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MULTIPLE-PEAK TRAVELING WAVES OF THE GRAY-SCOTT MODEL

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Abstract. We study a reaction-diffusion system which models the pre-mixed isothermal autocatalytic chemical reaction of order m ($m > 1$) between two chemical species, a reactant A and an auto-catalyst B, $A + mB \to (m+1)B$, and a linear decay $B \to C$, where C is an inert product. The special case of $m = 2$ is the much studied Gray-Scott model, but without feeding. We prove existence of multiple traveling waves which have distinctive number of local maxima or peaks. It shows a new and very distinctive feature of Gray-Scott type of models in generating rich and structurally different traveling pulses than related models in literature such as isothermal autocatalysis without decay, or a bio-reactor model with isothermal autocatalysis of order $m + 1$ with m-th order of decay.

1. Introduction

In this paper we study a reaction-diffusion system

$$U_t = DU_{xx} - UV^m, \qquad V_t = V_{xx} + UV^m - kV, \tag{1.1}$$

in $\mathbb{R}^1 \times (0, \infty)$, where $D > 0$, $m > 1$ and $k > 0$ are constants. It models an auto-catalytic chemical reaction of $(m + 1)$-order with a decay step:

$$A + mB \longrightarrow (m+1)B \quad \text{with rate } UV^m, \qquad B \longrightarrow C \quad \text{with rate } kV,$$

where C is an inert product. The special case of $m = 2$ is the popular Gray-Scott model without feeding.

The Gray-Scott model with feeding [16],

$$\begin{cases} u_t = u_{xx} - uv^2 + A(1-u), \\ v_t = dv_{xx} + uv^2 - kv, \end{cases} \tag{1.2}$$

has been studied intensively in the last two decades by many authors to demonstrate its rich structure in generating interesting pattern and complex dynamics [8, 9, 10, 11, 13, 19, 20, 22, 24, 26, 27, 28, 35]. Here d, A and k are positive constants with $k > A$. One of the most amazing phenomena is the self-replicating pulses, which are splitting traveling wave solutions. The splitting of the pulses is a complicated, not yet completely understood process. Past mathematical studies of traveling pulse use mainly numerical simulation and/or singular perturbation to justify the existence and to demonstrate stability [8, 9, 10, 11, 13, 26]. In our view, a rigorous proof of existence of traveling wave and in depth analysis of solution structure will help us to understand the system much better.

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The main purpose of the present work is to prove rigorously the existence of multiple traveling waves with large number of peaks (local maxima) of V. Our result shows the interesting interaction of competing terms of kinetics can generate very rich and complex patterns, even when the feeding is switched off. We hope this work provides an important step towards complete understanding of structure and stability of traveling waves of Gray-Scott model, with or without feeding.

By simple scaling, we can reduce the system in (1.1) to the special case $k = 1$, which we assume throughout. Morevoer, we use a different form from the standard formulation of Gray-Scott system for convenience of presentation. It is easy to see that any equilibrium point must be in the form of $(a, 0)$ with $a \in \mathbb{R}$, which promotes the following definition.

Traveling Wave: $(U(x,t), V(x,t))$ is called a traveling wave solution to (1.1) if $U(x,t) = U(z)$, $V(x,t) = V(z)$, with $z = x - Ct$ and U, V are positive functions with $(U, V, C) \in C^2(\mathbb{R}) \times C^2(\mathbb{R}) \times \mathbb{R}^+$ satisfying,

$$\begin{cases} DU'' + CU' = UV^m, & U' > 0 \qquad \text{in } \mathbb{R}, \\ V'' + CV' = V - UV^m, & V > 0 \qquad \text{in } \mathbb{R}, \\ U(-\infty) = h, \quad V(-\infty) = 0, & V(\infty) = 0, \qquad U(\infty) < \infty, \end{cases} \tag{1.3}$$

where h is a positive constant. It is easy to see that if (U, V, C) is a solution, then $C > 0$.

This paper is concerned with the multiplicity of solutions of (1.3). To do this, we focus on the situation when $h \gg 1$. For this we make the following change of scale and variables:

$$\varepsilon = h^{-\frac{m}{m-1}}, \quad U = [1 + \varepsilon u]h, \quad V = h^{-\frac{1}{m-1}} v, \quad C = c\varepsilon\,.$$

Then (1.3) is equivalent to finding $(u, v, c) \in C^2(\mathbb{R}) \times C^2(\mathbb{R}) \times (0, \infty)$ which satisfy

$$\begin{cases} Du'' + c\varepsilon u' = [1 + \varepsilon u]v^m, & u' > 0 \qquad \text{in } \mathbb{R}, \\ v'' + c\varepsilon v' = v - [1 + \varepsilon u]v^m, & v > 0 \qquad \text{in } \mathbb{R}, \\ u(-\infty) = 0, \quad v(-\infty) = 0, & v(\infty) = 0, \qquad u(\infty) < \infty. \end{cases} \tag{1.4}$$

When $\varepsilon = 0$, the equation for v becomes $v'' = v - v^m$ which can be solved explicitly. We shall use a singular perturbation method to tackle the multiple scale of (1.4) and shooting argument to analyze the case of small positive ε.

To describe our main result, we use notation $O(f) = O(1)f$ where $O(1)$ is a function or a constant that is bounded by a constant depending only on m and D. We suppress the dependence of solutions on the parameters ε and c. We introduce notation

$$G(s) = s^2 - \frac{2s_+^{m+1}}{m+1}, \quad \alpha = \frac{1}{m-1}, \quad M = \Big(\frac{m+1}{2}\Big)^{\alpha}, \quad \sigma = 4\int_0^M \sqrt{G(s)}ds, \quad \gamma = \frac{2\alpha}{D}\int_0^M \frac{s^m ds}{\sqrt{G(s)}}, \tag{1.5}$$

where $s_+ = \max\{s, 0\}$. Our main result is the following:

Theorem 1.1. *Let $m > 1$ and $D > 0$ be given constants.*

(i) There exist positive constants M_1, M_2, and M_3 that depend only on m and D such that for each $\varepsilon > 0$, (1.4) admits no solution if $c \geqslant \max\{\sqrt{M_1/\varepsilon}, M_2\}$ or if $c \leqslant \gamma - M_3\varepsilon$.

*(ii) For each sufficiently small positive ε, let N be the largest integer satisfying $1 \leqslant N \leqslant \varepsilon^{-1/4}$. For each positive integer $n \in [1, N]$, there exists a constant $c_n = n\gamma[1 + O(\varepsilon + [n-1]^2\varepsilon|\ln\varepsilon|)]$ such that when $c = c_n$, the system (1.4) admits a solution, unique up to a translation. The solution is an n-**hump solution** in the sense that $w := [1 + \varepsilon u]^{\alpha}v$ admits exactly n local maxima and $n - 1$ interior local minima. In addition, if denote the interior points of local minima of w by $\{a_i\}_{i=2}^{n-1}$*

(at which $w < 1/2$) and points of local maxima by $\{b_i\}_{i=1}^n$ with $-\infty = a_1 < b_1 < a_2 < b_2 < \cdots < b_n < a_{n+1} = \infty$, then

$$w(b_i) = M + O(i[n+1-i]\varepsilon), \quad G(w(a_{i+1})) = i(n-i)\sigma\gamma\varepsilon + O(i^2n^2\varepsilon^2|\ln\varepsilon|) \quad \forall\, i = 1, \cdots, n.$$

Furthermore, $\|w'^2 - G(w)\|_{L^\infty(\mathbb{R})} = O(n^2\varepsilon)$ and

$$\lim_{\varepsilon\searrow 0} w(b_i + z) = \lim_{\varepsilon\searrow 0} v(b_i + z) = W(z)$$

uniformly in $i = 1, \cdots, n$ and locally uniformly in $z \in \mathbb{R}$, where W is the unique solution of

$$W'' = W - W^m \quad \text{in } \mathbb{R}, \qquad W(0) = M, \qquad W'(0) = 0. \tag{1.6}$$

Remark. The method we use in proving Theorem 1.1 is novel and has several ingredients. Simply put, it is a combination of singular perturbation and geometric shooting argument by using an energy functional we constructed as a control device. In contrast, our earlier work [7], in which the existence of a traveling wave is prove for all $h > 0$, uses a shooting argument using speed C as the parameter. The success of our method depends on hard a priori estimates of change of solution when x goes through each cycle, from a local minimum to a local maximum and then to the next local minimum. The small parameter ε also plays an important role.

Remark. It is clear from Theorem 1.1 that there exist traveling wave solutions with large number of peaks of V. In particular, as $\varepsilon \to 0$, V has exactly the same number of peaks as w. The quantity w is the key function we work on. The following is a direct consequence of Theorem 1.1.

Corollary 1.2. *Suppose $\varepsilon > 0$ is very small and for each integer n satisfying $1 \leqslant n \leqslant \varepsilon^{-1/4}$ there exists a traveling wave solution of (1.4) with V having at least n peaks. Furthermore, at each local minimum point of V, the value of V is very small, at most of $O(\varepsilon^{1/4})$.*

For convenience of reader, we summarize the elementary properties of W in (1.6) below, which can be verified easily.

Proposition 1.3. *The solution W of (1.6) is positive in $(-\infty, \infty)$, even and decreasing for $z > 0$. In addition, $W(z) \to 0$ as $|z| \to \infty$.*

Note that (i) the non-existence of solution when $c \leqslant \gamma - M_3\varepsilon$ is trivial if ε is not small since $\gamma - M_3\varepsilon < 0$ when ε is not small; (ii) equation (1.6) admits a first integral $W'^2 = G(W)$; the solution is even and positive with $W(\pm\infty) = 0$. The one, two, and three-hump solutions are depicted on the v-v' phase plane in Fig 1.

The intimate connection of our model to that of (1.2) for the study of traveling wave can be justified by two observations. First, it was demonstrated in [9, 10, 11] using asymptotic analysis and computation that when $0 < A \ll 1$, $k = O(1)$, and $0 < d \ll 1$ ($D = 1/d \gg 1$), the traveling pulse does go through the self-replicating process. This justifies the drop of the feeding term. Second, once the traveling pulse is away from the homogeneous state and u is changing slowly, it is the interaction of kinetic terms in the v equation which is mainly responsible for the interesting behavior of the traveling pulse. Furthermore, from purely a mathematical point of view, if we go extreme and let both A and D be zero, then it is proved in [29] that no traveling wave exists.

But, there are important differences of our model with (1.2). First, $U(\infty)$ is unknown, and must be determined as part of the problem. Second, the traveling wave problem has positive solutions which

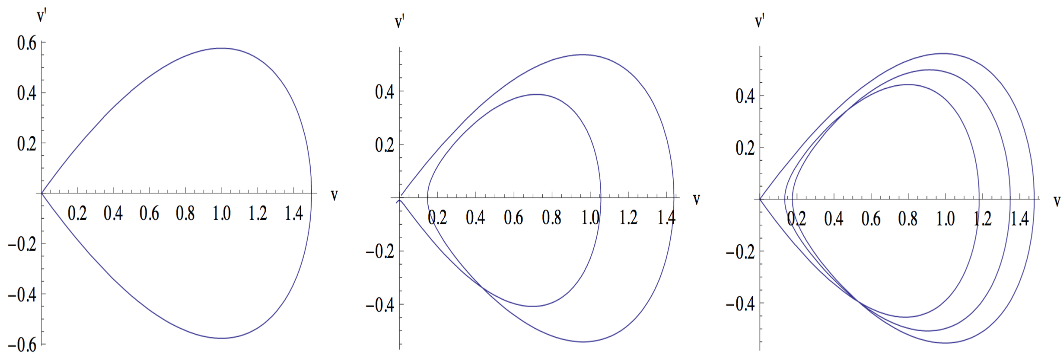

FIGURE 1. One, two, and three hump (loop) solutions on the v-v' phase plane; $m = 2, M = 1.5$.

are not traveling wave [7], which complicates the proof of main result. Third, the underlying traveling wave solution has a large number of oscillations, which has not been demonstrated for (1.2).

Next, we would like to make a comparison of our results with those of different but related systems in literature to demonstrate the unique feature of our system.

We start by looking at auto-catalytic chemical reaction of $(m + 1)$-order without decay,

$$U_t = DU_{xx} - UV^m, \qquad V_t = V_{xx} + UV^m, \tag{1.7}$$

which has been a popular model in the literature with global dynamics, [4, 23, 29], existence of traveling waves [3, 5, 15], stability of traveling waves [6, 15, 25] and spreading of local disturbance [30] being studied.

A traveling wave solution to (1.7) has both U and V be monotone functions. More importantly, as long as the traveling wave solution is concerned, (1.7) has a decisive deficiency in comparison with related experiments [36]. That is, any small amount of V locally added to the uniformly distributed U will generate a traveling wave. This was rigorously proved in [6].

Next, we compare a much more similar system in form to (1.1):

$$u_t = D_1 u_{xx} - uv^m, \qquad v_t = D_2 v_{xx} + uv^m - kv^m, \tag{1.8}$$

where $D_1, D_2 \geq 0$ and $m \geq 1$ was widely studied in the literature. For $m = 1$, Smith and Zhao [32] showed the existence for a special case of $D_1 = 0, D_2 > 0$; Huang [21] proved the existence for the general setting with $D_1, D_2 > 0$. For $m > 1$, the existence of traveling wave was first proved by Guo and Tsai for a special case [18] of $D_2 = 0$ and then by Tsai for the general case of $D_1, D_2 > 0$ [33].

There are a number of important differences between (1.1) and (1.8), and the approaches adopted in works such as [18, 21, 32, 33] are not applicable to the present case

First, the traveling wave problem of (1.8), like (1.7) is of mono-stable type, that is, for a fixed initial value at $-\infty$, there exists $C_0 > 0$ such that if $C > c_0$, there exists a traveling wave of speed C. But, (1.1) is different. In fact, it was proved in [7] that for a fixed initial value at $-\infty$, there exists $\hat{C} > 0$ such that if $C > \hat{C}$, there exists no traveling wave of speed C.

Second, an important fact of traveling wave problem of (1.8) is that for a traveling wave, v must be bell-shaped. But, the same cannot be said of (1.1) and the main result in this paper confirms a phenomenon first discovered by preliminary numerical simulation that V may oscillate many times before decaying to zero at ∞. In addition, a key insight of [21] and [18, 33] is that the behavior of solution to traveling wave problem of (1.8) with initial value h close to k is very different from that with h small, which allows a standard shooting argument using h. We do not have the same observation because the reaction terms in the second equation of (1.3) have very different property from that of (1.8). Finally, the solution structure of (1.3), having three classes of solutions [7] including one class of positive solutions which are not traveling waves, rather than two as in (1.8), is more complex than the corresponding traveling wave problem of (1.8).

We note in passing that existence of steady state in bounded domain for (1.1) was studied in [37, 38].

The primary tools we use in the present work, to establish the existence of multiple traveling waves, are a singular perturbation method, with multi-scaling of underlying variables and functions, in combination of a novel shooting argument. This work is a continuation of a previous study [7] in which the existence of a traveling wave is proved. But, the methods used here are completely new.

Furthermore, contrary to the classical single equation cases [2, 14]. or auto-catalytic chemical reaction without decay (1.7) or auto-catalytic chemical reaction of order $m+1$ with decay of order m, (1.8), where the set of traveling wave speed contains an interval $[c^*,\infty)$, with c^* depending on m, D and h, the set of wave speed in the current case of (1.3) is bounded as was proved in [7].

The organization of the rest of paper is as follows. In section 2 we perform some preliminary analysis. We prove an upper bound of speed c in section 3. In section 4, we show he existence of multiple traveling waves.

2. Preliminary

2.1. **Basic Setting.** For each constant $c \geqslant 0$, we consider the initial value problem, for $(u,v) = (u(x), v(x))$,

$$\begin{cases} Du'' + c\varepsilon u' = [1+\varepsilon u]v_+^m & \text{in } \mathbb{R}, \\ v'' + c\varepsilon v' = v - [1+\varepsilon u]v_+^m & \text{in } \mathbb{R}, \\ [v, v', u, u'] = [1, \lambda, 0, 0]e^{\lambda x} + O(1)e^{m\lambda x} & \text{as } x \to -\infty, \end{cases} \tag{2.1}$$

where λ is the positive root of $\lambda^2 + c\varepsilon\lambda = 1$ and $v_+ := \max\{v, 0\}$.

Lemma 2.1. *For each $c \geqslant 0$, problem (2.1), with $\lambda = \sqrt{1+\varepsilon^2c^2/4} - c\varepsilon/2$, admits a unique solution and the solution satisfies $u' > 0$ in $\mathbb{R}$. In addition, if (u,v) is a solution of (1.4), then up to a translation, it is the unique solution of (2.1).*

Proof. On the (v, v', u, u') phase space, $(0,0,0,0)$ is an equilibrium point. By a standard phase space analysis, there exists a unique trajectory leaving the equilibrium in the direction $[1, \lambda, 0, 0]$. Up to a translation, the associated solution has the asymptotic behavior described by the third equation in (2.1). We remark that $O(1)$ in (2.1) is indeed bounded by a constant depending only on m and D.

Denote by $(-\infty, Z)$ the maximal existence interval. Since $D(e^{c\varepsilon x/D}u')' = e^{c\varepsilon x/D}[1+\varepsilon u]v_+^m$, after integration we find that $u' > 0$ and $u > 0$ in the existence interval. In addition, u blows up (i.e., $Z < \infty$ and $U(Z) = \infty$) only if v_+ blows up. Next, from the equation for v, we find that v_+ cannot blow up; see Lemma 2.3 below. Thus $Z = \infty$, i.e., we have a unique solution of (2.1).

It is clear from (v, v', u, u') phase space analysis near the equilibrium $(0,0,0,0)$ that a solution of (1.4) must be a translation of the unique solution of (2.1). This completes the proof. □

In the sequel, (u,v) refers to the solution of (2.1) with (ε, c) dependence suppressed. We use $I = (-\infty, X)$ to represent the interval on which $v > 0$. It is easy to see that if $X < \infty$, then on $[X, \infty)$, $v' < 0$ and (u,v) is the solution of the linear system $v'' + c\varepsilon v' - v = 0, Du'' + cu' = 0$.

2.2. **Basic Estimates of v.**

First we investigate all critical points and possible oscillatory nature of v.

Lemma 2.2. *Suppose $z \in \mathbb{R}$ and $v'(z) = 0$. Then $v > 0$ in $(-\infty, z]$ and exactly one of the following holds:*

(i) *$v''(z) > 0$, so z is a point of local positive minimum;*
(ii) *$v''(z) < 0$, so z is a point of local positive maximum;*
(iii) *$v''(z) = 0$ and $v'''(z) < 0$, so v is strictly decreasing near z.*

As a consequence, the set $\{z \in \mathbb{R} \;:\; v'(z) = 0, v''(z) \neq 0\}$ can be arranged from small to large by $\{z_i\}_{i=1}^n$, where either $n = \infty$ or n is a positive integer. Set $z_0 = -\infty$ and $z_{n+1} = \infty$. Then for each integer i satisfying $0 \leqslant i \leqslant n/2$, $v' > 0$ in (z_{2i}, z_{2i+1}) and $v' \leqslant 0$ on $[z_{2i+1}, z_{2i+2}]$. Also $(-1)^i v''(z_i) > 0$ for $i = 1, \cdots, n$.

Proof. Since $v(\hat{x}) = 0$ implies that $v'(x) < 0$ in $[\hat{x}, \infty)$, we see that $v > 0$ in $(-\infty, z]$. Note that if $v'(z) = v''(z) = 0$, then $v'''(z) = -\varepsilon u' v^m < 0$. This completes the proof. □

Next, we establish an explicit upper bound of v.

Lemma 2.3. *Suppose $z \in [-\infty, \infty)$ is a point of local minimum of v. Then*

$$v(x) < \Big(\frac{m+1}{2[1+\varepsilon u(z)]}\Big)^{\frac{1}{m-1}} \qquad \forall\, x > z.$$

In particular, taking $z = -\infty$ we have $v(x) < M$ for all $x \in \mathbb{R}$, where M is as in (1.5).

Proof. Let $(z, \hat{z})$ be the maximal interval on which $v' > 0$. On $(z, \hat{z})$ we have $v > 0$ so for $x \in (z, \hat{z})$,

$$\frac{d}{dx}\Big(v'^2 + \frac{2[1+\varepsilon u(z)]v^{m+1}}{m+1} - v^2\Big) = -2c\varepsilon v'^2 + 2\varepsilon[u(z) - u(x)]v^m v' < 0.$$

After integration, we find that for $x \in (z, \hat{z}]$,

$$\Big([1+\varepsilon u(z)]v^{m-1}(x) - \frac{m+1}{2}\Big)v^2(x) < \Big([1+\varepsilon u(z)]v^{m-1}(z) - \frac{m+1}{2}\Big)v^2(z) \leqslant 0,$$

where the second inequality follows from the fact that $m > 1$ and $0 \geqslant v''(z) = v(z)[1 - (1 + \varepsilon u(z))v^{m-1}(z)]$. Hence, $[1+\varepsilon u(z)]v^{m-1}(x) < (m+1)/2$ for all $x \in (z, \hat{z}]$. As u is an increasing function, repeating the estimate at every local minimum of v we then obtain the assertion of the Lemma. □

2.3. **The Function $\rho = u'/[1 + \varepsilon u]$.**

Lemma 2.4. *Let $\rho = u'/[1+\varepsilon u]$ and $\bar{\rho}$ be the positive root of $\varepsilon(D\bar{\rho}^2 + c\bar{\rho}) = M^m$. Then*

$$D\rho' + \varepsilon(D\rho^2 + c\rho) = v_+^m \qquad \text{in } \mathbb{R}, \tag{2.2}$$

$$0 < \rho < \bar{\rho} \leqslant \sqrt{\frac{M^m}{\varepsilon D}}, \quad -\frac{M^m}{D} \leqslant \rho' \leqslant \frac{v_+^m}{D} \qquad \text{in } \mathbb{R}. \tag{2.3}$$

Proof. The differential equation for ρ follows from differentiation and the equation for u. Since $\rho(-\infty) = 0$ and 0 and $\bar{\rho}$ are sub and super solutions respectively, we find that $0 < \rho < \bar{\rho}$. This completes the proof. □

2.4. **The Function** $w = [1+\varepsilon u]^\alpha v$**.**

Let $\alpha = \frac{1}{m-1}$ and consider the function $w = [1+\varepsilon u]^\alpha v$. Direct differentiation gives

$$w'' + w_+^m - w = \eta_1 w' + \eta_2 w \qquad \text{in } \mathbb{R}, \tag{2.4}$$

where

$$\eta_1 = (2\alpha\rho - c)\varepsilon, \qquad \eta_2 = \frac{\alpha v_+^m}{D}\varepsilon + \Big\{\Big(1-\frac{1}{D}\Big)c\rho - (\alpha+1)\rho^2\Big\}\alpha\varepsilon^2. \tag{2.5}$$

Then, following lemma is obvious.

Lemma 2.5. *Let* $\alpha = \frac{1}{m-1}$ *and* $w = [1+\varepsilon u]^\alpha v$*. Then* w *satisfies* (2.4) *with* η_1, η_2 *given in* (2.5)*. In addition, by* $v \leqslant M$ *and* $0 < \rho \leqslant \bar\rho < \sqrt{M^m/\varepsilon D}$ *in* $\mathbb{R}$ *and* $\varepsilon(D\bar\rho^2 + c\bar\rho) = M^m$*, there hold the estimates*

$$-c\varepsilon \leqslant \eta_1 \leqslant 2\alpha\sqrt{\frac{M^m\varepsilon}{D}}, \qquad -\frac{\alpha(\alpha+1)M^m}{D}\,\varepsilon \leqslant \eta_2 \leqslant \alpha M^m \max\Big\{\frac{1}{D}, 1\Big\}\varepsilon. \tag{2.6}$$

3. An Upper Bound of c

In this section, we show that there is no traveling wave of fast speed.

Theorem 3.1. *There exist positive constants* M_1 *and* M_2 *that depend only on* m *and* D *such that for every* $\varepsilon > 0$*, if* $c \geqslant \max\{\sqrt{M_1/\varepsilon}, M_2\}$*, then the solution of* (2.1) *satisfies* $v > 0$ *in* $\mathbb{R}$ *and*

$$\lim_{x\to\infty}\Big(u,\rho,v,w\Big) = \Big(\infty,0,0,1\Big).$$

Consequently, (1.4) *admits no solution when* $c \geqslant \max\{\sqrt{M_1/\varepsilon}, M_2\}$.

Proof. We divide the proof into three steps.

Step 1: A Differential Inequality. Let $c > 0$ be a constant and (u, v) be the unique solution of (2.1). Set

$$K := \sup_{x\in\mathbb{R}} \eta_2(x), \qquad E := \frac{1}{2}w'^2 - \frac{1+K}{2}w^2 + \frac{w_+^{m+1}}{m+1} + \frac{c\varepsilon}{m+1}ww'.$$

In view of (2.6), we see that K is finite, so E is well-defined. Using (2.4) and (2.5) we derive that

$$E' + c\varepsilon E = \varepsilon w'^2\Big[2\alpha\rho - \frac{(m-1)c}{2(m+1)}\Big] + ww'\Big[\frac{2\alpha\rho c\varepsilon^2}{m+1} + \eta_2 - K\Big] - \frac{c\varepsilon w^2}{2}\Big[\frac{m-1}{m+1} + K - \frac{2\eta_2}{m+1}\Big].$$

Assume that

$$2\alpha\rho \leqslant \frac{(m-1)c}{4(m+1)}, \qquad 2\alpha\rho \leqslant \frac{(m-1)}{4\varepsilon}, \qquad K - \eta_2 \leqslant \frac{(m-1)c\varepsilon}{4(m+1)} \qquad \text{in } \mathbb{R}. \tag{3.1}$$

Then we obtain

$$E' + c\varepsilon E \leqslant \frac{(m-1)c\varepsilon}{4(m+1)}\Big\{-w'^2 + |ww'| - 2w^2\Big\} < 0 \quad \text{in } \mathbb{R}.$$

Step 2: A Necessary Condition for (3.1). First of all, since $0 < \rho \leqslant \bar\rho$ where $\bar\rho$ is the positive root of $\varepsilon(D\bar\rho^2 + c\bar\rho) = M^m$, we have

$$0 < 2\alpha\rho < \frac{2\alpha}{c}\Big[c\bar\rho + D\bar\rho^2\Big] = \frac{2\alpha M^m}{c\varepsilon}.$$

Thus, the first and second inequalities in (3.1) hold if we have

$$c^2 \geqslant \frac{8(m+1)\alpha M^m}{(m-1)\varepsilon} \qquad \text{and} \quad c \geqslant \frac{8\alpha M^m}{m-1}.$$

Next, we derive from (2.6) that

$$0 \leqslant K - \eta_2 \leqslant \max\Big\{1, \frac{1}{D}\Big\}\alpha(\alpha+2)\varepsilon M^m.$$

Thus, the third inequality in (3.1) holds if

$$c \geqslant \frac{4(m+1)\alpha(\alpha+2)M^m}{m-1}\max\Big\{1, \frac{1}{D}\Big\}.$$

In summary, the assumption (3.1) holds if $c \geqslant \max\{\sqrt{M_1/\varepsilon}, M_2\}$ where

$$M_1 = \frac{8(m+1)}{(m-1)^2}\Big(\frac{m+1}{2}\Big)^{\frac{m}{m-1}}, \quad M_2 = \frac{4(m+1)(2m-1)}{(m-1)^3}\Big(\frac{m+1}{2}\Big)^{\frac{m}{m-1}}\max\Big\{1, \frac{1}{D}\Big\}.$$

Step 3: Asymptotic Behavior as $x \to \infty$. Now assume that $c \geqslant \max\{\sqrt{M_1/\varepsilon}, M_2\}$. Then $E' + c\varepsilon E < 0$ in $\mathbb{R}$. Consequently, $E < 0$ in $\mathbb{R}$. Since $X < \infty$ would imply $E(X) > 0$, we must have $X = \infty$, i.e. $w > 0$ in $\mathbb{R}$. In addition, from $E < 0$ in $\mathbb{R}$ and $m > 1$, we derive that both w and w' are bounded.

As $x \to \infty$, there are only two possibilities: (i) $u(x) \to \infty$ or (ii) $u(x) \to u(\infty) < \infty$.

- (i) Suppose $\lim_{x\to\infty} u(x) = \infty$. Then $v = [1 + \varepsilon u]^{-\alpha} w \to 0$ as $x \to \infty$. Consequently, from the equation $D\rho' + \varepsilon[D\rho + c]\rho = v^m$ we derive that $\lim_{x\to\infty} \rho(x) = 0$. Otherwise, there exists $\varepsilon_0 > 0$ and a sequence $x_n \to \infty$ such that $\rho(x_n) \geq \varepsilon_0$ and $\rho'(x_n) \geq 0$, which contradicts the equation of ρ.
- (ii) Suppose $\lim_{x\to\infty} u(x) < \infty$. Then from $u' > 0$ and the equation $Du'' + c\varepsilon u' = [1 + \varepsilon u]v^m$ we derive that $\lim_{x\to\infty} u'(x) = 0$, $\lim_{x\to\infty} \rho(x) = 0$, and $\lim_{x\to\infty} v(x) = 0$.

Hence, in any case we have $\lim_{x\to\infty}(\rho(x)\ , v(x)) = (0, 0)$. Consequently, as $x \to \infty$,

$$w'' + c\varepsilon w' - w + w^m = 2\alpha\varepsilon\rho w' + \eta_2 w \longrightarrow 0.$$

Since $c > 0$ and w is bounded, there are only two possibilities: (1) $\lim_{x\to\infty} w(x) = 0$, (2) $\lim_{x\to\infty} w(x) = 1$.

Suppose $\lim_{x\to\infty} w(x) = 0$. Then using w-w' phase plane analysis for the saddle point $(0, 0)$, we find that

$$\lim_{x\to\infty}\frac{w'}{w} = \mu := \frac{-c\varepsilon - \sqrt{(c\varepsilon)^2 + 4}}{2} < -c\varepsilon.$$

This implies that as $x \to \infty$, $w + |w'| = O(1)e^{[1+o(1)]\mu x}$ and $|E| = O(1)e^{[2\mu+o(1)]x}$. However, from $(e^{c\varepsilon x}E)' = e^{c\varepsilon x}(E' + c\varepsilon E) < 0$, we derive that, for $x > 0$, $E(x)e^{c\varepsilon x} < E(0) < 0$, i.e., $|E(x)| > |E(0)|e^{-c\varepsilon x}$, contradicting $|E| = O(1)e^{[2\mu+o(1)]x}$ since $2\mu < -c\varepsilon$.

Thus, $\lim_{x\to\infty} w(x) = 0$ is impossible, so we must have $\lim_{x\to\infty} w(x) = 1$. As we already know that $\lim_{x\to\infty} v(x) = 0$, we must have $\lim_{x\to\infty} u(x) = \lim_{x\to\infty}(w/v)^{1/\alpha} = \infty$. This completes the proof. □

4. Existence of Multiple Traveling Waves for Small ε

In the sequel, we always assume that ε and c are parameters satisfying

$$0 < \varepsilon \ll 1, \qquad 0 \leqslant c < \max\{\sqrt{M_1/\varepsilon}, M_2\} = \sqrt{M_1/\varepsilon}. \tag{4.1}$$

We denote by (u, v) the solution of (2.1) and by $I = (-\infty, X)$ the maximal interval on which $v > 0$. On the w-w' phase plane, we call the trajectory between two neighboring local minima of w a **loop**.

Note that by the property of $G(s)$, for each $s \in [0, 1]$, there is a unique $s^* \in [1, M]$ such that $G(s^*) = G(s)$. We thus define s^* by the relation

$$G(s^*) = G(s), \quad s \in [0, 1], \quad s^* \in [1, M].$$

4.1. **Behavior of Solution in an Arbitrary Loop.**

Lemma 4.1. *Suppose $a \in [-\infty, X)$ is a point such that $w'(a) = 0$ and $w(a) \in [0, 1/2]$. Then a is a local minimum of w and there exist $z, b, \hat{z}, \hat{a}$ such that*

$$a < z < b < \hat{z} < \hat{a} \leqslant X,\ w(z) = 1,\ w' > 0 \text{ in } (a, b),$$
$$w(\hat{z}) = 1,\ w' < 0 < w \text{ in } (b, \hat{a}),\ w(\hat{a})w'(\hat{a}) = 0.$$

In addition, setting

$$r = w(a), \quad R = w(b), \quad \hat{r} = w(\hat{a}), \quad \varsigma = \Big(1 + c + \|\rho\|_{L^\infty((-\infty,\hat{a}])}\Big)\varepsilon\ , \tag{4.2}$$

we have $\varsigma = O(\sqrt{\varepsilon})$, and

$$w'^2 = \begin{cases} (1 + O(\varsigma))[G(w) - G(r)] & \text{in } [a, z], \\ (1 + O(\varsigma))[G(w) - G(R)] & \text{in } [z, \hat{z}], \\ (1 + O(\varsigma))[G(w) - G(\hat{r}) + w'^2(\hat{a})] & \text{in } [\hat{z}, \hat{a}], \end{cases}$$

$$G(R) = G(r) + O(\varsigma), \quad G(\hat{r}) = G(r) + O(\varsigma), \quad v'^2(\hat{a}) = O(\varsigma),$$
$$R = r^* + O(\varsigma), \quad \hat{r} = r + O(\sqrt{\varsigma}), \quad \int_a^{\hat{a}} w(y)dy = O(1).$$

Furthermore, exactly one of the following holds:

(i) $w(\hat{a}) > 0 = w'(\hat{a})$. *In this case $\hat{a}$ is a local positive minimum of w;*
(ii) $w(\hat{a}) = 0 > w'(\hat{a})$. *In this case, $\hat{a} = X < \infty$ and $w < 0$ in $[X, \infty)$;*
(iii) $w(\hat{a}) = w'(\hat{a}) = 0$. *In this case, $\hat{a} = X = \infty$ and the solution of (2.1) is a solution (1.4).*

Proof. Since $w''(a) = w(a)[1 + \eta_2 - w(a)^{m-1}]$, $w(a) \in [0, 1/2]$ and $\eta_2 = O(\varepsilon)$ (by (2.6)), a is a point of local minimum of w. We denote by (a, b) the maximal interval on which $w' > 0$. Later we shall show that $b < \infty$ and $w''(b) < 0$. We then define $(b, \hat{a})$ the maximal interval on which $w' < 0 < w$. We define $r, R, \hat{r}$, and ς as in (4.2). In view of (2.5), (2.6), (4.1), and (2.4), we see that $|\eta_1| + |\eta_2| = O(\varsigma)$, $\varsigma = O(\sqrt{\varepsilon})$, and

$$\frac{d}{dx}\Big(w'^2 - G(w)\Big) = 2(\eta_1 w' + \eta_2 w)w'. \tag{4.3}$$

We divide the rest of the proof into three steps.

Step 1. Integrating (4.3) over (a, y) with $y \in (a, x]$ and $x \in (a, b]$ we obtain

$$\begin{aligned} w'^2(y) - G(w(y)) + G(r) &= O(\varsigma)\Big(\max_{[a,y]} w'\Big) \int_a^y w'(z)dz + O(\varepsilon)\int_a^y w(z)w'(z)dz \\ &\leqslant \frac{1}{2}\Big(\max_{[a,x]} w'\Big)^2 + O(\varepsilon)(w^2(y) - r^2). \end{aligned}$$

In the above, we use the simple inequality that

$$O(\varsigma)\Big(\max_{[a,y]} w'\Big)\int_a^y w'(z)dz = O(\varsigma)\Big(\max_{[a,y]} w'\Big)(w(y) - w(r)) \le \frac{1}{2}\Big(\max_{[a,x]} w'\Big)^2 + O(\varsigma^2)(w(y) - w(r))^2.$$

Taking the supreme over $y \in [a, x]$ and using $w' > 0$ in (a, b) we then obtain

$$\frac{1}{2}\max_{[a,x]} w'^2 \quad \leqslant \quad \max_{[a,x]}\Big(G(w) - G(r) + O(\varepsilon)(w^2 - r^2)\Big) = O(1)[w^2 - r^2].$$

Therefore, on $[a, b]$

$$w'^2 = G(w) - G(r) + O(\varsigma)(w^2 - r^2).$$

This equation implies that $b < \infty$ and $w'(b) = 0$. In addition, setting $R = w(b)$ we have $G(R) = G(r) + O(\varsigma)$ and $R = r^* + O(\varsigma)$; there exists a unique $z \in (a, b)$ such that $w(z) = 1$.

We remark that the above argument can be used to show that $w + |w'| = O(1)$ in $(-\infty, X)$. On $[a, z]$, $r \leqslant w \leqslant 1$ so $w^2 - r^2 = O(1)[G(w) - G(r)]$. It then follows that

$$w'^2 = G(w) - G(r) + O(\varsigma)[w^2 - r^2] = [1 + O(\varsigma)][G(w) - G(r)] \quad \text{on} \quad [a, z].$$

Similarly, integrating (4.3) over (x, b) with $x \in [z, b]$ and using $R - w = O(1)[G(w) - G(R)]$ we obtain

$$w'^2 = G(w) - G(R) + O(\varsigma) \int_x^b w' dy = [1 + O(\varsigma)][G(w) - G(R)] \quad \text{on} \quad [z, b].$$

Finally,

$$\int_a^b w(x)dx = \int_a^b \frac{w(x)dw(x)}{w'(x)} = \int_r^1 \frac{[1 + O(\varsigma)]sds}{\sqrt{G(s) - G(r)}} + \int_1^R \frac{[1 + O(\varsigma)]sds}{\sqrt{G(s) - G(R)}} = O(1).$$

Step 2. At b, $w'(b) = 0$ and $w''(b) = R[1 + \eta_2 - R^{m-1}] < 0$, so b is a point of local maximum of w. Let $(b, \hat{a})$ be the maximal interval on which $w' < 0 < w$. Then $w(\hat{a})w'(\hat{a}) = 0$. Integrating (4.3) over (b, x) with $x \in (b, \hat{a})$ we obtain

$$w'^2 - [G(w) - G(R)] = O(\varsigma) \int_b^x |w'| dy = O(\varsigma)[R - w].$$

Hence, (i) if $\hat{r} := w(\hat{a}) = 0$, then $w'^2(\hat{a}) = -G(R) + O(\varsigma) = -G(r) + O(\varsigma)$, which implies, since $G(r) \geqslant 0$, that $G(r) = O(\varsigma)$ and $w'^2(\hat{a}) = O(\varsigma)$. (ii) If $w(\hat{a}) > 0$ then $w'(\hat{a}) = 0$ and $G(\hat{r}) = G(R) + O(\varsigma) = G(r) + O(\varsigma)$; this also implies that $\hat{r} = r + O(\sqrt{\varsigma})$. In any case, there exists $\hat{z} \in (b, \hat{a})$ such that $w(\hat{z}) = 1$. In addition,

$$1 \leqslant w \leqslant R, \qquad w'^2 = G(w) - G(R) + O(\varsigma)[R - w] = [1 + O(\varsigma)][G(w) - G(R)] \quad \text{on} \quad [b, \hat{z}].$$

Next, integrating (4.3) over $[x, \hat{a}]$ with $x \in [\hat{z}, \hat{a}]$ we obtain

$$w'^2(x) - \left\{w'^2(\hat{a}) + G(w) - G(\hat{r})\right\} \quad = \quad O(\varsigma) \max_{[x,\hat{a}]} |w'|(w - \hat{r}) + O(\varepsilon)(w^2 - \hat{r}^2).$$

Since $w' < 0 < w \leqslant 1$ in $[\hat{z}, \hat{a})$, this implies that for $x \in [\hat{z}, \hat{a}]$, $w^2 - \hat{r}^2 = O(1)[G(w) - G(\hat{r})]$ and

$$\max_{[x,\hat{a}]} w'^2 = O(1)\Big(w'^2(\hat{a}) + G(w(x)) - G(\hat{r})\Big).$$

Consequently,

$$w'^2 = [1 + O(\varsigma)]\Big\{w'^2(\hat{a}) + G(w) - G(\hat{r})\Big\} \qquad \text{on} \quad [\hat{z}, \hat{a}].$$

Hence,

$$\int_b^{\hat{a}} w(y)dy = \int_b^{\hat{a}} \frac{w(y)dw(y)}{w'(y)} = \int_{\hat{r}}^1 \frac{[1 + O(\varsigma)]sds}{\sqrt{G(s) - G(\hat{r}) + w'^2(\hat{a})}} + \int_1^R \frac{[1 + O(\varsigma)]sds}{\sqrt{G(s) - G(R)}} = O(1).$$

Step 3. There are three possibilities: (i) $\hat{a} < \infty$ and $w'(\hat{a}) = 0$, (ii) $\hat{a} < \infty$ and $w(\hat{a}) = 0$, (iii) $\hat{a} = \infty$.

(i) Suppose $\hat{a} < \infty$ and $w'(\hat{a}) = 0$. Then we must have $\hat{r} := w(\hat{a}) > 0$ (since otherwise we would have, by the uniqueness of initial value problem of the second order ode, $w \equiv 0$). That is, $\hat{a}$ is a local minimum.

(ii) Suppose $\hat{a} < \infty$ and $w(\hat{a}) = 0$. Then we must have $X = \hat{a}$ and $v < 0$ in (X, ∞).

(iii) Suppose $\hat{a} = \infty$. Then we must have $w'(\hat{a}) = 0$, $w''(\hat{a}) = 0$, and either $w(\hat{a}) = 0$ or $w(\hat{a}) = 1$. As $\hat{r} = r + O(\sqrt{\varsigma})$ we must have $w(\hat{a}) = 0$. The estimate above gives us $w' = -\sqrt{G(w)}[1 + O(\varsigma)]$ on $[\hat{z}, \infty)$. This implies that as $x \to \infty$, $w = O(1)e^{-[1+O(\varsigma)]x}$. Consequently, $v = w[1 + \varepsilon u]^{-\alpha} = O(1)e^{-[1+O(\varsigma)]x}$

as $x \to \infty$. Since $c > 0$, solving the differential equation $\rho' + \varepsilon[D\rho^2 + c\rho] = v^m = O(e^{-[m+O(\varsigma)]x})$ we find that $\rho = O(e^{-c\varepsilon x})$ as $x \to \infty$; from which we derive from $u' = [1 + \varepsilon u]\rho$ that $u(\infty) < \infty$. Thus the solution of (2.1) is a solution of (1.4). This completes the proof. □

Remark. It is clear from our proof that the condition $w(a) \in [0, 1/2]$ can be relaxed to $w \in [0, 1/2 + O(\sqrt{\varepsilon})]$ as long as $1/2 + O(\sqrt{\varepsilon}) < 1$ so that $w(a)$ is a local minimum.

4.2. **An Energy Functional.**

We define an energy functional by

$$\mathbf{E} := w'^2 - G(w) - \frac{2\alpha\varepsilon w_+^{m+2}}{(m+2)[1+\varepsilon u]^{m\alpha}D} - \varepsilon^2 w^2\Big[\alpha c\rho\big(1 - \frac{1}{D}\big) - \alpha(\alpha+1)\rho^2\Big].$$

Note that $\mathbf{E} = w'^2 - G(w) + O(\varepsilon)w^2$ and when $0 \leqslant w \leqslant 1$,

$$\mathbf{E} = w'^2 - [1 + O(\varepsilon)]G(w). \tag{4.4}$$

Direct differentiation together with the differential equation for w gives

$$\mathbf{E}' \quad = \quad 2(2\alpha\rho - c)w'^2\varepsilon + \Big[\frac{2\alpha^2 m\rho v_+^m}{(m+2)D} - \alpha c\big(1 - \frac{1}{D}\big)\rho' + 2\alpha(\alpha+1)\rho\rho'\Big]w^2\varepsilon^2. \tag{4.5}$$

Let $a_1 = -\infty$ be the "first local minimum" of w and b_1 be the first local maximum. We set $r_1 = w(a_1) = w(-\infty) = 0$ and $R_1 = w(b_1)$. Then by Lemma 4.1, $w' > 0$ in (a_1, b_1) and $R_1 = M + O(\varsigma)$.

Let $i = O(1)/\sqrt{\varepsilon}$ be a positive integer and assume that w has at least i local minima attained, from small to large, at $a_1, a_2, \cdots, a_i$, satisfying $r_j := w(a_j) \in [0, 1/2]$ for $j = 1, \cdots, i$. Then by Lemma 4.1, there exist i local maxima, attained, from small to large, at $b_1, b_2, \cdots, b_i$ with $R_j = w(b_j) = r_j^* + O(\varsigma_j)$, where $\varsigma_j = \varepsilon[1 + c + \|\rho\|_{L^\infty((-\infty, a_{j+1}))}]$ and (b_i, a_{i+1}) is the maximal interval on which $w' < 0 < w$. Set $r_{i+1} = w(a_{i+1})$. By Lemma 4.1, we have $G(r_{i+1}) = G(r_i) + O(\varsigma_i)$. We call $\gamma_i := \{(w(x), w'(x)) : x \in (a_i, a_{i+1})\}$ the ith loop of the trajectory on the w-w' phase plane. We observe the following:

(i) If $\mathbf{E}(a_{i+1}) = 0$, then $w(a_{i+1}) = 0$ and $w'(a_{i+1}) = 0$ so we have a solution of (2.1) with i loops.
(ii) If $\mathbf{E}(a_{i+1}) > 0$, then $w(a_{i+1}) = 0$ and $w'(a_{i+1}) < 0$. Consequently, $X = a_{i+1} < \infty$.
(iii) If $\mathbf{E}(a_{i+1}) < 0$, then $w'(a_{i+1}) = 0$ and $r_{i+1} := w(a_{i+1}) \in (0, 1/2 + O(\sqrt{\varepsilon})]$. Hence, $w(a_{i+1})$ is a local minimum of w and the trajectory has at least $i + 1$ loops on the phase plane.

In the sequel, we evaluate $\mathbf{E}(a_{i+1})$ in terms of $\mathbf{E}(a_i)$. For each positive integer n not too large, we shall find an appropriate $c = c_n > 0$ such that $\mathbf{E}(a_{i+1}) < 0$ for $1 \leqslant i < n$ and $\mathbf{E}(a_{n+1}) = 0$, i.e., the solution of (2.1) is a solution of (1.4) with exactly n loops; as a consequence, we obtain an n-hump traveling wave.

4.3. **The Difference $\mathbf{E}(a_{i+1}) - \mathbf{E}(a_i)$.**

To evaluate the difference $\mathbf{E}(a_{i+1}) - \mathbf{E}(a_i)$, we continue to use the setting in the previous subsection. Integrating (4.5) over (a_i, a_{i+1}) we find that

$$\mathbf{E}(a_{i+1}) - \mathbf{E}(a_i) = \big\{[2\alpha\rho(b_i) - c]\sigma_i + K_i + L_i\big\}\,\varepsilon,$$

where

$$\begin{aligned}\sigma_i &:= 2\int_{a_i}^{a_{i+1}} w'^2(y)dy,\\ K_i &:= 4\alpha\int_{a_i}^{a_{i+1}}[\rho(y)-\rho(b_i)]w'^2(y)dy,\\ L_i &:= \alpha\varepsilon\int_{a_i}^{a_{i+1}}\Big[\frac{2\alpha m\rho v^m}{D(m+2)}-c\Big(1-\frac{1}{D}\Big)\rho'+2(\alpha+1)\rho\rho'\Big]w^2(y)dy.\end{aligned}$$

Firstly, since $D\rho' < v_+^m < w_+^m$, after integration we find that, for $x\in(-\infty,a_{i+1}]$,

$$0<\rho(x)\leqslant\frac{1}{D}\int_{-\infty}^{x} w^m(y)dy = O(1)\sum_{j=1}^{i}\int_{a_j}^{a_{j+1}} w(y)dy = O(i).$$

Consequently,

$$\begin{aligned}\varsigma_i &= \varepsilon\big[c+1+\|\rho\|_{L^\infty((-\infty,a_{i+1}))}\big] = O([c+i]\varepsilon),\\ L_i &= O(1)\varepsilon\int_{a_i}^{a_{i+1}}\big[c+i\big]w(y)dy = O([c+i]\varepsilon).\end{aligned}$$

Secondly, in other to evaluate σ_i and K_i, we define, for $j=1,\cdots,i+1$, $x\in\mathbb{R}$,

$$p_j(x)=\int_{a_j}^{x} w'^2(y)dy,\qquad k_j(x)=\int_{a_j}^{x} p_j(y)dy\geqslant 0,\quad q_j(x)=\int_{a_j}^{x} p_j(y)w^m(y)dy\geqslant 0.$$

Note that for $x\in[a_i,b_i]$, $p_i(x)=O(1)[w(x)-r_i]$ and by Lemma 4.1,

$$\begin{aligned}\int_{a_i}^{x}[w(y)-r_i]dy &= \int_{r_i}^{\min\{w(x),1\}}\frac{O(1)(s-r_i)ds}{\sqrt{G(s)-G(r_i)}}+\int_{\min\{w(x),1\}}^{w(x)}\frac{O(1)ds}{\sqrt{G(s)-G(R_i)}}\\ &= O(1)[w(x)-r_i].\end{aligned}$$

After a similar calculation for $x\in[b_i,a_{i+1}]$ and using

$$\sqrt{G(w)-G(r_{i+1})+w'(a_{i+1})^2}\geqslant\sqrt{G(w)-G(r_{i+1})},$$

we find that

$$\int_{a_i}^{b_i}[k_i(x)+q_i(x)]dx=O(1),\qquad \int_{b_i}^{a_{i+1}}[k_{i+1}(x)+q_{i+1}(x)]dx=O(1). \tag{4.6}$$

Now we calculate K_i, which we write as

$$\frac{K_i}{4\alpha} = \int_{a_i}^{a_{i+1}}[\rho-\rho(b_i)]w'^2dx=\int_{a_i}^{b_i}[\rho-\rho(b_i)]p_i'\,dx-\int_{a_{i+1}}^{b_i}[\rho-\rho(b_1)]p_{i+1}'\,dx.$$

Using integration by parts and the differential equation for ρ,

$$\begin{aligned}\int_{a_i}^{b_i}[\rho-\rho(b_i)]p_i'\,dx &= -\int_{a_i}^{b_i} p_i\rho'\,dx = -\int_{a_i}^{b_i} p_i\Big(\frac{w^m}{D[1+\varepsilon u]^{m\alpha}}-\frac{\varepsilon[D\rho^2+c\rho]}{D}\Big)dx\\ &= -\int_{a_i}^{b_i}\Big(\frac{q_i'}{D[1+\varepsilon u]^{m\alpha}}-\frac{\varepsilon[D\rho^2+c\rho]k_i'}{D}\Big)dx\\ &= -\frac{q_i(b_i)}{D[1+\varepsilon u(b_i)]^{m\alpha}}+\frac{k_i(b_i)\varepsilon[D\rho^2(b_i)+c\rho(b_i)]}{D}\\ &\quad -\int_{a_i}^{b_i}\Big(\frac{\varepsilon m\alpha q_i\rho}{[1+\varepsilon u]^{m\alpha}}+\frac{\varepsilon\rho'[2D\rho+c]k_i}{D}\Big)dx\\ &= -\frac{q_i(b_i)}{D[1+\varepsilon u(b_i)]^{m\alpha}}+\frac{k_i(b_i)\varepsilon[D\rho^2(b_i)+c\rho(b_i)]}{D}+O(\varsigma_i),\end{aligned}$$

where in the last equation, we have used (4.6), $\rho'=O(1)$, and the definition of ς_i. After a similar calculation for the integral over $[b_i,a_{i+1}]$ and using $\varepsilon[D\rho^2+c\rho]=O(i\varsigma_i)$ we find that

$$K_i=O(1)\Big\{|q_i(b_i)-q_{i+1}(b_i)|+|k_i(b_i)-k_{i+1}(b_i)|i\varsigma_i\Big\}+O(\varsigma_i).$$

Using Lemma 4.1 we find that

$$\frac{p_i(x)}{[1+O(\varsigma_i)]} = \left\{\int_{r_i}^{\min\{w(x),1\}}\sqrt{G(s)-r_i}ds+\int_{\min\{w(x),1\}}^{w(x)}\sqrt{G(s)-G(R_i)}ds\right\}\quad \forall\, x\in[a_i,b_i],$$

$$\begin{aligned}k_i(b_i) &= [1+O(\varsigma_i)]\left\{\int_{r_i}^{1}\frac{\int_{r_i}^t\sqrt{G(s)-G(r_i)}ds}{\sqrt{G(t)-G(r_i)}}dt\right\}\\ &+ [1+O(\varsigma_i)]\left\{\int_1^{R_i}\frac{\int_{r_i}^1\sqrt{G(s)-G(r_i)}ds+\int_1^t\sqrt{G(s)-G(R_i)}}{\sqrt{G(t)-G(R_i)}}dt\right\},\end{aligned}$$

$$\begin{aligned}k_{i+1}(b_i) &= [1+O(\varsigma_i)]\left\{\int_{r_{i+1}}^{1}\frac{\int_{r_{i+1}}^t\sqrt{G(s)-B}ds}{\sqrt{G(t)-B}}dt\right\}\\ &+ [1+O(\varsigma_i)]\left\{\int_1^{R_i}\frac{\int_{r_{i+1}}^1\sqrt{G(s)-B}ds+\int_1^t\sqrt{G(s)-G(R_i)}}{\sqrt{G(t)-G(R_i)}}dt\right\},\end{aligned}$$

$$\begin{aligned}q_i(b_i) &= [1+O(\varsigma_i)]\left\{\int_{r_i}^{1}\frac{t^m\int_{r_i}^t\sqrt{G(s)-G(r_i)}ds}{\sqrt{G(t)-G(r_i)}}dt\right\}\\ &+ [1+O(\varsigma_i)]\left\{\int_1^{R_i}\frac{t^m(\int_{r_i}^1\sqrt{G(s)-G(r_i)}ds+\int_1^t\sqrt{G(s)-G(R_i)}ds)}{\sqrt{G(t)-G(R_i)}}dt\right\},\end{aligned}$$

$$\begin{aligned}q_{i+1}(b_i) &= [1+O(\varsigma_i)]\left\{\int_{r_{i+1}}^{1}\frac{t^m\int_{r_{i+1}}^t\sqrt{G(s)-B}ds}{\sqrt{G(t)-B}}dt\right\}\\ &+ [1+O(\varsigma_i)]\left\{\int_1^{R_i}\frac{t^m(\int_{r_{i+1}}^1\sqrt{G(s)-B}ds+\int_1^t\sqrt{G(s)-G(R_i)}ds)}{\sqrt{G(t)-G(R_i)}}dt\right\},\end{aligned}$$

where $B = G(r_{i+1}) - w'(a_{i+1})^2$. Since $B = G(r_i) + O(\varsigma_i)$ and $r_{i+1}w'(a_{i+1}) = 0$, by an elementary algebraic computation, one can show that $k_i(b_i) - k_{i+1}(b_i) = O(\varsigma_i \ln \varsigma_i)$ so $i\varsigma_i|k_i(b) - k_{i+1}(b_i)| = O(i\varsigma_i \ln \varsigma_i)\varsigma_i = O(\varsigma_i)$. Similarly, one can show that $q_i(b_i) - q_{i+1}(b_i) = O(\varsigma_i \ln \varepsilon)$. Thus,

$$K_i = \frac{4\alpha[q_{i+1}(b_i) - q(b_i)]}{[1 + \varepsilon u(b_i)]^{m\alpha}} + O(\varsigma_i) = O(\varsigma_i|\ln \varepsilon|).$$

Finally, we use Lemma 4.1 to obtain

$$\begin{aligned} \sigma_i &= 2[1 + O(\varsigma_i)]\Big\{ \int_{r_i}^1 \sqrt{G(s) - G(r_i)}ds + \int_{r_{i+1}}^1 \sqrt{G(s) - B}ds + 2\int_1^{R_i} \sqrt{G(s) - G(R_i)}ds \Big\} \\ &= \mathbf{A}(r_i) + O(\varsigma_i|\ln \varepsilon|), \end{aligned}$$

where

$$\mathbf{A}(r) := 4\int_r^{r^*} \sqrt{G(s) - G(r)}ds \qquad \forall\, r \in [0, 1]. \tag{4.7}$$

Hence, we have the following:

Lemma 4.2. *Suppose a_i is the ith local minimum of w, $w(a_i) \in [0, 1/2]$, and b_i is the ith local maximum. Let $r_i = w(a_i)$ and (b_i, a_{i+1}) be the maximum interval on which $w' < 0 < w$. Then*

$$\mathbf{E}(a_{i+1}) - \mathbf{E}(a_i) = \Big\{ \Big(2\alpha\rho(b_i) - c\Big)\Big(\mathbf{A}(r_i) + O([c + i]\varepsilon|\ln \varepsilon|)\Big) + O([c + i]\varepsilon|\ln \varepsilon|) \Big\}\varepsilon.$$

4.4. Evaluation of $\rho(b_1)$ and the Minimal Speed Wave.

Note that $a_1 = -\infty$, $r_1 = 0$, and $R_1 = M + O([c + 1]\varepsilon)$. For $x \in (-\infty, b_1]$, integrating $D\rho' < w^m$ over $(-\infty, x]$ we obtain, when $x \leqslant b_1$,

$$\begin{aligned} 0 < \rho(x) &< \int_{-\infty}^x \frac{w^m(y)}{D}dy = O(1)\int_0^{\min\{w(x), M\}} \frac{s^m\, ds}{\sqrt{G(s)}} = O(1)w^m(x), \\ \int_{-\infty}^x \rho(y)dy &= O(1)\int_{-\infty}^x w^m(y)dy = O(1), \\ \int_{-\infty}^x [D\rho^2 + c\rho]dy &= O(1 + c)\int_{-\infty}^x \rho(y)dy = O(1 + c). \end{aligned}$$

Consequently, since $\rho' = u'/(1 + \varepsilon u)$, we have, for $x \in (-\infty, b_1)$,

$$\varepsilon u(x) = e^{\varepsilon \int_{-\infty}^x \rho(y)dy} - 1 = e^{O(\varepsilon)} - 1 = O(\varepsilon);$$

Hence, integrating $D\rho' = v^m - \varepsilon[D\rho^2 + c\rho]$ we obtain

$$\begin{aligned} \rho(b_1) &= \int_{-\infty}^{b_1} \frac{w^m(y)}{D[1 + \varepsilon u]^{m\alpha}}dy - \varepsilon\int_{-\infty}^{b_1} \Big[\rho^2 + \frac{c}{D}\rho\Big]dy \\ &= \int_{-\infty}^{b_1} \frac{w^m(y)}{D}dy + O(\varepsilon[1 + c]) \\ &= [1 + O(\varsigma_1)]\int_0^M \frac{s^m}{D\sqrt{G(s)}}ds + O(\varepsilon[1 + c]) \\ &= \frac{\gamma}{2\alpha} + O([c + 1]\varepsilon), \\ \mathbf{A}(r_1) &= \mathbf{A}(0) = 4\int_0^M \sqrt{G(s)}ds = \sigma. \end{aligned}$$

Thus, we have the following:

Lemma 4.3. *Let σ and γ be as in* (1.5). *Then*

$$\mathbf{E}(a_2) = \Big\{(\gamma - c)\Big(\sigma + O([c+1]\varepsilon|\ln\varepsilon|)\Big) + 4\alpha[q_2(b_1) - q_1(b_1)] + O([c+1]\varepsilon)\Big\}\varepsilon. \tag{4.8}$$

Now we are ready to prove the following:

Theorem 4.4. *Assume that $0 < \varepsilon \ll 1$. Then there exists $c_1 = \gamma + O(\varepsilon)$ such that* (1.4) *admits a* **(one hump)** *solution when $c = c_1$. In addition, if* (1.4) *admits a solution, then $c > \gamma - O(\varepsilon)$. Consequently, the minimal wave speed of* (1.4) *is $c_{\min} = \gamma + O(\varepsilon)$.*

Proof. We know that $q_1(b_1) - q_2(b_1) = O(\varepsilon|\ln\varepsilon|)$. By (4.8), we see that there exist positive constants K and ε_0 which depend only on m and D such that when $0 < \varepsilon < \varepsilon_0$, the following holds:

(i) By continuity and intermediate value theorem, there exists $c_1 = \gamma + O(\varepsilon|\ln\varepsilon|)$ such that $\mathbf{E}(a_2) = 0$; this implies that the solution of (2.1) is a solution of (1.4) when $c = c_1$. Upon noting that $\mathbf{E}(a_2) = 0$ implies that $r_1 = r_2 = 0$, so $q_1(b_1) - q_2(b_1) = O(\varepsilon)$. It then follows from (4.8) with $\mathbf{E}(a_2) = 0$ that $c_1 = \gamma + O(\varepsilon)$.

(ii) If $\sqrt{M_1/\varepsilon} \geqslant c > \gamma + K\varepsilon$, then $\mathbf{E}[a_2] \leqslant |O(\varepsilon^2|\ln\varepsilon|)|$, which implies that $B = G(r_2) - w'^2(r_2) > -|O(\varepsilon^2|\ln\varepsilon|)|$, so $q(r_2) - q(r_1) < |O(\varepsilon^2|\ln\varepsilon|)|$. Consequently, $\mathbf{E}(a_2) < -\sigma\gamma K\varepsilon^2/2 < 0$. Thus, $w'(a_2) = 0$ and $w(a_2) > 0$, so the trajectory admits at least two loops.

(iii) If $0 \leqslant c < \gamma - K\varepsilon$, then $\mathbf{E}(a_2) \geqslant -|O(\varepsilon^2|\ln\varepsilon|)|$, which implies that $q(b_2) - q(b_1) \geqslant -|O(\varepsilon^2|\ln\varepsilon|)|$. It then implies that $\mathbf{E}(a_i) \geqslant \sigma K\gamma\varepsilon^2/2$. Hence, $w(a_2) = 0$ and $w'(a_2) < 0$. This means that there is no traveling wave solution of (1.4) with speed $c \in [0, \gamma - K\varepsilon]$.

This completes the proof of Theorem 4.4. ☐

Remark 4.1. Taking $M_3 = \max\{K, \varepsilon_0/\gamma\}$ we see that (1.4) admits no solution if $c \leqslant \gamma - M_3\varepsilon$, since when $\varepsilon \geqslant \varepsilon_0$, $\gamma - M_2\varepsilon \leqslant 0$ and (1.4) admits no solution when $c \leqslant 0$.

4.5. **Two-Hump Solution.**

Now assume that $c \in [\frac{3}{2}\gamma, \frac{5}{2}\gamma]$ and $0 < \varepsilon \ll 1$. We see from (4.8) that $\mathbf{E}(a_2) < -\sigma\gamma\varepsilon/3$. This implies that $w'(a_2) = 0$. By (4.4), we find that $G(r_2) = -[1 + O(\varepsilon)]\mathbf{E}(a_2) > \sigma\gamma\varepsilon/4$ so $r_2 := w(a_2) > \sqrt{\sigma\gamma\varepsilon/5}$. Consequently, by Lemma 4.1, with $R_1 = G(b_1)$ and $R_2 = G(b_2)$,

$$\begin{aligned}
b_2 - b_1 &= \int_{b_1}^{a_2} \frac{dw(y)}{w'(y)} + \int_{a_2}^{b_2} \frac{dw(y)}{w'(y)} \\
&= \int_{r_2^*}^{R_1} \frac{ds}{w'(s)} + 2\int_{r_2}^{r_2^*} \frac{ds}{w'(s)} + \int_{r_2^*}^{R_2} \frac{ds}{w'(s)} \\
&= O(1) + O(1)\int_{r_2}^{r_2^*} \frac{ds}{\sqrt{G(s) - G(r_2)}} = O(1)|\ln\varepsilon|, \\
\varepsilon u(b_2) &= e^{\varepsilon\int_{-\infty}^{b_1}\rho(y)dy + \varepsilon\int_{b_1}^{b_2}\rho(y)dy} - 1 = e^{O(\varepsilon) + O(\varepsilon[b_2 - b_1])} - 1 = O(\varepsilon|\ln\varepsilon|).
\end{aligned}$$

Also, using $G(r_2) = -[1 + O(\varepsilon)]\mathbf{E}(a_2) = [\sigma(c - \gamma) + O(\varepsilon)]\varepsilon$ we find that

$$\begin{aligned}
\rho(b_2) - \rho(b_1) &= \int_{b_1}^{b_2} \frac{w^m}{D[1 + \varepsilon u]^{m\alpha}} dy - \varepsilon\int_{b_1}^{b_2}\Big[\rho^2 + \frac{c}{D}\rho\Big]dy \\
&= \frac{1 + O(\varepsilon|\ln\varepsilon|)}{D}\int_{b_1}^{b_2} w^m(y)dy + O(\varepsilon|b_2 - b_1|)
\end{aligned}$$

$$\begin{aligned} &= \frac{2+O(\varepsilon|\ln\varepsilon|)}{D}\int_{r_2}^{r_2^*}\frac{s^m ds}{\sqrt{G(s)-G(r_2)}}+O(\varepsilon|\ln\varepsilon|)\\ &= \frac{2}{D}\int_0^M \frac{s^m}{\sqrt{G(s)}}ds+O(\varepsilon|\ln\varepsilon|)=\frac{\gamma}{\alpha}+O(\varepsilon|\ln\varepsilon|),\\ \mathbf{A}(r_2) &= \mathbf{A}(0)+O(G(r_2)|\ln\varepsilon|=\sigma+O(\varepsilon|\ln\varepsilon|). \end{aligned}$$

It then follows that

$$\begin{aligned} \mathbf{E}(a_3) &= \Big\{[2\alpha\rho(b_2)-c][\mathbf{A}(r_2)+O(\varepsilon|\ln\varepsilon|)]+[2\alpha\rho(b_1)-c][\mathbf{A}(r_1+O(\varepsilon)]+O(\varepsilon|\ln\varepsilon|)\Big\}\varepsilon\\ &= \Big\{(4\gamma-2c)\sigma+O(\varepsilon|\ln\varepsilon|)\Big\}\varepsilon, \end{aligned}$$

By using the same analysis as above, we then obtain the following:

Lemma 4.5. *When $0<\varepsilon\ll 1$, there exists $c_2=2\gamma+O(\varepsilon|\ln\varepsilon|)$ such that when $c=c_2$, $\mathbf{E}(a_2)<0$ and $\mathbf{E}(a_3)=0$. Consequently, when $c=c_2$, (1.4) admits a two-hump solution in the sense that w admits exactly two local maxima. In addition, if $c\in[\frac{5}{2}\gamma,\varepsilon^{-1/2}]$, then $\mathbf{E}(a_2)<0$ and $\mathbf{E}(a_3)<0$.*

4.6. **Multi-Hump Traveling Waves and Proof of Theorem 1.1.**

Let n be an integer satisfying $3\leqslant n\leqslant\varepsilon^{-1/4}$. Assume that $c\in[(n-\frac{1}{2})\gamma,(n+\frac{1}{2})\gamma]$. We then know that

$$\rho(b_1)=\frac{\gamma}{2\alpha}+O(c\varepsilon),\qquad \mathbf{E}(a_2)=\Big\{(\gamma-c)\sigma+O(c^2\varepsilon|\ln\varepsilon|)\Big\}\varepsilon.$$

For induction, we assume that $i\in[1,n-1]$ is an integer and there holds the estimates

$$\rho(b_i)=\frac{(2i-1)\gamma}{2\alpha}+O(ic^2\varepsilon|\ln\varepsilon|),\qquad -\mathbf{E}(a_{i+1})=i\Big[(i\gamma-c)\sigma+O(ic^2\varepsilon|\ln\varepsilon|)\Big]\varepsilon. \tag{4.9}$$

Then $w'(a_{i+1})=0$ and $G(r_{i+1})=-[1+O(\varepsilon)]\mathbf{E}(a_{i+1})>\sigma\gamma\varepsilon/4$. Hence,

$$\begin{aligned} b_{i+1}-b_i &= O(1)\int_{r_{i+1}}^{r_{i+1}^*}\frac{ds}{\sqrt{G(s)-G(r_{i+1})}}=O(|\ln\varepsilon|),\\ \int_{-\infty}^{b_{i+1}}\rho(y)dy &= \int_{-\infty}^{b_1}\rho(y)dy+O(1)\sum_{j=1}^{i}j[b_{j+1}-b_j]=O(1)i^2|\ln\varepsilon|,\\ \varepsilon u(b_{i+1}) &= e^{\varepsilon\int_{-\infty}^{b_{i+1}}\rho(y)dy}-1=O(i^2\varepsilon|\ln\varepsilon|)=O(c^2\varepsilon|\ln\varepsilon|),\\ \rho(b_{i+1})-\rho(b_i) &= \int_{b_i}^{b_{i+1}}\frac{w^m(y)dy}{D[1+\varepsilon u]^{m\alpha}}-\varepsilon\int_{b_i}^{b_{i+1}}\Big[\rho^2+\frac{c}{D}\rho\Big]dy\\ &= O(c^2\varepsilon|\ln\varepsilon|)+\int_{r_{i+1}}^{r_{i+1}^*}\frac{2s^m}{\sqrt{G(s)-G(r_{i+1})}}ds\\ &= O(c^2\varepsilon|\ln\varepsilon|)+\frac{\gamma}{\alpha}+O(1)G(r_{i+1})\ln G(r_{i+1})\\ &= \frac{\gamma}{\alpha}+O(c^2\varepsilon|\ln\varepsilon|), \end{aligned}$$

where we use the fact that $G(a_{i+1})=-[1+O(\varepsilon)]\mathbf{E}(a_{i+1})=O(ci\varepsilon)=O(c^2\varepsilon)$. Hence,

$$\begin{aligned} \rho(b_{i+1}) &= \frac{(2i-1)\gamma}{2\alpha}+O(ic^2\varepsilon|\ln\varepsilon|)+\frac{\gamma}{\alpha}+O(c^2\varepsilon|\ln\varepsilon|)=\frac{(2i+1)\gamma}{2\alpha}+O([i+1]c^2\varepsilon|\ln\varepsilon|),\\ \mathbf{A}(r_{i+1}) &= \mathbf{A}(0)+O(1)G(r_{i+1})|\ln\varepsilon|=\sigma+O(ci\varepsilon|\ln\varepsilon|). \end{aligned}$$

Consequently,

$$\begin{aligned}\mathbf{E}(a_{i+2}) &= \mathbf{E}(a_{i+1}) + \Big\{[2\alpha\rho(b_{i+1}) - c]\Big(\mathbf{A}(r_{i+1}) + O(c\varepsilon|\ln\varepsilon|)\Big) + O(c\varepsilon|\ln\varepsilon|)\Big\}\varepsilon \\ &= \mathbf{E}(a_{i+1}) + \Big\{[(2i+1)\gamma - c)]\sigma + O([2i+1]c^2\varepsilon|\ln\varepsilon|)\Big\}\varepsilon \\ &= \Big[[\{i^2+2i+1\}\gamma - (i+1)c]\sigma + O(\{i^2+2i+1\}c^2\varepsilon|\ln\varepsilon|)\Big]\varepsilon \\ &= [i+1]\Big\{([i+1]\gamma - c])\sigma + O([i+1]c^2\varepsilon|\ln\varepsilon|)\Big\}\varepsilon.\end{aligned}$$

Thus, by mathematical induction, (4.9) holds for $i = 1, \cdots, n$. Consequently, there exists $c_n = n\gamma[1 + O(n^2\varepsilon|\ln\varepsilon|)]$ such that $\mathbf{E}(a_{i+1}) < 0$ for $i = 1, \cdots, n-1$ and $\mathbf{E}(a_{n+1}) = 0$. That is, when $c = c_n$, (1.4) admits a solution with n humps. This completes the proof of Theorem 1.1.

5. Discussion

The stability is always the central problem in the study of traveling wave. Previous works of Doelman et al [9]-[11], Ei et al [13] and Muratov and Osipov [26, 27] are on Gray-Scott with feeding, not the present model. Because the formal analysis has been done by those authors, not rigorous proof, we do not have clear indication of what to expect for our case.

In addition, in comparison of well established theory of stability of travelling waves for scalar reaction-diffusion equations, there are very serious challenge to systems. For example, very powerful tools such as maximum principle and calculus of variation are not applicable to our system. Furthermore, we show the complex structure of travelling wave problem in this work, but we do not know what are the exactly speeds of travelling waves.

In what follows. we present some numerical computation done to study traveling waves.

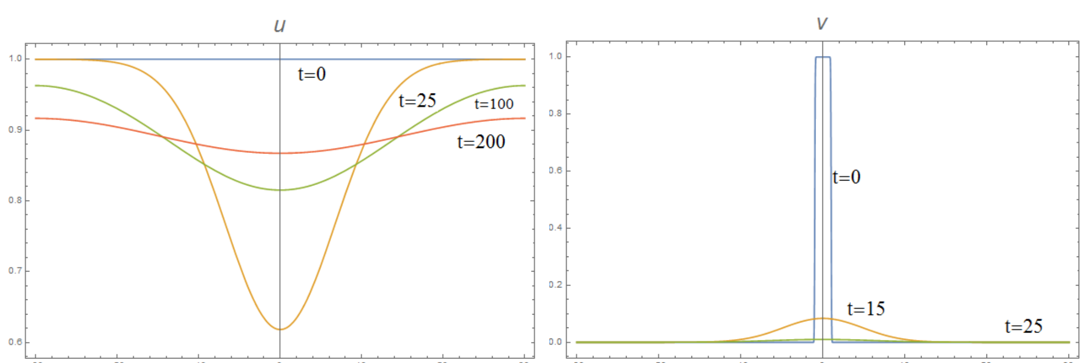

FIGURE 2. System (1.1) with $m = 2$, $D = 4$ and $k = 0.2$, $U(x, 0) \equiv 1$, $V(x, 0) = 1$ in $[-1, 1]$, and $V(x, 0) = 0$ outside $[-1, 1]$.

The computation is done with periodic boundary conditions on the interval $[-50, 50]$ as spatial domain. It can be seen from our computational result that the solution of (1.1) can either evolve to a single-peak or double-peak traveling wave when the parameter k is changed, but with the same initial values. Although ultimately v converges to zero because of the role of linear decay in v, and the boundary effect, it is a good indication that the solution can converge to either a single-peak or double-peak traveling wave with identical initial value.

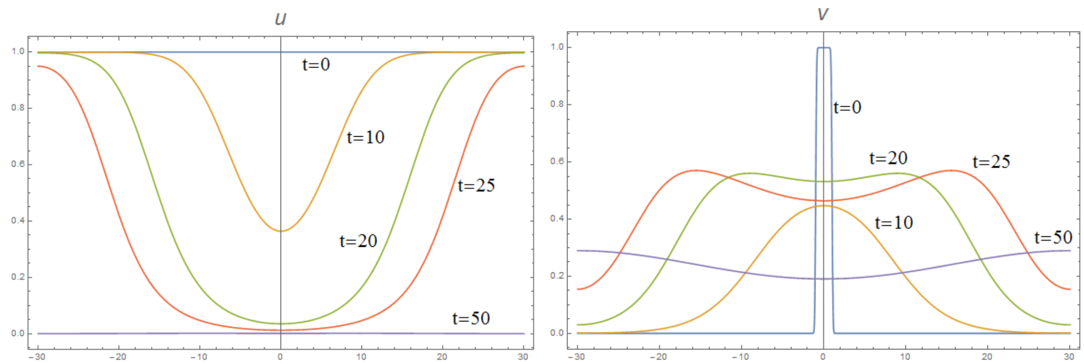

FIGURE 3. System (1.1) with $m = 2$, $D = 4$ and $k = 0.05$, $U(x, 0) \equiv 1$, $V(x, 0) = 1$ in $[-1, 1]$, and $V(x, 0) = 0$ outside $[-1, 1]$.

What we can conclude is that the stability is far more complex than anything we know so far for similar but different type of problems.

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