/\delta_j)} \quad (\text{I.25})$$

$$\leq 8d\varepsilon_1^2 + \frac{\varepsilon}{2} \quad (\text{I.26})$$

$$\leq \frac{\varepsilon^2}{4 \ln(1/\delta_1)} + \frac{\varepsilon}{2} \quad (\text{I.27})$$

$$\leq \varepsilon, \quad (\text{I.28})$$

by our choices of  $\varepsilon_j, \delta_j$  and the assumption that  $\varepsilon \leq 8 \ln(1/\delta_1) = 8 \ln(2d/\delta)$ .

**Bias:** Let  $\hat{\nu} = (\hat{\nu}_1, \dots, \hat{\nu}_d)$  denote the output of the non-private version of Algorithm 13 where  $\mathcal{P}_{1D}$  in line 8 is replaced by the (noiseless) sum of  $z \in Z_j^i$ . Then, Lemma 57 tells us that

$$\|\hat{\nu} - \nu\|^2 \leq 200d \left( \frac{m}{s} \gamma_k^{2/k} + \gamma_k^2 \left( \frac{2}{\tau} \right)^{2k-2} \right)$$

with probability at least  $1 - \zeta$ , if  $m = \lceil 20 \log(4d/\zeta) \rceil$ . Thus,

$$\begin{aligned} b^2 = \|\mathbb{E}\tilde{\nu} - \nu\|^2 &= \|\mathbb{E}\hat{\nu} - \nu\|^2 \leq \mathbb{E}\|\hat{\nu} - \nu\|^2 \\ &\leq 200d \left( \frac{m}{s} \gamma_k^{2/k} + \gamma_k^2 \left( \frac{2}{\tau} \right)^{2k-2} \right) (1 - \zeta) + 2 \sup(\|\hat{\nu}\|^2 + \|\nu\|^2) \zeta \\ &\leq 200d \left( \frac{m}{s} \gamma_k^{2/k} + \gamma_k^2 \left( \frac{2}{\tau} \right)^{2k-2} \right) + 4\tau^2 d\zeta, \end{aligned}$$

since  $\tau \geq L \geq \|\nu\|$  by assumption. Then choosing

$$\zeta = \frac{1}{\tau^2} \left[ \left( \frac{1}{s} \gamma_k^{2/k} + \gamma_k^2 \left( \frac{2}{\tau} \right)^{2k-2} \right) \right]$$

and noting that  $\zeta$  is polynomial in all parameters (so that  $m = \tilde{\mathcal{O}}(1)$ ) implies

$$b^2 \leq \mathbb{E}\|\hat{\nu} - \nu\|^2 = \tilde{\mathcal{O}} \left( d \left( \frac{1}{s} \gamma_k^{2/k} + \gamma_k^2 \left( \frac{2}{\tau} \right)^{2k-2} \right) \right). \quad (\text{I.29})$$

**Variance:** We have

$$\mathbb{E}\|N\|^2 \leq 2 \left[ \underbrace{\mathbb{E}\|\tilde{\nu} - \hat{\nu}\|^2}_{\text{(a)}} + \underbrace{\mathbb{E}\|\hat{\nu} - \mathbb{E}\hat{\nu}\|^2}_{\text{(b)}} \right]. \quad (\text{I.30})$$

We bound (a) as follows. For any  $j \in [d], i \in [m]$ , denote  $\hat{\nu}_j^i := \frac{m}{s} \sum_{z \in Z_j^i} z$  (c.f. line 8 of Algorithm 13), the mean of the (noiseless)  $s/m$  clipped  $j$ -th coordinates of data in group  $i$ . Denote  $\tilde{\nu}_j^i$  as in Algorithm 13, which is the same as  $\hat{\nu}_j^i$  except that the summation over  $Z_j^i$  is

replaced by  $\mathcal{P}_{1D}(Z_j^i)$ . Also, denote  $\hat{\nu}_j = \text{median}(\hat{\nu}_j^1, \dots, \hat{\nu}_j^m)$  and  $\tilde{\nu}_j = \text{median}(\tilde{\nu}_j^1, \dots, \tilde{\nu}_j^m)$ . Then for any  $j \in [d]$ , we have

$$\begin{aligned}
|\hat{\nu}_j - \tilde{\nu}_j|^2 &= |\text{median}(\hat{\nu}_j^1, \dots, \hat{\nu}_j^m) - \text{median}(\tilde{\nu}_j^1, \dots, \tilde{\nu}_j^m)|^2 \\
&\leq \left( \sum_{i=1}^m |\hat{\nu}_j^i - \tilde{\nu}_j^i| \right)^2 \\
&= \left( \sum_{i=1}^m \left| \frac{m}{s} \left[ \sum_{z \in Z_j^i} z - \mathcal{P}_{1D}(Z_j^i) \right] \right| \right)^2 \\
&\leq \left( \frac{m}{s} \sum_{i=1}^m \left| \mathcal{P}_{1D}(Z_j^i) - \sum_{z \in Z_j^i} z \right| \right)^2 \\
&\leq \frac{m^2}{s^2} m \sum_{i=1}^m \left| \mathcal{P}_{1D}(Z_j^i) - \sum_{z \in Z_j^i} z \right|^2.
\end{aligned}$$

Now using Lemma 56 (part c), we get

$$\begin{aligned}
\mathbb{E}[|\hat{\nu}_j - \tilde{\nu}_j|^2] &\leq \frac{m^4}{s^2} \mathbb{E} \left[ \left| \mathcal{P}_{1D}(Z_j^i) - \sum_{z \in Z_j^i} z \right|^2 \right] \\
&\lesssim \frac{m^4 \tau^2}{s^2 \varepsilon_j^2} \ln(1/\delta_j) \\
&\lesssim \frac{m^4 \tau^2 d \ln^2(d/\delta)}{s^2 \varepsilon^2}.
\end{aligned}$$

Thus, summing over  $j \in [d]$ , we get

$$\mathbb{E}[\|\hat{\nu} - \tilde{\nu}\|^2] \lesssim \frac{m^4 \tau^2 d \ln^2(d/\delta)}{s^2 \varepsilon^2}. \tag{I.31}$$

Next, we bound ⑥:

$$\begin{aligned}
\text{⑥} &= \mathbb{E} \|\hat{\nu} - \mathbb{E}\hat{\nu}\|^2 \\
&\leq 2 [\mathbb{E} \|\hat{\nu} - \nu\|^2 + \|\nu - \mathbb{E}\hat{\nu}\|^2] \\
&\leq 2 [\mathbb{E} \|\hat{\nu} - \nu\|^2 + b^2] \\
&= \tilde{\mathcal{O}} \left( d \left( \frac{1}{s} \gamma_k^{2/k} + \gamma_k^2 \left( \frac{2}{\tau} \right)^{2k-2} \right) \right),
\end{aligned}$$

by the bias bound Eq. (I.29). Therefore,

$$\mathbb{E} \|N\|^2 = \tilde{\mathcal{O}} \left( \frac{\tau^2 d \ln^2(d/\delta)}{s^2 \varepsilon^2} + d \left( \frac{1}{s} \gamma_k^{2/k} + \gamma_k^2 \left( \frac{2}{\tau} \right)^{2k-2} \right) \right).$$

□