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RECOVERY OF AN INITIAL TEMPERATURE OF A ONE-DIMENSIONAL BODY FROM FINITE TIME-OBSERVATIONS

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ABSTRACT. Under the Dirichlet boundary setting, Aryal and Karki (2022) studied an inverse problem of recovering an initial temperature profile from known temperature measurements at a fixed location of a one-dimensional body and at linearly growing finitely many later times within a bounded interval. This paper studies the problem under the Neumann boundary conditions. That is, under this boundary setting, we suitably select a fixed location x_0 on the body of length π and construct finitely many times $t_k, k = 1, 2, 3, \ldots, n$ that grow linearly with k and are in $[0, T]$ such that from the temperature measurements taken at x_0 and at these n times, we recover the initial temperature profile $f(x)$ with a desired accuracy, provided f is in a suitable subset of $L^2[0, \pi]$.

1. Introduction

We begin by considering the temperature distribution $u(x, t)$ of a thin uniform one-dimensional homogeneous rod of length π, in which the heat propagates through the rod, as described by the initial-boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u}{\partial t}(0, t) = \frac{\partial u}{\partial t}(\pi, t) = 0, \quad u(x, 0) = f(x), \tag{1.1}$$

where f is assumed to be in a suitable subset of $L^2([0, \pi])$ which will be discussed soon. Given such an f, the problem (1.1) has a solution that can be represented as

$$u(x, t) = \sum_{k=0}^{\infty} e^{-k^2 t} \hat{f}_k \cos kx, \tag{1.2}$$

where $\hat{f}_k$ is the k^{th} Fourier cosine coefficient of f. On the other hand, if temperature measurements $u(x_0, t_k)$ taken at a fixed location x_0 on the rod and at finitely many later times $t_1 < t_2 < t_3 < \cdots < t_k < \cdots < t_n$ are known, can we recover f with a certain accuracy? This type of inverse problem is highly ill-posed unless further assumptions on f, x_0 and t_k are made. It has been widely studied in recent times (see [1, 2, 6, 8, 9, 10, 11]) and has applications in many areas (see [3, 7, 12, 13]).

Under the Dirichlet boundary conditions (i. e., $u(0, t) = u(0, \pi) = 0$), DeVore and Zuazua ([6]) have studied an analogous problem by assuming that f is in

$$\mathcal{F}_r = \left\{ f \in L^2([0, \pi]) : \sum_{j=1}^{\infty} j^{2r} |\hat{f}_j|^2 \leq 1 \right\}, \quad r > 0. \tag{1.3}$$

They have selected x_0 in $[0, \pi]$ such that x_0/π is an algebraic number of order 2 to avoid vanishing temperature measurements at any time by skipping the nodal set of eigenfunctions $\sin kx$, $k = 1, 2, 3, \ldots,$

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at x_0 (in the Fourier sine series representation of $u(x,t)$ in the case of Dirichlet boundary conditions) and defined $t_k = \left(2\sqrt{2}\right)^{k-1} t_1,\ k = 1, 2, \ldots$ with an arbitrarily chosen (and fixed) $t_1 > 0$. Using the n temperature measurements $u(x_0, t_j),\ j = 1, 2, \ldots, n$ for a large n, they constructed an approximation to f in $L^2([0,\pi])$ with an accuracy of order n^{-r}.

Aryal and Karki ([2]) have selected n forward times $t_k,\ k = 1, 2, 3, \ldots, n$ so that t_1 and the next $n-1$ times

$$t_k = (n+k-1)t_1 \tag{1.4}$$

for $k = 2, 3, \ldots, n$ lie within a bounded interval $[0, T]$ by slightly compromising on the choice of f from the closed subset $\mathcal{F}_r$ of $L^2([0,\pi])$ to a smaller subset

$$\mathcal{B}_r = \left\{ f \in L^2([0,\pi]) : \sum_{k=1}^{\infty} (k+2)^{2r} |\hat{f}_k|^2 \le 1 \right\}, \tag{1.5}$$

where $r \ge p^*$ for some $p^* > 0$, and obtained the same near optimal rate of accuracy of order n^{-r}. Their time selection requires only linear growth which is far more controlled than exponential growth and can be done within a predetermined time-bound compared to the selection without such a bound. Their result is a significant improvement over the result of DeVore and Zuazua. Particularly, when studying this recovery process using a numerical approach, which is highly suggested by the nature of the problem, the waiting time between any two successive temperature measurements taken in a controlled growth within a finite time interval is much more realistic than in rapid exponential growth without a definite bound. Additionally, they require to place a thermometer at x_0 to take the body temperature at t_k's in such a way that x_0/π is an algebraic number of order 1 (instead of 2), that is,

$$dist\left(\frac{x_0}{\pi}, \left\{0, \frac{1}{k}, \frac{2}{k} \ldots, \frac{k-1}{k}, 1\right\}\right) \ge \frac{c}{k} \tag{1.6}$$

for all $k = 1, 2, 3, \ldots$ and for some constant c. This allows $\sin kx_0 \ne 0$ for all $k = 1, 2, 3, \ldots, n$.

In this paper, we will use the approach of Aryal and Karki ([2]) and establish a result corresponding to the initial-boundary value problem described in (1.1). For this, we consider a finite time interval $[0, T]$ and select n forward times t_1 and t_k, $k = 2, 3, \ldots, n$ as described in (1.4). We assume that f is in the following closed subset of $L^2([0,\pi])$:

$$\mathcal{C}_r = \left\{ f \in L^2([0,\pi]) : \sum_{k=0}^{\infty} (k+e)^{2r} |\hat{f}_k|^2 \le 1 \right\}, \tag{1.7}$$

where $r \ge p^*$ for some $p^* > 0$. We choose the body location at x_0 in $[0,\pi]$ so that $\cos kx_0 \ne 0$ for all $k = 0, 1, 2, \ldots,$ for which we assume

$$dist\left(\frac{2x_0}{\pi}, \left\{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, 1\right\}\right) \ge \frac{c}{k} \tag{1.8}$$

for all $k = 1, 2, 3, \ldots$ and for some constant c. This implies that

$$dist\left(kx_0, \left\{\frac{1}{2}\pi, \frac{3}{2}\pi, \ldots, \frac{2k-1}{2}\pi\right\}\right) \ge c\pi$$

for all $k = 1, 2, 3 \ldots.$ Since $|\cos kx_0| = \left|\sin |\frac{2j-1}{2}\pi - kx_0|\right|$ for all $j = 1, 2, 3, \ldots, k$ and for all $k = 1, 2, 3, \ldots,$ the last inequality gives

$$|\cos kx_0| \ge c'$$

for all $k = 1, 2, 3 \ldots$ and for some $c' > 0$. Choosing $d = \min\{1, c'\}$, we obtain

$$|\cos kx_0| \ge d \tag{1.9}$$

for all $k = 0, 1, 2, 3 \ldots$. This shows that such a selection of the location x_0 allows us to avoid the nodal set of eigenfunctions $\cos kx$, $k = 0, 1, 2, 3, \ldots$, at x_0, thereby giving us nonvanishing temperature measurements at x_0 and at any time t.

Finally, for any forward time sequence $0 < t_1 < t_2 < \ldots$, the corresponding discrete temperature measurements $u(x_0, t_k),\ k = 1, 2, \ldots$ are sufficient to determine f uniquely. Indeed, consider a holomorphic function

$$F(z) = \sum_{j=0}^{\infty} z^{j^2} \hat{f}_j \cos jx_0,\ z \in \mathbb{D},$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Since $F(z_k) = u(x_0, t_k),\ k = 1, 2, \ldots$ where $z_k = e^{-t_k},\ k = 1, 2, \ldots$ are in $\mathbb{D}$ and $\{z_k\}_{k=1}^{\infty}$ converges in $\mathbb{D}$, we can uniquely determine F and hence $\hat{f}_j$, $j = 0, 1, 2, \ldots$. Due to this fact, we can expect to construct a unique approximation to f from our n temperature measurements $u(x_0, t_k)$, $k = 1, 2, 3, \ldots, n$.

2. Recovery of Initial Temperature

2.1. **Bounds on optimal error of approximation.** From [1, 2, 6, 11], we will briefly recall a measurement algorithm that we will use in our approximation process and then a lower bound on the optimal error of the approximation. This algorithm is developed by using the theory of manifold width (see [4, 5]) and is indeed an encoder coupled with a decoder. An encoder is a continuous function that maps each element of a compact subspace $\mathcal{C}$ of $L^2([0, \pi])$ into a point in $\mathbb{R}^n$, and a decoder is a continuous function that maps each point $y \in \mathbb{R}^n$ into an element of $L^2([0, \pi])$. In particular, an encoder is a continuous function a_n mapping $f \in \mathcal{C}_r$ into $a_n(f) = (u_1, \ldots, u_n)$ where u_j are the corresponding n temperature measurements $u_j = u_j(x_0, t_j)$, $j = 1, 2, \ldots n$, and a decoder is a continuous function M_n mapping $(u_1, \ldots, u_n) \in \mathbb{R}^n$ into an approximation $\bar{f}_n$ to f. Thus, the point $M_n(a_n(f))$ in the n-dimensional manifold $\{M_n(y) \in L^2([0, \pi]) : y \in \mathbb{R}^n\}$ is viewed as $\bar{f}_n$. The optimal error of approximation to f is defined as

$$\delta_{a_n, M_n}(\mathcal{C}_r, L^2([0, \pi])) = \sup_{f \in \mathcal{C}_r} ||f - \bar{f}_n||_{L^2([0,\pi])}. \tag{2.1}$$

Referring to [2], one can obtain a lower bound for this optimal error (also see [6]) as follows.

Theorem 2.1. *For the measurement algorithm with an encoder $a_n : f \mapsto (u_1, u_2, \ldots, u_n)$ and a decoder $M_n : (u_1, u_2, \ldots, u_n) \mapsto \bar{f}_n$ as described above, we have*

$$\delta_{a_n, M_n}(\mathcal{C}_r, L^2([0, \pi])) \geq C n^{-r} \tag{2.2}$$

where C is a constant depending on r only.

Using the measurement algorithm, one may also expect to deduce an upper bound of the form $C'(r)n^{-r}$ for the optimal error $\delta_{a_n, M_n}(\mathcal{C}_r, L^2([0, \pi]))$. The lower bound for the optimal error given by (2.1) is derived from using a lower bound for the manifold width, where the manifold width is the infimum of $\delta_{a_n, M_n}(\mathcal{C}_r, L^2([0, \pi]))$ over all encoders a_n and decoders M_n that correspond to the choice of n samples. On the other hand, there is no upper bound for this manifold width for a measurement algorithm like ours. This posses a challenge to deduce an upper bound of the form we have expected. However, in the next section we will construct an approximation to f with a similar upper bound for its L^2-error.

2.2. **An approximation with a near optimal rate of accuracy.** The main result of this paper is the following.

Theorem 2.2. *Given $T > 0$, choose $t_1 \in (0, T]$ and $n \in \mathbb{N}$ such that t_j, $j = 2, 3, 4, \ldots, n$ as given in (1.4) are in $[0, T]$. Let $f \in \mathcal{C}_r$ where $\mathcal{C}_r$ is as described in (1.7). Fix x_0 in $[0, \pi]$ as described in (1.8). Then for the known n temperature measurements $u(x_0, t_j), j = 1, 2, ..., n$, there exists $\bar{f}_n$ in $L^2([0, \pi])$ such that*

$$||f - \bar{f}_n||_{L^2([0,\pi])} \leq Cn^{-r} \tag{2.3}$$

where C is a constant depending on x_0, r and t_1.

Based on the assumptions of Theorem 2.2, we first establish a couple of estimates.

Lemma 2.3. *If $c_j = \hat{f}_j \cos jx_0$, $j = 0, 1, 2, \ldots,$ then*

$$|c_j| \leq (j + e)^{-r}, \; j = 0, 1, 2, \ldots \tag{2.4}$$

Proof: For each $j = 0, 1, 2, \ldots,$ we get

$$|c_j| \leq |\hat{f}_j| \leq \left(\sum_{k \geq j} |\hat{f}_k|^2 \right)^{\frac{1}{2}} \leq (j + e)^{-r} \left[\sum_{k=0}^{\infty} (k + e)^{2r} |\hat{f}_k|^2 \right]^{\frac{1}{2}}.$$

Since $f \in \mathcal{C}_r$, we obtain from (1.7) that

$$|c_j| \leq (j + e)^{-r}$$

for all $j = 0, 1, 2, \ldots$ ■

With $c_j = \hat{f}_j \cos jx_0$, $j = 0, 1, 2, \ldots,$ we have

$$u(x_0, t_k) = \sum_{j=0}^{\infty} c_j e^{-j^2 t_k}, \quad k = 1, 2, 3, \ldots, n$$

from which we obtain

$$c_k = u(x_0, t_k) e^{k^2 t_k} - \sum_{j=0}^{k-1} c_j e^{(k^2 - j^2) t_k} - \sum_{j=k+1}^{\infty} c_j e^{(k^2 - j^2) t_k}. \tag{2.5}$$

Define $\bar{c}_k$, an approximation to c_k, as $\bar{c}_0 = c_0 = \hat{f}_0$ and

$$\bar{c}_k = u(x_0, t_k) e^{k^2 t_k} - \sum_{j=0}^{k-1} (j + e)^{-r} e^{(k^2 - j^2) t_k} - \sum_{j=k+1}^{n} (j + e)^{-r} e^{(k^2 - j^2) t_k} \tag{2.6}$$

for all $k = 1, 2, 3, \ldots, n$.

Lemma 2.4. *For the n future times $t_k, k = 1, 2, ..., n$ as in Theorem 2.2, consider a real number p^* such that*

$$p^* = T \left(\frac{T}{2t_1} + \frac{3}{2} \right)^2$$

and $r \geq p^*$. Then

$$|c_k - \bar{c}_k| \leq C(t_1) k e^{-(2k+1) t_k}, k = 0, 1, 2, ..., n. \tag{2.7}$$

Proof: For $k=0$, the inequality holds trivially. For $k=1,2,...,n$, we use (2.4), (2.5), (2.6) and (2.7) to obtain

$$\begin{aligned}
&|c_k-\bar{c}_k| \\
&=\left|\sum_{j=0}^{k-1}\left((j+e)^{-r}-c_j\right)e^{(k^2-j^2)t_k}-\sum_{j=k+1}^{n}\left((j+e)^{-r}-c_j\right)e^{(k^2-j^2)t_k}-\sum_{j=n+1}^{\infty}c_je^{(k^2-j^2)t_k}\right| \\
&\le 2\sum_{j=0}^{k-1}(j+e)^{-r}\,e^{(k^2-j^2)t_k}+2\sum_{j=k+1}^{n}(j+e)^{-r}\,e^{(k^2-j^2)t_k}+\sum_{j=n+1}^{\infty}(j+e)^{-r}\,e^{(k^2-j^2)t_k} \\
&\le 2\sum_{j=0}^{k-1}(j+e)^{-r}\,e^{(k^2-j^2)t_k}+2\sum_{j=k+1}^{\infty}(j+e)^{-r}\,e^{(k^2-j^2)t_k} \\
&\le 2\sum_{j=0}^{k-1}(j+e)^{-r}\,e^{(k^2-j^2)t_k}+2\,(k+3)^{-r}\sum_{j=k+1}^{\infty}e^{(k^2-j^2)t_k}.
\end{aligned}$$

Since

$$\sum_{j=k+1}^{\infty}e^{(k^2-j^2)t_k}\le\sum_{j=k+1}^{\infty}e^{-(j+k)t_k}=\sum_{j=0}^{\infty}e^{-(2k+1+j)t_k}\le e^{-(2k+1)t_k}s(t_1)$$

where

$$s(t_1)=\sum_{j=0}^{\infty}e^{-jt_1}$$

and $(k^2-j^2+2k+1)t_k\le(n+1)^2T\le p^*$ for all $j=0,1,2,\ldots,k-1$, we have

$$\begin{aligned}
|c_k-\bar{c}_k| &\le 2\sum_{j=0}^{k-1}(j+e)^{-r}\,e^{(k^2-j^2)t_k}+2\,(k+3)^{-r}\,e^{-(2k+1)t_k}s(t_1) \\
&=2e^{-(2k+1)t_k}\left[\sum_{j=0}^{k-1}(j+e)^{-r}\,e^{(k^2-j^2+2k+1)t_k}+(k+3)^{-r}\,s(t_1)\right] \\
&\le 2e^{-(2k+1)t_k}\left[\sum_{j=0}^{k-1}e^{-r}e^{p^*}+s(t_1)\right] \\
&=2ke^{-(2k+1)t_k}\left[1+s(t_1)\right] \\
&=C(t_1)ke^{-(2k+1)t_k}
\end{aligned}$$

for all $k=1,2,3,\ldots,n$. ■

Proof of Theorem 2.2. As we have $\hat{f}_k=c_k/\cos kx_0$ for $k=0,1,2,\ldots$, define $\hat{\bar{f}}_k=\bar{c}_k/\cos kx_0$ for $k=0,1,2,...,n$. Then

$$|\hat{f}_k-\hat{\bar{f}}_k|\le\frac{1}{|\cos kx_0|}|c_k-\bar{c}_k|,\quad k=0,1,2,...,n.$$

Using (1.9) and (2.7) in this inequality, we get

$$|\hat{f}_k-\hat{\bar{f}}_k|\le C(x_0,t_1)ke^{-(2k+1)t_k},\quad k=0,1,2,...,n. \tag{2.8}$$

As an approximation to f, define

$$\bar{f}_n(x) = \sum_{k=0}^{n} \hat{\bar{f}}_k \cos kx.$$

Then, by assuming $\hat{\bar{f}}_k = 0$ for all $k \geq n+1$, we obtain

$$\begin{aligned}
||f - \bar{f}_n||^2_{L^2([0,\pi])} &= \left|\left|\sum_{k=0}^{\infty}(\hat{f}_k - \hat{\bar{f}}_k)\cos kx\right|\right|^2_{L^2[0,\pi]} \\
&= \frac{\pi}{2}\sum_{k=1}^{\infty}\left|\hat{f}_k - \hat{\bar{f}}_k\right|^2 \\
&= \frac{\pi}{2}\left[\sum_{k=1}^{n}\left|\hat{f}_k - \hat{\bar{f}}_k\right|^2 + \sum_{k=n+1}^{\infty}\left|\hat{f}_k\right|^2\right] \\
&\leq \frac{\pi}{2}\left[\sum_{k=1}^{n}\left|\hat{f}_k - \hat{\bar{f}}_k\right|^2 + n^{-2r}\sum_{k=n+1}^{\infty}(k+e)^{2r}\left|\hat{f}_k\right|^2\right] \\
&\leq \frac{\pi}{2}\left[\sum_{k=1}^{n}\left|\hat{f}_k - \hat{\bar{f}}_k\right|^2 + n^{-2r}\right]
\end{aligned}$$

and hence

$$\frac{2}{\pi}||f - \bar{f}_n||^2_{L^2} \leq \sum_{k=1}^{n}|\hat{f}_k - \bar{f}_k|^2 + n^{-2r}. \tag{2.9}$$

From (2.8) and (2.9), we obtain

$$\begin{aligned}
\frac{2}{\pi}||f - \bar{f}_n||^2_{L^2([0,\pi])} &\leq C^2(x_0,t_1)\sum_{k=1}^{n}k^2 e^{-2(2k+1)t_k} + n^{-2r} \\
&= C^2(x_0,t_1)\sum_{k=1}^{n}\frac{k^2}{e^{2(2k+1)kt_1}} + n^{-2r} \\
&\leq \frac{C^2(x_0,t_1)}{e^{6nt_1}}\sum_{k=1}^{n}k^2 + n^{-2r} \\
&= \frac{C^2(x_0,t_1)}{e^{6nt_1}}\cdot\frac{n(n+1)(2n+1)}{6} + n^{-2r} \\
&= C^2(x_0,t_1)\cdot\frac{n^3}{6e^{6nt_1}}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + n^{-2r}.
\end{aligned} \tag{2.10}$$

For each pair of r and t_1, the convergence of

$$\left\{\frac{m^{2r+3}}{6e^{6mt_1}}\left(1+\frac{1}{m}\right)\left(2+\frac{1}{m}\right)\right\}$$

leads us to obtain

$$\frac{n^{2r+3}}{6e^{6mt_1}}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \leq c(r,t_1) \tag{2.11}$$

for some constant $c(r,t_1)$. From (2.10) and (2.11),

$$\frac{2}{\pi}||f - \bar{f}_n||^2_{L^2} \leq C(x_0,t_1,r)n^{-2r}$$

and therefore,

$$||f - \bar{f}_n||_{L^2} \leq Cn^{-r}$$

where C is a constant depending on x_0, t_1 and r.

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