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CONVOLUTION INEQUALITIES WITH PROBABILITY DISTRIBUTIONS

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Abstract

There are many results related to inequalities linked to convolutions. We can create a new probability distribution from well-known probability distributions. One of the classical method is addition. If we want to find the probability distribution of the sum of two independent probability random variables then we need to find the convolution of their distributions. In this paper, I computed the upper bound of the convolution of several several independent random variables: Normal Distributions and Exponential Distributions. See the paper for correct style for mathematics.

1. Introduction

In mathematics, there are many ways to approach a problem. In those vari-ous attempts, there is always the possibility to arrive at the same, correct solu-tion. Probability theory is not any different; however, the approach to the problem could make it complicated or simplistic. The goal here was to take a complicated probability theory concept and attempt to find a way to simplify the calculation. With mathematics, encountering complicated data sets is something that occurs frequently. When a mathematician is presented with one of these data sets, he often performs multiple operations that will make it more complex. For example, imagine the first data set is the probabilities that the random variable X1 takes on given values, and the second set is the probabilities of another random variable X2 taking on given values. From these sets we can, through brute force and many calculations, determine the probabilities of X1 + X2 equaling anything; however, if at all possible we would like to avoid these tedious computations. To further look at these random variables, we consider a nickel and a penny. These two different coins are flipped for a data set 50 times. We then record each

time the coins land on heads versus each time the coins land on tails. After recording the results, we would calculate the probability of the addition of the two variables. It is important to re-alize that these variables could be anything, not just coins. Currently, we only have certain convolution knowledge in the area of Probability Theory.
In many papers, people computed the convolution inequalities and applications[4], [1]. This limits our ability to perform these more complicated calculations in a manner that makes the result worth discovering. To circle back, by the completion of this project, we hoped to obtain a new form or new problem that would make those tedious computations easier to manage. Additionally, the discovery of a new convolution inequality could change the way that Probability Theory is taught and allow new found knowledge of calculating random variables. In the beginning of our research, we hypothesized that we could determine new upper and lower bounds to serve the convolution inequality integral for computations. Our research worked to first find the upper bound of convolution with two specific probabilistic distributions and then find the numerical value of upper bound. We chose exponential distribution and normal distribution for our study. With these distributions we will check the inequality in [2] such that

Theorem 1.1. Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\mathbb{R}} f(x)f(x+t)dxdt \leq 0.91\|f\|_1\|f\|_2 \tag{1}$$

Moreover, the constant cannot be replaced by 0.8.

What follows is our result for this inequality and the upper bound with probabilistic distributions.

2. Preliminaries

In this section, I introduce the definitions of key terms relevant to our research. Most definitions are written in [3].

Definition 2.1. The sample space Ω is the set of all the possible outcomes of the experiment. Elements of Ω are called sample points and typically denoted by ω.

Definition 2.2. A probability measure P on a sample space Ω is a real-valued function with three properties:

(1) $P(A) \geq 0$, for $A \subseteq \Omega$
(2) $P(\Omega) = 1,\ P(\emptyset) = 0$
(3) For the disjoint sequence of events (i.e. $A_i \cap A_j =$ for $i \neq j$) then:

$$P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n).$$

Definition 2.3 (Random Variable). Let **Ω** be a sample space. A Random Variable, X is a function from **Ω** into the real numbers.

Definition 2.4 (Probability Mass Function). The probability mass function (PMF), of a discrete random variable is X is the function p (or p_X) defined by p(k) = P (X = k) for possible values k of X

Definition 2.5 (Probability Density Function). Let X be a random variable. If

a function f satisfies $P(x \leq b) = \int_{-\infty}^{b} f(x)dx$ for all real values, b, then f is the

probability density function (PDF) of X.

Definition 2.6 (Exponential Distribution). Let $0 < \lambda < \infty$. A random variable X has the exponential distribution with the parameter λ if X has density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

on the real line. Abbreviate this by $X \sim Exp(\lambda)$. The $Exp(\lambda)$ distribution is also called the exponential distribution with rate λ.

Definition 2.7 (Normal Distribution). A random variable Z has standard normal distribution (also called standard Gaussian distribution) if Z has the density function

$$\psi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}}$$

on the real line. Abbreviate this as $Z \sim N(0, 1)$.

Definition 2.8 (Convolution). If X and Y are independent continuous random variables with density functions f_X and f_Y then the density function of $X + Y$ is

$$f_{X+Y}(z) = f_X * f_Y(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx = \int_{-\infty}^{\infty} f_X(z-x) f_Y(x)dx.$$

Let X be a random variable with the density function $f_X(x)$ where $x \in \mathbb{R}$. Now if you consider the random variable $-X$, the density function of $-X$ is $f_X(-x)$ where $x \in \mathbb{R}$. And the convolution of these two random variable is

$$f_{X+(-X)}(z) = \int_{\mathbb{R}} f_X(x) f_X(x-z)dx.$$

Thus this form implies the statement in the Theorem1.1 came from the convolution of two random variables.

Definition 2.9 (L^p space and L^p norm)**.** Let (X, A, μ) be a measure space and $1 \leq p < \infty$. The space $L^p(X)$ is the set of equivalence classes of measurable functions $f : X \to \mathbb{R}$, for which $\int_X |f|^p d\mu < \infty$, when these measurable functions are equivalent if they are equal, μ-a.e. The L^p norm is defined by $||f||_p = (\int_X |f|^p d\mu)^{\frac{1}{p}}$.

3. Inequalities on Convolution

3.1. Exponential Distributions. In this section, I work with the probability density function of Exponential Distributions.

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

with a parameter $0 < \lambda < \infty$.

Since the function f(x) is non-negative for all $x \in \mathbb{R}$,

$$\begin{aligned} \|f\|_1 &= \int_{-\infty}^{\infty} |f(x)| \ dx \\ &= \int_{-\infty}^{\infty} f(x) \ dx = 1 < \infty. \end{aligned}$$

Since $0 < \lambda < \infty$, we have

$$
\begin{aligned}
\|f\|_2^2 &= \int_{-\infty}^{\infty} |f(x)|^2 \; dx \\
&= \int_{-\infty}^{\infty} f^2(x) \; dx \\
&= \int_{0}^{\infty} \lambda^2 e^{-2\lambda x} \; dx \\
&= \lim_{b\to\infty} \int_{0}^{b} \lambda^2 e^{-2\lambda x} \; dx \\
&= \lim_{b\to\infty} \frac{{}^{2}}{2\lambda} - \frac{\lambda^2 e^{-2\lambda b}}{2\lambda} = \frac{\lambda}{2} < \infty.
\end{aligned}
$$

Hence we can conclude that

$$f \in L^1 \cap L^2.$$

Now, we compute the convolution for

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

$$\int_{t=-\frac{1}{2}}^{t=\frac{1}{2}} \int_{x=-\infty}^{x=\infty} f(x) \cdot f(x+t) dx dt$$

$f(x)$ is 0 for $x < 0$, thus the inner integral bounds can be $[0, \infty)$

$$\int_{t=-\frac{1}{2}}^{t=1} \int_{x=0}^{x=\infty} f(x) \cdot f(x+t) dx dt$$

If $-\frac{1}{2} \le t \le 0$ and $x \le -t$ then $f(x+t)$ is 0. Therefore, the outer integral can be split as $[-\frac{1}{2}, 0] \cup [0, \frac{1}{2}]$.

$$= \int_{t=-\frac{1}{2}}^{t=0} \int_{x=0}^{x=\infty} f(x)f(x+t)dxdt + \int_{t=0}^{t=\frac{1}{2}} \int_{x=0}^{x=\infty} f(x)f(x+t)dxdt$$

$$= \int_{t=-\frac{1}{2}}^{t=0} \int_{x=-t}^{x=\infty} f(x)f(x+t)dxdt + \int_{t=0}^{t=\frac{1}{2}} \int_{x=0}^{x=\infty} f(x)f(x+t)dxdt$$

$$= \int_{t=-\frac{1}{2}}^{t=0} \lim_{b\to\infty} \int_{x=-t}^{b} \lambda^2 e^{-\lambda x} \cdot e^{\lambda x - \lambda t} dxdt + \int_{t=0}^{t=\frac{1}{2}} \lim_{b\to\infty} \int_{x=0}^{x=\infty} \lambda^2 e^{-\lambda x} e^{-\lambda x - \lambda t} dxdt$$

$$= \int_{t=-\frac{1}{2}}^{0} \lim_{b\to\infty} \left[-\frac{\lambda}{2} e^{-2\lambda x - \lambda t} \right]_{x=-t}^{x=b} dt + \int_{t=0}^{t=\frac{1}{2}} \lim_{b\to\infty} \left[-\frac{\lambda}{2} e^{-2\lambda x - \lambda t} \right]_{x=0}^{x=b} dt$$

$$= \int_{t=-\frac{1}{2}}^{0} (0 + \frac{\lambda}{2} e^{\lambda t}) dt + \int_{t=0}^{t=\frac{1}{2}} (0 + \frac{\lambda}{2} e^{-\lambda t}) dt$$

$$= \left[\frac{1}{2} e^{\lambda t} \right]_{-\frac{1}{2}}^{0} + \left[-\frac{\lambda}{2} e^{-\lambda t} \right]_{0}^{\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{1}{2} e^{-\frac{\lambda}{2}} + -\frac{\lambda}{2} e^{-\frac{\lambda}{2}} + \frac{1}{2}$$

$$= 1 - e^{-\frac{\lambda}{2}}.$$

Thus, we have $\|f\|_1 = 1$ and $\|f\|_2 = \sqrt{\frac{\lambda}{2}}$. Hence from 1.1, we have $1 - e^{-\frac{\lambda}{2}} \le c \cdot \sqrt{\frac{\lambda}{2}}$ for every $\lambda > 0$.

Example. If $\lambda = 1$ then

$$\frac{1 - e^{-\frac{\lambda}{2}}}{\sqrt{\frac{\lambda}{2}}} = \frac{1 - e^{-\frac{1}{2}}}{\sqrt{\frac{1}{2}}} \approx 0.56.$$

Hence we can conclude that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\mathbb{R}} f(x)f(x+t)dxdt \le 0.6\|f\|_1\|f\|_2.$$

This means that the Theorem 1.1 holds for Exponential Distributions.

3.2. **Normal Distributions.** In this section, I work with the probability density function of Standard Normal Distributions. $N(0,1)$.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

Then this function satisfies the following two conditions:
Since the function $f(x)$ is non-negative for all $x \in \mathbb{R}$,

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)| \, dx$$

$$= \int_{-\infty}^{\infty} f(x) \, dx = 1 < \infty.$$

Also, we also know that

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$$

and for any $a \in \mathbb{R}$

$$\int_{-\infty}^{\infty} e^{-(x-a)^2} dy = \sqrt{\pi}$$

Hence,

$$\begin{aligned} ||f||_2^2 &= \int_{-\infty}^{\infty} |f(x)|^2 \, dx \\ &= \int_{-\infty}^{\infty} f^2(x) \, dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-x^2} \, dx = \frac{1}{2\sqrt{\pi}} < \infty. \end{aligned}$$

From the above computation, we can conclude that

$$f \in L^1 \cap L^2.$$

Now let's consider left side of the inequality (1): So begins the solution of the inequality with the use of the normal distributions above.

$$\begin{aligned} &\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+t)^2} dxdt \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-x^2-xt-\frac{1}{2}t^2} dxdt \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-(x^2+xt)-\frac{1}{2}t^2} dxdt \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-(x+\frac{1}{2})^2-\frac{1}{2}t^2+\frac{1}{4}t^2} dxdt \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{4}t^2} \cdot e^{-(x+\frac{1}{2}t)^2} dxdt \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2\pi} e^{-\frac{1}{4}t^2} \int_{-\infty}^{\infty} e^{-(x+\frac{1}{2}t)^2} dxdt \end{aligned}$$

We recognize that the function with respect to x is equivalent to $\sqrt{\pi}$ and will be using substitution as $u = \frac{t}{2}$.

$$\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2\sqrt{\pi}} \cdot e^{-\frac{1}{4}t^2} dt \\
&\quad = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2\sqrt{\pi}} e^{-(\frac{t}{2})^2} dt \\
&\quad = \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{2\sqrt{\pi}} e^{-u^2} \cdot 2du \\
&\quad = \frac{1}{\sqrt{\pi}} \int_{-\frac{1}{4}}^{\frac{1}{4}} e^{-u^2} du \\
&\quad = \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{1}{4}} e^{-u^2} du \\
&\quad = \frac{2}{\sqrt{\pi}} \cdot \operatorname{erf}(\frac{1}{4})
\end{aligned}$$

where

$$\operatorname{erf}(x) = \int_0^x e^{-t^2} dt$$

.

Now we will compute the right hand side of the inequality (1). Since $\|f\| \; = 1$ and $\|f\|_2^2 = \frac{1}{2\sqrt{\pi}}$, we have

$$\|f\|_1 \|f\|_2 = \sqrt{\frac{1}{2\sqrt{\pi}}}.$$

Hence we can conclude the following Theorem.

Theorem 3.1. For probability density function of Standard Normal Distribution, $f(x)$.we have the following inequality:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\infty}^{\infty} f(x) f(x+t) dx dt \leq c \|f\|_1 \|f\|_2, \tag{2}$$

for any $c \geq \frac{\sqrt{2}\operatorname{erf}(\frac{1}{4})}{(\pi)^{1/4}}$.

Example. Since

$$\frac{\sqrt{2}\operatorname{erf}(\frac{1}{4})}{\pi^{1/4}} \approx 0.293.$$

Hence we can conclude that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\mathbb{R}} f(x) f(x+t) dx dt \leq 0.3 \|f\|_1 \|f\|_2.$$

This means that the Theorem 1.1 holds for Standard Normal Distributions.

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