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SOLUTION OF FRACTIONAL MODIFIED KAWAHARA EQUATION: A SEMI-ANALYTIC APPROACH

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Abstract. The present study introduces and explores the novel application of the Fractional Residual Power Series Method as a highly efficient and reliable approach for solving the challenging non-linear, time-fractional Kawahara and modified Kawahara equations. These equations, which are fifth-order, non-linear partial differential equations, are of significant importance in modeling shallow water waves. In contrast to existing methods such as the Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM), and Adomian Decomposition Method (ADM), our research uniquely demonstrates the superior performance of the Fractional Residual Power Series Method. We emphasize that our approach not only outperforms these conventional techniques but also offers ease of implementation. This novel methodology promises to advance the field of mathematical modeling, providing a powerful tool for solving complex problems in the field of science and engineering.

1. Introduction

In 1972, Kawahara[7] was the first to propose the partial differential equation describing the behavior of solitary waves as follows,

$$\frac{\partial u}{\partial t} + au^m \frac{\partial u}{\partial x} + b\frac{\partial^3 u}{\partial x^3} - \lambda \frac{\partial^5 u}{\partial x^5} = 0, \tag{1.1}$$

where a, b and λ are some specific arbitrary constants. Equation (1.1) arises while modeling wave theory, and scales down to Kortewag-de Vries(KdV) equation [23, 35] for $b = 1$ and $\lambda = 0$. Moreover, the second term of (1.1) is the convective part, and the third term is the dispersive part. Karpman and Vanden-Brock[18] proposed that the fifth order term of equation (1.1) shows critical significance for solitary stability.

Obtaining a solution effectively and accurately due to the involvement of fifth-order derivative terms is not easy. The exact and approximate solution of the space fractional Kawahara equation has already been studied by several authors and a considerable amount of work has been done in this direction. These analytical and numerical methods involve a direct method based on the Jacobi elliptic functions[7], Differential transformation method[35], Iterative Laplace transform method[9] based on Atangana-Baleanu derivative, Fractional complex transform[36], Multiquadric Radial basis functions(MQ-RBF) method[8], Predictor-corrector and RBF-QR method[8], Variational iteration method[28], Homotopy perturbation method[28], and Tanh method[38].

Fractional calculus permits the derivative and integration of arbitrary orders and it has grown popularity in the recent decades in many fields such as physical science[26], electromagnetic theory[13],

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epidemics models[21], fluid mechanics[22], electrical networks[19], shallow water problems[33], biological sciences[2], diffusive transport[31], and many more.

Easiest way to obtain solutions of non-linear fractional differential equations using the power series method was first proposed by Jordan mathematician Arqub[4] in 2013. Several attempts have been made to solve fractional differential equations(FDEs) and Fractional Partial Differential Equations(FPDEs)[6, 10, 11]. Recently, Khirsariya et al.[24] gave a new semi-analytic method to obtain the solution of partial fractional differential equations and also solved various fractional order models using hybrid techniques [25–27]. In 2021, Tariq et al.[37] obtained an analytic approximate solution to non-linear temporal conformable fractional foam drainage equation, Kumar et al.[20] investigated the approximate analytical solution of fractional Bi-Hamiltonian Boussinesq system, Jena et al.[15] investigated time-fractional fuzzy vibration equation of large membranes. Whereas in 2020, Hasan et al.[14] introduced a solution for linear time fractional Swift-Hohenberg equation, Khalauta et al.[17] presented fractional Bratu-type equation, Dunnimit et al.[12] gave analytic approach to deal with fractional logistic equations.

Present research showcases the Fractional Residual Power Series Method(FRPSM) to solve non-linear FPDE application to the time-fractional Kawahara equation. FRPSM is one of the superior methods for fractional-type ordinary and partial differential equations, which is established from the generalized form of the Taylor series. Because of the convergent nature of the FRPSM against other methods, it approaches straight to the problem by considering appropriate initial conditions[29].

In perspective on the above literature, the next section comprises of preliminaries followed by an overview of FRPSM and its application to solve the time-fractional Kawahara equation. Further, the comparison of results with VIM, ADM, and HPM is given. The last section gives details about the results and discussion related to the stability of the solutions obtained.

2. Preliminaries

Definition 2.1. [30] The Riemann-Liouville version of the fractional integral operator of order α is given by

$$I_t^{\alpha}[u(x,t)] = {}_aD_t^{-\alpha}[u(x,t)] = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}u(x,\tau)d\tau. \tag{2.1}$$

Definition 2.2. [32] For the function $u(x,t)$, the Caputo time-fractional derivative of order $\alpha > 0$, is defined as

$$D_t^{\alpha}[u(x,t)] = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int\limits_0^t (t-\tau)^{n-\alpha-1}\frac{\partial^n u(x,\tau)}{\partial \tau^n}d\tau, & n-1<\alpha<n, \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in N. \end{cases} \tag{2.2}$$

Definition 2.3. [32, 34] One parameter Mittag-Leffler function with $\alpha > 0$ is defined as

$$E_{\alpha}(z) = \sum_{\kappa=0}^{\infty}\frac{z^{\kappa}}{\Gamma(\alpha\kappa+1)}, \text{ for } |z|<1;\ z\in C. \tag{2.3}$$

Theorem 2.1. *[1] Let the FPS representation at $t = t_0$ for the function u of the form*

$$u(t) = \sum_{m=0}^{\infty} c_m(t-t_0)^{m\alpha}, t_0 \le t < t_0 + R, \tag{2.4}$$

where R is the radius of convergence.

If $D^{m\alpha}u(t)$, $m = 0, 1, 2, 3, ...$ are continuous on $(t_0, t_0 + R)$ then the coefficient c_m are given by the formula

$$c_m = \frac{D^{m\alpha}u(t_0)}{\Gamma(1 + m\alpha)},\ m = 0, 1, 2, ...$$

where $D^{m\alpha} = D^{\alpha}.\, D^{\alpha}.\, D^{a} \ldots D^{a}\, (m\,\text{times})$.

Theorem 2.2. *[1] Let the FPS representation at $t = t_0$ for the function u is expressed by the form*

$$u(x,t) = \sum_{m=0}^{\infty} f_m(x)(t - t_0)^{m\alpha}, \tag{2.5}$$

where $x \in \mathbb{R}, t_0 \le t < t_0 + R, 0 \le n - 1 < \alpha \le n$.
If $D^{m\alpha}u(x,t)$, $m = 0, 1, 2, 3, ...$ are continuous on $\mathbb{R} \times (t_0, t_0 + R)$ then the coefficient c_m are given by

$$f_m(x) = \frac{D^{m\alpha}u(x,t_0)}{\Gamma(1 + m\alpha)},\ m = 0, 1, 2, ...$$

where $D^{m\alpha} = D^{\alpha} \cdot D^{\alpha} \cdot D^{\alpha} \cdot \cdots \cdot D^{\alpha}\ (m\,\text{times})$.

Thus, the generalized Taylor series formula for fractional power series at $t = t_0$ can be expressed as

$$u(x,t) = \sum_{m=0}^{\infty} \frac{D^{m\alpha}u(x,t_0)}{\Gamma(1 + m\alpha)}(t - t_0)^{m\alpha}, \tag{2.6}$$

where $x \in \mathbb{R}$, $t_0 \le t < t_0 + R$, $0 \le n - 1 < \alpha \le n$.

If $\alpha = 1$, then the classical Taylor series is obtained as

$$u(x,t) = \sum_{m=0}^{\infty} \frac{D^{m}u(x,t_0)}{\Gamma(1 + m)}(t - t_0)^{m}, \text{where } x \in \mathbb{R},\ t_0 \le t < t_0 + R. \tag{2.7}$$

Corollary 2.3. *[3] Suppose that $u(x,y,t)$ has a multiple fractional power series representation at $t = t_0$ of the form*

$$u(x,y,t) = \sum_{m=0}^{\infty} f_m(x,y)(t - t_0)^{m\alpha}, \tag{2.8}$$

where $(x,y) \in \mathbb{R} \times \mathbb{R}, t_0 \le t < t_0 + R, 0 \le n - 1 < \alpha \le n$.

If $D^{m\alpha}u(x,y,t)$, $m = 0, 1, 2, 3, ...$are continuous on $\mathbb{R} \times \mathbb{R} \times (t_0, t_0 + R)$ then the coefficient c_m are given by

$$f_m(x,y) = \frac{D^{m\alpha}u(x,y,t_0)}{\Gamma(1 + m\alpha)},\ m = 0, 1, 2, \ldots.$$

3. Analysis of Fractional Residual Power Series Method

To illustrate the essential concept of FRPSM, the generalized fractional differential equation of non-linear form is considered as follows:

$$D_t^{\alpha}[u(x,t)] = R(u) + N(u), \tag{3.1}$$

where $R(u)$ and $N(u)$ are linear and non-linear terms respectively subject to initial conditions,

$$u(x,0) = f_0(x) = f(x) \text{ and } D_t^{(n-1)\alpha}[u(x,0)] = f_{n-1}(x). \tag{3.2}$$

The FRPSM presents the solution for (3.1) at $t = 0$,

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \tag{3.3}$$

where $0 < \alpha \leq 1, 0 \leq t < R, -\infty < x < \infty$.

Let $u_k(x,t)$ denote as $k^{th}-$ truncated series

$$u_k(x,t) = \sum_{n=0}^{k} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \tag{3.4}$$

where $0 < \alpha \leq 1, 0 \leq t < R, -\infty < x < \infty, k = 1, 2, 3, \ldots$.

The solution of FDE (3.1) namely $u(x,t)$ has to satisfy initial conditions as given in (3.2). Moreover applying $t = 0$ in equation (3.3), we obtain

$$u_0(x,t) = u(x,0) = f_0(x) = f(x). \tag{3.5}$$

Using (3.4), we have

$$u_1(x,t) = f(x) + f_1(x) \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \tag{3.6}$$

and in general,

$$u_k(x,t) = f(x) + f_1(x) \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \sum_{n=2}^{k} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \tag{3.7}$$

where $k = 2, 3, 4, \ldots.$

Subsequently, using FRPSM we can evaluate $f_n(x), n = 1, 2, 3, ..., k$ in the equation (3.7).

Now, we define a residual function to generalized FDE (3.1) as,

$$\text{Res}[u(x,t)] = D_t^{\alpha}[u(x,t)] - R(u) - N(u). \tag{3.8}$$

Thus, $k^{th}-$ residual function is

$$\text{Res}[u_k(x,t)] = D_t^{\alpha}[u_k(x,t)] - R(u_k) - N(u_k). \tag{3.9}$$

As mentioned in[4], we can easily see that

$$\lim_{k\to\infty} \text{Res}[u_k(x,t)] = \text{Res}[u(x,t)] = 0,$$

and

$$D_t^{n\alpha}\left(\text{Res}[u(x,t)]\right) = 0. \tag{3.10}$$

In Caputo's sense, fractional differentiation is

$$D_t^{n\alpha}(\text{Res}[u(x,0)]) = D_t^{n\alpha}(\text{Res}[u_k(x,0)]) = 0; \quad n = 0, 1, 2, ..., k. \tag{3.11}$$

To evaluate $f_i(x)$ where $i = 1, 2, ...$ we calculate for $k = 1, 2, ...,$ in (3.7) then replace it in (3.9), operating $D_t^{(k-1)\alpha}$ on both the sides, we have $f_i(x)$ where $i = 1, 2, ...,$ using

$$D_t^{(k-1)\alpha}(\text{Res}[u_k(x,0)]) = 0, \quad k = 1, 2, 3, ... \tag{3.12}$$

Substitution of $f_i(x)$, $i = 1, 2, \ldots$ in (3.3) provide us with the series solution of (3.1). The convergence criteria of FRPSM are given in [1].

4. Solution of Time-fractional Kawahara and modified Kawahara equations

Example 4.1. Solution of fractional Kawahara equation

To demonstrate the non-linear homogeneous time-fractional Kawahara equation with $a = 1, b = 1, m = 1$ and $\lambda = 1$ in (1.1), we used the concept of FRPSM, let us examine the equation,

$$D_t^{\alpha}[u(x,t)] + uu_x + u_{xxx} - u_{xxxxx} = 0, 0 < \alpha \leq 1, t > 0, x \in R, \tag{4.1}$$

with initial conditions,

$$u(x,0) = \frac{105}{169}\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right). \tag{4.2}$$

The exact solution[29] at ($\alpha = 1$) of (4.1), is given by,

$$u(x,t) = \frac{105}{169}\operatorname{sech}^4\left(\frac{1}{2\sqrt{13}}\left(x - \frac{36}{169}t\right)\right). \tag{4.3}$$

Explicating the residual function for (4.1) as

$$\operatorname{Res}[u(x,t)] = D_t^{\alpha}[u(x,t)] + uu_x + u_{xxx} - u_{xxxxx} = 0, \tag{4.4}$$

thus, for the k^{th} residual function $\operatorname{Res}[u_k(x,t)]$,

$$\operatorname{Res}[u_k(x,t)] = D_t^{\alpha}[u_k] + u_k u_{kx} + u_{kxxx} - u_{kxxxxx} = 0. \tag{4.5}$$

For $k = 1$, equations (3.7) and (4.5) yields,

$$\begin{aligned}\operatorname{Res}[u_1(x,t)] =& f_1 + ff_x + f_{xxx} - f_{xxxxx} + (ff_{1x} + f_1 f_x + f_{1xxx}\\ &- f_{1xxxxx})\frac{t^{\alpha}}{\Gamma(\alpha+1)} + f_1 f_{1x}\frac{t^{2\alpha}}{\Gamma(\alpha+1)^2},\end{aligned} \tag{4.6}$$

using initial condition (4.2), we have

$$f_1(x) = \frac{7560}{28561\sqrt{13}}\tanh\left(\frac{x}{2\sqrt{13}}\right)\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right). \tag{4.7}$$

Similarly, for $k = 2$ we have,

$$\operatorname{Res}[u_2(x,t)] = D_t^{\alpha}u_2 + u_2 u_{2x} + u_{2xxx} - u_{2xxxxx} = 0. \tag{4.8}$$

Now, from (3.7) at $k = 2$,

$$u_2(x,t) = f(x) + f_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{4.9}$$

and

$$\begin{aligned}\operatorname{Res}[u_2(x,t)] =& f_1 + ff_x + f_{xxx} - f_{xxxxx}\\ &+ (f_2 + ff_{1x} + f_1 f_x + f_{1xxx} - f_{1xxxxx})\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\ &+ (f_2 f_x + f_x f_{2x} + f_{2xxx} - f_{2xxxxx})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f_1 f_{1x}\frac{t^{2\alpha}}{\Gamma(\alpha+1)^2}\\ &+ (f_2 f_{1x} + f_{2x} f_1)\frac{t^{3\alpha}}{\Gamma(\alpha+1)(2\alpha+1)} + f_2 f_{2x}\frac{t^{4\alpha}}{\Gamma(2\alpha+1)^2}.\end{aligned} \tag{4.10}$$

Taking D_t^{α} on both side and calculating the equation $D_t^{\alpha}(\operatorname{Res}[u_2(x,0)]) = 0$, then we get

$$f_2 = -ff_{1x} - f_1 f_x - f_{1xxx} + f_{1xxxxx}. \tag{4.11}$$

using the known values, we achieve

$$f_2(x) = \frac{68040}{62748517}\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\left(4 - 5\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right). \tag{4.12}$$

And, for $k = 3$ we have

$$\operatorname{Res}[u_3(x,t)] = D_t^{\alpha}u_3 + u_3u_{3x} + u_{3xxx} - u_{3xxxxx} = 0. \tag{4.13}$$

Now, from (3.7) at $k = 3$,

$$u_3(x,t) = f(x) + f_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \tag{4.14}$$

and

$$\begin{aligned}\operatorname{Res}[u_3(x,t)] =& f_1 + ff_x + f_{xxx} - f_{xxxxx}\\ &+ (f_2 + ff_{1x} + f_1f_x + f_{1xxx} - f_{1xxxxx})\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\ &+ (f_3 + f_2f_x + ff_{2x} + f_{2xxx} - f_{2xxxxx})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &+ f_1f_{1x}\frac{t^{2\alpha}}{\Gamma(\alpha+1)^2} + (ff_{3x} + f_3f_x + f_{3xxx} - f_{3xxxxx})\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\ &+ (f_1f_{2x} + f_2f_{1x})\frac{t^{3\alpha}}{\Gamma(\alpha+1)(2\alpha+1)} + f_2f_{2x}\frac{t^{4\alpha}}{\Gamma(2\alpha+1)^2}\\ &+ (f_3f_{1x} + f_1f_{3x})\frac{t^{4\alpha}}{\Gamma(\alpha+1)(3\alpha+1)}\\ &+ (f_2f_{3x} + f_3f_{2x})\frac{t^{5\alpha}}{\Gamma(2\alpha+1)(3\alpha+1)} + f_3f_{3x}\frac{t^{6\alpha}}{\Gamma(3\alpha+1)^2}.\end{aligned} \tag{4.15}$$

Again, taking $D_t^{2\alpha}$ on both side and calculating the equation $D_t^{2\alpha}(\operatorname{Res}[\mathrm{u}_3(x,0)]) = 0$, then we get

$$f_3 = -f_2f_x - ff_{2x} - f_{2xxx} + f_{2xxxxx} - f_{1x}f_{1xxx}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}, \tag{4.16}$$

substituting the values of f, f_1 and f_2, leads to

$$f_3(x) = \frac{9525600}{10604499373}\operatorname{sech}^8\left(\frac{x}{2\sqrt{13}}\right)\tanh\left(\frac{x}{2\sqrt{13}}\right)\left(4 - 5\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right). \tag{4.17}$$

By repeating the same process next approximation will be

$$f_4(x) = \frac{4762800\sqrt{13}}{137858491849}\operatorname{sech}^8\left(\frac{x}{2\sqrt{13}}\right)\left(-32 + 86\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right) - 55\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\right). \tag{4.18}$$

Similarly, the values of $f_5, f_6, ...$ can be obtained.

Substituting values of $f_i,\ i = 1, 2, 3, 4, \ldots$, into equation (3.7), we obtained the approximate solution of time-fractional Kawahara equation (4.1).

$$
\begin{aligned}
&u(x,t) = \\
&\frac{105}{169}\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right) + \frac{7560}{28561\sqrt{13}}\tanh\left(\frac{x}{2\sqrt{13}}\right)\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} \\
&+ \frac{68040}{62748517}\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\left(4-5\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&+ \frac{9525600}{10604499373}\operatorname{sech}^8\left(\frac{x}{2\sqrt{13}}\right)\tanh\left(\frac{x}{2\sqrt{13}}\right)\left(4-5\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\
&+ \frac{4762800\sqrt{13}}{137858491849}\operatorname{sech}^8\left(\frac{x}{2\sqrt{13}}\right)\left(-32+86\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)-55\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\right)\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \\
&+\dots. \qquad (4.19)
\end{aligned}
$$

Table 1. The Absolute Error in the solution of Kawahara Equation (4.1) by RPSM method, VIM method[28], HPM method[28], and ADM method[5].

x	t	$\mid u_{exact} - u_{rpsm} \mid$	$\mid u_{exact} - u_{vim} \mid$	$\mid u_{exact} - u_{hpm} \mid$	$\mid u_{exact} - u_{adm} \mid$
-20	0.1	1.58361E-11	3.97274E-11	1.01469E-08	6.04582E-09
	0.2	8.28473E-11	3.13089E-10	4.04332E-08	2.67826E-08
	0.3	5.38679E-10	1.05406E-09	9.06246E-08	7.30563E-08
	0.4	2.82485E-09	2.49070E-09	1.60493E-07	1.14693E-07
	0.5	7.28460E-09	4.85110E-09	2.49809E-07	6.02567E-07
-10	0.1	8.29485E-10	2.46593E-09	1.53191E-06	4.34952E-08
	0.2	4.83572E-09	1.92434E-08	6.11064E-06	9.53984E-08
	0.3	1.63892E-08	6.48336E-08	1.37103E-05	3.03457E-07
	0.4	7.04823E-08	1.53132E-07	2.43056E-05	7.24586E-07
	0.5	3.47678E-07	2.98332E-07	3.78709E-05	2.19450E-06
0	0.1	2.00732E-10	1.74165E-10	5.42153E-06	1.49535E-08
	0.2	8.48329E-10	1.77754E-09	2.16847E-05	5.04568E-08
	0.3	3.18473E-09	9.03168E-09	4.87858E-05	2.38543E-07
	0.4	9.48372E-09	2.82581E-08	8.67179E-05	6.45683E-07
	0.5	4.03846E-08	6.90256E-08	1.35472E-04	1.39024E-06
10	0.1	6.03826E-10	2.41376E-09	1.54059E-06	5.83492E-08
	0.2	2.94573E-09	1.94624E-08	6.17975E-06	1.00438E-07
	0.3	9.89347E-09	6.58172E-08	1.39436E-05	4.89372E-07
	0.4	4.28345E-08	1.56456E-07	2.48584E-05	9.45935E-07
	0.5	1.38277E-07	3.06352E-07	3.89507E-05	5.34853E-06
20	0.1	2.28467E-11	3.98825E-11	1.02266E-08	4.93485E-09
	0.2	8.68913E-11	3.16631E-10	4.10637E-08	9.56823E-09
	0.3	5.03682E-10	1.07259E-09	9.27540E-08	3.48934E-08
	0.4	2.00384E-09	2.54866E-09	1.65539E-07	7.03948E-08
	0.5	7.28395E-09	4.99298E-09	2.59666E-07	2.94583E-07

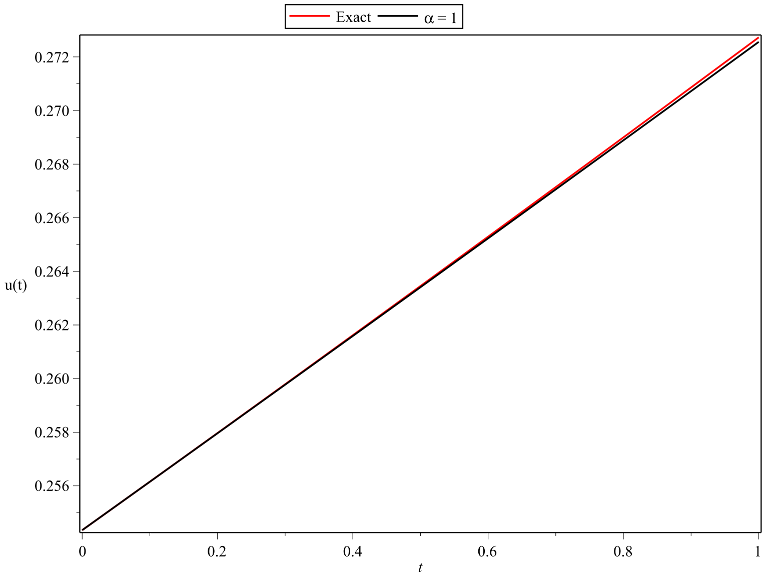


FIGURE 1. Comparison of solution of Fractional Kawahara Equation at fractional order of $\alpha = 1$ with Exact solution at $x = 5$.

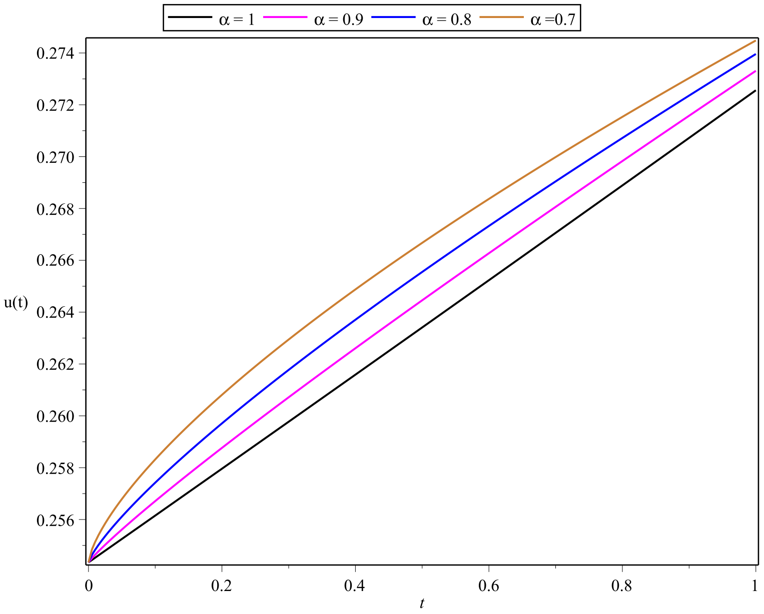
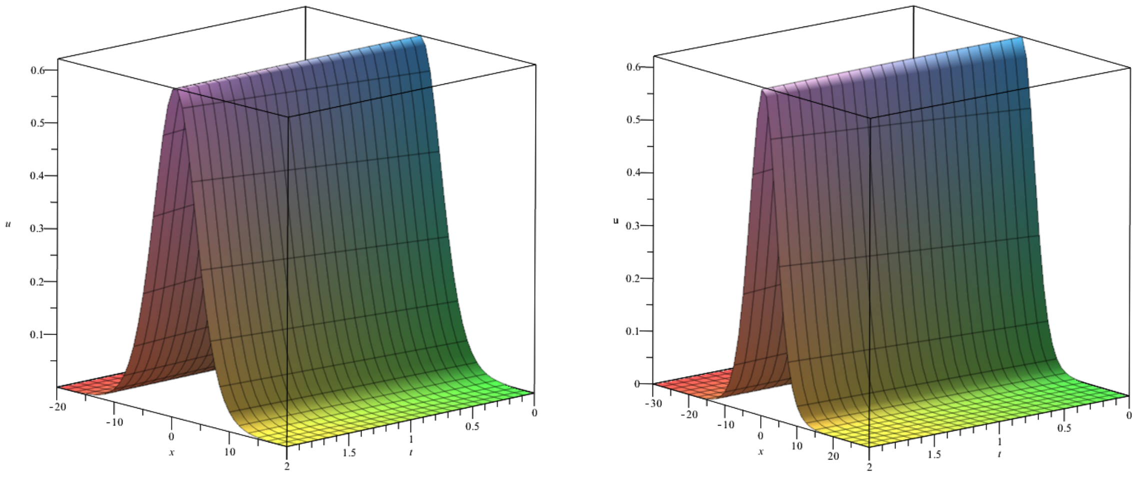


FIGURE 2. Comparison of solution of Fractional Kawahara Equation with different fractional order of α at $x = 5$.



(A) $\alpha = 1$ (B) Exact solution

FIGURE 3. Solution of Kawahara equation as (4.1) by FRPSM with order $\alpha = 1$ and Exact solution.

Example 4.2. Solution of fractional Kawahara equation

To demonstrate the non-linear homogeneous time-fractional Kawahara equation with $a = 1, b = 1, m = 1$ and $\lambda = 1$ in (1.1), let us examine the equation,

$$D_t^{\alpha}[u(x,t)] + uu_x + u_{xxx} - u_{xxxxx} = 0, 0 < \alpha \leq 1, t > 0, x \in R, \tag{4.20}$$

with initial conditions,

$$u(x,0) = -\frac{72}{169} + \frac{420}{169} \frac{\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)}{\left(1 + \operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)^2}. \tag{4.21}$$

The exact solution[39] at ($\alpha = 1$) of (4.20), is given by,

$$u(x,t) = -\frac{72}{169} + \frac{420}{169} \frac{\operatorname{sech}^2\left(\frac{1}{2\sqrt{13}}\left(x + \frac{36}{169}t\right)\right)}{\left(1 + \operatorname{sech}^2\left(\frac{1}{2\sqrt{13}}\left(x + \frac{36}{169}t\right)\right)\right)^2}. \tag{4.22}$$

By implementing the same methodology of FRPSM, we leads to the approximate solution of time-fractional Kawahara equation (4.20),

$$
\begin{aligned}
&u(x,t) = \\
&-\frac{72}{169}+\frac{420}{169}\frac{\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)}{\left(1+\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)^2} \\
&+\frac{420\sqrt{13}}{2197}\frac{\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\tanh\left(\frac{x}{2\sqrt{13}}\right)\left(\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)-1\right)}{\left(1+\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)^3}\frac{t^{\alpha}}{\Gamma(1+\alpha)} \\
&+\frac{210}{2197}\frac{\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\left(1-11\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)+10\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right)-\operatorname{sech}^6\left(\frac{x}{2\sqrt{13}}\right)\right)}{\left(1+\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)^4}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&-\frac{420\sqrt{13}}{28561}\frac{\operatorname{sech}^3\left(\frac{x}{2\sqrt{13}}\right)\tanh\left(\frac{x}{2\sqrt{13}}\right)\left(1-14\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)+26\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right)-7\operatorname{sech}^6\left(\frac{x}{2\sqrt{13}}\right)\right)}{\left(1+\operatorname{sech}^2\left(\frac{x}{2\sqrt{13}}\right)\right)^5}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\
&+\ldots.
\end{aligned}
\tag{4.23}
$$

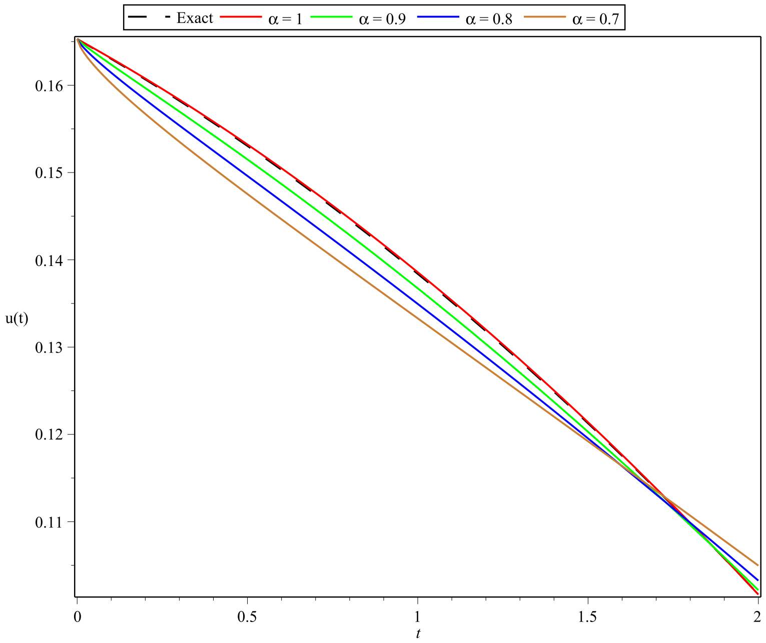


FIGURE 4. Comparison of solution of Fractional Kawahara Equation (4.20) with different fractional order of α with exact solution at $x = 5$.

Example 4.3. Solution of fractional Modified Kawahara equation

Table 2. The Absolute Error in the solution of Kawahara Equation (4.20) by RPSM method, VIM method, HPM method, and ADM method.

x	t	$\mid u_{exact} - u_{rpsm} \mid$	$\mid u_{exact} - u_{vim} \mid$	$\mid u_{exact} - u_{hpm} \mid$	$\mid u_{exact} - u_{adm} \mid$
-20	0.1	7.95296E-10	7.82552E-09	7.95395E-09	3.88526E-08
	0.2	1.61492E-09	1.56394E-08	1.61571E-08	6.88312E-08
	0.3	2.45886E-09	2.34416E-08	2.46153E-08	9.88098E-08
	0.4	3.32711E-09	3.12321E-08	3.33346E-08	2.87882E-07
	0.5	4.21967E-09	3.90108E-08	4.23207E-08	5.87664E-07
-10	0.1	4.53022E-10	4.52809E-09	4.52965E-09	4.61140E-08
	0.2	9.06454E-10	9.05602E-09	9.05996E-09	6.48881E-08
	0.3	1.36030E-09	1.35838E-08	1.35875E-08	8.36620E-08
	0.4	1.81456E-09	1.81115E-08	1.81090E-08	2.24359E-07
	0.5	2.26924E-09	2.26392E-08	2.26209E-08	4.12097E-07
0	0.1	2.00000E-10	2.00000E-10	2.00000E-10	2.00000E-10
	0.2	2.00000E-10	2.00000E-10	2.00000E-10	2.00000E-10
	0.3	1.20000E-09	1.20000E-09	1.20000E-09	1.20000E-09
	0.4	2.90000E-09	2.90000E-09	2.90000E-09	2.90000E-09
	0.5	7.40000E-09	7.40000E-09	7.40000E-09	7.40000E-09
10	0.1	4.52616E-10	4.52828E-09	4.52558E-09	1.85653E-08
	0.2	9.04828E-10	9.05680E-09	9.04371E-09	5.97907E-08
	0.3	1.35664E-09	1.35856E-08	1.35510E-08	8.10158E-08
	0.4	1.80806E-09	1.81146E-08	1.80440E-08	2.22406E-07
	0.5	2.25908E-09	2.26440E-08	2.25193E-08	4.34650E-07
20	0.1	7.70965E-10	7.83709E-09	7.71064E-09	3.88950E-08
	0.2	1.51759E-09	1.56857E-08	1.51839E-08	7.89160E-08
	0.3	2.23988E-09	2.35457E-08	2.24256E-08	1.89369E-07
	0.4	2.93782E-09	3.14172E-08	2.94417E-08	3.89577E-07
	0.5	3.61140E-09	3.92999E-08	3.62381E-08	5.89784E-07

Let us consider the non-linear homogeneous time-fractional Modified Kawahara equation with $a = 1, b = p, \lambda = -q$ and $m = 2$ in equation (1.1) as

$$D_t^{\alpha} u + u^2 u_x + p u_{xxx} + q u_{xxxxx} = 0, \tag{4.24}$$

with the initial condition,

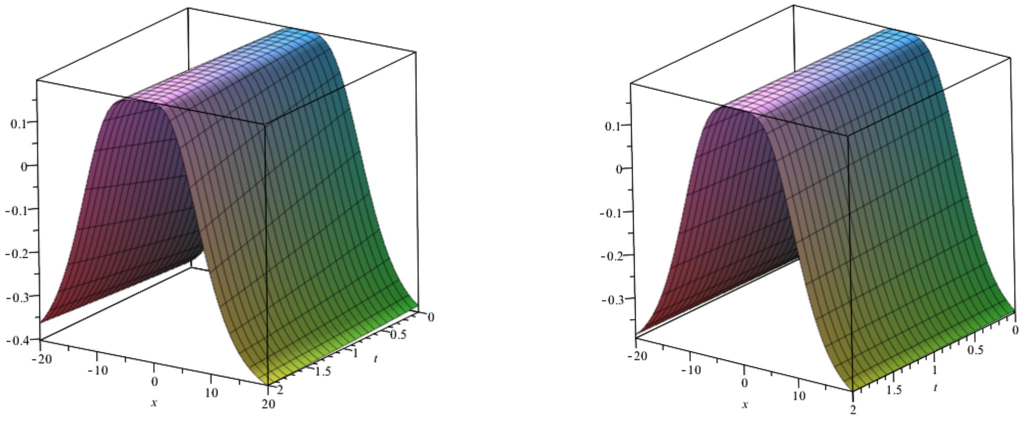
$$u(x,0) = \frac{3p}{\sqrt{-10q}} sech^2(kx), \text{ where } k = \frac{1}{2}\sqrt{\frac{-p}{5q}}. \tag{4.25}$$

The exact solution of (4.24) for spacial case $\alpha = 1, p > 0$, and $q < 0$ is

$$u(x,t) = \frac{3p}{\sqrt{-10q}} sech^2 \left[k \left(x - \left(\frac{25q - 4p^2}{25q} \right) t \right) \right]. \tag{4.26}$$

Illuminating the residual function for (4.24) as

$$\text{Res}[u(x,t)] = D_t^{\alpha} + u^2 u_x + p u_{xxx} + q u_{xxxxx} = 0, \tag{4.27}$$



(A) $\alpha = 1$

(B) Exact solution

FIGURE 5. Solution of Kawahara equation (4.20) by FRPSM with order $\alpha = 1$ and Exact solution.

thus, for the k^{th} residual function $\text{Res}[u_k(x,t)]$ defined as

$$\text{Res}[u_k(x,t)] = D_t^\alpha u_k + u_k^2 u_{kx} + p u_{kxxx} + q u_{kxxxxx} = 0. \tag{4.28}$$

For $k = 1$, equation (3.7) and (4.28) yields,

$$\begin{aligned}\text{Res}[u_1(x,t)] =& f_1 + f^2 f_x + p f_{xxx} + q f_{xxxxx} \\ &+ \left(f^2 f_{1x} + 2 f f_1 f_x + p f_{1xxx} + q f_{1xxxxx}\right) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ &+ \left(2 f f_1 f_{1x} + f_1^2 f_x\right) \frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2} \\ &+ f_1^2 f_{1x} \frac{t^{3\alpha}}{(\Gamma(1+\alpha))^3},\end{aligned} \tag{4.29}$$

using initial condition (4.25), we get

$$f_1(x) = \frac{-6\sqrt{2} p^{\frac{7}{2}}}{125 q^2} \tanh(kx) \operatorname{sech}^2(kx). \tag{4.30}$$

Similarly, for $k = 2$ we have

$$\text{Res}[u_2(x,t)] = D_t^\alpha u_2 + u_2^2 u_{2x} + p u_{2xxx} + q u_{2xxxxx} = 0. \tag{4.31}$$

Now, from equations (3.7) and (4.31) yields

$$\begin{aligned}\operatorname{Res}[u_2(x,t)] =& f_1 + f^2 f_x + p f_{xxx} + q f_{xxxxx}\\ &+ \left(f_2 + f^2 f_{1x} + 2 f f_1 f_x + p f_{1xxx} + q f_{1xxxxx}\right) \frac{t^{\alpha}}{\Gamma(1+\alpha)}\\ &+ \left(2 f f_{1x} f_1 + f_1^2 f_x\right) \frac{t^{2\alpha}}{\left(\Gamma(1+\alpha)\right)^2}\\ &+ \left(2 f f_2 f_x + f^2 f_{2x} + p f_{2xxx} + q f_{2xxxxx}\right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\ &+ \left(2 f_1 f_2 f_x + 2 f f_2 f_{1x} + 2 f f_1 f_{2x}\right) \frac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\\ &+ f_1^2 f_{1x} \frac{t^{3\alpha}}{\left(\Gamma(1+\alpha)\right)^3} + \left(f_x f_2^2 + 2 f f_2 f_{2x}\right) \frac{t^{4\alpha}}{\left(\Gamma(1+2\alpha)\right)^2}\\ &+ \left(2 f_1 f_2 f_{1x} + f_1^2 f_{2x}\right) \frac{t^{4\alpha}}{\left(\Gamma(1+\alpha)\right)^2 \Gamma(1+2\alpha)}\\ &+ f_{1x} f_2^2 \frac{t^{5\alpha}}{\Gamma(1+\alpha)\left(\Gamma(1+2\alpha)\right)^2} + f_2^2 f_{2x} \frac{t^{6\alpha}}{\left(\Gamma(1+2\alpha)\right)^2},\end{aligned} \tag{4.32}$$

Operating D_t^{α} on both side and computing the equation $D_t^{\alpha}(\operatorname{Res}[u_2(x,0)]) = 0$, we have

$$f_2 = -f^2 f_{1x} - 2 f f_1 f_x - p f_{1xxx} - q f_{1xxxxx}. \tag{4.33}$$

Using equation (4.25) and (4.30),

$$f_2(x) = \frac{12\sqrt{2} p^{\frac{11}{2}}}{15625 q^3} \sqrt{\frac{-5p}{q}} \operatorname{sech}^4(kx)\left(2\cosh^2(kx) - 3\right). \tag{4.34}$$

Similarly, the values of $f_3, f_4, \dots$ can be obtained.

Substituting values of $f_i,\ i = 1, 2, 3, \dots,$ into equation (3.7) yields

$$\begin{aligned}u(x,t) =& \frac{3p}{\sqrt{-10q}} sech^2(kx) + \frac{-6\sqrt{2} p^{\frac{7}{2}}}{125 q^2} \tanh(kx) \operatorname{sech}^2(kx) \frac{t^{\alpha}}{\Gamma(1+\alpha)}\\ &+ \frac{12\sqrt{2} p^{\frac{11}{2}}}{15625 q^3} \sqrt{\frac{-5p}{q}} \operatorname{sech}^4(kx)\left(2\cosh^2(kx) - 3\right) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots.\end{aligned} \tag{4.35}$$

Example 4.4. Solution of fractional Modified Kawahara equation

Let us consider the non-linear homogeneous time-fractional Modified Kawahara equation with $a = 1, b = p, \lambda = -q$ and $m = 2$ in equation (1.1) as

$$D_t^{\alpha} u + u^2 u_x + p u_{xxx} + q u_{xxxxx} = 0, \tag{4.36}$$

with the initial condition,

$$u(x,0) = \frac{3p}{\sqrt{-10q}} sech^2(k_2 x), \text{ where } k_2 = \frac{1}{2}\sqrt{\frac{-p}{5q}}. \tag{4.37}$$

The exact solution[40] of (4.36) for spacial case $\alpha = 1, p > 0$, and $q < 0$ is

$$u(x,t) = \frac{3p}{\sqrt{-10q}} sech^2\left[k_2\left(x - \left(\frac{25q - 4p^2}{25q}\right) t\right)\right]. \tag{4.38}$$

By implementing the same methodology of FRPSM, we obtain the approximate solution of time-fractional modified Kawahara equation (4.36),

Table 3. The Absolute Error in the solution of modified Kawahara Equation (4.24) by RPSM method, VIM method[16], HPM method[16] and ADM method[5].

x	t	$\mid u_{exact} - u_{rpsm} \mid$	$\mid u_{exact} - u_{vim} \mid$	$\mid u_{exact} - u_{hpm} \mid$	$\mid u_{exact} - u_{adm} \mid$
-20	0.1	8.51590E-07	2.14652E-05	2.14652E-05	4.56893E-06
	0.2	2.54525E-06	4.16408E-05	4.16408E-05	7.78534E-06
	0.3	4.73921E-06	6.05908E-05	6.05908E-05	1.45834E-05
	0.4	5.79092E-06	7.83757E-05	7.83757E-05	4.57893E-05
	0.5	4.94688E-06	9.50531E-05	9.50531E-05	8.29452E-05
-10	0.1	6.94927E-05	1.79880E-03	1.79880E-03	7.40245E-04
	0.2	1.95225E-04	3.49407E-03	3.49407E-03	9.52489E-04
	0.3	3.43100E-04	5.09051E-03	5.09051E-03	2.58935E-03
	0.4	4.61860E-04	6.59263E-03	6.59263E-03	5.45924E-03
	0.5	8.52801E-04	8.00473E-03	8.00473E-03	8.04589E-03
0	0.1	2.42340E-05	6.50131E-04	6.50131E-04	1.58935E-04
	0.2	5.25436E-05	2.59710E-03	2.59710E-03	3.85782E-04
	0.3	8.26822E-05	5.83065E-03	5.83065E-03	6.02482E-04
	0.4	3.33831E-04	1.03338E-02	1.03338E-02	9.45839E-04
	0.5	6.77303E-04	1.60832E-02	1.60832E-02	3.49305E-03
10	0.1	8.96249E-06	1.90724E-03	1.90724E-03	2.04824E-04
	0.2	2.80261E-05	3.92803E-03	3.92803E-03	4.24364E-04
	0.3	6.44425E-05	6.06769E-03	6.06769E-03	6.34567E-04
	0.4	7.06877E-05	8.33178E-03	8.33178E-03	9.00523E-04
	0.5	1.72788E-04	1.07261E-02	1.07261E-02	2.56345E-03
20	0.1	1.31295E-07	2.28219E-05	2.28219E-05	5.23589E-06
	0.2	2.99425E-07	4.70715E-05	4.70715E-05	8.98472E-06
	0.3	2.82314E-06	7.28231E-05	7.28231E-05	3.56394E-05
	0.4	7.15552E-06	1.00156E-04	1.00156E-04	6.13824E-05
	0.5	8.95801E-06	1.29151E-04	1.29151E-04	8.95723E-05

$$
\begin{aligned}
&u(x,t) = \\
&\frac{6p}{\sqrt{-10q}} \frac{\operatorname{sech}(k_2x)}{(1+\operatorname{sech}(k_2x))} \\
&- \frac{6\sqrt{2}p^{\frac{3}{2}}}{q} \frac{\operatorname{sech}(k_2x)\tanh(k_2x)}{(1+\operatorname{sech}(k_2x))^2} \frac{t^{\alpha}}{\Gamma(1+\alpha)} \\
&+ \frac{6p^2}{5\sqrt{-10}q^{\frac{3}{2}}} \frac{\operatorname{sech}(k_2x)\left(1+\operatorname{sech}(k_2x)-2\tanh^2(k_2x)\right)}{(1+\operatorname{sech}(k_2x))^3} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&+ \frac{6p^{\frac{5}{2}}}{25\sqrt{2}q^2} \frac{\operatorname{sech}(k_2x)\tanh(k_2x)\left(-5-4\operatorname{sech}(k_2x)+\operatorname{sech}^2(k_2x)+6\tanh^2(k_2x)\right)}{(1+\operatorname{sech}(k_2x))^4} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\
&+ \ldots. \qquad (4.39)
\end{aligned}
$$

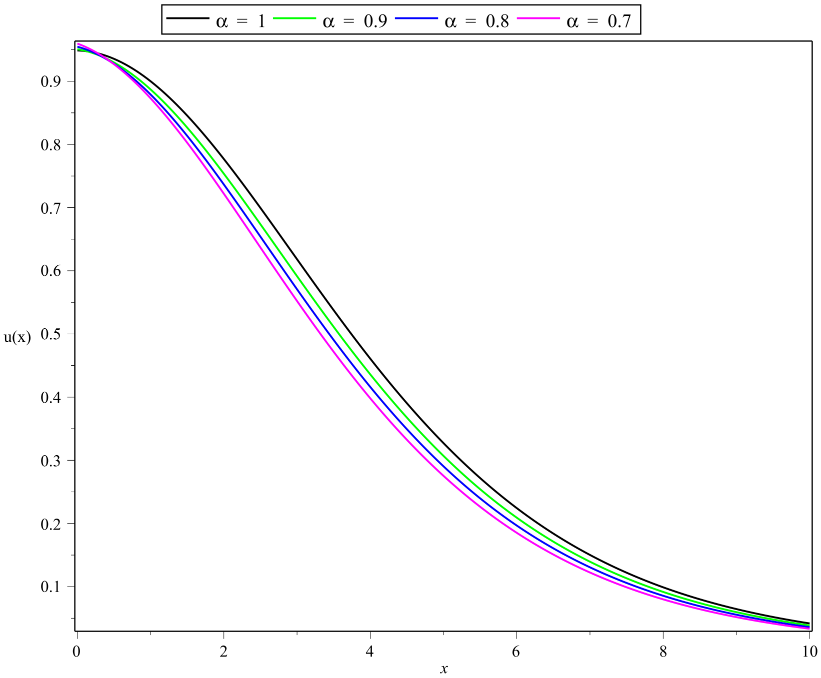


Figure 6. Comparison of solution of Time-fractional Modified Kawahara Equation with different fractional order of α at time $t = 1$.

5. Result

The present paper showcases the approximate analytical solution obtained by using FRPSM for the time-fractional Kawahara and Modified Kawahara equations. A comparison of solutions obtained from FRPSM, VIM, HPM, and Exact solution is observed in Tables – 1,2,3, and 4. These tables show the absolute errors for different time t and distance x at $\alpha = 1$. The behaviour of the equation using different fractional orders i.e., $\alpha = 0.7, 0.8, 0.9$ along with $\alpha = 1$ and exact solution are seen in Figures – 1,2,4,6,7, and 9. Also, Figures – 3,5,8, and 10 describe the geometrical behavior of the solutions obtained by FRPSM and the exact solution respectively. The results obtained state the convergence of the method.

6. Conclusion

The present work exhibits the solution of fractional ordered Kawahara and modified Kawahara equations using Fractional Residual Power Series Method (FRPSM). Our analysis, as depicted in Figures – 1,4,7, and 9, unequivocally illustrates that the solutions obtained for the Kawahara and Modified Kawahara equations with order $\alpha = 1$ closely approximates to the exact solutions. Also, the comparison shown in Tables – 1 and 2 show that the FRPSM solves the time-fractional Kawahara equation much more accurately than the Homotopy Perturbation Method (HPM) and the Adomian Decomposition Method (ADM). Notably, it yields results almost equivalent in accuracy to the Variation Iteration Method (VIM). Tables – 3 and 4 show that the FRPSM is better than VIM, HPM, and ADM at solving the time-fractional modified Kawahara equation. This shows that the FRPSM is more accurate than VIM, HPM, and ADM. The only limitation of RPSM is that, one can not use this method to obtain the generalized solution of fractional differential equations. The outcomes and insights garnered

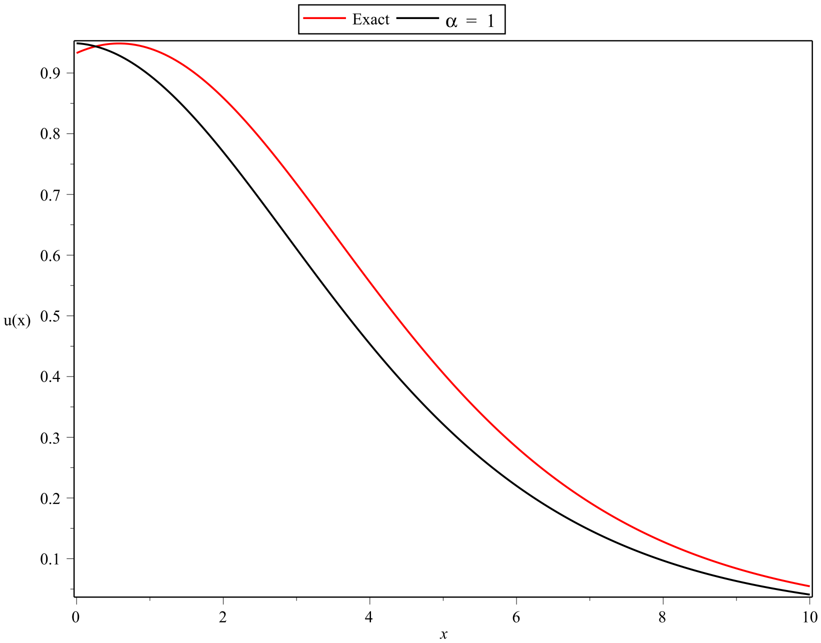
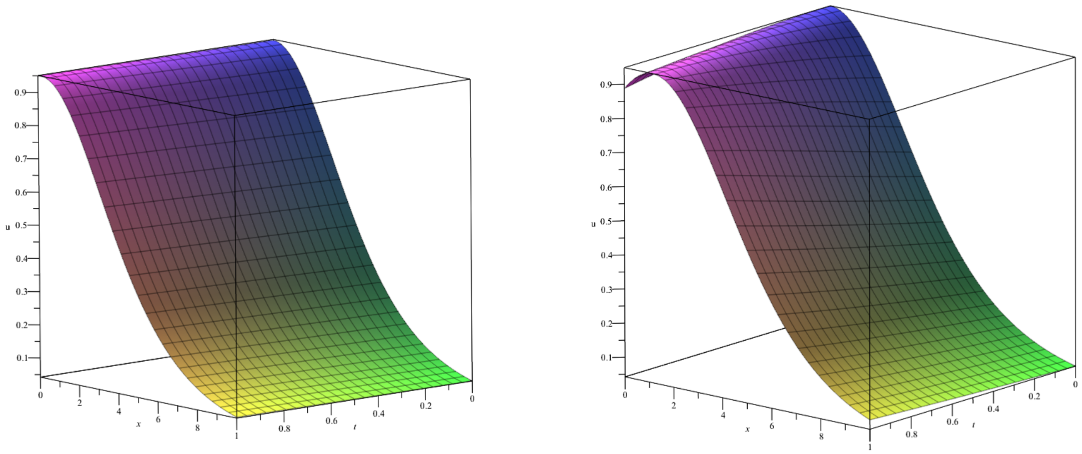


FIGURE 7. Comparison of solution of Fractional Modified Kawahara Equation with fractional order of $\alpha = 1$ with an exact solution at $t = 1$.



(A) $\alpha = 1$

(B) Exact solution

FIGURE 8. Solution of Modified Kawahara equation (4.24) by FRPSM with order $\alpha = 1$ and Exact solution.

throughout this article have profound implications for understanding and addressing complex non-linear physical problems. The remarkable accuracy and efficiency of the FRPSM make it a valuable tool for

TABLE 4. The Absolute Error in the solution of modified Kawahara Equation (4.36) by RPSM method, VIM method, HPM method, and ADM method.

x	t	$\mid u_{exact} - u_{rpsm} \mid$	$\mid u_{exact} - u_{vim} \mid$	$\mid u_{exact} - u_{hpm} \mid$	$\mid u_{exact} - u_{adm} \mid$
-20	0.1	4.76456E-09	4.76530E-07	1.53243E-07	1.35453E-07
	0.2	9.50163E-09	9.50753E-07	6.23543E-07	6.34343E-07
	0.3	1.42176E-08	1.42375E-06	1.26435E-06	2.53453E-06
	0.4	1.89185E-08	1.89656E-06	4.63456E-06	5.33366E-06
	0.5	2.36101E-08	2.37022E-06	8.84245E-06	7.22210E-06
-10	0.1	3.98788E-08	3.98842E-06	1.27345E-06	1.23843E-07
	0.2	7.95333E-08	7.95761E-06	4.85632E-06	5.38434E-07
	0.3	1.19010E-07	1.19155E-05	7.83475E-06	8.80034E-07
	0.4	1.58355E-07	1.58697E-05	1.23854E-05	3.45454E-06
	0.5	1.97611E-07	1.98279E-05	3.58237E-05	6.44565E-06
0	0.1	1.63646E-08	1.63646E-06	4.33563E-07	3.11456E-07
	0.2	6.51157E-08	6.51157E-06	7.74384E-07	5.33387E-07
	0.3	1.45228E-07	1.45228E-05	1.84754E-06	2.23454E-06
	0.4	2.55007E-07	2.55007E-05	3.55465E-06	6.58473E-06
	0.5	3.92107E-07	3.92107E-05	6.99343E-06	8.44584E-06
10	0.1	4.01523E-08	4.01576E-06	1.39453E-06	2.37436E-07
	0.2	8.06291E-07	8.06719E-06	4.53453E-06	6.84473E-07
	0.3	1.21484E-07	1.21628E-05	7.53463E-06	9.53453E-07
	0.4	1.62772E-07	1.63114E-05	1.23992E-05	3.56564E-06
	0.5	2.04551E-07	2.05219E-05	4.53444E-05	7.00349E-06
20	0.1	4.79878E-09	4.79952E-07	3.44848E-07	7.11234E-08
	0.2	9.63886E-09	6.64476E-07	5.11234E-07	2.00456E-07
	0.3	1.45277E-08	1.45476E-06	8.45454E-07	5.22203E-07
	0.4	1.94732E-08	1.95203E-06	2.34854E-06	8.00345E-07
	0.5	2.44835E-08	2.45756E-06	5.11456E-06	2.33456E-06

solving various challenging differential equations, both in the time-fractional and space-time-fractional domains. We anticipate that the findings and methodologies presented herein will not only enhance our understanding of fractional differential equations but will also pave the way for their application to diverse natural phenomena. In light of the promising results achieved in this study, we can extend this approach to tackle similar types of space-time-fractional differential equations that arise in various natural phenomena. By doing so, we aim to further advance the applicability and utility of the FRPSM in solving intricate real-world problems.

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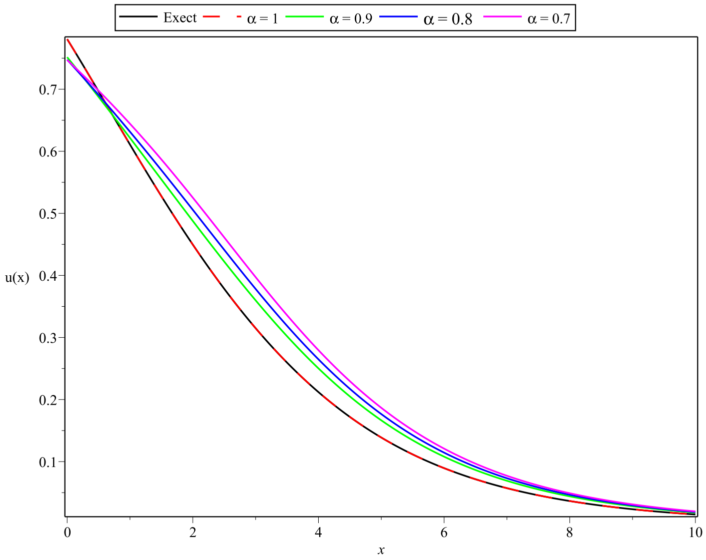
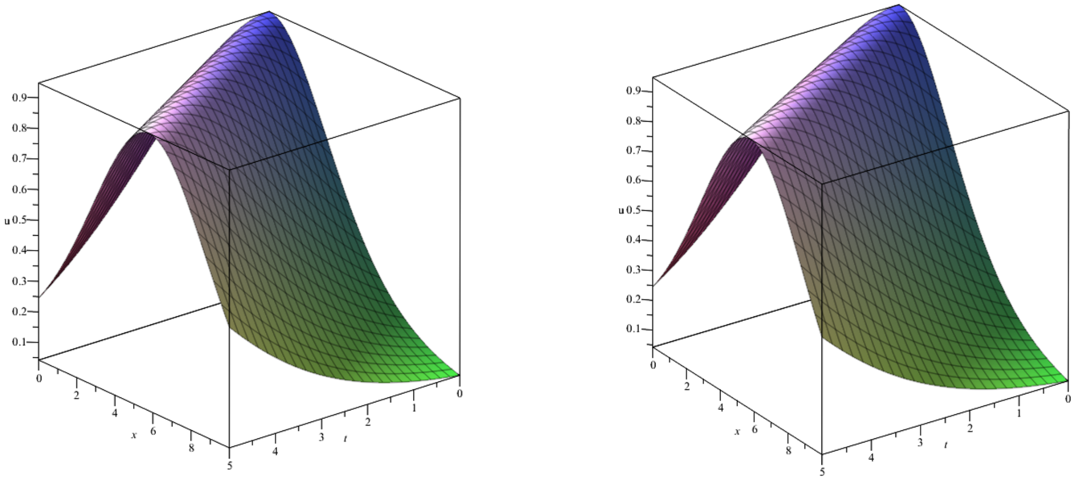


Figure 9. Comparison of solution of Fractional Modified Kawahara Equation as equation (4.36) with different fractional order of α with exact solution at $t = 1$.



(A) $\alpha = 1$

(B) Exact solution

Figure 10. Solution of Modified Kawahara equation (4.36) by FRPSM with order $\alpha = 1$ and Exact solution.

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