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RELATIVE OPERATOR ENTROPY PROPERTIES RELATED TO SOME WEIGHTED METRICS

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Abstract. In recent decades, intensive research has been devoted to the study of various operator entropies. In this work, we investigate the properties of the parametrized relative operator entropy $S_p(A \mid B)$ acting on positive definite matrices with respect to weighted Hellinger and Alpha Procrustes distances. In particular, we investigated estimation of the distance between the entropy $S_p(A \mid B)$ and certain standard means.

1. Introduction

Let $\mathbb{M}_n$ be the space of $n \times n$ matrices over $\mathbb{R}$ endowed by an inner product $\langle . \rangle$, and I the identity matrix. A matrix $B \in \mathbb{M}_n$ is said positive (denoted by $B \geq 0$) if the inner product $\langle Bx, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$, and it is positive definite (denoted by $B > 0$) if $\langle Bx, x\rangle > 0$ for any $x \in \mathbb{R}^n \backslash \{0\}$. $\mathbb{P}_n$ will stand for the cone of all symmetric positive definite matrices in $\mathbb{M}_n$.

As a deduction of the positivity of matrices, a partial order is defined on $\mathbb{P}_n$ as follows

$$\forall\, A, B \in \mathbb{P}_n \quad A \leq B \Longleftrightarrow B - A \geq 0.$$

For a quantum mechanical system described by a density matrix ρ, i.e. ρ is positive with $tr\rho = 1$, von Neuman [19] defined the entropy, as an extension of the concept of Gibbs entropy [14], by the following formula:

$$\mathrm{S}(\rho) = -tr(\rho \log \rho).$$

$tr(A)$ stands for the trace of the matrix A, which is a linear form and satisfies the following cyclicity property [28, 29]

$$tr(ABC) = tr(BCA) = tr(CAB), \tag{1.1}$$

for any matrices A, B and C from $\mathbb{M}_n$.

To estimate the amount of information purely transmitted through a channel from the initial system to the final one, Shannon [23] introduced the entropy by setting for a discrete random variable X with a probability distribution $\{p_i\}_i$,

$$S_s(X) = -\sum_{i=1}^{n} p_i \log p_i. \tag{1.2}$$

It is important to note that von Neumann entropy generalizes Shannon entropy. This last one represents a fundamental tool that caused an enormous change in studying many fields like physical quantum systems. It has allowed the development of modern communication.

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Later the quantum relative entropy was introduced by Umegaki [26], by setting for two states

$$\mathbb{S}(\rho,\sigma) = tr\big(\rho(\log\rho - \log\sigma)\big), \tag{1.3}$$

where ρ and σ are two hermitian densities.

Relative entropy has its origin in the attempt to quantify the difficulty of discriminating probability distributions, which can be viewed as a distance between them. Its matrix equivalent can be used in the same way to quantify distances between positive matrices.

In [27], the authors extended the relative entropy (1.3) to any two positive operators A and B as follows:

$$S_e(A,B) = tr\big(A\left(\log A - \log B\right)\big).$$

The notion of entropy, formulated by von Neumann, has also been extended by Nakamura and Umegaki [18] to give the operator entropy defined for a positive operator A on a Hilbert space by

$$S(A) = -A\log A. \tag{1.4}$$

In 1989, Fujii and Kamei [6, 7] gave another extension for the notion of the relative operator entropy as a relative version of the entropy defined in (1.4). Namely, for two definite positive operators A and B acting on Hilbert space, the relative operator entropy denoted by $S(A|B)$ is

$$S(A|B) = A^{\frac{1}{2}}\log\big(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\big)A^{\frac{1}{2}}. \tag{1.5}$$

Afterwards, a parameterized extension of the relative operator entropy was stated by Furuta in [9] in the following manner:

$$S_p(A|B) = A^{\frac{1}{2}}\big(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\big)^p\log\big(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\big)A^{\frac{1}{2}}, \quad p\in\mathbb{R}. \tag{1.6}$$

This last generalization is to be understood by the following result [8]

$$\lim_{p\to 0} S_p(A\mid B) = S_0(A\mid B) = S(A\mid B).$$

These operator entropies have several algebraic properties that can be found in a large literature, such as [3, 6, 8, 9, 11, 13, 17, 20, 21]. We limit ourselves to recalling the following properties, which will be used in the sequel

$$B\geq A \Longrightarrow S(A\mid B)\geq 0,$$

$$S_p(A\mid B)\leq S_q(A\mid B), \forall B\geq A \text{ and } 0\leq p\leq q\leq 1, \tag{1.7}$$

$$S_p(A\mid B)\leq A\sharp B, \forall B\geq A \text{ and for all } p\in\left[0,\frac{e-2}{2e}\right], \tag{1.8}$$

$$S_p(A\mid B)\leq B \text{ for all } B\geq A \text{ and } p\in\left[0,\frac{e-1}{e}\right], \tag{1.9}$$

$$S_p(A\mid B)\geq A \text{ for all } B\geq eA \text{ and } p\in[0,1], \tag{1.10}$$

where

$$A\sharp B := A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}$$

refers to the geometric mean of the positive definite matrices A and B.

Generalized entropies offer an alternative framework for measuring information content. In particular, they can be used to study the properties of standard entropy from a broad perspective. The development of this general framework is very promising. Indeed, generalized versions of the entropy operator make it possible to define new measurement tools for quantifying certain informational quantities. In [10], using the relative entropy of the Tsallis operator, the authors recently defined a family of coherence quantifiers that represent a fundamental component of quantum physics and a crucial tool in quantum information processing.

Many symmetric positive definite matrix applications require an answer to the fundamental question: how can we measure the distance between two such matrices?
This question arises, for example, in verifying the convergence of an optimization procedure, or in designing algorithms. In this respect, several attempts have been made to give formal meaning to this distance. Given the specificities of the set of positive definite matrices, different approaches have been implemented for this purpose. Some authors use the notion of divergence, while others insist on using the notion of metric in the usual sense [2, 25].

In some areas, the notion of fidelity is commonly used. It measures the degree of overlap between two states of a system as it evolves over time. In quantum information theory, a version of fidelity is expressed as a function of the Bures-Wasserstein distance, which will be recalled below [15].

Inspired by some concepts defined in [4], the authors undertook recently a study for relative operator entropy and for Tsallis operator entropy with respect to various Hellinger metric [12, 13].

In the current paper, we are interested in investigating further geometric properties for the relative operator entropies $S(A \mid B)$ and its extension $S_p(A \mid B)$ with respect to weighted Hellinger metric and Alpha Procrustes distance, introduced very recently in [5] and [16] respectively. For $\alpha > 0$ and for positive definite matrices A and B, weighted Hellinger metric is defined as follows

$$d_{1,\alpha}(A, B) = \frac{1}{\alpha} d_1\left(A^{2\alpha}, B^{2\alpha}\right), \tag{1.11}$$

where

$$d_1(A, B) := \sqrt{trA + trB - 2tr\left(A^{\frac{1}{2}} B^{\frac{1}{2}}\right)}$$

refers to Bhattacharya metric [24] widely used in quantum information theory.

Alpha Procrustes distance is defined by the following expression

$$d_{2,\alpha}(A, B) = \frac{1}{\alpha} d_2\left(A^{2\alpha}, B^{2\alpha}\right), \tag{1.12}$$

where

$$d_2(A, B) := \sqrt{trA + trB - 2tr\left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)^{\frac{1}{2}}}$$

is the Bures-Wasserstein distance [1], closely related to optimal transport theory.

The distance $d_{2,\alpha}$ represents a unified formulation [16] linking Bures-Wasserstein and log Euclidean distances defined on $\mathbb{P}_n$.

In [5], the authors proved that the distances $d_{1,\alpha}$ and $d_{2,\alpha}$ are equivalent on $\mathbb{P}_n$. More precisely, for any $A, B \in \mathbb{P}_n$, we have

$$d_{2,\alpha}(A, B) \leq d_{1,\alpha}(A, B) \leq \sqrt{2}\, d_{2,\alpha}(A, B). \tag{1.13}$$

The rest of the paper is organized as follows. In section 2, we investigate monotonicity related to the metrics $d_{1,\alpha}$ and $d_{2,\alpha}$, as well as localization of $S_p(A \mid B)$. In section 3, the focus is on the estimation of the distance between the relative operator entropy and some standard means of two positive matrices.

2. Monotonicity and localization results

In this section, we study the variation of the map

$$p \longmapsto d_{i,\alpha}\big(A, S_p(A \mid B)\big)\ (1 \leq i \leq 2)$$

for two positive definite matrices A and B. Throughout this paper, we stand

$$C := A^{\frac{-1}{2}} B A^{\frac{-1}{2}}.$$

We begin by stating the following three lemmas, which will be necessary to establish our main results.

Lemma 2.1. *Let x be a strictly positive number. The map $\alpha \mapsto x^\alpha$ is increasing on $(0, 1]$.*

Proof. It can be deduced by the use of some classical tools of real analysis. □

In [22, Proposition 1], Petz pointed out the following result concerning the monotonicity of the trace function.

Lemma 2.2. *Let $A, B \in \mathbb{M}_n$ be two symmetric matrices and $f : \mathbb{R} \to \mathbb{R}$ be an increasing continuous function. We have,*

$$A \le B \Longrightarrow tr[f(A)] \le tr[f(B)].$$

Using Lemma 2.2, we obtain straightforward the following result.

Lemma 2.3. *Let A, B and X be three matrices from $\mathbb{P}_n$. We have*

$$A \le B \Longrightarrow tr(X\,A) \le tr(X\,B).$$

The monotonicity of the map $X \longrightarrow d_{1,\alpha}(A, X)$ is provided in the following proposition.

Proposition 2.4. *Let A, M and N be three matrices from $\mathbb{P}_n$ such that $A \le M \le N$. For all $\alpha \in (0, 1]$ we have*

$$d_{1,\alpha}(A, M) \le d_{1,\alpha}(A, N).$$

Proof. By the monotonocity of the map $\alpha \longmapsto x^\alpha$ on $(0, 1]$, we have

$$A^\alpha \le M^\alpha \le N^\alpha.$$

This gives

$$0 \le M^\alpha - A^\alpha \le N^\alpha - A^\alpha.$$

The use of Lemma 2.2 allows us to write

$$tr\,(M^\alpha - A^\alpha)^2 \le tr\,(N^\alpha - A^\alpha)^2\,.$$

Combining this last inequality with the cyclicity property of the trace function (1.1), we obtain the desired result. □

Now, we are in a position to state our first result about monotonicity of the map $p \longrightarrow d_{1,\alpha}\big(A, S_p(A \mid B)\big)$.

Theorem 2.5. *Let A and B be two positive definite matrices such that $B \ge eA$. For all $\alpha \in (0, 1]$, we have*

$$d_{1,\alpha}\big(A, S_p(A \mid B)\big) \le d_{1,\alpha}\big(A, S_q(A \mid B)\big), \tag{2.1}$$

for all $0 \le p \le q \le 1$.

Proof. Combining inequalities (1.7) and (1.10), we obtain

$$A \le S_p(A \mid B) \le S_q(A \mid B),\ \forall\, 0 \le p \le q \le 1.$$

By applying Proposition 2.4, we get the desired inequality (2.1). □

In the following proposition, we point out the monotonicity of the map $X \longrightarrow d_{2,\alpha}(A, X)$.

Proposition 2.6. *Let A, B and M be three matrices from $\mathbb{P}_n$ such that $I \le A^{-1} \le B \le M$. For all $\alpha \in \big(0, \frac{1}{2}\big]$, We have*

$$d_{2,\alpha}(A, B) \le d_{2,\alpha}(A, M). \tag{2.2}$$

Proof. Let $A^{-1} \leq B \leq M$. The monotonicity of the map $\alpha \longmapsto x^{2\alpha}$ on $\left(0, \frac{1}{2}\right]$ ensures that $A^{-2\alpha} \leq B^{2\alpha} \leq M^{2\alpha}$.
Hence,

$$2I \leq \left(A^{\alpha} B^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}} + \left(A^{\alpha} M^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}}.$$

Employing Lemma 2.3, we get

$$2 \times tr\left[\left(A^{\alpha} M^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}} - \left(A^{\alpha} B^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}}\right] \leq tr\left[\left(\left(A^{\alpha} M^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}} + \left(A^{\alpha} B^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}}\right)\left(\left(A^{\alpha} M^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}} - \left(A^{\alpha} B^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}}\right)\right].$$

By the invariance of trace under cyclic permutations, we obtain

$$\begin{aligned} 2 \times tr\left[\left(A^{\alpha} M^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}} - \left(A^{\alpha} B^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}}\right] &\leq tr\left(A^{\alpha} M^{2\alpha} A^{\alpha} - A^{\alpha} B^{2\alpha} A^{\alpha}\right) \\ &= tr\left(A^{2\alpha}\left(M^{2\alpha} - B^{2\alpha}\right)\right) \\ &\leq tr\left(M^{2\alpha} - B^{2\alpha}\right). \end{aligned}$$

Thus, the inequality (2.2) is deduced. □

The following theorem deals with the monotonicity of the map $p \longmapsto d_{2,\alpha}\left(A, S_p(A \mid B)\right)$.

Theorem 2.7. *Let A and B be two positive definite matrices such that $A \leq I$ and $B \geq A^{\frac{1}{2}} \exp(A^{-2}) A^{\frac{1}{2}}$. For all $\alpha \in \left(0, \frac{1}{2}\right]$ and for all $p, q \in [0\,,\,1]$ with $p \leq q$, we have the following inequality*

$$d_{2,\alpha}\left(A, S_p(A \mid B)\right) \leq d_{2,\alpha}\left(A, S_q(A \mid B)\right). \tag{2.3}$$

Proof. Let $A \leq I$ and $B \geq A^{\frac{1}{2}} \exp(A^{-2}) A^{\frac{1}{2}}$.
Using the monotonicity of the logarithm function on $\mathbb{P}_n$, and $C \geq \exp(A^{-2})$, we get

$$\log C \geq A^{-2}.$$

Hence,

$$S(A \mid B) \geq A^{-1}.$$

From inequality (1.7), we obtain for all $0 < p \leq q \leq 1$,

$$A^{-1} \leq S_p(A \mid B) \leq S_q(A \mid B).$$

Finally, employing Proposition 2.6, we get the inequality (2.3). □

In the rest of this section, we investigate the position of $S_p(A \mid B)$ with respect to the sphere centered at A with radius $d_{i,\alpha}(A, B)(1 \leq i \leq 2)$.

Theorem 2.8. *Let $A, B \in \mathbb{P}_n$ such that $B \geq eA$. For all $\alpha \in (0, 1]$ and for all $p \in \left[0, \frac{e-1}{e}\right]$, we have*

$$d_{1,\alpha}\left(A, S_p(A \mid B)\right) \leq d_{1,\alpha}(A, B). \tag{2.4}$$

Proof. Combining the inequalities (1.9) and (1.10) we get, for all $p \in \left[0, \frac{e-1}{e}\right]$,

$$A \leq S_p(A \mid B) \leq B.$$

So, by Proposition 2.4, we obtain the desired inequality (2.4). □

Theorem 2.9. *Let $A, B \in \mathbb{P}_n$ such that $A \leq I$ and $B \geq A^{\frac{1}{2}} \exp(A^{-2}) A^{\frac{1}{2}}$. We have*

$$d_{2,\alpha}\big(A, S_p(A \mid B)\big) \leq d_{2,\alpha}(A, B), \tag{2.5}$$

for all $\alpha \in \left(0, \frac{1}{2}\right]$ and $p \in \left[0, \frac{e-1}{e}\right]$.

Proof. The condition $B \geq A^{\frac{1}{2}} \exp(A^{-2}) A^{\frac{1}{2}}$, combined with the inequality (1.9), allows us to write that for all $p \in \left[0, \frac{e-1}{e}\right]$,

$$A^{-1} \leq S_p(A \mid B) \leq B.$$

Using Proposition 2.6, we get the inequality (2.5). □

Remark 2.1. Taking in Theorem 2.8 $p = 0$ and $B = I$, we obtain:

$$d_{1,\alpha}\big(A, S(A)\big) \leq d_{1,\alpha}(I, A).$$

Remark 2.2. If the condition stated for the matrix B in theorems 2.5, 2.7, 2.8 and 2.9 is not satisfied, the inequalities (2.1), (2.3), (2.4) and (2.5) are no longer true. The following example justifies this statement.

Example 2.1. Consider the following two positive definite symmetric matrices

$$A = \begin{pmatrix} 0.2 & 0.1 & 0 \\ 0.1 & 0.7 & 0.1 \\ 0 & 0.1 & 0.3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0.26 & 0.13 & 0 \\ 0.13 & 0.26 & 0.13 \\ 0 & 0.13 & 0.13 \end{pmatrix}.$$

We have $A < I$ and $B = 1.3\,A < A^{\frac{1}{2}} \exp(A^{-2}) A^{\frac{1}{2}}$. Calculations with Matlab software give the following values:

- $d_{1,0.75}(A, S_{0.3}(A|B)) = 0.7566 > d_{1,0.75}(A, S_{0.4}(A|B)) = 0.7470 > d_{1,0.75}(A, B) = 0.2692.$
- $d_{2,0.6}(A, S_{0.35}(A|B)) = 0.8958 > d_{2,0.6}(A, S_{0.45}(A|B)) = 0.8830 > d_{2,0.6}(A, B) = 0.29.$

3. Distances involving $S_p(A \mid B)$ and some operator means

The current section is devoted to estimate distances involving $S_p(A \mid B)$ and geometric, arithmetic and harmonic means.

Theorem 3.1. *Let A and B be two positive definite matrices such that $B \geq A$. For all $0 < \alpha \leq 1$ and $p \in \left[0, \frac{e-2}{2e}\right]$, the following inequality is satisfied*

$$d_{1,\alpha}\big(A \sharp B, S_p(A \mid B)\big) \leq d_{1,\alpha}\left(B, S_p(A \mid B)\right). \tag{3.1}$$

Proof. Let $B \geq A$. By the inequality (1.8) and using that $A \sharp B \leq B$, we can write

$$0 < S_p(A \mid B) \leq A \sharp B \leq B.$$

Using Proposition, 2.4 we deduce the inequality (3.1). □

Remark 3.1. Taking in Theorem 3.1, $B = I$ and $p = 0$, we obtain:

$$d_{1,\alpha}\big(A^{\frac{1}{2}}, S(A)\big) \leq d_{1,\alpha}(I, S(A)). \tag{3.2}$$

It is important to note that the inequality (3.2) is also true for a suitably chosen state A. In particular, we can replace S by the Shanon entropy S_s recalled in (1.2) by considering a discrete random probability distribution.

Another localization of $S_p(A \mid B)$ using the distance $d_{1,\alpha}$ is given in the following theorem.

Theorem 3.2. *Let A and B be two positive definite matrices such that $B \geq 16A$. For all $\alpha \in \left[\frac{1}{2}, 1\right]$ and $p \in \left[0, \frac{e-2}{2e}\right]$, we have*

$$d_{1,\alpha}\big(A\sharp B, S_p(A \mid B)\big) \leq \frac{1}{2} d_{1,\alpha}(A, B). \tag{3.3}$$

Proof. Let $B \geq 16A$. So $C \geq 16I$ and $C^{\frac{1}{2}} \leq \dfrac{1}{4}C$. Thus, $A\sharp B \leq \frac{1}{4}B$.

This combined with the inequalities (1.9), (1.10) and using the monotonicity of the map $\alpha \longmapsto x^{\alpha}$ on $\left[\frac{1}{2}, 1\right]$, we can write

$$\frac{1}{2}A^{\alpha} \leq A^{\alpha} \leq S_p^{\alpha}(A \mid B) \leq (A\sharp B)^{\alpha} \leq \frac{1}{4^{\alpha}}B^{\alpha} \leq \frac{1}{2}B^{\alpha}.$$

So,

$$0 < (A\sharp B)^{\alpha} - S_p^{\alpha}(A \mid B) \leq \frac{1}{2}B^{\alpha} - \frac{1}{2}A^{\alpha}.$$

Employing Lemma 2.2 by taking the square function, we obtain

$$\frac{1}{\alpha^2} tr\left((A\sharp B)^{\alpha} - S_p^{\alpha}(A \mid B)\right)^2 \leq \frac{1}{4\alpha^2} tr\left(B^{\alpha} - A^{\alpha}\right)^2,$$

which is equivalent to the inequality (3.3). □

Using Theorem 3.2, we get the following corollary.

Corollary 3.3. *Let A and B be two positive definite matrices such that $B \geq 16A$. For all $\alpha \in \left[\frac{1}{2}, 1\right]$ and $p, q \in \left[0, \frac{e-2}{2e}\right]$, we have the following inequality*

$$d_{1,\alpha}\big(S_p(A \mid B), S_q(A \mid B)\big) \leq d_{1,\alpha}(A, B). \tag{3.4}$$

Proof. Considering the conditions stated in the corollary and using the triangular inequality and the inequality (3.3), we get

$$\begin{aligned} d_{1,\alpha}\big(S_p(A \mid B), S_q(A \mid B)\big) &\leq d_{1,\alpha}\big(A\sharp B, S_p(A \mid B)\big) + d_{1,\alpha}\big(A\sharp B, S_q(A \mid B)\big) \\ &\leq \frac{1}{2}d_{1,\alpha}(A, B) + \frac{1}{2}d_{1,\alpha}(A, B) \\ &\leq d_{1,\alpha}(A, B). \end{aligned}$$

□

Remark 3.2. With the conditions of Theorem 3.1, the inequalities (3.1) and (1.13) lead to

$$d_{2,\alpha}\big(A\sharp B, S_p(A \mid B)\big) \leq \sqrt{2}\, d_{2,\alpha}\left(B, S_p(A \mid B)\right). \tag{3.5}$$

Theorem 3.4. *Let A and B be two positive definite matrices such that $B \geq 64A$. For all $\alpha \in \left[\frac{1}{2}, 1\right]$ and $p \in \left[0, \frac{e-2}{2e}\right]$, we have the following inequality*

$$d_{2,\alpha}\big(A\sharp B, S_p(A \mid B)\big) \leq \frac{1}{2} d_{2,\alpha}(A, B). \tag{3.6}$$

Proof. If $B \geq 64A$, then $C \geq 64I$ and $C^{\frac{1}{2}} \leq \dfrac{1}{8}C$.
So,

$$A\sharp B \leq \frac{1}{8}B.$$

We have for all $p \in \left[0, \frac{e-2}{2e}\right]$,

$$\frac{1}{8}A \leq S_p(A \mid B) \leq A\sharp B.$$

By the monotonicity of the map $\alpha \longmapsto x^\alpha$ on $\left[\frac{1}{2}, 1\right]$, we obtain

$$\frac{1}{2\sqrt{2}}A^\alpha \leq S_p^\alpha(A \mid B) \leq (A\sharp B)^\alpha \leq \frac{1}{2\sqrt{2}}B^\alpha,$$

which gives

$$0 \leq (A\sharp B)^\alpha - S_p^\alpha(A \mid B) \leq \frac{1}{2\sqrt{2}}B^\alpha - \frac{1}{2\sqrt{2}}A^\alpha.$$

Applying Lemma 2.2 with the square function, we get

$$tr\left((A\sharp B)^\alpha - S_p^\alpha(A \mid B)\right)^2 \leq \frac{1}{8}\left(B^\alpha - A^\alpha\right)^2.$$

Consequently, using the inequality (1.13), we can write

$$\begin{aligned} d_{2,\alpha}\big(A\sharp B, S_p(A \mid B)\big) &\leq d_{1,\alpha}\big(A\sharp B, S_p(A \mid B)\big) \\ &\leq \frac{1}{2\sqrt{2}}d_{1,\alpha}(A, B) \\ &\leq \frac{1}{2}d_{2,\alpha}(A, B). \end{aligned}$$

This ends the proof. □

The following result gives an estimation of the distance $d_{2,\alpha}\big(S_p(A \mid B), S_q(A \mid B)\big)$ for any $p, q \in \left[0, \frac{e-2}{2e}\right]$.

Corollary 3.5. *Let A and B be two positive definite matrices such that $B \geq 64A$. For all $\alpha \in \left[\frac{1}{2}, 1\right]$ and $p, q \in \left[0, \frac{e-2}{2e}\right]$, we have the following inequality*

$$d_{2,\alpha}\big(S_p(A \mid B), S_q(A \mid B)\big) \leq d_{2,\alpha}(A, B). \tag{3.7}$$

Proof. Employing the inequality (3.6) and the triangular inequality, we have under the conditions cited above

$$\begin{aligned} d_{2,\alpha}\big(S_p(A \mid B), S_q(A \mid B)\big) &\leq d_{2,\alpha}\big(A\sharp B, S_p(A \mid B)\big) + d_{2,\alpha}\big(A\sharp B, S_q(A \mid B)\big) \\ &\leq \frac{1}{2}d_{2,\alpha}(A, B) + \frac{1}{2}d_{2,\alpha}(A, B) \\ &\leq d_{2,\alpha}(A, B). \end{aligned}$$

□

To determine a localization of $S_p(A \mid B)$ with respect to the arithmetic mean, we need the following lemma.

Lemma 3.6. *Let x be a positive real number. We have the following inequality*

$$x^p \log x \leq \frac{1+x}{2}, \text{ for all } p \in [0, e^{-1}].$$

Proof. By studying the function defined on $[1, \infty)$ by $h(x) = \dfrac{1+x}{2} - x^{\frac{1}{e}} \log x$, we get

$$h(x) > 0, \text{ for all } x \geq 1.$$

Using the fact that the map $p \longmapsto x^p \log x$ is increasing on $[0, e^{-1}]$, we deduce that

$$x^p \log x \leq \frac{1+x}{2}$$

holds for any $p \in [0, e^{-1}]$. □

Theorem 3.7. *Let A and B be two positive definite matrices such that $B \geq A$. For all $\alpha \in [0,1]$ and $p \in [0, e^{-1}]$, we have the following inequality*

$$d_{1,\alpha}\big(A\nabla B, S_p(A \mid B)\big) \leq d_{1,\alpha}\left(B, S_p(A \mid B)\right), \tag{3.8}$$

where $A\nabla B := \frac{A+B}{2}$ stands for the arithmetic mean of the matrices A and B.

Proof. Let $B \geq A$. We have $0 < S(A \mid B) \leq S_p(A \mid B)$.
So, by virtue of Lemma 3.6, we have

$$S_p(A \mid B) \leq A\nabla B.$$

Employing Proposition 2.4, we get the desired inequality (3.8). □

Corollary 3.8. *Let A and B be two positive definite matrices such that $B \geq A$. The following inequality*

$$d_{2,\alpha}\big(A\nabla B, S_p(A \mid B)\big) \leq \sqrt{2} d_{2,\alpha}\left(B, S_p(A \mid B)\right) \tag{3.9}$$

holds for all $\alpha \in (0,1]$ and $p \in [0, e^{-1}]$.

Proof. Using the inequalities (1.13) and (3.8), we deduce the desired inequality (3.9). □

For the particular case $\alpha = 1$, we have the following result.

Proposition 3.9. *Let A and B be two positive definite matrices such that $B \geq eA$. For all $p \in [0\,,\, e^{-1}]$, we have*

$$d_{1,1}\big(A\nabla B, S_p(A \mid B)\big) \leq \frac{1}{2} d_{1,1}(A, B). \tag{3.10}$$

Proof. Using inequality (1.10) and Lemma 3.6, we can write for all $B \geq eA$,

$$\begin{aligned} \frac{A+B}{2} + \frac{1}{2}A &= A + \frac{1}{2}B \\ &\leq S_p(A \mid B) + \frac{1}{2}B, \end{aligned}$$

which implies

$$0 < \frac{A+B}{2} - S_p(A \mid B) \leq \frac{1}{2}(B - A).$$

Employing Lemma 2.2 with the square function, we obtain

$$tr\left(\frac{A+B}{2} - S_p(A \mid B)\right)^2 \leq \frac{1}{4} tr\,(B-A)^2,$$

which is equivalent to the inequality (3.10). □

To point out the results involving the harmonic mean, the following lemma will be useful.

Lemma 3.10. *The function r defined by $r(x) = \log x - 2\left(1 + x^{-1}\right)^{-1}$ is strictly increasing on $(0, \infty)$. Moreover, there exists a unique real number β such that $r(\beta) = 0$ and $5.40 < \beta < 5.41$.*

Proof. It is easy to check that r is strictly increasing on $(0, \infty)$.
Since r is continuous on $(0, \infty)$, it establishes a bijection from $(0, \infty)$ onto $\big(\lim_{x \downarrow 0} r(x), \lim_{x \to \infty} r(x)\big) = (-\infty, +\infty)$.
Then, there exists a unique number β verifying $r(\beta) = 0$.
By noticing that $r(5.40) \times r(5.41) < 0$, we get $5.40 < \beta < 5.41$. □

Theorem 3.11. *Let A and B be two positive definite matrices such that $B \geq \beta A$. For $\alpha \in (0,1]$ and for all $p \in \left[0, \frac{e-1}{e}\right]$, the following inequality holds*

$$d_{1,\alpha}\big(\mathcal{H}(A,B), S_p(A \mid B)\big) \leq d_{1,\alpha}\big(B, S_p(A \mid B)\big), \tag{3.11}$$

where β is the fixed real number defined in Lemma 3.10 and $\mathcal{H}(A,B) := 2\left(A^{-1}+B^{-1}\right)^{-1}$ stands for the harmonic mean of A and B.

Proof. If $B \geq \beta A$ then $C \geq \beta I$ and $2\left(I + C^{-1}\right)^{-1} > 0$. From Lemma 3.10, we have

$$\log C \geq 2\left(I + C^{-1}\right)^{-1}.$$

Multiplying both the right-hand side and the left-hand side of this inequality by $A^{\frac{1}{2}}$, we obtain

$$S(A \mid B) \geq \mathcal{H}(A,B).$$

Thus, by employing the inequalities (1.7) and (1.9), we get for all $p \in \left[0, \frac{e-1}{e}\right]$

$$B \geq S_p(A \mid B) \geq S(A,B) \geq \mathcal{H}(A,B).$$

Using Proposition 2.4, we get the inequality (3.11). □

To establish another estimation of the distance $d_{1,\alpha}\left(\mathcal{H}(A,B), S_p(A \mid B)\right)$, we need the following result.

Lemma 3.12. *The function defined by $f(x) = \frac{1}{4}x - x^{\frac{1}{4}}\log x$ is strictly increasing on $[34,\infty)$. Moreover, there exists a unique real number λ such that $34.14 < \lambda < 34.15$ verifying $f(\lambda) = 0$.*

Proof. The proof is similar to that for Lemma 3.10, and is therefore omitted here. □

Theorem 3.13. *Let A and B be two positive definite matrices such that $B \geq \lambda A$. For all $\alpha \in \left(0, \frac{1}{2}\right]$ and $p \in \left[0, \frac{1}{4}\right]$, the following inequality is verified*

$$d_{1,\alpha}\big(\mathcal{H}(A,B), S_p(A \mid B)\big) \leq \frac{1}{2} d_{1,\alpha}(A,B), \tag{3.12}$$

where λ is the fixed number defined in Lemma 3.12.

Proof. Combining Lemmas 3.10 and 3.12, we can write that, for all $B \geq \lambda A$ and for all $p \in \left[0, \dfrac{1}{4}\right]$,

$$A \leq \mathcal{H}(A,B) \leq S_p(A \mid B) \leq \frac{1}{4}B.$$

Using the monotonocity of the map $\alpha \longmapsto x^\alpha$ on $\left[0, \frac{1}{4}\right]$, we get

$$\frac{1}{2}A^\alpha \leq \mathcal{H}^\alpha(A,B) \leq S_p^\alpha(A \mid B) \leq \frac{1}{4^\alpha}B^\alpha \leq \frac{1}{2}B^\alpha.$$

Hence,

$$0 < S_p^\alpha(A \mid B) - \mathcal{H}^\alpha(A,B) \leq \frac{1}{2}\left(B^\alpha - A^\alpha\right).$$

Employing Lemma 2.2, by taking the square function, we obtain

$$tr\left(S_p^\alpha(A \mid B) - \mathcal{H}^\alpha(A,B)\right)^2 \leq \frac{1}{4} tr\left(B^\alpha - A^\alpha\right)^2,$$

which is equivalent to the inequality (3.12). □

The following result extends the inequality (3.4) to any $\alpha \in (0,1]$.

Corollary 3.14. *Let A and B be two positive definite matrices such that $B \geq \lambda A$. For all $\alpha \in \left(0, \frac{1}{2}\right]$ and $p, q \in \left[0, \frac{1}{4}\right]$, we have the following inequality:*

$$d_{1,\alpha}\big(S_p(A \mid B), S_q(A \mid B)\big) \leq d_{1,\alpha}(A, B). \tag{3.13}$$

Proof. Using the triangular inequality for the distance $d_{1,\alpha}$ and the inequality (3.12), the result is straightforward. □

To estimate the distance $d_{2,\alpha}\left(Sp(A \mid B), \mathcal{H}(A, B)\right)$, we need the following lemma.

Lemma 3.15. *The function defined by $g(x) = \frac{1}{8}x - x^{\frac{1}{8}} \log x$ is strictly increasing on $[51, \infty)$, and there exists a unique $51.67 < \lambda_1 < 51.68$ such that $g(\lambda_1) = 0$.*

Proof. The result can be shown in the same way as in the proof of Lemma 3.10. □

Theorem 3.16. *Let $A, B \in \mathbb{P}_n$ such that $B \geq \lambda_1 A$. For all $\alpha \in \left(0, \frac{1}{2}\right]$ and $p \in \left[0, \frac{1}{8}\right]$, we have*

$$d_{2,\alpha}\big(\mathcal{H}(A, B), S_p(A \mid B)\big) \leq \frac{1}{2} d_{2,\alpha}(A, B), \tag{3.14}$$

where λ_1 is the real number defined in Lemma 3.15.

Proof. Let $B \geq \lambda_1 A$. According to Lemma 3.15, we have

$$\mathcal{H}(A, B) \leq S_p(A \mid B) \leq \frac{1}{8} B, \text{ for all } p \in \left[0, \frac{1}{8}\right].$$

Employing the monotonicity of the map $\alpha \longmapsto x^\alpha$ on $\left(0, \dfrac{1}{2}\right]$, we can write

$$\frac{1}{2\sqrt{2}} A^\alpha \leq \mathcal{H}^\alpha(A, B) \leq S_p^\alpha(A \mid B) \leq \frac{1}{8^\alpha} B^\alpha \leq \frac{1}{2\sqrt{2}} B^\alpha.$$

So,

$$0 < S_p^\alpha(A \mid B) - \mathcal{H}^\alpha(A, B) \leq \frac{1}{2\sqrt{2}} \left(B^\alpha - A^\alpha\right).$$

Using Lemma 2.2 with the square function, we obtain

$$tr\left(S_p^\alpha(A \mid B) - \mathcal{H}^\alpha(A, B)\right)^2 \leq \frac{1}{8} tr\left(B^\alpha - A^\alpha\right)^2.$$

Whence, we get

$$\begin{aligned} d_{2,\alpha}\big(\mathcal{H}(A, B), S_p(A \mid B)\big) &\leq d_{1,\alpha}\big(\mathcal{H}(A, B), S_p(A \mid B)\big) \\ &\leq \frac{1}{2\sqrt{2}} d_{1,\alpha}(A, B). \end{aligned}$$

According to the inequality (1.13), we deduce

$$d_{2,\alpha}\big(\mathcal{H}(A, B), S_p(A \mid B)\big) \leq \frac{1}{2} d_{2,\alpha}(A, B).$$

□

The following corollary provides an estimation of $d_{2,\alpha}\big(S_p(A \mid B), S_q(A \mid B)\big)$ for any $p, q \in \left[0, \frac{1}{8}\right]$.

Corollary 3.17. *Let $A, B \in \mathbb{P}_n$ such that $B \geq \lambda_1 A$. For all $\alpha \in \left(0, \frac{1}{2}\right]$ and $p, q \in \left[0, \frac{1}{8}\right]$, we have*

$$d_{2,\alpha}\big(S_p(A \mid B), S_q(A \mid B)\big) \leq d_{2,\alpha}(A, B). \tag{3.15}$$

Proof. Employing triangular inequality and the inequality (3.14), we deduce the desired result. □

Remark 3.3. If the conditions stated for the matrix B or for the scalar p are not verified in theorems 3.2, 3.4, 3.13 and 3.16, then the inequalities (3.3), (3.6), (3.13) and (3.14) are no longer true. The following example explains this fact.

Example 3.1. Let us consider the following positive definite symmetric matrix

$$A = \begin{pmatrix} 0.7 & 0.1 & 0 \\ 0.1 & 0.2 & -0.05 \\ 0 & -0.05 & 0.1 \end{pmatrix}.$$

Computing with Matlab software, we get the following values.

B	p	α	$d_{1,\alpha}(A\sharp B, S_p)$	$d_{2,\alpha}(A\sharp B, S_p)$	$d_{1,\alpha}(S_p, \mathcal{H})$	$d_{2,\alpha}(S_p, \mathcal{H})$	$\frac{1}{2}\ d_i(A, B)$
$7A$	0.75	0.7	3.0639	3.0639	3.6847	3.6847	1.8158
$20A$	0.8	0.55	7.9795	7.9795	9.4762	9.4762	3.6762
$30A$	0.7	0.8	14.4745	14.4745	16.7533	16.7533	7.3448
$36A$	0.6	0.9	14.7177	14.7177	17.5125	17.5125	10.5656
$55A$	0.65	0.75	17.5614	17.5614	20.7764	20.7764	10.8854

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