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Article

The Extended TOPSIS Method for MAGDM with Probabilistic Hesitant Fermatean Fuzzy Linguistic Term Set

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Abstract: To tackle the uncertainties in language-based evaluation and the limitations of traditional methods in capturing expert preferences within complex decision-making scenarios, this study introduces an extended TOPSIS method for MAGDM with PHFFLTSs. This novel method leverages the advantages of probabilistic linguistic term sets, hesitant fuzzy linguistic term sets, and Fermatean fuzzy sets by constructing a probabilistic hesitant Fermatean fuzzy linguistic term decision matrix. This matrix enables experts to articulate their confidence in various linguistic terms while retaining a hesitant stance towards multiple terms, offering a more nuanced and adaptable reflection of decision-making uncertainties and fuzziness. The paper details the methodological steps, encompassing the creation of a "seven-value" linguistic term set, formulation of the probabilistic hesitant Fermatean fuzzy linguistic term decision matrix, integration of multiple decision matrices, normalization, attribute weight calculation, identification of positive and negative ideal solutions, and proximity coefficient computation. These steps collectively enable a thorough evaluation and ranking of alternatives. Example analysis and comparative analysis reveal that this method surpasses the extended TOPSIS method for MAGDM with PHIFLTSs in both discrimination power and decision quality. It presents a new research avenue and theoretical framework for fuzzy multi-attribute decision-making, enriching the field's theoretical underpinnings and providing a robust tool for addressing increasingly complex and uncertain decision challenges.

Keywords: Multiple Attribute Group Decision Making; Probabilistic Linguistic Term Sets; Hesitant Fuzzy Linguistic Term Sets; Fermatean Fuzzy Sets; Probabilistic Hesitant Fermatean Fuzzy Linguistic Term Sets; TOPSIS

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1. Introduction

In complex decision-making environments, decision makers often face the problem of uncertainty in linguistic evaluation information, and traditional decision-making methods are difficult to accurately characterize experts' preferences. To address this challenge, this study innovatively integrates the probabilistic hesitant Fermatean fuzzy linguistic term set and the TOPSIS method to construct a novel multi-attribute group decision making model.

The fuzzy set theory was first proposed by Zadeh in 1965 to deal with uncertainty and fuzzy problems by assigning an object a membership degree level between 0 and 1 through a membership degree function [1]. With the increasing complexity of decision-making problems, Atanassov proposed intuitionistic fuzzy sets, which added two metrics, non-membership degree and hesitation, to effectively deal with the uncertainty of decision-making information [2]. However, intuitionistic fuzzy sets have the limitation that the sum of membership degree and non-membership degree is equal to 1. Yager proposed Pythagorean fuzzy sets, which deal with membership degree and non-membership degree by means of sum of squares, making them more suitable for practical problems with complex information [3]. Subsequently, Senapati introduced the Fermatean fuzzy set, which deals with the membership degree and non-membership degree by means of cubic sum, further enriches the information capacity of fuzzy sets, and is widely used in the field of multi-criteria decision making [4]. In the actual decision-making process, it is often difficult for decision makers to give definite values of the membership degree or non-membership degree to describe their decision psychology. For this reason, Torra proposed hesitant fuzzy sets, which allow the membership degree or non-membership degree of elements in a set to be represented by a set of numerical values, and are suitable for decision making environments that are more complex, have incomplete information, and have a diversity of preferences [5]. Beg and Rodriguez introduced hesitant intuitionistic fuzzy sets, studied the hesitant fuzzy linguistic approach and its computational framework, and established a hesitant fuzzy linguistic-based multicriteria decision making model for dealing with complex and incomplete information. criteria decision making model, which provides a new methodology for dealing with complex uncertainty problems [6-7]. Liu Weifeng et al. further proposed the Pythagorean hesitant fuzzy set, which describes a more complex decision-making environment in which the number of elements changes depending on the degree of hesitation [8]. In order to quantify uncertainty, Pang et al. proposed Probabilistic Linguistic Term Sets (PLTS), which combines probability theory with language by assigning a probability value to each linguistic term to quantify the degree of preference assessed by the decision maker, which makes the decision making process more in line with human intuitive thinking, and enhances the representation of information uncertainty [9].

Subsequent studies have been carried out under this theoretical framework, and the main results include: 1) various algorithms and related metrics have been proposed for multi-attribute decision making problems in hesitant-fuzzy linguistic environments [10-19]; 2) the hesitant intuitionistic fuzzy linguistic term set theoretical system has been further improved by using the new algorithms [20-21]; and 3) probabilistic linguistic term-sets are combined with fuzzy sets, which are applied to a variety of multi-attribute group decision making problems [22-26].

In summary, the existing theoretical and applied research on multi-attribute group decision making have achieved rich results, but there are still shortcomings: 1) multi-attribute group decision making research is mostly based on intuitionistic fuzzy set environments, which are seldom combined with the Fermatean fuzzy set; 2) in practical decision making, experts tend to evaluate decision making solutions using uncertain linguistic terms, and it is difficult to express their evaluations with definite numerical values, the It is not possible to adequately capture vagueness and uncertainty.

This paper innovatively introduces the probabilistic hesitant Fermatean fuzzy linguistic term set (PHFFLTS), a concept that amalgamates the characteristics of probabilistic linguistic term sets, hesitant fuzzy linguistic term sets, and Fermatean fuzzy sets. This integration enables a more holistic and flexible representation of experts' uncertainties and fuzzy evaluations during the decision-making process. The rationale behind this innovation is elaborated as follows: Firstly, the incorporation of probability allows experts to articulate their confidence levels in various linguistic terms during the evaluation phase. This approach not only enhances the granularity of the evaluations but also aligns the results more closely with the inherent uncertainties of real-world decision-making scenarios. Secondly, the inherent properties of hesitant fuzzy linguistic term sets empower experts to maintain a hesitant stance towards multiple linguistic terms during their assessments. This means that experts can contemplate several potential evaluation outcomes concurrently, rather than being constrained to select a single, definitive linguistic term. Such hesitant evaluations are particularly vital in complex decision-making contexts, as they authentically mirror experts' cognitive processes when grappling with uncertain information. Lastly, the introduction of Fermatean fuzzy sets provides a robust mathematical foundation for the PHFFLTS. By incorporating a non-membership degree function, Fermatean fuzzy sets augment the expressive capacity of traditional fuzzy sets, enabling more nuanced and precise handling of uncertainties and fuzziness. The fusion of Fermatean fuzzy sets with probabilistic hesitant fuzzy linguistic term sets yields the PHFFLTS. This innovation not only bolsters the expressive power of the linguistic term set but also elevates its applicability in multi-attribute decision-making problems. Specifically, the method entails constructing a decision matrix, integrating multiple decision matrices, normalization, calculating attribute weights, determining positive and negative ideal solutions, and computing proximity coefficients to achieve a comprehensive assessment and ranking of alternative solutions. The results from instance verification and comparative analysis demonstrate that this method surpasses the extended TOPSIS method for MAGDM with PHIFLTSs [23] in terms of discrimination and decision outcomes, offering a fresh research perspective and theoretical toolkit for the field of fuzzy multi-attribute decision-making.

The rest of the paper is organized as follows: Part 2 is the preliminaries and introduces the previous research results; Part 3 introduces the definition of probabilistic hesitant Fermatean fuzzy linguistic term set; Part 4 constructs the extended TOPSIS method for MAGDM with probabilistic hesitant Fermatean fuzzy linguistic term set and gives the specific decision-making steps; Part 5 gives an example analysis that illustrates the feasibility and validity of the proposed method; Part 6 is a concluding analysis.

2. Preliminaries

This section provides a brief introduction to Fermatean fuzzy sets, hesitant fuzzy linguistic term sets, and probabilistic linguistic term sets.

2.1. Fermatean Fuzzy Sets and Related Definitions

Definition 1 [4] Fermatean fuzzy set F is defined on a non-empty set X, where the elements have the following form:

$$F = \{\langle x, \alpha_F(x), \beta_F(x) \mid x \in X \rangle\} \tag{1}$$

where, $\alpha_F(x): X \to [0,1]$ and $\beta_F(x): X \to [0,1]$ represent the membership degree and the non-membership degree of each element in the fuzzy set. For all $x \in X$, it satisfies $0 \le (\alpha_F(x))^3 + (\beta_F(x))^3 \le 1$.

For any Fermatean fuzzy set F and $x \in X$, $\pi_F(x) = \sqrt[3]{1-(\alpha_F(x))^3-(\beta_F(x))^3}$ is called the uncertainty of x to F.

Definition 2 [4] Let $F_1=(\alpha_{F_1},\beta_{F_1}), F_2=(\alpha_{F_2},\beta_{F_2})$ be two Fermatean fuzzy sets, then the Fermatean fuzzy set of Euclidean distance is:

$$d(F_1,F_2)=\sqrt{\frac{1}{2}\left[(\alpha_{F_1}^3-\alpha_{F_2}^3)^2+\left(\beta_{F_1}^3-\beta_{F_2}^3\right)^2+\left(\pi_{F_1}^3-\pi_{F_2}^3\right)^2\right]} \tag{2}$$

Definition 3 [4] Let $F=\left(\alpha_F,\beta_F\right)$ be a Fermatean fuzzy set, the fractional formula is defined as:

$$score\left(F\right)=\alpha_F^3-\beta_F^3 \tag{3}$$

Let $F_1=(\alpha_{F_1},\beta_{F_1})$ and $F_2=(\alpha_{F_2},\beta_{F_2})$ are two Fermatean fuzzy sets, if $s(F_1)\succ s(F_2)$, then $F_1 \succ F_2$; if $s(F_1)\prec s(F_2)$, then $F_1 \prec F_2$. If $s(F_1)=s(F_2)$, then $F_1=F_2$.

2.2. Hesitant Fuzzy Linguistic Term Sets, Probabilistic Linguistic Term Sets and Related Definitions

Definition 4 [27] Let $S=\{s_g \mid g=0,1,\ldots,\tau\}$ be a linguistic term set, $a_i \in A, i=1,2,\ldots,N$, the mathematical form of a hesitant fuzzy linguistic term set on A is:

$$H_s=\{\langle a_i,h_s(a_i)\rangle \mid a_i \in A\}$$

where $h_s(a_i)$ is the set of elements in S, $h_s(a_i)$ is a continuous column of possible values in the linguistic term set S, $h_s(a_i)=\{s_{\varphi l}(a_i) \mid s_{\varphi l}(a_i)\in S, l=1,2,\ldots,L(a_i)\}$. $\varphi l \in \{0,1,\ldots,\tau\}$ is the subscript of the linguistic term $s_{\varphi l}(a_i)$, $L(a_i)$ is the number of linguistic terms in $h_s(a_i)$. For simplicity, $h_s(a_i)$ is called hesitant fuzzy linguistic number, H_s is the set of all hesitant fuzzy linguistic numbers on the linguistic term set.

Definition 5 [28-29] Let $S=\{s_\beta \mid \beta=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ be the set of linguistic terms, under S which the linguistic expressions generated by the text free grammar G_H is $ll \in S_{ll}$, where, S_{ll} is the set of all linguistic expressions, then can be transformed into the set of hesitant fuzzy linguistic terms H_S by means of the transformation function $E_{G_H}: S_{ll} \to H_S$:

$$E_{G_H}\{s_\beta \mid s_\beta \in S\},\ E_{G_H}(s_\beta \le s_\alpha)=\{s_\beta \mid s_\beta \in S, s_\beta \le s_\alpha\}$$

$$E_{G_H}(s_\beta \prec s_\alpha)=\{s_\beta \mid s_\beta \in S, s_\beta \prec s_\alpha\},\ E_{G_H}(s_\beta \ge s_\alpha)=\{s_\beta \mid s_\beta \in S, s_\beta \ge s_\alpha\}$$

$$E_{G_H}(s_\beta \succ s_\alpha)=\{s_\beta \mid s_\beta \in S, s_\beta \succ s_\alpha\},\ E_{G_H}(s_\alpha \le s_\beta \le s_\gamma)=\{s_\beta \mid s_\beta \in S, s_\alpha \le s_\beta \le s_\gamma\}$$

Definition 6 [9] Let $S=\{s_i \mid i=0,1,\ldots,\tau\}$ be a set of linguistic terms, a probabilistic linguistic term set related to probability can be defined as:

$$S(p)=\{S^k(p^{(k)}) \mid S^{(k)}\in S, p^{(k)}\ge 0, k=1,2,\ldots\#S(p), \sum_{k=1}^{\#S(p)} p^{(k)} \le 1\} \tag{4}$$

Here, $p^{(k)}$ is the probability of linguistic terms, $\#S(p)$ is the number of different linguistic terms $S(p)$ in the set of linguistic terms.

3. Probabilistic Hesitant Fermatean Fuzzy Linguistic Term Set

Definition 7 $X=\{x_1,x_2,\ldots,x_n\}$ is the domain, and let $S=\{s_i \mid i=0,1,\ldots,\tau\}$ be an ordered set of linguistic terms, a Probabilistic Hesitant Fermatean Fuzzy Linguistic Term Set (PHFLTS) on the domain X can be defined as:

$$H_s(F)=\{<x_i,\alpha_s^i(p),\beta_s^i(p)>|\,x_i\in X,i=1,2,\dots,n\}$$

Here, $\alpha_s^i(p)$ represents the membership degree of x_i to the set $H_s(F)$, and $\beta_s^i(p)$ represents the non-membership degree of x_i to the set $H_s(F)$.

$$\alpha_s^i(p)=\{l_m^i(p_m^i)\,|\,l_m^i\in S,p_m^i\geq 0,i=1,2,\dots\#i_m,\sum_{i=1}^{\#i_m}p_m^i\leq 1\}$$
$$\beta_s^i(p)=\{l_n^i(p_n^i)\,|\,l_n^i\in S,p_n^i\geq 0,i=1,2,\dots\#i_n,\sum_{i=1}^{\#i_n}p_n^i\leq 1\} \quad (5)$$

where, respectively, $l_m^i(p_m^i)$ and $l_n^i(p_n^i)$ represent the i probabilistic linguistic term with probability of p_m^i and p_n^i. $\#i_m$ and $\#i_n$ represent the total number of linguistic terms. And the membership degree and non-membership degree satisfy:

$$S_0\leq\max(\alpha_s^i(p))\leq S_{2\tau},S_0\leq\min(\alpha_s^i(p))\leq S_{2\tau}, \text{ and } \min(\alpha_s^i(p))\leq\max(\alpha_s^i(p))$$
$$S_0\leq\max(\beta_s^i(p))\leq S_{2\tau},S_0\leq\min(\beta_s^i(p))\leq S_{2\tau}, \text{ and } \min(\beta_s^i(p))\leq\max(\beta_s^i(p))$$

4. The Extended TOPSIS Method for MAGDM with PHFFLTSs

4.1. Problem Description

There are m alternatives $A_i(i=1,2,\dots,m)$ that form the set of alternatives; there are n attributes $C_j(j=1,2,\dots,n)$ that form the set of attributes, and the weight vector of the attributes is $W=(w_1,w_2,\dots,w_n)$, where, $w_j\in[0,1]$ and $\sum_{j=1}^{n}w_j=1$; there are k experts that form the set of experts $D=(D_1,D_2,\dots,D_k)$.

4.2. Decision Making Steps

The specific steps of the extended TOPSIS method for MAGDM with PHFFLTSs are as follows:

Step 1: Establish the linguistic term set S for the membership degree A_{ij} (or non-membership degree B_{ij}) of the alternative A_i with respect to attribute C_j.

Let $S=\{s_i\,|\,i=0,1,\dots,\tau\}$ be the linguistic term set, where $s_0,s_1,\dots,s_\tau$ characterize the membership degree A_{ij} (or non-membership degree B_{ij}) of the alternative A_i with respect to attribute C_j. The larger τ is, the higher the membership degree A_{ij} (or non-membership degree B_{ij}) of the alternative A_i with respect to attribute C_j is; the smaller τ is, the lower the membership degree A_{ij} (or non-membership degree B_{ij}) of A_i with respect to attribute C_j is.

For example, S is a linguistic term set,

$$S=\{s_0=extremely\ \ low,s_1=very\ \ low,s_2=low,s_3=medium,s_4=high,s_5=very\ \ high,s_6=extremely\ \ high\}$$

where the linguistic term s_0 indicates that the membership degree A_{ij} (or non- membership degree B_{ij}) of the alternative A_i with respect to attribute C_j is extremely low, and so on.

Step 2: Establish the probabilistic hesitant Fermatean fuzzy linguistic term decision matrix of each expert.

According to Definition 7, the expert evaluates the membership degree A_{ij} (or non-membership degree B_{ij}) of the alternative A_i with respect to the attributes C_j to obtain the probabilistic hesitant Fermatean fuzzy linguistic term decision matrix:

$$H^k = \left[\langle A_{ij}^k(p_{ij}), B_{ij}^k(p_{ij})\rangle\right]_{m\times n}$$

where $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$.

$$H^k = \begin{bmatrix} (A_{11}^k(p_{ij}), B_{11}^k(p_{ij})) & (A_{12}^k(p_{ij}), B_{12}^k(p_{ij})) & \cdots & (A_{1n}^k(p_{ij}), B_{1n}^k(p_{ij})) \\ (A_{21}^k(p_{ij}), B_{21}^k(p_{ij})) & (A_{22}^k(p_{ij}), B_{22}^k(p_{ij})) & \cdots & (A_{2n}^k(p_{ij}), B_{2n}^k(p_{ij})) \\ \vdots & \vdots & \ddots & \vdots \\ (A_{m1}^k(p_{ij}), B_{m1}^k(p_{ij})) & (A_{m2}^k(p_{ij}), B_{m2}^k(p_{ij})) & \cdots & (A_{mn}^k(p_{ij}), B_{mn}^k(p_{ij})) \end{bmatrix}$$

where $A_{ij}^k(p_{ij}) = \left\{\alpha_{m_{ij}}(p_{ij}), \beta_{n_{ij}}(p_{ij})\right\}$ and $B_{ij}^k(p_{ij}) = \left\{\alpha_{m_{ij}}^*(p_{ij}), \beta_{n_{ij}}^*(p_{ij})\right\}$ represent the membership degree and non-membership degree respectively.

Step 3: Multiple decision matrices are integrated into a probabilistic hesitant Fermatean fuzzy linguistic term decision matrix [21].

$$H = \left\langle A_{ij}^k(p_{ij}), B_{ij}^k(p_{ij})\right\rangle$$

$$\begin{aligned} \alpha_{m_{ij}}(p_{ij}) &= \min\left\{\min_{k=1}^{l}(\max\ A_{ij}^k(p_{ij})), \max_{k=1}^{l}(\min\ A_{ij}^k(p_{ij}))\right\} \\ \beta_{n_{ij}}(p_{ij}) &= \max\left\{\min_{k=1}^{l}(\max\ A_{ij}^k(p_{ij})), \max_{k=1}^{l}(\min\ A_{ij}^k(p_{ij}))\right\} \\ \alpha_{m_{ij}}^*(p_{ij}) &= \min\left\{\min_{k=1}^{l}(\max\ B_{ij}^k(p_{ij})), \max_{k=1}^{l}(\min\ B_{ij}^k(p_{ij}))\right\} \\ \beta_{n_{ij}}^*(p_{ij}) &= \max\left\{\min_{k=1}^{l}(\max\ B_{ij}^k(p_{ij})), \max_{k=1}^{l}(\min\ B_{ij}^k(p_{ij}))\right\} \end{aligned} \tag{6}$$

We can calculate the maximum and minimum values of membership degree and non-membership degree in the decision matrix. $A_{ij}^k(p_{ij})$ is computed by $p_{ij}\times r_{ij}^l, l = 1,2,\ldots,\#l_{ij}^k(p_{ij})$. Where the subscript r_{ij}^l denoting the linguistic term set $l-th$, p_{ij} is the corresponding probability.

Step 4: Normalize the integrated matrix.

The probabilities of membership degree and non-membership degree in this integrated matrix are normalized by applying Equation (7) to obtain the final integrated criteria matrix H^*.

$$\begin{aligned} p_m^{*(i)} &= \frac{p_m^{*(i)}}{\sum_{i=1}^{\#i_m(p)} p_m^{*(i)}}, \forall i = 1,2,\ldots,\#i_m(p) \\ p_n^{*(i)} &= \frac{p_n^{*(i)}}{\sum_{i=1}^{\#i_n(p)} p_n^{*(i)}}, \forall i = 1,2,\ldots,\#i_n(p) \end{aligned} \tag{7}$$

Step 5: Calculate the weight w_j of the corresponding attribute values [21].

$$\begin{aligned} w_j &= \frac{\sum_{i=1}^{m}\sum_{q\neq i}^{m} D(h_{ij}, h_{qj})}{\sum_{j=1}^{m}\sum_{i=1}^{m}\sum_{q\neq i} D(h_{ij}, h_{qj})} \\ &= \frac{\sum_{i=1}^{m}\sum_{q\neq i}^{m}\left(\sqrt{\frac{1}{\#l_{ij}(p_{ij})}\sum_{k_1=1}^{\#l_{ij}(p_{ij})}(p_{ij}^{k_1}r_{ij}^{k_1} - p_{qj}^{k_1}r_{qj}^{k_1})^2} + \sqrt{\frac{1}{\#l_{ij}^{'}(p_{ij})}\sum_{k_2=1}^{\#l_{ij}^{'}(p_{ij})}(p_{ij}^{'k_2}r_{ij}^{'k_2} - p_j^{'k_2}r_j^{'k_2})^2}\right.}{\sum_{j=1}^{m}\sum_{i=1}^{m}\sum_{q\neq i}\left(\sqrt{\frac{1}{\#l_{ij}(p_{ij})}\sum_{k_1=1}^{\#l_{ij}(p_{ij})}(p_{ij}^{k_1}r_{ij}^{k_1} - p_{qj}^{k_1}r_{qj}^{k_1})^2} + \sqrt{\frac{1}{\#l_{ij}^{'}(p_{ij})}\sum_{k_2=1}^{\#l_{ij}^{'}(p_{ij})}(p_{ij}^{'k_2}r_{ij}^{'k_2} - p_j^{'k_2}r_j^{'k_2})^2}\right.} \end{aligned} \tag{8}$$

where m represents the number of alternatives $A_i (i = 1,2,...,m)$, n represents the number of attributes $C_j (j = 1,2,...,n)$, and $q \neq i$.

Step 6: Determine the positive and negative ideal solutions among the alternatives.

Positive ideal solution: For benefit-type attributes, the membership degree takes the maximum value in the attribute column, and the non-membership degree takes the minimum value in that column. For cost-type attributes, the membership degree takes the minimum value in the attribute column, and the non-membership degree takes the maximum value in that column. Negative ideal solution: For benefit-type attributes, the membership degree takes the minimum value in the attribute column, and the non-membership degree takes the maximum value in that column. For cost-type attributes, the membership degree takes the maximum value in the attribute column, and the non-membership degree takes the minimum value in that column. The formulas for the positive ideal solution and negative ideal solution of benefit-type attributes are as follows:

$$A^+ = \left\langle \alpha^+(p) = (\{\alpha_1^+(p), \alpha_2^+(p), ..., \alpha_n^+(p)), \beta^+(p) = (\beta_1^+(p), \beta_2^+(p), ..., \beta_n^+(p))\} \right\rangle \tag{9}$$

where

$$\alpha^+(p) = \{(\alpha_j^{(k_1)})^+ \mid k_1 = 1,2,...,\#\alpha_{ij}(p)\}, (\alpha_j^{(k_1)})^+ = s_{\max_i}\{p_{ij}^{(k_1)} r_{ij}^{(k_1)}\}, k_1 = 1,2,...,\#\alpha_{ij}(p), j = 1,2,...,n$$

$$\beta^+(p) = \{(\beta_j^{(k_1)})^+ \mid k_1 = 1,2,...,\#\beta_{ij}(p)\}, \ (\beta_j^{(k_1)})^+ = s_{\min_i}\{p_{ij}^{(k_1)} r_{ij}^{(k_1)}\}, k_1 = 1,2,...,\#\beta_{ij}(p), j = 1,2,...,n$$

$$A^- = \left\langle \{\alpha^{*+}(p) = (\{\alpha_1^{*+}(p), \alpha_2^{*+}(p), ..., \alpha_n^{*+}(p)), \beta^{*+}(p) = (\beta_1^{*+}(p), \beta_2^{*+}(p), ..., \beta_n^{*+}(p))\} \right\rangle \tag{10}$$

where

$$\alpha^{*+}(p) = \{(\alpha_j^{(*k_1)})^+ \mid k_1 = 1,2,...,\#\alpha_{ij}^*(p)\}, (\alpha^{*+}(p))^+ = s_{\min_i}\{p_{ij}^{(k_1)} r_{ij}^{(k_1)}\}, k_1 = 1,2,...,\#\alpha_{ij}^*(p), j = 1,2,...,n$$

$$\beta^{*+}(p) = \{(\beta_j^{(*k_1)})^+ \mid k_1 = 1,2,...,\#\beta_{ij}^*(p)\}, (\beta^{*+}(p))^+ = s_{\max_i}\{p_{ij}^{(k_1)} r_{ij}^{(k_1)}\}, k_1 = 1,2,...,\#\chi_{ij}^*(p), j = 1,2,...,n$$

Step 7: Calculate the distances between alternative A_i and the positive ideal solution A^+, as well as between alternative A_i and the negative ideal solution A^-.

The optimal alternative should simultaneously satisfy the two conditions of having the shortest distance to the positive ideal solution A^+ and the longest distance to the negative ideal solution A^-. Given that the lengths of the elements in the linguistic term sets corresponding to the alternatives, the positive ideal solution, and the negative ideal solution are all the same, the distance between alternative A_i and the positive ideal solution A^+ can be determined in the following way [4]:

$$D(A_i, A^+) = \frac{1}{2}\sum_{j=1}^{n} w_j \sqrt{\frac{1}{2}\left[(a_{ij}^3 - (a_j^+)^3)^2 + (\beta_{ij}^3 - (\beta_j^+)^3)^2 + (\pi_{ij}^3 - (\pi_j^+)^3)^2\right]} \tag{11}$$

The distance between alternative A_i and the positive ideal solution A^- can be determined in the following way:

$$D(A_i, A^-) = \frac{1}{2}\sum_{j=1}^{n} w_j \sqrt{\frac{1}{2}\left[(a_{ij}^3 - (a_j^-)^3)^2 + (\beta_{ij}^3 - (\beta_j^-)^3)^2 + (\pi_{ij}^3 - (\pi_j^-)^3)^2\right]} \tag{12}$$

Step 8: Determine the minimum value of the distances between the alternatives and the positive ideal solution and the maximum value of the distances between the alternatives and the negative ideal solution.

From the results of Step 7, we can obtain the minimum value $D_{\min}(A_i, A^+)$ of the distances between the alternatives and the positive ideal solution and the maximum value $D_{\max}(A_i, A^-)$ of the distances between the alternatives and the negative ideal solution. The smaller the deviation degree between the alternative and the positive ideal solution is, the better it is, and the larger the deviation degree between the alternative and the negative ideal solution is, the better it is.

$$D_{\min}(A_i, A^+) = \min_{1\le i\le m} D(A_i, A^+), D_{\max}(A_i, A^-) = \max_{1\le i\le m} D_{\max}(A_i, A^-) \quad (13)$$

Step 9: Calculate the proximity coefficient cl for each alternative A_i.

This is used to measure how close each alternative is to the ideal solution; a larger proximity coefficient indicates that the alternative is superior.

$$cl(A_i) = \frac{D(A_i, A^-)}{D_{\max}(A_i, A^-)} - \frac{D(A_i, A^+)}{D_{\min}(A_i, A^+)} \quad (14)$$

Step 10: Rank the alternatives according to the value of the proximity coefficient $cl(A_i)$; the higher the value of the proximity coefficient is, the better the alternative is; conversely the worse the alternative is.

5. Example Analysis

5.1. Description of the Problem

In this paper, an example from the literature [23] is used as a reference to illustrate the feasibility and effectiveness of the extended TOPSIS method for MAGDM with PHFFLTSs.

Situation description [23]: A group of seven peoples $m_l (l = 1,2,...,7)$ need to invest their savings in a most profitable way. They considered five possible alternatives: A_1 is the real estate, A_2 is the stock market, A_3 is the short-term treasury bonds, A_4 is the National Savings Plan, and A_5 is the insurance companies. To determine best option, the following attributes are considered: C_1 is a risk factor, C_2 is the growth, C_3 is the quick returns, and C_4 is the complex documentation requirements. Based upon their knowledge and experience, they provide their opinion.

5.2. Decision Making Process and Results

Step 1: Establish the "seven-valued" linguistic term set S for the membership degree A_{ij} (or non-membership degree B_{ij}) of alternative A_i with respect to attribute C_j.

In this paper, the "seven-valued" linguistic terms are as follows: s_0 ="extremely low"; s_1 ="very low"; s_2="low"; s_3="medium"; s_4="high"; s_5="very high"; s_6="extremely high". Then,

$$S = \{s_0 = extremely\ \ low, s_1 = very\ \ low, s_2 = low, s_3 = medium, s_4 = high, s_5 = very\ \ high, s_6 = extremely\ \ high\}$$

Step 2: Create the probabilistic hesitant Fermatean fuzzy linguistic term decision matrixes.

Seven experts were divided into three groups to establish three probabilistic hesitant Fermatean fuzzy linguistic term decision matrixes, and the experts evaluated the membership degree A_{ij} (or non-membership degree B_{ij}) of the alternative A_i about the attributes C_j to obtain the probabilistic hesitant Fermatean fuzzy linguistic term decision matrixes (Table 1-Table 3).

Table 1. Decision matrix H_1 (m_1, m_2, m_3).

A_i	C_1	C_2
A_1	$(\{s_3(0.14), s_4(0.14), s_5(0.42)\}, \{s_4(0.14), s_5(0.28)\})$	$(\{s_4(0.14), s_5(0.42)\}, \{s_0(0.42), s_1(0.28)\})$
A_2	$(\{s_1(0.28), s_2(0.14)\}, \{s_3(0.42), s_4(0.14)\})$	$(\{s_3(0.14), s_4(0.14), s_5(0.28)\}, \{s_4(0.14), s_5(0.14)\})$
A_3	$(\{s_0(0.14)\}, \{s_0(0.28), s_1(0.28), s_2(0.28)\})$	$(\{s_3(0.14), s_4(0.28)\}, \{s_1(0.14), s_2(0.14)\})$
A_4	$(\{s_5(0.42), s_6(0.28)\}, \{s_3(0.28)\})$	$(\{s_1(0.14), s_2(0.14)\}, \{s_3(0.14), s_4(0.14)\})$
A_5	$(\{s_6(0.14)\}, \{s_0(0.28)\})$	$(\{s_4(0.14), s_5(0.14)\},\{s_3(0.14), s_4(0.14), s_5(0.28)\})$
A_i	C_3	C_4
A_1	$(\{s_1(0.28)\}, \{s_3(0.28), s_4(0.28)\})$	$(\{s_1(0.28), s_2(0.14)\}, \{s_3(0.28), s_4(0.28)\})$
A_2	$(\{s_3(0.14), s_4(0.28)\}, \{s_0(0.42), s_1(0.28))$	$(\{s_4(0.14), s_5(0.28)\}, \{s_1(0.28), s_2(0.28)\})$
A_3	$(\{s_5(0.42), s_6(0.14)\}, \{s_3(0.28))$	$(\{s_1(0.42), s_2(0.14)\}, \{s_2(0.28), s_3(0.42), s_4(0.28)\})$
A_4	$(\{s_1(0.42), s_2(0.42)\}, \{s_3(0.42), s_4(0.28)\})$	$(\{s_3(0.14), s_4(0.14), s_5(0.28)\}, \{s_4(0.14), s_5(0.14)\})$
A_5	$(\{s_0(0.14), s_1(0.28)\}, \{s_2(0.28), s_3(0.42)\})$	$(\{s_4(0.14), s_5(0.28)\}, \{s_1(0.14), s_2(0.14)\})$

Table 2. Decision matrix H_2 (m_4, m_5).

A_i	C_1	C_2
A_1	$(\{s_5(0.42), s_6(0.0.28)\}, \{s_3(0.28), s_4(0.14)\})$	$(\{s_5(0.42), s_6(0.28)\}, \{s_2(0.28), s_3(0.28)\})$
A_2	$(\{s_5(0.14), s_6(0.14)\}, \{s_2(0.28), s_3(0.42)\})$	$(\{s_5(0.14), s_6(0.14)\}, \{s_2(0.28), s_3(0.28), s_4(0.28)\})$
A_3	$(\{s_2(0.14)\}, \{s_0(0.28), s_1(0.28)\})$	$(\{s_1(0.14), s_2(0.14)\}, \{s_3(0.14), s_4(0.14)\})$
A_4	$(\{s_5(0.42), s_6(0.28)\}, \{s_1(0.28)\})$	$(\{s_3(0.14), s_4(0.28)\}, \{s_0(0.28), s_1(0.28), s_2(0.14)\})$
A_5	$(\{s_4(0.28), s_5(0.14)\}, \{s_1(0.28), s_2(0.28)\})$	$\{s_3(0.14), s_4(0.14)\},\{s_1(0.14), s_2(0.28), s_3(0.42)\}$
A_i	C_3	C_4
A_1	$(\{s_0(0.14), s_1(0.28)\}, \{s_3(0.28), s_4(0.28)\})$	$(\{s_3(0.14), s_4(0.14)\}, \{s_1(0.14), s_2(0.14)\})$
A_2	$(\{s_4(0.28), s_5(0.28)\}, \{s_0(0.42), s_1(0.28))$	$(\{s_5(0.28), s_6(0.14)\}, \{s_3(0.14)\})$
A_3	$(\{s_4(0.28), s_5(0.42)\}, \{s_1(0.28), s_2(0.14))$	$(\{s_1(0.42)\}, \{s_2(0.28), s_3(0.42)\})$
A_4	$(\{s_5(0.14), s_6(0.14)\}, \{s_2(0.28), s_3(0.42), s_4(0.28)\})$	$(\{s_4(0.42), s_5(0.42)\}, \{s_0(0.14)\})$
A_5	$(\{s_1(0.28), s_2(0.14)\}, \{s_3(0.42), s_4(0.28)\})$	$(\{s_5(0.28), s_6(0.28)\}, \{s_3(0.28)\})$

Table 3. Decision matrix H_3 (m_6, m_7).

A_i	C_1	C_2
A_1	$(\{s_4(0.28), s_5(0.42)\}, \{s_4(0.14), s_5(0.28)\})$	$(\{s_5(0.42), s_6(0.28)\}, \{s_3(0.28)\})$
A_2	$(\{s_3(0.14), s_4(0.14)\}, \{s_1(0.14), s_2(0.28), s_3(0.42)\})$	$(\{s_1(0.28), s_2(0.28)\}, \{s_3(0.28), s_4(0.28)\})$
A_3	$(\{s_0(0.14)\}, \{s_2(0.28), s_3(0.14), s_4(0.14)\})$	$(\{s_5(0.28), s_6(0.14)\}, \{s_3(0.14)\})$
A_4	$(\{s_4(0.14), s_5(0.42)\}, \{s_3(0.42), s_4(0.28)\})$	$(\{s_4(0.28), s_5(0.14)\}, \{s_4(0.28)\})$
A_5	$(\{s_3(0.14), s_4(0.28)\}, \{s_0(0.28), s_1(0.28), s_2(0.28)\})$	$\{s_1(0.28), s_2(0.28)\},\{s_2(0.28), s_3(0.42), s_4(0.28)\}$
A_i	C_3	C_4
A_1	$(\{s_3(0.14), s_4(0.14)\}, \{s_1(0.14), s_2(0.14)\})$	$(\{s_4(0.14), s_5(0.14), s_6(0.14)\}, \{s_3(0.28), s_4(0.28)\})$
A_2	$(\{s_5(0.28), s_6(0.14)\}, \{s_3(0.42))$	$(\{s_3(0.14), s_4(0.28)\}, \{s_1(0.28), s_2(0.28)\})$
A_3	$(\{s_4(0.28), s_5(0.42)\}, \{s_4(0.28)\})$	$(\{s_6(0.14)\}, \{s_3(0.42), s_4(0.28)\})$
A_4	$(\{s_0(0.14), s_1(0.42), s_2(0.42)\}, \{s_2(0.28), s_3(0.42)\})$	$(\{s_3(0.28), s_4(0.42), s_5(0.42)\}, \{s_1(0.28), s_2(0.28)\})$
A_5	$(\{s_2(0.14), s_3(0.14)\}, \{s_3(0.42), s_4(0.28)\})$	$(\{s_6(0.28)\}, \{s_0(0.28)\})$

Step 3: The three decision matrices are integrated into a probabilistic hesitant Fermatean fuzzy linguistic term decision matrix.

The maximum and minimum values of the membership degree and non-membership degree in the decision matrices are computed. Where, $A_{ij}^{k}(p_{ij})$ is computed by $p_{ij} \times r_{ij}^{l}, l = 1,2,...,\#l_{ij}^{k}(p_{ij})$; the subscripts r_{ij}^{l} indicating the $l-th$ linguistic term set ; p_{ij} are the corresponding probabilities. The maximum and minimum values of the membership degree and non-membership degree are summarized according to Equation (6), and the three decision matrices are integrated into a probabilistic hesitant Fermatean fuzzy linguistic term decision matrix (Table 4).

Taking the membership degree in C_{11}^{k} as an example, the integrated membership degree is $\{s_6(0.28), s_5(0.42)\}$:

$$\alpha_{m_{ij}}(p_{ij}) = \min\left\{\min_{k=1}^{l}(s_5(0.42), s_5(0.42), s_5(0.42)), \max_{k=1}^{l}(s_3(0.14), s_4(0.28), s_6(0.28)\right\} = s_6(0.28)$$

$$\beta_{n_{ij}}(p_{ij}) = \max\left\{\min_{k=1}^{l}(s_5(0.42), s_5(0.42), s_5(0.42)), \max_{k=1}^{l}(s_3(0.14), s_4(0.28), s_6(0.28)\right\} = s_5(0.42)$$

Table 4. The integrated decision matrix H .

A_i	C_1	C_2
A_1	$(\{s_6(0.28), s_5(0.42)\}, \{s_4(0.14), s_3(0.28)\})$	$(\{s_5(0.42), s_6(0.28)\}, \{s_1(0.28), s_3(0.28)\})$
A_2	$(\{s_2(0.14), s_5(0.14)\}, \{s_3(0.42), s_4(0.14)\})$	$(\{s_2(0.28), s_5(0.14)\}, \{s_3(0.28), s_5(0.14)\})$
A_3	$(\{s_0(0.14), s_2(0.14)\}, \{s_1(0.28), s_3(0.14)\})$	$(\{s_2(0.14), s_6(0.14)\}, \{s_2(0.14), s_3(0.14)\})$
A_4	$(\{s_5(0.42), s_6(0.28)\}, \{s_1(0.28), s_3(0.28)\})$	$(\{s_2(0.14), s_5(0.14)\}, \{s_2(0.14), s_4(0.28)\})$
A_5	$(\{s_6(0.14), s_6(0.14)\}, \{s_0(0.28), s_1(0.28)\})$	$\{s_2(0.28), s_4(0.14)\},\{s_2(0.28), s_3(0.42)\}$
A_i	C_3	C_4
A_1	$(\{s_1(0.28), s_3(0.14)\}, \{s_2(0.14), s_3(0.28)\})$	$(\{s_2(0.14), s_4(0.14)\}, \{s_2(0.14), s_3(0.28)\})$
A_2	$(\{s_4(0.28), s_4(0.28)\}, \{s_1(0.28), s_3(0.42))$	$(\{s_4(0.28), s_6(0.14)\}, \{s_3(0.14), s_3(0.14)\})$
A_3	$(\{s_4(0.28), s_5(0.42)\}, \{s_2(0.14), s_4(0.28)\})$	$(\{s_1(0.42), s_6(0.14)\}, \{s_3(0.42), s_4(0.28)\})$
A_4	$(\{s_2(0.42), s_5(0.14)\}, \{s_3(0.42), s_4(0.28)\})$	$(\{s_4(0.42), s_5(0.28)\}, \{s_0(0.14), s_4(0.14)\})$
A_5	$(\{s_1(0.28), s_2(0.14)\}, \{s_3(0.42), s_4(0.28)\})$	$(\{s_5(0.28), s_6(0.28)\}, \{s_0(0.28), s_3(0.28)\})$

Step 4: Normalize the integrated matrix.

The probabilities of membership degree and non-membership degree in this integrated matrix are normalized by applying Equation (7) to obtain the final integrated criteria matrix H^* (Table 5).

Take for example the membership degree in C_{11}^{k}: 0.28 / (0.42+0.28) = 0.40, 0.42 / (0.42+0.28) = 0.60, the linguistic term for the membership degree after normalization is $\{s_6(0.40), s_5(0.60)\}$.

Step 5: Calculate the weight w_j of the attributes C_j.

The weights of the corresponding attribute values are calculated by applying Equation (8). In this paper, to facilitate the comparison between the extended TOPSIS method for MAGDM with PHIFLTSs [23] and the extended TOPSIS method for MAGDM with PHFFLTSs, the weights are used in line with the literature [23].

$$W = (0.2715, 0.2219, 0.2445.0.2621)$$

Step 6: Determine the positive ideal solution and negative ideal solution among the alternatives.

We calculate the positive ideal solution and negative ideal solution of the decision matrix according to Equations (9) and (10). Take C_{11}^{k} as an example, because C_1 is the risk factor belongs to the cost type attribute, the maximum value in the membership degree is $\{3,3\}$, the minimum value is $\{0,1\}$; the maximum value in the non-membership degree is $\{2.25,2.01\}$, the minimum value is $\{0,0.5\}$; so under the attribute C_1 of the positive ideal solution is $\langle\{0,1\},\{2.25,2.01\}\rangle$, the negative ideal solution is $\langle\{3,3\},\{0,0.5\}\rangle$ (Table 6).

Table 5. The normalized decision matrix H^*.

A_i	C_1	C_2
A_1	$(\{s_6(0.40),s_5(0.60)\},\{s_4(0.33),s_3(0.67)\})$	$(\{s_5(0.60),s_6(0.40)\},\{s_1(0.50),s_3(0.50)\})$
A_2	$(\{s_2(0.50),s_5(0.50)\},\{s_3(0.75),s_4(0.25)\})$	$(\{s_2(0.67),s_5(0.33)\},\{s_3(0.67),s_5(0.33)\})$
A_3	$(\{s_0(0.50),s_2(0.50)\},\{s_1(0.67),s_3(0.33)\})$	$(\{s_2(0.50),s_6(0.50)\},\{s_2(0.50),s_3(0.50)\})$
A_4	$(\{s_5(0.60),s_6(0.40)\},\{s_1(0.50),s_3(0.50)\})$	$(\{s_2(0.50),s_5(0.50)\},\{s_2(0.33),s_4(0.67)\})$
A_5	$(\{s_6(0.50),s_6(0.50)\},\{s_0(0.50),s_1(0.50)\})$	$\{s_2(0.67),s_4(0.33)\},\{s_2(0.4),s_3(0.6)\}$
A_i	C_3	C_4
A_1	$(\{s_1(0.67),s_3(0.33)\},\{s_2(0.33),s_3(0.67)\})$	$(\{s_2(0.50),s_4(0.50)\},\{s_2(0.33),s_3(0.67)\})$
A_2	$(\{s_4(0.50),s_4(0.50\},\{s_1(0.40),s_3(0.60))$	$(\{s_4(0.67),s_6(0.33)\},\{s_3(0.50),s_3(0.50)\})$
A_3	$(\{s_4(0.40),s_5(0.60)\},\{s_2(0.33),s_4(0.67)\})$	$(\{s_1(0.75),s_6(0.25)\},\{s_3(0.60),s_4(0.40)\})$
A_4	$(\{s_2(0.75),s_5(0.25\},\{s_3(0.60),\ s_4(0.40)\})$	$(\{s_4(0.6),s_5(0.4)\},\{s_0(0.5),s_4(0.5)\})$
A_5	$(\{s_1(0.67),s_2(0.33)\},\{s_3(0.60),s_4(0.40)\})$	$(\{s_5(0.50),s_6(0.50)\},\{s_0(0.50),s_3(0.50)\})$

Table 6. The maximum and minimum value.

C_1	Membership degree	Non-membership degree
A_1	$\{2.4,3\}$	$\{1.32,2.01\}$
A_2	$\{1,2.5\}$	$\{2.25,2.01\}$
A_3	$\{0,1\}$	$\{0.67,0.99\}$
A_4	$\{3,2.4\}$	$\{0.5,1.5\}$
A_5	$\{3,3\}$	$\{0,0.5\}$

The positive ideal solution and negative ideal solution are:

$$A^+ = (\langle\{0,1\},\{2.25,2.01\}\rangle,\langle\{3,3\},\{0.5,1.5\}\rangle,\langle\{2,3\},\{0.4,1.6\}\rangle,\langle\{2.68,3\},\{0,1.5\}\rangle$$
$$A^- = (\langle\{3,3\},\{0,0.5\}\rangle,\langle\{1,1.32\},\{2.01,2.68\}\rangle,\langle\{0.67,0.66\},\{1.8,2.68\}\rangle,\langle\{0.75,1.5\},\{1.8,2.01\}\rangle$$

Step 7: Calculate the distances between alternative A_i and the positive ideal solution A^+, as well as between alternative A_i and the negative ideal solution A^-.

$$D(A_1,A^+)=1.8854, D(A_2,A^+)=1.7503, D(A_3,A^+)=2.0383, D(A_4,A^+)=2.2191, D(A_5,A^+)=2.4591$$
$$D(A_1,A^-)=1.7452, D(A_2,A^-)=2.0166, D(A_3,A^-)=2.0445, D(A_4,A^-)=1.6773, D(A_5,A^-)=1.2842$$

Step 8: Determine the minimum value of the distances between the alternatives and the positive ideal solution and the maximum value of the distances between the alternatives and the negative ideal solution.

From the results of Step 7, we can obtain the minimum value $D_{\min}(A_i,A^+)$ of the distances between the alternatives and the positive ideal solution, and the maximum value $D_{\max}(A_i,A^-)$ of the distances between the alternatives and the negative ideal solution. The smaller the deviation degree between the alternative and the positive ideal solution is, the better it is, and the larger the deviation degree between the alternative and the negative ideal solution is, the better it is.

$$\underset{\min}{D}(A_i,A^+)=1.7503, \underset{\max}{D}(A_i,A^-)=2.0445$$

Step 9: Calculate the proximity coefficient cl and rank the alternatives.

We calculate the proximity coefficients cl of the alternatives according to Equation (14), according to Equation (14) the proximity coefficients of the alternatives can be calculated as respectively:

$$cl(A_1) = -0.2235, cl(A_2) = -0.0138, cl(A_3) = -0.1645, cl(A_4) = -0.4474, cl(A_5) = -0.0128$$

Step 10: The ranking of the alternatives can be derived based on the values of cl as follows:

$$A_5 \succ A_2 \succ A_3 \succ A_1 \succ A_4$$

5.3. Comparative Analysis

To illustrate the feasibility and effectiveness of the extended TOPSIS method for MAGDM with PHFFLTSs (Method 1) in this paper, the decision-making results will be compared with those of the extended TOPSIS method for MAGDM with PHIFLTSs (Method 2). The decision results of Method 1 are: the proximity coefficients are -0.2235, -0.0138, -0.1645, -0.4475 and -0.0128, respectively, and the ranking of the alternatives is $A_5 \succ A_2 \succ A_3 \succ A_1 \succ A_4$. From the literature [23], the decision results of Method 2 are: the proximity coefficients are -0.4486, -0.8876, -1.0924, -0.0912 and -0.0519, respectively, and the ranking of the alternatives is $A_5 \succ A_4 \succ A_1 \succ A_2 \succ A_3$. This demonstrates the feasibility of Method 1.

To further examine the effectiveness of Method 1, we calculate the discrimination [30] of Method 1 for alternatives, and the discrimination of Method 2 for alternatives, and the larger the discrimination is, the better the performance of method is. The calculation results show that the discrimination of Method 1 for alternatives is 0.7129, and the discrimination of Method 2 for alternatives is 0.7091. The discrimination of Method 1 for alternatives is higher than that of Method 2, which means that the performance of Method 1 is better than that of Method 2, indicating the effectiveness of Method 1.

6. Conclusions

This paper proposes the extended TOPSIS method for MAGDM with PHFFLTSs in response to the limitations of the existing researches. The results of the example analysis illustrate the feasibility and validity of the extended TOPSIS method for MAGDM with PHFFLTSs as proposed in this paper. The potential marginal contributions of this paper are as follows: 1) Introduction of the probabilistic hesitant Fermatean fuzzy linguistic term set. This paper innovatively introduces the concept of the probabilistic hesitant Fermatean fuzzy linguistic term set, which combines the characteristics of probabilistic linguistic term sets, hesitant fuzzy linguistic term sets, and Fermatean fuzzy sets. This concept enables a more comprehensive and flexible representation of experts' uncertain and fuzzy evaluations during the decision-making process. This innovation provides a new theoretical tool for addressing complex multi-attribute group decision-making problems. 2) Instance verification and comparative analysis to demonstrate method effectiveness and superiority. Through specific case studies, the paper validates the feasibility and effectiveness of the proposed method. Simultaneously, by comparing it with the extended TOPSIS method for MAGDM with PHIFLTSs, the paper showcases the superiority of the new method in terms of discrimination and decision outcomes. This comparison not only proves the practicality of the new method but also provides strong support for its broader application. 3) Providing a new research perspective for the field of fuzzy multi-attribute decision-making. The research in this paper not only enriches the theoretical framework of fuzzy multi-attribute decision-making but also offers a fresh perspective and methodology for the field. By introducing the probabilistic hesitant Fermatean fuzzy linguistic term set, it opens possibilities for addressing more complex and uncertain decision-making problems, thereby advancing the development of the fuzzy multi-attribute decision-making domain.

The research of this paper will provide a new way and idea for the problem of determining the membership degree and non-membership degree of fuzzy set theory in practice, and it is of great practical significance to convert the linguistic terms into the form of numerical values, which can reflect the ambiguities and hesitations of the experts' assessment more accurately. The research in this paper also further improves the theoretical system of hesitant fuzzy set and expands its trial scope and method. Firstly, in the example study of evaluating alternatives, the decision matrix and linguistic term in this paper are fewer, which has certain limitations, and the subsequent study will further consider introducing more expert evaluations and linguistic term to enhance the objectivity and persuasiveness of multi-attribute evaluation. Secondly, in this paper, the subscripts of linguistic terms in membership degree or non-membership degree are multiplied by probability to merge and deal with probabilistic hesitant fuzzy information, however, membership degree and probability are two different dimensions of information, and future research will consider a more reasonable way to deal with probabilistic information.

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