

THE MINIMAL INVASION SPEED OF TWO COMPETING SPECIES IN A HOMOGENEOUS ENVIRONMENT

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ABSTRACT. Biological invasion has become an important element of global changes. In this paper, we use a reaction-diffusion system to discuss the minimal invasion speed of two competing species in the homogeneous environment. The general condition for the minimum invasion speed is obtained by applying the theory of propagation dynamics. Then the explicit conditions are derived by constructing upper solutions. The analytical results are corroborated by simulations of the considered reaction-diffusion system. Our results reveal the impact of the diffusion rate, growth rate, competitiveness of the species, as well as the carrying capacity of the environment, on the invasion speed, which provides an effective method for preventing biological invasion and controlling existing biological invasion.

1. Introduction

Biological invasion, global warming and habitat loss constitute the three most pressing environmental problems in the 21st century. The purpose of this work is to propose effective methods to prevent or control biological invasions through the analysis of mathematical models. The following population model of Lotka-Volterra type

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - b_1 u - a_1 v), \\ v_t = d_2 v_{xx} + r_2 v(1 - b_2 v - a_2 u) \end{cases} \quad (1.1)$$

describes the dynamical behavior of two species who have spatial isotropic diffusion and compete for the same resources (see, for example, [6]). Using this model, Okubo *et al.* [7] characterized the invasion of the externally introduced grey squirrel in Britain, that is, the interaction between the externally introduced grey squirrel and the indigenous red squirrel. In [3], the authors tested Lotka-Volterra model (1.1) with biological data from experiments with two competing species of *Drosophila*. However, an exact fit was not expected. By applying a curvilinear regression approach, they found that analytical models of competition were fitted. By statistical and biological criteria, they chose the best one that is nowadays called the Gilpin-Ayala model as follows:

$$\begin{cases} \bar{u}_t = d_1 \Delta \bar{u} + r_1 \bar{u} (1 - b_1 \bar{u} - a_1 \bar{v}^2), & t > 0, x \in \mathbb{R}, \\ \bar{v}_t = d_2 \Delta \bar{v} + r_2 \bar{v} (1 - b_2 \bar{v} - a_2 \bar{u}^2), & t > 0, x \in \mathbb{R} \end{cases} \quad (1.2)$$

with initial data

$$\bar{u}(x, 0) = u_0^*(x) \geq 0, \quad \bar{v}(x, 0) = v_0^*(x) \geq 0, \quad (1.3)$$

which has a cubic competition term, and thus stands for an extension of the Lotka-Volterra competition model. Here $\bar{u}(x, t)$ and $\bar{v}(x, t)$ measure the densities of two competing species \bar{u} and \bar{v} at location x at

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time t , respectively; Δ is the one-dimensional Laplacian operator $\frac{\partial}{\partial x^2}$; d_i , r_i and $1/b_i$, $i = 1, 2$ represent the diffusive rates, the net growth rates and the carrying capacities; a_1 and a_2 are the nonnegative competition coefficients of \bar{v} and \bar{u} , respectively. Since all of the coefficients are constant, the model (1.2) describes the dynamical behaviors of two species \bar{v} and \bar{u} living in a homogeneous environment. As in [13], by using the scaling

$$\begin{aligned} \sqrt{\frac{r_1}{d_1}}x &\rightarrow x, \quad r_1t \rightarrow t, \\ b_1\bar{u}(x, t) &\rightarrow \bar{u}(x, t), \quad b_2v(x, t) \rightarrow v(x, t), \\ d &= \frac{d_2}{d_1}, \quad r = \frac{r_2}{r_1}, \quad \frac{a_1}{b_2^2} \rightarrow a_1, \quad \frac{a_2}{b_1^2} \rightarrow a_2, \end{aligned} \quad (1.4)$$

we transform (1.2) into a equivalent non-dimensional system

$$\begin{cases} \bar{u}_t = \Delta\bar{u} + \bar{u}(1 - \bar{u} - a_1\bar{v}^2), & t > 0, x \in \mathbb{R}, \\ \bar{v}_t = d\Delta\bar{v} + r\bar{v}(1 - \bar{v} - a_2\bar{u}^2), & t > 0, x \in \mathbb{R}. \end{cases} \quad (1.5)$$

It is easy to check that the system (1.5) has an invariant region $\mathbb{W} = \{(\bar{u}, \bar{v}) | 0 \leq \bar{u} \leq 1, 0 \leq \bar{v} \leq 1\}$. Through a simple computation, we obtain three cases:

Case i: If

$$a_1 > 1, \quad a_2 < \frac{1}{3}, \quad (1.6)$$

then (1.5) has three constant steady states

$$q_1 = (0, 0), \quad q_2 = (0, 1), \quad q_3 = (1, 0),$$

where q_2 is stable, and q_1, q_3 are unstable.

Case ii: If

$$a_1 < 1, \quad a_2 < \frac{1}{3}, \quad (1.7)$$

then it has four constant steady states

$$q_1 = (0, 0), \quad q_2 = (0, 1), \quad q_3 = (1, 0), \quad \bar{q}_4 = (u_1, v_1),$$

where $\bar{q}_4 \in \mathbb{W}$ satisfies

$$\begin{cases} \frac{d\bar{u}}{dt} = \bar{u}(1 - \bar{u} - a_1\bar{v}^2), \\ \frac{d\bar{v}}{dt} = r\bar{v}(1 - \bar{v} - a_2\bar{u}^2), \end{cases} \quad (1.8)$$

which is termed as the kinetic system of (1.5). Moreover, \bar{q}_4 is stable, and q_1, q_2, q_3 are unstable.

Case iii: If

$$a_1 > 1, \quad a_2 > 1, \quad (1.9)$$

then it has four constant steady states

$$q_1 = (0, 0), \quad q_2 = (0, 1), \quad q_3 = (1, 0), \quad \underline{q}_4 = (u_2, v_2),$$

where $\underline{q}_4 \in \mathbb{W}$ satisfying (1.8), and q_2, q_3 are stable, and q_1, \underline{q}_4 are unstable.

When **Case i** or **Case ii** happens, (1.5) is called monostable; while **Case iii** appears, (1.5) is called bistable. In [10], by applying the method of energy function, the author proved that the speed sign of the bistable traveling wave connecting q_2 to q_3 is the same as that of $a_2 - a_1$, which was also mentioned in [4]. In this paper, under **Case i** we use the model (1.5) to study the minimal invasion speed of

the two species \bar{v} and \bar{u} . Therefore, throughout this work we will always assume that the competition coefficients a_1 and a_2 satisfy

$$a_1 > 1, \quad a_2 < \frac{1}{3}, \tag{1.10}$$

which indicates that the species \bar{v} is a strong competitor, and the species \bar{u} is a weak competitor.

The rest of this paper is organized as follows: In Section 2, based on the theory of propagation dynamics, we present the general theory of the minimal invasion speed. Section 3 presents the explicit conditions for the minimal invasion speed. In Section 4, we give numerical simulation and discuss the simulation results. Section 5 offers a summary and directions for future research.

2. A general theory for the minimal invasion speed

We concentrate on discussing that the species \bar{v} invades the region of the species \bar{u} , that is, the colonized state $(0, 1)$ of \bar{u} invades the uncolonized state $(1, 0)$ of \bar{u} . The opposite case can be handled in the same way. In order to mathematically describe this invasion process and determine the minimal invasion speed, we look for a traveling wave solution connecting $(0, 1)$ and $(1, 0)$ of (1.5) and its minimal wave speed.

Definition 2.1. A traveling wave solution is a solution to the system (1.5) of the form $(\bar{u}, \bar{v})(x, t) = (\bar{U}, \bar{V})(z)$ with $z = x - ct$, where z is called the wave variable, (\bar{U}, \bar{V}) is the wave profile, and c is a positive constant and termed as the wave speed or the invasion speed. We say it connects the states $(0, 1)$ and $(1, 0)$ if $(\bar{U}, \bar{V})(-\infty) = (0, 1)$, $(\bar{U}, \bar{V})(\infty) = (1, 0)$.

For convenience, let us use the transform $u = 1 - \bar{u}$, $v = \bar{v}$ to change (1.5) into a cooperative system

$$\begin{cases} u_t = \Delta u + (1 - u)(a_1 v^2 - u), \\ v_t = d\Delta v + rv[1 - v - a_2(1 - u)^2], \end{cases} \tag{2.1}$$

with the initial data

$$u_0 = 1 - u_0^*(x) \geq 0, \quad v_0 = v_0^*(x) \geq 0, \quad x \in \mathbb{R}. \tag{2.2}$$

Correspondingly, the equilibria q_1 , q_2 and q_3 become

$$\alpha = (1, 0), \quad \beta = (1, 1), \quad \mathbf{0} = (0, 0), \tag{2.3}$$

and then β is stable while α and $\mathbf{0}$ are unstable. Thus, the traveling wave solution (\bar{U}, \bar{V}) connecting $(0, 1)$ and $(1, 0)$ of (1.5) is just the traveling wave solution (U, V) connecting β and $\mathbf{0}$ of (2.1). By (2.1), the wave profile (U, V) satisfies

$$\begin{cases} U'' + cU' + (1 - U)(a_1 V^2 - U) = 0, \\ dV'' + cV' + rV[1 - V - a_2(1 - U)^2] = 0, \end{cases} \tag{2.4}$$

and the boundary conditions

$$(U, V)(-\infty) = \beta, \quad (U, V)(\infty) = \mathbf{0}. \tag{2.5}$$

Now linearizing (2.4) at the unstable state $\mathbf{0}$ yields a linear system

$$\begin{cases} U'' + cU' - U = 0, \\ dV'' + cV' + r(1 - a_2)V = 0, \\ U(\infty) = V(\infty) = 0. \end{cases} \tag{2.6}$$

Then the characteristic equations of (2.6) are

$$\mu^2 - c\mu - 1 = 0, \quad \text{and} \quad d\mu^2 - c\mu + r(1 - a_2) = 0.$$

It is easy to check that (2.6) has real positive solutions if and only if $c \geq c_0$, where

$$c_0 = 2\sqrt{dr(1 - a_2)}. \tag{2.7}$$

Obviously, c_0 and

$$\bar{\mu} = \mu(c_0) = \sqrt{\frac{r(1 - a_2)}{d}} \tag{2.8}$$

satisfy

$$d\bar{\mu}^2 - c_0\bar{\mu} + r(1 - a_2) = 0. \tag{2.9}$$

We denote the minimal wave speed by c_{min} , that is,

$$c_{min} = \inf\{c \mid (2.4) - (2.5) \text{ has a nonnegative solution}\}. \tag{2.10}$$

Here c_{min} is attained in the set of (2.10), which can be proved by using the similar method to Remark 3.1 in [8]. By this and Theorem 2.2 in [13] we have the following proposition.

Proposition 2.2. *Assume that (1.10) is satisfied. Then there exists a positive constant c_{min} such that (2.1) has a traveling wave solution $(U, V)(x - ct)$ connecting β and $\mathbf{0}$ for $c \geq c_{min}$; Furthermore, $c_{min} \geq c_0$.*

This result can also be obtained by using the same methods as in [2, 5, 11, 12]. By convention, if $c_{min} > c_0$, then the minimal wave speed is nonlinearly selective; Otherwise, i.e., $c_{min} = c_0$, the minimal wave speed is linearly selective. Obviously, c_0 is the minimal invasion speed. Hence we expect that linear selection can be realized. Based on Theorem 3.1, Theorem 3.2 and Theorem 3.3 in [13], to obtain the minimal invasion speed it is sufficient to find upper solutions with speed c_0 . To this end, we give the definition of upper solutions.

Definition 2.3. (see Section 6 in [1]) By an upper solution to system (2.4)-(2.5), we mean that a pair of continuous functions $(\bar{U}, \bar{V})(z)$ are twice-differentiable on $z \in (-\infty, \infty)$ except at finite number of points $z_i, i = 1, \dots, n$ and satisfy

$$\begin{cases} \bar{U}'' + c\bar{U}' + (1 - \bar{U})(a_1\bar{V}^2 - \bar{U}) \leq 0, \\ d\bar{V}'' + c\bar{V}' + r\bar{V}(1 - \bar{V} - a_2(1 - \bar{U})^2) \leq 0, \\ (\bar{U}, \bar{V})(-\infty) \geq (1, 1), \quad (\bar{U}, \bar{V})(\infty) \geq (0, 0), \end{cases} \tag{2.11}$$

for $z \neq z_i, i = 1, \dots, n$, and $(\bar{U}', \bar{V}')(z_i^-) \geq (\bar{U}', \bar{V}')(z_i^+)$ for $i = 1, \dots, n$.

The following lemma gives the general conditions for the linear selection of the minimal wave speed, which is Theorem 3.1 in [13] proved by using the comparison principle.

Lemma 2.4. *Assume that, for a given $\bar{c} \geq c_0$, there exists a positive and non-increasing upper solution $(\bar{U}, \bar{V})(z), z = x - \bar{c}t$ of system (2.4)-(2.5) satisfying*

$$\liminf_{z \rightarrow -\infty} (\bar{U}, \bar{V})(z) > (0, 0), \quad \lim_{z \rightarrow \infty} (\bar{U}, \bar{V})(z) = (0, 0). \tag{2.12}$$

Then we have that $c_{min} \leq \bar{c}$. Particularly, if $\bar{c} = c_0$, then the minimal wave speed is linearly selective, i.e., the minimal invasion speed is $c_0 = 2\sqrt{dr(1 - a_2)}$.

3. Explicit conditions for the minimal invasion speed

In this section, based on the Lemma 2.4, we derive the conditions for obtaining the minimal invasion speed represented by the system parameters, implying how the diffusion rate, the growth rate, the carrying capacity and the competitive strength affects the minimal invasion speed, respectively.

Theorem 3.1. *For the fixed parameters d, r, a_1 and a_2 , suppose that (1.10) holds and that*

$$\begin{cases} d \geq \frac{1}{2} \\ r \geq \frac{d(a_1 - 1)}{(1 + 2d)(1 - a_2)} \\ a_1^2 \leq \frac{2(1 - a_2)}{a_2} \end{cases} \tag{3.1}$$

Then the minimal wave speed is linearly selective, and thus the minimal invasion speed is equal to c_0 .

Proof. Define a pair of functions $(\bar{U}, \bar{V})(z)$ by

$$\bar{V} = \frac{1}{1 + e^{\bar{\mu}z}}, \quad \bar{U} = \begin{cases} 1, & z \leq z_0, \\ a_1 \bar{V}, & z \geq z_0, \end{cases}$$

where $\bar{\mu} = \bar{\mu}(c_0)$ and z_0 satisfy $a_1 \bar{V}(z_0) = 1$. It is easy to check that

$$\bar{V}' = -\bar{\mu}\bar{V}(1 - \bar{V}), \quad \bar{V}'' = \bar{\mu}^2\bar{V}(1 - \bar{V})(1 - 2\bar{V}).$$

By substituting (\bar{U}, \bar{V}) into the first equation of (2.4), when $z \leq z_0$, we have

$$\bar{U}'' + c_0\bar{U}' + r_1(1 - \bar{U})(a_1\bar{V}^2 - \bar{U}) = 0.$$

When $z \geq z_0$, we obtain

$$\bar{U}'' + c_0\bar{U}' + r_1(1 - \bar{U})(a_1\bar{V}^2 - \bar{U}) = a_1\bar{V}(1 - \bar{V})F(\bar{V}), \tag{3.2}$$

where

$$F(\bar{V}) = (a_1 - 2\bar{\mu}^2)\bar{V} + \bar{\mu}^2 - c_0\bar{\mu} - 1.$$

If $F'(\bar{V}) = a_1 - 2\bar{\mu}^2 \geq 0$, by using the second inequality of (3.1), we can obtain

$$F(\bar{V}) \leq F(1) = \frac{(1 + 2d)(1 - a_2)}{d} \left[\frac{d(a_1 - 1)}{(1 + 2d)(1 - a_2)} - r \right] \leq 0.$$

On the other hand, if $F'(\bar{V}) < 0$, based on the first inequality of (3.1), we have

$$F(\bar{V}) \leq F(0) = r \frac{(1 - a_2)(1 - 2d)}{d} - 1 \leq 0,$$

Thus $U'' + c_0U' + (1 - \bar{U})(a_1\bar{V}^2 - \bar{U}) \leq 0$.

By substituting (\bar{U}, \bar{V}) into the second equation of (2.4), when $z \leq z_0$, by using the fact that $\bar{V}(z) \geq \bar{V}(z_0) = 1/a_1$ and $a_1 > 1$, and the third inequality of (3.1), we obtain

$$\begin{aligned}
 & d\bar{V}'' + c_0\bar{V}' + r\bar{V}[1 - \bar{V} - a_2(1 - \bar{U})^2] \\
 &= \bar{V}(1 - \bar{V})(ra_2 - 2d\bar{\mu}^2\bar{V}) \\
 &\leq \bar{V}(1 - \bar{V}) \left[ra_2 - 2d\bar{\mu}^2 \cdot \frac{1}{a_1} \right] \\
 &= r \frac{a_2}{a_1} \bar{V}(1 - \bar{V}) \left[a_1 - \frac{2(1 - a_2)}{a_2} \right] \\
 &\leq r \frac{a_2}{a_1} \bar{V}(1 - \bar{V}) \left[a_1^2 - \frac{2(1 - a_2)}{a_2} \right] \leq 0.
 \end{aligned} \tag{3.3}$$

From the third inequality in (3.1), when $z \geq z_0$, and by using $0 \leq \bar{V}(z) \leq 1$, it follows that

$$\begin{aligned}
 & d\bar{V}'' + c_0\bar{V}' + r\bar{V}[1 - \bar{V} - a_2(1 - U)^2] \\
 &= \bar{V}^2 [(-2d\bar{\mu}^2 - ra_2 + 2ra_1a_2) + (2d\bar{\mu}^2 - ra_1^2a_2)\bar{V}] \\
 &= r\bar{V}^2 \left[(-2 + a_2 + 2a_1a_2) + a_2 \left(\frac{2(1 - a_2)}{a_2} - a_1^2 \right) \bar{V} \right] \\
 &\leq r\bar{V}^2 \left[(-2 + a_2 + 2a_1a_2) + a_2 \left(\frac{2(1 - a_2)}{a_2} - a_1^2 \right) \cdot 1 \right] \\
 &= -ra_2\bar{V}^2(1 - a_1)^2 \leq 0.
 \end{aligned} \tag{3.4}$$

Thus, (\bar{U}, \bar{V}) is an upper solution to system (2.4)-(2.5). Then, by Lemma 2.4, the desired result follows. \square

This theorem together with (1.4) shows that if the invader \bar{v} 's diffusion rate and growth rate are, respectively, greater than or equal to $1/2$ and $d(a_1 - 1)/(1 + 2d)(1 - a_2)$ times of those of the native species \bar{u} , and the competitiveness of the alien species \bar{v} is no more than $\sqrt{2(1/a_2 - 1)}$, then the minimum invasion speed can be obtained (i.e., invasion success).

Theorem 3.2. *Let the parameters d, r, a_1 , and a_2 be fixed. Assume that (1.10) is satisfied and that*

$$a_1 \leq 1 + \frac{2r(1 - a_2)}{d} \min\{2(d - 1), 3\}, \tag{3.5}$$

or

$$1 + \frac{6r(1 - a_2)}{d} \leq a_1 \leq 1 + \frac{r(1 + 2d)(1 - a_2)}{d}. \tag{3.6}$$

Then the linear selection of the minimal wave speed is realized. Hence the minimal invasion speed is equal to c_0 .

Proof. Let (\bar{U}, \bar{V}) be given by

$$\bar{U} = \left(\frac{1}{1 + e^{\bar{\mu}z}} \right)^2, \quad \bar{V} = \frac{1}{1 + e^{\bar{\mu}z}}.$$

Then, we have

$$\bar{U}' = -2\bar{\mu}\bar{V}^2(1 - \bar{V}), \quad \bar{U}'' = 2\bar{\mu}^2\bar{V}^2(1 - \bar{V})(2 - 3\bar{V}).$$

By substituting these expressions into system (2.4), and by using (3.5), it follows that

$$\begin{aligned} & \bar{U}'' + c_0 \bar{U}' + (1 - \bar{U}) \left(a_1 \bar{V}^2 - \bar{U} \right) \\ &= \bar{V}^2 (1 - \bar{V}) \left(2\bar{\mu}^2 (2 - 3\bar{V}) - 2c_0 \bar{\mu} + (1 + \bar{V})(a_1 - 1) \right) \\ &= \bar{V}^2 (1 - \bar{V}) \left(4\bar{\mu}^2 - 2c_0 \bar{\mu} + a_1 - 1 + \bar{V}(a_1 - 1 - 6\bar{\mu}^2) \right) \\ &= \bar{V}^2 (1 - \bar{V}) \left(a_1 - 1 - \frac{4r(d-1)(1-a_2)}{d} + \bar{V} \left(a_1 - 1 - \frac{6r(1-a_2)}{d} \right) \right) \leq 0. \end{aligned}$$

On the other hand, if the inequalities in (3.6) are satisfied, then we also obtain

$$\begin{aligned} & \bar{U}'' + c_0 \bar{U}' + (1 - \bar{U}) \left(a_1 \bar{V}^2 - \bar{U} \right) \\ &= \bar{V}^2 (1 - \bar{V}) \left(a_1 - 1 - \frac{4r(d-1)(1-a_2)}{d} + \bar{V} \left(a_1 - 1 - \frac{6r(1-a_2)}{d} \right) \right) \\ &\leq \bar{V}^2 (1 - \bar{V}) \left(a_1 - 1 - \frac{4r(d-1)(1-a_2)}{d} + a_1 - 1 - \frac{6r(1-a_2)}{d} \right) \\ &= 2\bar{V}^2 (1 - \bar{V}) \left(a_1 - 1 - \frac{r(1+2d)(1-a_2)}{d} \right) \leq 0. \end{aligned}$$

Since $\bar{U} = \bar{V}^2 \leq 1$, we obtain

$$\begin{aligned} & d\bar{V}'' + c_0 \bar{V} + r\bar{V} (1 - \bar{V} - a_2(1 - \bar{U})^2) \\ &\leq d\bar{V}'' + c_0 \bar{V} + r\bar{V} (1 - \bar{V} - a_2(1 - \bar{V})^2) \\ &= \bar{V}^2 (1 - \bar{V})(ra_2 - 2d\bar{\mu}^2) = r\bar{V}^2 (1 - \bar{V})(3a_2 - 2) \leq 0. \end{aligned}$$

Hence (\bar{U}, \bar{V}) is an upper solution of system (2.4)-(2.5). Then, by Lemma 2.4, the proof is completed. \square

The combination of Theorem 3.2 and (1.4) indicates that

- (i) if the invader \bar{v} 's diffusion rate is greater than that of the native species \bar{u} , then the minimal invasion speed is obtained provided that v 's competitive strength can not exceed

$$1 + \frac{2r(1-a_2)}{d} \min\{2(d-1), 3\},$$

that is, in view of (1.10),

$$a_1 \in \left(1, 1 + \frac{2r(1-a_2)}{d} \min\{2(d-1), 3\} \right];$$

and

- (ii) for any given the invader \bar{v} 's positive diffusion rate, the minimal invasion speed is realized when \bar{v} 's competitiveness is within a certain range of values, that is,

$$a_1 \in \left[1 + \frac{6r(1-a_2)}{d}, 1 + \frac{r(1+2d)(1-a_2)}{d} \right].$$

These results are consistent with our established intuition of regarding that the more competitive it is, the more successful it is. Hence they are very instructive conclusions.

4. Numerical study

In this section, we give two numerical examples to corroborate the results of Theorems 3.1 and 3.2 in the previous section by direct simulations of the system (1.5). By Proposition 2.2, Theorems 3.1 and 3.2, we know that the system (1.5) has a traveling wave solution connecting $(0, 1)$ and $(1, 0)$ with wave speed $c = c_{\min} = c_0$ if the conditions (1.10) and (3.1) or (3.5) or (3.6) are satisfied. This implies that the invader \bar{v} moves towards the territory of the native species \bar{u} with a minimal invasion speed and eventually occupies the habitats of \bar{u} , that is, \bar{v} successfully invades and survives at the slowest speed c_0 , while \bar{u} dies out at the same speed.

In the following two examples, the initial data of (1.5) is set to

$$\bar{u}(x, 0) = \frac{1}{1 + e^{-x}}, \quad \bar{v}(x, 0) = \frac{1}{1 + e^x}. \quad (4.1)$$

Example 4.1. We intend to demonstrate the results of Theorem 3.1. The system parameters are taken as

$$d = 1, \quad r = \frac{4}{3}, \quad a_1 = 2, \quad a_2 = 0.25. \quad (4.2)$$

It is easy to check that the conditions (1.10) and (3.1) are satisfied. In view of Theorem 3.1, we have that (1.5) has a traveling wave solution connecting $(0, 1)$ and $(1, 0)$ with $c_{\min} = c_0$, and

$$c_0 = 2\sqrt{dr(1 - a_2)} = 2.$$

This outcome of the evolution of \bar{u} and \bar{v} can be observed in Fig. 1 as follows:

(i) The three dimensional figures (a) in Fig. 1 show the traveling wave (\bar{U}, \bar{V}) connecting $(0, 1)$ and $(1, 0)$, which indicates that the species \bar{u} changes from 1 to 0 and \bar{v} transforms from 0 to 1. This means that \bar{u} will die out and the species \bar{v} will approach a limiting state 1 as time increases.

(ii) The top views (b) in Fig. 1 clearly show that, as time progresses, the species \bar{v} is gradually moving toward the territory of the species \bar{u} , and the species \bar{u} is retreating.

(iii) As seen in figure (c) in Fig.1, the X coordinates of the points A and B obviously indicates that both of \bar{u} 's speed $c_{\bar{u}}$ and \bar{v} 's speed $c_{\bar{v}}$ are 2, i.e., $c_{\bar{u}} = c_{\bar{v}} = 2$.

In summary, the analytical predictions of Theorem 3.1 is in agreement with full numerical simulations of the original system (1.5).

Example 4.2. We now aim to show the results of Theorem 3.2. The system parameters are chosen as

$$d = 1.5, \quad r = 0.5, \quad a_1 = 1.5, \quad a_2 = 0.25. \quad (4.3)$$

By a simple computation, we find that the conditions (1.10) and (3.5) are satisfied. Theorem 3.2 implies that (1.5) admits a traveling wave solution connecting $(0, 1)$ and $(1, 0)$ with $c_{\min} = c_0$, and

$$c_0 = 2\sqrt{dr(1 - a_2)} = 1.5.$$

The development of \bar{u} and \bar{v} is shown in Fig. 2. Likewise, as observed in Fig. 2, we can get the similar numerical results (i) – (iii) to Example 4.1, which are also completely consistent with the prediction of Theorem 3.2.

5. Conclusions

In this work, we have proposed and analyzed a technique to obtain the minimal invasion speed of exotic species for the case where $a_1 > 1, a_2 < \frac{1}{3}$. This technique relies on the use of a mathematical model with two competing species. Based on this technique, we have derived general and explicit conditions for the the minimal invasion and also illustrated these conditions by direct simulations of the original model. Evidently, our research results are very useful for the development of measures to

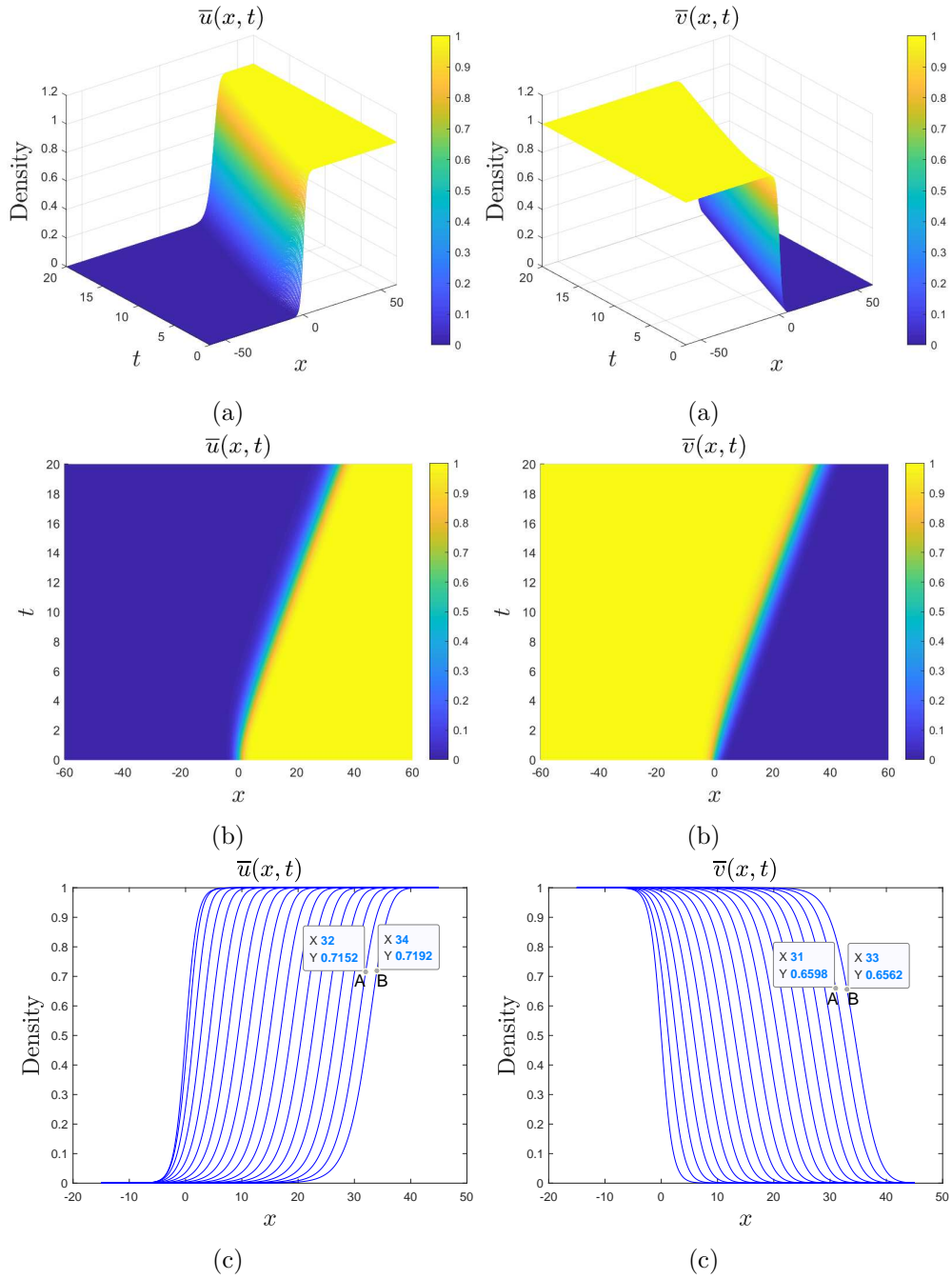


FIGURE 1. Evolution of the system (1.5) with the initial data (4.1) and system parameters being set as $d = 1$, $r = \frac{4}{3}$, $a_1 = 2$, $a_2 = 0.25$. The left and right panels depict, respectively, the spatiotemporal development of the components \bar{u} and \bar{v} . (a): the three dimensional figures. (b): the top views of (a). (c): the time snapshots for unit time interval.

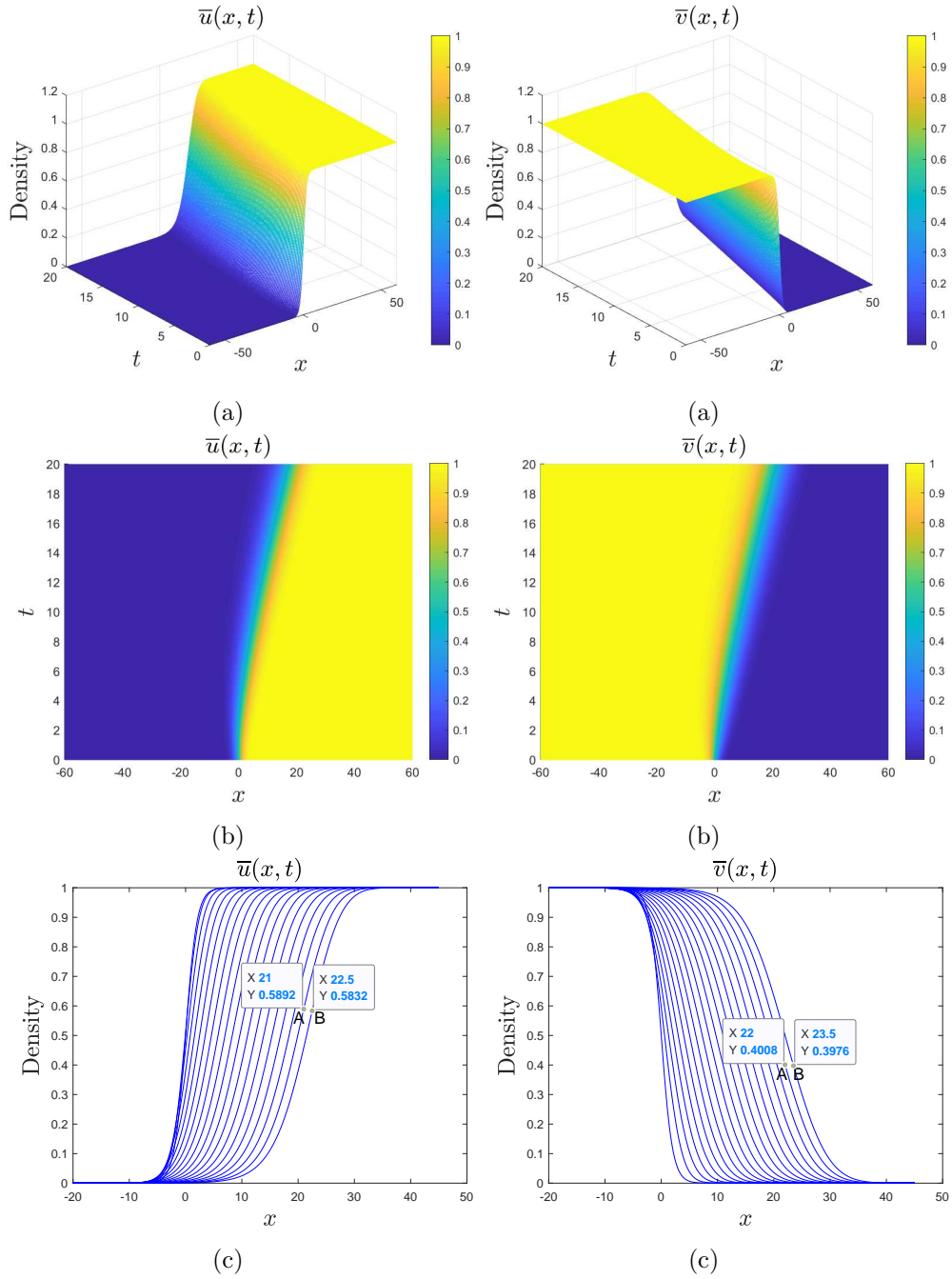


FIGURE 2. Evolution of the system (1.5) with the initial data (4.1) and system parameters being set as $d = 1.5$, $r = 0.5$, $a_1 = 1.5$, $a_2 = 0.25$. The left and right panels depict, respectively, the spatiotemporal development of the components \bar{u} and \bar{v} . (a): the three dimensional figures. (b): the top views of (a). (c): the time snapshots for unit time interval.

control or prevent biological invasion. However, we must point out that our explicit conditions are not optimal. We believe that it is possible to find different upper solutions to obtain conditions that do not contain each other.

The idea proposed in this work may be naturally extended in diverse directions. First, to find an optimal condition, one can discuss a full classification of the minimal wave speed c_{min} in terms of coefficients of the considered system. Second, motivated by the known results on the linear determinacy of the classical Lotka-Volterra competition model [1, 9, 12], where authors considered the case when $0 < a_1 < 1 < a_2$, we think it also interesting to discuss the minimal invasion speed of **Case ii** (i.e., $a_1 < 1, a_2 < \frac{1}{3}$) and compare the results with those of the classical Lotka-Volterra model. A third extension is to study the minimal invasion speed using a model with periodic coefficients, that is, the diffusion rates, the carrying rates, the competitive strength and the growth rates are periodic functions of time t or space x , the latter implying a spatially heterogeneous environment. Finally, a challenging possibility is to generalize our model to a setting where there are both cooperative and competitive species.

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