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LYAPUNOV-SCHMIDT REDUCTION IN THE STUDY OF BIFURCATION OF PERIODIC TRAVELLING WAVE SOLUTIONS OF A PERTURBED (1+1)-DIMENSIONAL DISPERSIVE LONG WAVE EQUATION

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Abstract. In this paper, the Lyapunov-Schmidt reduction is used to investigate the bifurcation of periodic travelling wave solutions of a perturbed (1+1)-dimensional dispersive long wave equation. We demonstrate that the bifurcation equation corresponding to the original problem is supplied by a nonlinear system of two cubic algebraic equations. As the bifurcation parameters change, this system has only one, three, or five regular real solutions. The linear approximation of the solutions to the main problem has been discovered.

1. Introduction

Many researchers have focused their attention on water wave equations. The nonlinear wave equations are used to describe the dynamic behavior of nonlinear waves in shallow water. The qualities of these waves can be investigated to determine what these waters are. One of these characteristics is that these waves do not change as a result of reciprocal interactions and collisions. Darboux transformation, the extended tanh-function approach, homogeneous balancing method, and Jacobi elliptic function method [10] *etc* have all been used to study long wave equations in shallow waters. Medhat *et al.* investigated the $(1+1)-$dimensional dispersive long wave equation utilizing the characteristic function method to search for traveling wave solutions in [3]. They reduced the system of nonlinear partial differential equations to a system of nonlinear ordinary differential equations that they solved using the shooting method in conjunction with the Runge-kutta scheme. In [4], Wang *et al.* developed a nonlinear transformation of the dispersive long wave equations in $(2+1)-$dimensions utilizing the homogeneous balancing method to get exact solutions. In [2], Chen and Wang successfully constructed new complexiton solutions for the $(1+1)-$dimensional dispersive long wave equation using two different Riccati equations, including various combinations of trigonometric periodic and hyperbolic function solutions, various combinations of trigonometric periodic and rational function solutions, and various combinations of hyperbolic and rational function solutions. In [6] Lu *et al.* have studied the following $(2+1)-$dimensional nonlinear dispersive long wave system

$$\begin{cases} v_{ty} + w_{xx} + \frac{1}{2}(v^2)_{xy} = 0, \\ w_t + (v + vw + v_{xy})_x = 0, \end{cases} \tag{1.1}$$

where $v(x,y,t)$ is the horizontal velocity, $w(x,y,t)$ represents the wave altitude above the undisturbed water surface, t denotes the time, and (x,y) stands for the spreading plane. By using the inverse

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spectral transform, the system of equations (1.1) can be used to solve nonlinear evolutional problems in two spatial dimensions. Finding new forms of approximation solutions for system (1.1) is thus of essential importance in fluid dynamics. There are numerous papers on these equations [5, 7, 9, 11]. If $x = y$, the system (1.1) can be reduced to a $(1+1)-$dimensional nonlinear dispersive long wave system as follows:

$$\begin{cases} v_{tx} + w_{xx} + \dfrac{1}{2}(v^2)_{xx} = 0, \\ w_t + (v + vw + v_{xx})_x = 0, \end{cases} \tag{1.2}$$

where $v(x,t)$ and $w(x,t)$ represent the horizontal velocity and height of water waves respectively. The system (1.2) can be used to represent the evolution of the horizontal velocity composition $v(x,t)$ and height $w(x,t)$ of water waves propagating in both directions at an infinte narrow channel of finite constant depth. In this paper, I use the local Lyapunov-Schmidt method to investigate the system of equations (1.2) perturbed $(1+1)-$dimensional described by

$$\begin{cases} v_{tx} + w_{xx} + \dfrac{1}{2}(v^2)_{xx} = 0, \\ w_t + (v + vw + v_{xx})_x + \tilde{\varepsilon}_1((x-ct)v)_x + \tilde{\varepsilon}_2 v_{xx} = \tilde{\varepsilon}\psi_x(x,t), \end{cases} \tag{1.3}$$

where c denotes the wave speed, $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$ and $\tilde{\varepsilon}$ are sufficiently small parameters,v_{xx} is backward diffusion and $\psi(x,t)$ is a differentiable function.

We will need the some following Definitions 1-2 and Theorem 1.1.

Definition 1.1. [8] The Nonlinear operator $\mathcal{F} : \tilde{U} \subset \tilde{X} \to \tilde{Y}$ is called Fredholm if the first Fréchet derivative $d\mathcal{F}(\tilde{x})$ is a Fredholm for every $\tilde{x} \in \tilde{U}$. The index of the nonlinear Fredholm operator $\mathcal{F}$ is equal to the index of the linear operator $d\mathcal{F}(\tilde{x})$.

Definition 1.2. [8] The discriminate (bifurcation) set Σ of equation $\mathcal{F}(v, \varrho, \varepsilon_1, \varepsilon_2) = b$ is defined to be the union of all $\varrho = \bar{\varrho}$ for which this equation has degenerate solution $\bar{v} \in \tilde{X}$:

$$\mathcal{F}(\bar{v}, \bar{\varrho}, \varepsilon_1, \varepsilon_2) = b, \qquad Codim\Big(Im\frac{\partial \mathcal{F}}{\partial v}(\bar{v}, \bar{\varrho}, \varepsilon_1, \varepsilon_2)\Big) > 0.$$

Theorem 1.1. [1] *(Finite dimensional reduction) Assume that $\tilde{X}$ and $\tilde{Y}$ be real Banach spaces and $\mathcal{F} : \tilde{X} \times \mathbb{R} \to \tilde{Y}$ is a C^1 map defined in a neighborhood $\tilde{U}$ of a point $(\tilde{x}_0, \sigma_0)$ with range in $\tilde{Y}$ such that $\mathcal{F}(\tilde{x}_0, \sigma_0) = 0$ and $\mathcal{F}_{\tilde{x}}(\tilde{x}_0, \sigma_0)$ is a linear Fredholm operator. Then all solutions $(\tilde{x}, \sigma)$ of $\mathcal{F}(\tilde{x}, \sigma) = 0$ near $(\tilde{x}_0, \sigma_0)$ (with σ fixed) are in one-to-one correspondence with the solutions of a finite-dimensional system of $\tilde{N}_1$ real equations in a finite number $\tilde{N}_0$ of real variables. Furthermore, $\tilde{N}_0 = dim(ker\tilde{L})$ and $\tilde{N}_1 = dim(coker\tilde{L})$, $(\tilde{L} = \mathcal{F}_{\tilde{x}}(\tilde{x}_0, \sigma_0))$.*

To obtain the travelling wave solutions of (1.3), we first consider the travelling wave solutions in the form $v(x,t) = v(\zeta)$, $w(x,t) = w(\zeta)$, $\quad \zeta = x - ct$. Therefore (1.3) reduces to

$$\begin{cases} -cv'' + w'' + \dfrac{1}{2}(v^2)'' = 0, \\ -cw' + (v + vw + v'')' + \tilde{\varepsilon}_1(\zeta v)' + \tilde{\varepsilon}_2 v'' = \tilde{\varepsilon}\psi'. \end{cases} \tag{1.4}$$

We can integrate the first equation in (1.4) once, with the integral constant setting to zero, and then integrate it again and set the integral constant to zero; and also integrate the second equation of (1.4) and set the integration constant to zero . We then obtain

$$\begin{cases} -cv+\dfrac{v^2}{2}+w=0, \\ -cw+v+vw+v''+\tilde{\varepsilon}_1\zeta v+\tilde{\varepsilon}_2 v'=\tilde{\varepsilon}\psi. \end{cases} \tag{1.5}$$

Substituting the first equation is substituted into the second equation of (1.5) results in

$$v''+(\varrho+\varepsilon_1\zeta)v+\varepsilon_2 v'+\frac{3}{2}cv^2-\frac{1}{2}v^3=\varepsilon\psi(\zeta) \tag{1.6}$$

where, $\varrho = 1-c^2$, $\varepsilon_1 = \tilde{\varepsilon_1}$ and $\varepsilon_2 = \tilde{\varepsilon_1}$, in this work we assume that $|c| < 1$. Setting periodic conditions for functions v and w is convenient, that is $v(\zeta) = v(\zeta + 2\pi)$ and $w(\zeta) = w(\zeta + 2\pi)$. It is sufficient to explore the bifurcation of periodic travelling wave solutions of system (1.3) to investigate the bifurcation of periodic travelling wave solutions of equation (1.6).

2. Reduction to bifurcation equation

In this part, we want to discover the bifurcation equation that corresponds to the equation (1.6). We express the local Lyapunov-Schmidt reduction in the equation (1.6) in the form of an operator equation to implement it:

$$\mathcal{F}(v,\varrho,\varepsilon_1,\varepsilon_2)=v^{''}+(\varrho+\varepsilon_1\zeta)v+\varepsilon_2 v^{'}+\frac{3}{2}cv^2-\frac{1}{2}v^3=\varepsilon\psi(\zeta) \tag{2.1}$$

where, $\mathcal{F}:\tilde{X}\to\tilde{Y}$, $\tilde{X}=\Pi_2([0,2\pi],\mathbb{R})$ is the space of all periodic continuous functions that have derivative of order at most two and $\tilde{Y}=\Pi_0([0,2\pi],\mathbb{R})$ is the space of all periodic continuous functions. The behavior of the nonlinear operator $\mathcal{F}$ near the point $(0,\varrho,0,0)$ is known to be similar to that of the linear operator $\mathcal{A}=d\mathcal{F}(0,\varrho,0,0)=\frac{d^2}{d\zeta^2}+\varrho$, ($d\mathcal{F}$ is the Fréchet derivative of the operator $\mathcal{F}$). The linearized equation provides an important set of solutions for researching bifurcation solutions

$$\mathcal{A}\tilde{h}=0, \quad \tilde{h}\in\tilde{X}.$$

As a result, the periodic solutions of the linearized equation are

$$\tilde{h}_q(\zeta)=\kappa_q sin(q\zeta)+\omega_q cos(q\zeta).$$

A small change in the parameter ϱ causes a bifurcation along the modes

$$\tilde{h}_1(\zeta)=\kappa_1 sin(\zeta), \quad \tilde{h}_2(\zeta)=\omega_1 cos(\zeta)$$

In the Hilbert space $\mathcal{H}=\mathcal{L}_2([0,2\pi],\mathbb{R})$, the functions $\tilde{h_1}(\zeta)$ and $\tilde{h_2}(\zeta)$ are eigenfunctions of the operator $\mathcal{A}$. In $\mathcal{H}$, eigenfunctions are known to form an orthonormal system. From this, we can deduce that $\parallel \tilde{h_j} \parallel_{\mathcal{H}}=1$ and $\kappa_1,\omega_1=\sqrt{2}$ when $j=1,2$. As a result, $\mathcal{N}=ker(\mathcal{A})=\{\tilde{h_1},\tilde{h_2}\}$ are obtained. When the operator $\mathcal{A}$ is Fredholm and the spaces $\tilde{X}$ and $\tilde{Y}$ are Banach spaces, we get the following decomposition

$$\tilde{X}=\mathcal{N}\oplus\mathcal{N}^{\perp}, \qquad \mathcal{N}^{\perp}=\{\tilde{v}\in\tilde{X}:\tilde{v}\perp\mathcal{N}\},$$

$$\tilde{Y}=\mathcal{N}\oplus\mathcal{M}^{\perp}, \qquad \mathcal{M}^{\perp}=\{\tilde{g}\in\tilde{Y}:\tilde{g}\perp\mathcal{N}\}.$$

Accordingly, there exist projections $\mathcal{P}:\tilde{X}\to\mathcal{N}$ and $I-\mathcal{P}:\tilde{X}\to\mathcal{N}^{\perp}$ such that $\mathcal{P}v=\tilde{v}_1$, $(\mathcal{I}-\mathcal{P})v=\tilde{v}_2$. Then every element $v\in\tilde{X}$ can be represented in the unique form

$$v=\tilde{v}_1+\tilde{v}_2,\ \tilde{v}_1=\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2\in\mathcal{N},\ \ \tilde{v}_2\in\mathcal{N}^{\perp},\ \zeta_j=\langle v,\tilde{h}_j\rangle_{\mathcal{H}},\ j=1,2.$$

Similarly, there exists projections $\mathcal{Q} : \tilde{Y} \to \mathcal{N}$ and $I - \mathcal{Q} : \tilde{Y} \to \mathcal{M}^{\perp}$ such that $\mathcal{F}_1(v, \varrho, \varepsilon_1, \varepsilon_2) = \mathcal{Q}\mathcal{F}(v, \varrho, \varepsilon_1, \varepsilon_2)$, $\mathcal{F}_2(v, \varrho, \varepsilon_1, \varepsilon_2) = (I - \mathcal{Q})\mathcal{F}(v, \varrho, \varepsilon_1, \varepsilon_2)$ and every element $\mathcal{F}(v, \varrho, \varepsilon_1, \varepsilon_2) \in \tilde{Y}$ is represented in the unique form

$$\mathcal{F}(v, \varrho, \varepsilon_1, \varepsilon_2) = \mathcal{F}_1(v, \varrho, \varepsilon_1, \varepsilon_2) + \mathcal{F}_2(v, \varrho, \varepsilon_1, \varepsilon_2),$$

with

$$\mathcal{F}_1(v, \varrho, \varepsilon_1, \varepsilon_2) = \sum_{j=1}^{2} \langle \mathcal{F}(v, \varrho, \varepsilon_1, \varepsilon_2), \tilde{h}_j \rangle_{\mathcal{H}} \tilde{h}_j \in \mathcal{N}, \quad \text{and} \quad \mathcal{F}_2(v, \varrho, \varepsilon_1, \varepsilon_2) \in \mathcal{M}^{\perp},$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathcal{H}$. Since $\psi \in \tilde{Y}$ it follows that $\psi = \mathcal{Q}\psi + (I - \mathcal{Q})\psi$, $\quad \mathcal{Q}\psi = \vartheta_1 \tilde{h}_1 + \vartheta_2 \tilde{h}_2 \in \mathcal{N}$, $(I - \mathcal{Q})\psi \in \mathcal{M}^{\perp}$. Therefore, equation (2.1) can be written as

$$\mathcal{F}(\zeta_1 \tilde{h}_1 + \zeta_2 \tilde{h}_2 + \tilde{v}_2, \varrho, \varepsilon_1, \varepsilon_2) = \beta_1 \tilde{h}_1 + \beta_2 \tilde{h}_2 \tag{2.2}$$

where $\beta_1 = \varepsilon \vartheta_1$, $\quad \beta_2 = \varepsilon \vartheta_2$. Equation (2.2) can be separated into a system of two equations

$$\begin{cases} \mathcal{F}_1(\zeta_1 \tilde{h}_1 + \zeta_2 \tilde{h}_2 + \tilde{v}_2, \varrho, \varepsilon_1, \varepsilon_2) = \mathcal{Q}\psi = \beta_1 \tilde{h}_1 + \beta_2 \tilde{h}_2, \\ \mathcal{F}_2(\zeta_1 \tilde{h}_1 + \zeta_2 \tilde{h}_2 + \tilde{v}_2, \varrho, \varepsilon_1, \varepsilon_2) = (I - \mathcal{Q})\psi. \end{cases}$$

If the operator $d\mathcal{F}_2$ is an isomorphism at the point $(0, \varrho, 0, 0)$, then there exists in infinite Banach space a smooth function $\theta : \mathcal{N} \to \mathcal{N}^{\perp}$ in such a way that

$$\mathcal{F}_2(\zeta_1 \tilde{h}_1 + \zeta_2 \tilde{h}_2 + \theta(\tilde{v}_1, \varrho, \varepsilon_1, \varepsilon_2), \varrho, \varepsilon_1, \varepsilon_2) = (I - \mathcal{Q})\psi.$$

To study the solutions of equation (2.1) in the neighbourhood of the point $(0, \varrho, 0, 0)$, it is sufficient to investigate the solutions of equation

$$\mathcal{F}_1(\zeta_1 \tilde{h}_1 + \zeta_2 \tilde{h}_2 + \theta(\tilde{v}_1, \varrho, \varepsilon_1, \varepsilon_2), \varrho, \varepsilon_1, \varepsilon_2) = \beta_1 \tilde{h}_1 + \beta_2 \tilde{h}_2. \tag{2.3}$$

Equation (2.3) is a nonlinear system of algebraic equations with $2 = dim(ker(\mathcal{A}))$ variables and $2 = dim(coker(\mathcal{A}))$ equations.

3. Analysis of bifurcation of system (2.3)

We will demonstrate in this section that the nonlinear system of equation (2.3) is provided by two nonlinear cubic algebraic equations.

Theorem 3.1. *The bifurcation equation*

$$\mathcal{F}_1(\zeta_1 \tilde{h}_1 + \zeta_2 \tilde{h}_2 + \theta(\tilde{v}_1, \varrho, \varepsilon_1, \varepsilon_2), \varrho, \varepsilon_1, \varepsilon_2) = \beta_1 \tilde{h}_1 + \beta_2 \tilde{h}_2.$$

corresponding to the equation (2.1) is given by the following system

$$\begin{cases} \zeta_1^3 + \zeta_1 \zeta_2^2 + \kappa_1 \zeta_1 + \kappa_2 \zeta_2 + ... = \rho_1, \\ \zeta_2^3 + \zeta_1^2 \zeta_2 + \kappa_3 \zeta_1 + \kappa_1 \zeta_2 + ... = \rho_2. \end{cases} \tag{3.1}$$

Proof. Equation (2.1) can be expressed as follows:

$$\begin{aligned}\mathcal{F}(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2+\theta(\tilde{v}_1,\varrho,\varepsilon_1,\varepsilon_2),\varrho,\varepsilon_1,\varepsilon_2) &= \mathcal{A}(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2+\theta(\tilde{v}_1,\varrho,\varepsilon_1,\varepsilon_2))\\ &\quad+\mathcal{B}(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2+\theta(\tilde{v}_1,\varrho,\varepsilon_1,\varepsilon_2))\\ &\quad+\mathcal{T}(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2+\theta(\tilde{v}_1,\varrho,\varepsilon_1,\varepsilon_2))\\ &=\mathcal{A}(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2)+\varepsilon_1\zeta(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2)+\varepsilon_2(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2)'\\ &\quad+\frac{3}{2}c(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2)^2-\frac{1}{2}(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2)^3+..\end{aligned}$$

where,

$$\mathcal{B}(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2+\theta(\tilde{v}_1,\varrho,\varepsilon_1,\varepsilon_2))=\varepsilon_1\zeta(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2)+\varepsilon_2(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2)',$$

$$\mathcal{T}(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2+\theta(\tilde{v}_1,\varrho,\varepsilon_1,\varepsilon_2))=\frac{3}{2}c(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2)^2-\frac{1}{2}(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2)^3.$$

and the dots denote the terms that comprise the element $\theta(\tilde{v}_1,\varrho,\varepsilon_1,\varepsilon_2)$. Hence,

$$\begin{aligned}&\mathcal{F}_1(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2+\theta(\tilde{v}_1,\varrho,\varepsilon_1,\varepsilon_2),\varrho,\varepsilon_1,\varepsilon_2)\\ &=\sum_{i=1}^{2}\langle\mathcal{F}(\zeta_1\tilde{h}_1+\zeta_2\tilde{h}_2+\theta(\tilde{v}_1,\varrho,\varepsilon_1,\varepsilon_2),\varrho,\varepsilon_1,\varepsilon_2),\tilde{h}_j\rangle_{\mathcal{H}}\tilde{h}_j+...\\ &=\beta_1\tilde{h}_1+\beta_2\tilde{h}_2.\end{aligned}$$

Because the operator $\mathcal{A}$ is self-adjoint we obtain $\mathcal{A}\tilde{h}_j=\nu_j(\varrho)\tilde{h}_j$, ($\nu_j(\varrho)$ are smooth functions, $j=1,2$). By employing the linear property of inner product and computing some definite integrals, we obtain

$$\zeta_1^3+\zeta_1\zeta_2^2+\kappa_1\zeta_1+\kappa_2\zeta_2+...=\rho_1,$$

$$\zeta_2^3+\zeta_1^2\zeta_2+\kappa_3\zeta_1+\kappa_1\zeta_2+...=\rho_2.$$

where, $\kappa_1=-\frac{4\sqrt{2}}{3}(\vartheta+\varepsilon_1\pi)$, $\kappa_2=\frac{2\sqrt{2}}{3}(\varepsilon_1+2\varepsilon_2)$, $\kappa_3=\frac{2\sqrt{2}}{3}(\varepsilon_1-2\varepsilon_2)$, $\vartheta=\nu_1(\varrho)=\nu_2(\varrho)$, $\rho_1=-\frac{8}{3\sqrt{2}}\beta_1$ and $\rho_2=-\frac{8}{3\sqrt{2}}\beta_2$. □

System (3.1) is the symmetric contact equivalent of system

$$\begin{aligned}\zeta_1^3+\zeta_1\zeta_2^2+\kappa_1\zeta_1+\kappa_2\zeta_2&=\rho_1,\\ \zeta_2^3+\zeta_1^2\zeta_2+\kappa_3\zeta_1+\kappa_1\zeta_2&=\rho_2.\end{aligned}\tag{3.2}$$

Contact equivalence indicates that the bifurcation set of the system (3.1) is comparable to the bifurcation set of the system (3.2), hence studying the bifurcation set of the system (3.2) is adequate. On the surface described by the following equation, the solutions of system (3.2) are degenerate.

$$3\zeta_1^4+6\zeta_1^2\zeta_2^2+3\zeta_2^4+4\kappa_1{\zeta_1}^2+4\kappa_1{\zeta_2}^2-2\kappa_2\zeta_1\zeta_2-2\kappa_3\zeta_1\zeta_2+\kappa_1^2-\kappa_2\kappa_3=0.\tag{3.3}$$

Because equation (3.3) has four solutions in terms of the variable ζ_2, the bifurcation set of system (3.2) is the union of four curves. By adopting the equations in system (3.2), we can demonstrate the bifurcation set of system (3.2) in the $\rho_1-\rho_2$ plane for $\kappa_1=-1.5,\kappa_2=0.3,\kappa_3=0.5$ and for $\kappa_1=-0.7,\kappa_2=0.3,\kappa_3=0.5$ in Figure 1 and Figure 2 respectively.

Note that $\Upsilon=\mathbb{R}^5\backslash\Sigma$ is the union of three open subsets of the bifurcation set, $\Upsilon=R_1\cup R_2\cup R_3$ such that if $(\rho_1,\rho_2)\in R_1$ then system (3.2) has five regular real solutions with topological indices $1,-1,1,-1,1$, if $(\rho_1,\rho_2)\in R_2$ there is three regular real solutions for system (3.2) with topological

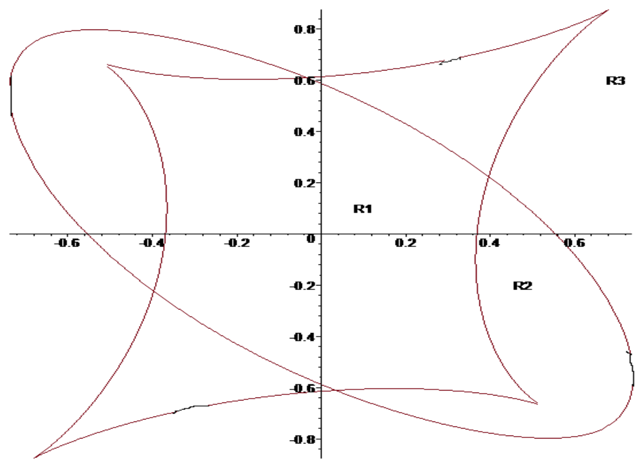

FIGURE 1. Bifurcation set of system (3.2) for $\kappa_1 = -1.5, \kappa_2 = 0.3, \kappa_3 = 0.5$.

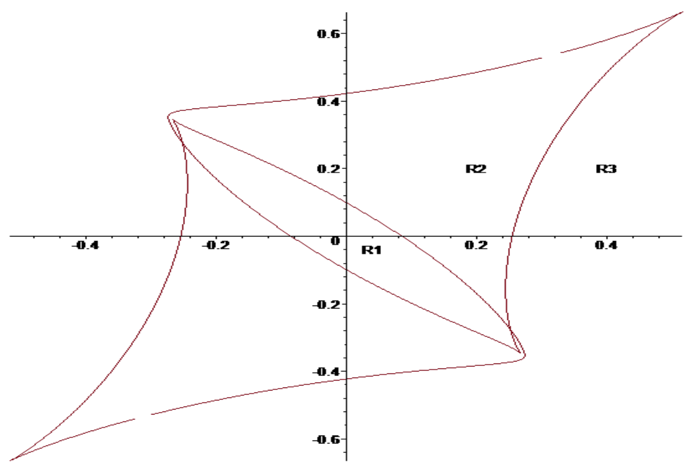

FIGURE 2. Bifurcation set of system (3.2) for $\kappa_1 = -0.7, \kappa_2 = 0.3, \kappa_3 = 0.5$.

indices $1, -1, 1$, and when $(\rho_1, \rho_2) \in R_3$ the there is only one regular real solution for system (3.2) with topological index is equal to 1. In polar coordinates,

$$\zeta_1 = \nu cos\varphi, \quad \zeta_2 = \nu sin\varphi$$

system (3.2) takes the following form:

$$\begin{cases} \nu(\nu^2 cos\varphi + \kappa_1 cos\varphi + \kappa_2 sin\varphi) - \rho_1 = 0, \\ \nu(\nu^2 sin\varphi + \kappa_3 cos\varphi + \kappa_1 sin\varphi) - \rho_2 = 0. \end{cases} \tag{3.4}$$

On the surface described by the following equation, the solutions set is degenerate.

$$3\nu^4 + (4\kappa_1 - (\kappa_2 + \kappa_3) sin2\varphi)\nu^2 + \kappa_1^2 - \kappa_2\kappa_3 = 0. \tag{3.5}$$

We get when we solve equation (3.5) for the variable ν

$$\nu^2 = \frac{\sqrt{(4\kappa_1 - (\kappa_2 + \kappa_3) sin2\varphi)^2 - 12(\kappa_1^2 - \kappa_2\kappa_3)}}{6}.$$

assuming the following assumptions

$$\kappa_2 = \nu(1+cos\varphi), \quad \kappa_3 = \nu(1-cos\varphi)$$

we get

$$\kappa_1 = \nu(-2\nu \pm \sqrt{\nu^2 + \nu sin2\varphi + sin^2\varphi})$$

$$\rho_1 = \nu^2(\frac{1}{2}sin2\varphi + cos\varphi\sqrt{\nu sin2\varphi - cos^2\varphi + \nu^2 + 1} - \nu cos\varphi + sin\varphi)$$

$$\rho_2 = \nu^2(sin\varphi\sqrt{\nu sin2\varphi - cos^2\varphi + \nu^2 + 1} - \nu sin\varphi + cos\varphi - cos^2\varphi)$$

The preceding parametrization results in a bifurcation set of system (3.4), which yields the following surface in Figure 3:

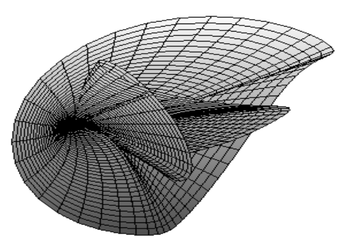

FIGURE 3. Bifurcation set of system (3.4).

According to theorem (1.1), there is a one-to-one correspondence between the solutions of equation (2.1) and the solutions of system (3.2) such that if the point $(\bar{\zeta_1}, \bar{\zeta_2})$ is a solution of system (3.2) then the point $\bar{v} = \bar{\zeta_1}\tilde{h}_1 + \bar{\zeta_2}\tilde{h}_2 + \theta(\bar{\zeta_1}\tilde{h}_1 + \bar{\zeta_2}\tilde{h}_2, \varrho, \varepsilon_1, \varepsilon_2)$ is a solution to equation (2.1).

Conclusions. The bifurcation of periodic travelling wave solutions of a perturbed (1+1)−Dimensional Dispersive Long Wave Equation has been explored in this article using local Lyapunov-Schmidt method. We demonstrate that the reduced equation corresponding to equation (2.1) is given by a system of two nonlinear cubic algebraic equations using the finite dimensional reduction theorem. All of the topological and analytical properties of equation (2.1) are present in this system. The system bifurcation diagram (3.2) has been discovered for four specific parameter values. The number of regular real solutions of system (3.2) has been found to be one, three, or five in each region of the bifurcation diagram.

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