# SOME PROPERTIES OF GRADED OVERRINGS

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Honor's Thesis

**Abstract**

In this thesis, we introduce graded notions of injective extensions and injective domains. We prove that for a graded going-down domain $R$, if a graded overring $T$ has the property that $R \subseteq T$ is a graded injective extension, then $T$ is a graded going-down domain.

## Introduction

We begin by recalling standard definitions found in undergraduate algebra texts such as [2] or [1].

**Definition 1.1.** ***A ring*** $\langle R, +, \cdot \rangle$ *is a set* $R$ *together with two binary operations + and ·,which we call addition and multiplication, defined on* $R$ *such that the following axioms are satisfied:*

$\mathscr{R}_1$. $\langle R, + \rangle$ *is an abelian group.*

$\mathscr{R}_2$. *Multiplication is associative.*

$\mathscr{R}_3$. *For all,* $a, b, c \in R$ ***the left distributive law,***

$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ *and the* ***right distributive law*** $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ *hold.*

**Definition 1.2.** *A ring in which the multiplication is commutative is a* ***commutative ring****.*

**Definition 1.3.** *A ring with a multiplicative identity element is a* ***ring with unity;*** *the multiplicative identity element 1 is called "****unity****."*

**Definition 1.4.** *If a and b are two nonzero elements of a ring* $R$ *such that,* $ab = 0$*, then* $a$ *and* $b$ *are* ***divisors of*** $0$ (or $0$ ***divisors***).

**Definition 1.5.** An ***integral domain*** $D$ *is a commutative ring with unity* $1 \neq 0$ *that contains no divisors of* $0$.

Let $\mathbb{Z}$ be the ring of integers defined by the usual multiplication and addition. A nonzero integer $a \in \mathbb{Z}$ is a divisor of $0$ if there exists a nonzero integer b such that $ab = 0.$. In the integers, $ab = 0.$ if and only if $a = 0$ or $b = 0$ hence, we know that $\mathbb{Z}$ is an integral domain. However, $\mathbb{Z}_6 = \{0, 1, 2, 3, \dot{4}, 5\}$ the set defined by addition and multiplication modulus $6$ is not an integral domain, since $2 \cdot 3 = 6 \equiv 0(\mod 6)$.

**Definition 1.6.** *Let* $R$ *be an integral domain, then we say* $K$ *is the fraction field of* $R$ *if*:

$$K = \{\frac{r}{s} : r \in R, s \in S = R \backslash \{0\}\}.$$

An example of a fraction field would be $\mathbb{Q}$ the set of rationals over the set of integers $\mathbb{Z}$. We see that the rational numbers can be defined such that $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$.

**Definition 1.7.** *Let* $R$ *be an integral domain. An* ***overring*** *of* $R$ *is a ring* $S$ *sitting between* $R$ *and its fraction field* $K$*. That is,* $R \subseteq S \subseteq K$.

**Definition 1.8.** *A* ***multiplicatively closed subset*** *is a subset* $S$ *of a ring* $R$ *if it satisfies the following:*

1. $1 \in S$, and
2. For any elements $x, y \in S$, then $x \cdot y \in S$.

**Definition 1.9.** *Let* $R$ *be an integral domain and* $S$ *a multiplicatively closed subset in* $R$*. The* ***localization*** *of* $R$ *with respect to* $S$ *is the set*

$$S^{-1}R = \{\frac{r}{s} : r \in R, s \in S\}.$$

Similarly, we can use $\mathbb{Z}$ to illustrate the idea of localizations. Consider $\mathbb{Z}$ with the usual multiplication and addition to be a commutative ring. Let $S = \{2^n : n \in \mathbb{N}\}$. Then $S$ is a multiplicatively closed subset in $\mathbb{Z}$ since for any $n, m \in \mathbb{Z}$, we have $2^n \cdot 2^m = 2^{n+m}$, where any element of this localization would be of the form $\frac{m}{2^n}$ , where $m \in \mathbb{Z}$. For example, an element of this fraction field would be of the form $\frac{3}{2^2} = \frac{3}{4}$ would be in $S^{-1}\mathbb{Z}$.

**Definition 1.10.** *Let $R$ be a ring. A subset $\mathfrak{a}$ is called an ideal if*

*(1)* $0 \in \mathfrak{a}$

*(2) whenever* $a, b \in \mathfrak{a}$, *also* $a + b \in \mathfrak{a}$, *and*

*(3) whenever* $x \in R$ *and* $a \in \mathfrak{a}$, *also* $xa \in \mathfrak{a}$.

**Definition 1.11.** *A **proper nontrivial ideal** of a ring $R$ is an ideal $\mathfrak{a}$ such that* $\mathfrak{a} \neq R$ and $\mathfrak{a} \neq \{0\}$ .

**Definition 1.12.** *An ideal $\mathfrak{p}$ is called **prime** if its complement $R - \mathfrak{p}$ is multiplicative, or equivalently, if $1 \notin \mathfrak{p}$ and if $xy \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ , for all $x, y \in R$*

**Definition 1.13.** *The prime spectrum of a commutative ring $R$, also referred to as the spectrum and denoted*

$$Spec\ (R) = \{p : p\ is\ a\ prime\ ideal\ of\ R\ \}'$$

*is the set of all prime ideals of $R$.*

The prime spectrum forms a topological space under the Zariski topology.

**Definition 1.14.** *Let $R$ be a commutative ring. Let the set of all prime*

*ideals within the ring that contain a fixed ideal be a closed set. These closed sets form the **Zariski topology**.*

**Definition 1.15.** *Let $R$ be a ring. An ideal m is said to be maximal if $\mathfrak{m}$ is proper and if there is no proper ideal $\mathfrak{a}$ with $\mathfrak{m} \subsetneq \mathfrak{a}$.*

All maximal ideals in a commutative ring with unity are prime ideals.

**Definition 1.16.** *A ring extension $R \subseteq S$ satisfies going-down if whenever $P_1 \subseteq P_2$ where $P_1, P_2 \in Spec(R)$ and $Q_2 \in Spec(S)$ with $Q_2 \cap R = P_2$, there exists $Q_1 \in Spec(S)$ with $Q_1 \subseteq Q_2$ and $Q_1 \cap R = P_1$.*

**Definition 1.17.** *Let $R$ be an integral domain. Then $R$ is a **going-down domain**, denoted GD domain, if every overring satisfies going-down.*

The following definition and theorems of injective extensions is as defined in [3].

**Definition 1.18.** *Suppose $R \subset T$ is a ring extension. A ring extension is a **injective extension**, denoted i-extension, if $Spec(T) \to Spec(R)$ is injective.*

**Definition 1.19.** *A domain $R$ is an **injective domain**, denoted i-domain, if $R \subseteq T$ is an i-extension for each domain $T$ containing $R$ and contained in the quotient field of $R$.*

**Theorem 1.20.** *Let $R$ be GD domain. If $R \subset T$ is an i-extension, then $T$ is GD domain.*

## 2. GRADED ANALOGS

We present the following definitions and theorems elementary to the student of graded ring theory that will be utilized in the following section. This information can be found in a standard text on graded ring theory.

A torsion-free abelian group is an abelian group which has no nonzero torsion elements such that the identity element is the only element with finite order.

**Definition 2.1.** *Let G be a torsion-free abelain group. An integral domain* $R$ *is a* ***G-graded domain*** *if*

$$R = \bigoplus_{\alpha \in G} R_\alpha$$

*where each* $R_\alpha$ *is an additive subgroup of* $R$ *and* $R_\beta R_\gamma \subseteq R_{\beta+\gamma}$ *for all* $\beta, \gamma \in G$.

Throughout this section, we assume that $R$ is a G-graded domain.

**Definition 2.2.** *A nonzero element* $r \in R_\alpha$ *is* ***homogeneous*** *of degree* $\alpha$.

**Definition 2.3.** *Every element* $r \in R$ *has a unique decomposition* $r = \sum_{\alpha \in G} r_\alpha$, $r_\alpha \in R_\alpha$ *, where all but finitely many* $r_\alpha = 0$. *The homogeneous elements ra are the homogeneous components of* $r$.

The ring $R = \mathbb{Z}[x]$ is a Z-graded domain, with $R_\alpha = \mathbb{Z}x^\alpha$ for $\alpha \geq 0$ and $R_\alpha = 0$ for $\alpha < 0$.

**Definition 2.4.** *The domain* $R_H$, *where* $H = \{x \in R | x \neq 0$ *is homogeneous}, is called the* ***graded (or homogenous) fraction field*** *of* $R$ *where*

$$(R_H)_\alpha = \{\frac{f}{g} : f, g \in H, g \neq 0 \text{ and } \deg(f) - \deg(g) = \alpha\}$$

For the G-graded domain $\mathbb{Z}[x]$, we have the graded fraction field: $\mathbb{Q}[x, x^{-1}]$.

**Definition 2.5.** *An ideal $I$ of $R$ is a **graded ideal** of $R$ if $I$ is generated by homogeneous elements contained in $I$.*

**Definition 2.6.** *A **graded prime** ideal of $P$ of $R$ is a graded ideal such that for all homogeneous $x, y \in R$, if $xy \in P$, implies $x \in P$ or $y \in P$.*

**Definition 2.7.** *The **graded prime spectrum** of a ring $R$, also referred to as the graded spectrum and denoted*

$$gr\text{-}Spec\ (R) = \{p : p\ is\ a\ graded\ prime\ ideal\ of\ R\ \}$$

*is the set of all graded prime ideals of $R$.*

**Definition 2.8.** *A graded ideal is a **graded maximal ideal*** if it is maximal *with respect to the graded ideals of $R$.*

**Definition 2.9.** *A graded domain with exactly one graded maximal ideal is a graded local domain, denoted gr-local.*

**Definition 2.10.** *An overring $S = \bigoplus_{\alpha \in G} S_\alpha$ of $R$ that is a subring of $R_H$ where $S_\alpha = S \cap (R_H)_\alpha$ is a **graded overring** of $R$.*

The following definition of the graded going-down property is as defined in [4].

**Definition 2.11.** *Let $R \subseteq T$ be an extension of graded domains. We say that $R \subseteq T$ satisfies **graded going-down**, denoted gr-GD if, whenever $P_0 \subseteq P$ are homogeneous prime ideals of $R$ and $Q$ is a homogeneous*

*prime ideal of $T$ such that $Q \cap R = P$, there exists a homogeneous prime ideal $Q_0$ of $T$ such that $Q_0 \subseteq Q$ and $Q_0 \cap R = P_0$.*

The following theorem was proven in [4, Theorem 3.12]. We utilize Theorem 2.12 to prove our result Theorem 3.3.

**Theorem 2.12.** *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded domain. Then the following are equivalent:*

*(i) $R$ is a grGD domain;*
*(ii) $R \subseteq R[u]$ satisfies grGD for each homogeneous element $u \in R_H$;*
*(iii) $R \subseteq V$ satisfies grGD for each gr-valuation overring $V$;*
*(iiii) $R \subseteq T$ satisfies grGD for each graded domain $T$ containing $R$ as a graded subring.*

## 3. GRADED INJECTIVE EXTENSIONS

We provide graded analogs of i-extension and i-domains, defined in [3, Section 2].

**Definition 3.1.** *Suppose $R \subset T$ is a graded extension. A graded extension is a **graded injective extension**, denoted gr-i-extension, if $gr\text{-}Spec(T) \to gr\text{-}Spec(R)$ is injective.*

**Definition 3.2.** *A graded domain $R$. A **graded injective domain**, or gr-i-domain, if $R \subset T$ is a gr-i-extension for each domain $T$ containing $R$ and contained in the gr-quotient field of $R$.*

**Theorem 3.3.** *Let $R$ be grGD and $T$ be a graded overring. If $R \subseteq T$ is a gr-i-extension, then $T$ is grGD.*

**Proof.** By Theorem 2.12, it suffices to show that $T \subset V$ satisfies grGD for each gr-valuation overring $V$ of $T$. Let $P \subsetneq M$ be graded primes of $T$

and $N$ a graded prime of $V$ contracting to $M$. By hypothesis, there exists a prime $Q \subsetneq N$ such that $Q \cap R = P \cap R$. Thus, $Q \cap T$ and $P$ have the same contraction in $R$, and so are equal. ■

## REFERENCES

[1] M. F. Atiayh and I. G. Macdonald. *Introduction to Commutative Algebra*. Addison-Wesley Professional, 1 edition, 1969.

[2] John B. Fraleigh. *A First Course in Abstract Algbera*. Addison-Wesley Professional, 7 edition, 2002.

[3] IRA J. Papick. Topologically defined classes of going-down domains. *Transactions of the American Mathematical Society*, 219(1976), 1975.

[4] Parviz Sahandi and Nematolla Shirmohammadi. On graded going down domains. arXiv: 2306.16067 [math.AC], 2023.