

MARKOVIAN PHASE-TYPE SINGLE WORKING VACATION QUEUE WITH BREAKDOWN AND STANDBY SERVER

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ABSTRACT. We consider a single server queueing model with three service types, each with three phases, where customers arrive according to a Poisson process and the service time of each customer follows a Hyper-Erlang distribution. This study aims to analyze the impact of interruptions, such as breakdowns, repairs, and single working vacations, on the performance of the queueing system. The standby server activates whenever the main server experiences a breakdown. Using the matrix geometric method, we examine the steady-state probability vector of the model. Our results demonstrate the impact of these interruptions on system performance measures and provide a cost analysis. Finally, we present numerical examples to illustrate the practical applications of our findings in an image compression system.

1. INTRODUCTION

Queueing theory has emerged as a powerful tool for analyzing the behavior of complex systems that requires the processing of incoming data in a sequential manner. Although single-server queueing models have received a great deal of attention in service systems, real-world applications often require advanced techniques to take consideration of specific practical issues. Levy and Yachiali [14] have introduced queueing systems with single and multiple vacations. The literature on vacation models is rapidly expanding, with significant contributions from [28, 5]. Recent advancements in vacation queueing systems are summarized in [9, 24]. In 2007, Li and Tian [15] introduced the concept of a vacation interruption policy and analyzed an $M/M/1$ queue. Further, [16] explored an $M/M/R$ queue with vacations, where the server continues to provide service at varying rates during the vacation period, rather than halting service entirely.

Seenivasan and Chandiraleka [25] explored a queueing model in which the server experiences multiple working vacations and is also subject to breakdowns during service. Houalef et al. [8] designed an $M/M/c$ queueing system under a Markovian environment, featuring servers that can balk and renege while operating under both synchronous single and multiple working vacation policies. Azhagappan and Deepa [1] conducted a time-dependent analysis of an $M/M/1$ queueing model that includes single and multiple working vacations, balking, and vacation interruptions. Baba [2] investigated an $M/PH/1$ queue with phase-type working vacations and vacation interruptions, where the vacation duration follows a phase-type distribution. In this scenario, the server resumes its usual busy period after a service completion without finishing the vacation, which is termed as vacation interruption.

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Gao and Liu [6] examined an $M/G/1$ queueing system with a single working vacation and vacation interruption, utilizing a Bernoulli schedule. Xu et al. [32] investigated an $M/M/1$ queue with a single working vacation and set-up times, employing a quasi birth-and-death process and matrix-geometric solution approach. Yang et al. [33] analyzed the F -policy $M/M/1/K$ queueing system with a single working vacation, deriving the steady-state probability vectors through a matrix-analytic approach. Goswami's thesis [7] provided a performance analysis of working vacation queueing models in communication systems.

TABLE 1. Comparison of this work with other relevant studies

Author	Model	Breakdown	Vacation	Standby server	Findings
Baba [2]	$M/PH/1$	✗	✓	✗	Steady-state probabilities Waiting time
Kumar and Chandan [11]	$M^X/E_K/1$	✗	✗	✗	Steady-state probabilities Cost analysis
Lv et al. [17]	$M/M/1$	✓	✗	✓	Steady-state probabilities Performance measures
Marin and Buló [18]	$M/PH/1$	✗	✗	✗	Product-form approximation
Seenivasan and Chandiraleka [25]	$M/M/1$	✓	✓	✗	Steady-state probabilities Performance measures
Vijayalakshmi et al. [29]	$M/M/1/N$	✓	✗	✗	Steady-state probabilities performance measures Cost analysis
Xu et al.[30]	$M/PH/1$	✗	✗	✗	Steady-state probabilities Performance measures
Our proposed model	$M/PH/1$	✓	✓	✓	Steady state probabilities Performance measures Cost analysis

In terms of the analysis of queueing modelling study by [3, 10], queueing models with standbys support have become important for balancing the effectiveness and availability of real-time systems. The queueing modelling of the machine maintenance problem with standbys was initially considered by Taylor and Jackson [27]. According to Natarajan [20], reliability and availability of a single unit queueing system with standbys and numerous repair facilities have been investigated. Two types of failure and standby service station for the $M/M/1$ queueing model explored by Lv et al. [17]. A $MAP/PH/1$ type queueing model including server vacations, breakdowns, repairs, and backup servers was examined by Chakravarthy et al. [4]. Through the use of a variety of system performance indicators, such as the tail probabilities of the sojourn duration for various scenarios, they carried out a qualitative assessment of the model in steady-state.

Xu et al. [30] proposed a method for analyzing the Wireless Network Protocol using the Hyper-Erlang distribution. Orlik et al. [23] introduced the Hyper-Exponential distribution to model wireless network nodes in the context of wireless networks and mobile computing. Xuan et al. [31] presented the work on two stage mixed queue with application. Kumar and Chandan [11] studied the bulk arrival and Erlang two phase queueing model with N - policy and analysed expected cost for the model. Hypo and Hyper - exponential service time distribution has analysed and the form of R matrix discussed for product-form approximation in [18]. Cost analysis and comparative study has carried out for a queueing

system in [19, 29].

In order to overcome the limitations of traditional queueing models and address the complexities of modern service systems, we present a comprehensive queueing model that incorporates multiple types of service within a single server configuration. Our objective is to develop an adaptable framework capable of effectively managing the inherent complexities of service systems to achieve greater performance, minimized service interruptions, and improved quality of service. As demonstrated in comparison table 1, our model significantly outperforms traditional models, validating our approach and achieving the intended goals of this research by using the matrix geometric method [12, 13, 22].

The layout of this article is as follows: The description of the mathematical model and the creation of the model's matrix are presented in section 2 and 3 respectively. In section 4, we describe the stability condition and presents an analysis of the steady-state probability vector for the proposed model. In section 5, the performance measures are examined. Section 6 presents the real life implementation of the proposed model and covers several numerical examples and graphical representations in section 7. Section 8 presents the cost analysis of the model and the conclusion is provided in section 9.

The following conventional notations are used throughout the paper:

- \otimes refers the kronecker product.
- e is the column vector whose entries are ones and its dimension will be taken accordingly.
- I_n indicates the identity matrix of order n .

2. DESCRIPTION OF THE MATHEMATICAL MODEL

- We assume that the customers arrive into the system according to Poisson process.
- The system has a single server with three types of service and the service process follows Hyper-Erlang distribution. When the customers enter the system, they can select any type of service and the server provide services in phases for each type.
- The main server goes on a single working vacation when the system is empty and provides service at a reduced rate whenever the customers arrive into the system at that time. The duration of working vacation is modeled with an exponential distribution with rate α . The main server may get breakdown during the busy or working vacation period with rate ψ and is sent for repair immediately, and at that moment the standby server instantaneously takes over the service. When a repair is completed, the very next moment the main server takes over the service from the standby server. Also, make a note that the standby server does not breakdown. The repair times are exponentially distributed with rate ϵ .
- The duration of service offered by the main server in busy period, working vacation and also the standby server are considered to be Hyper-Erlang distribution denoted by $(\delta \otimes \beta, \gamma \otimes V)$, $(\delta \otimes \beta, \gamma \otimes \tau V)$ and $(\delta \otimes \beta, \gamma \otimes \zeta V)$, of order n respectively. Note that the service rate of the main server in busy period, working vacation period and the standby server are given by η , $\tau\eta$ and $\zeta\eta$ respectively, where $\eta = [(\delta \otimes \beta)(-\gamma \otimes V)^{-1}e]^{-1}$, $\delta = (\delta_1, \delta_2, \delta_3)$, $\beta = (1, 0, 0)$.

$$\gamma = \begin{bmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{bmatrix},$$

- Moreover, we denote V^0 as the column vector satisfying $Ve_3 + V^0 = 0$.

$$V = \begin{bmatrix} -\mu_1 & \mu_1 & 0 \\ 0 & -\mu_2 & \mu_2 \\ 0 & 0 & -\mu_3 \end{bmatrix}, \quad V^0 = \begin{bmatrix} 0 \\ 0 \\ \mu_3 \end{bmatrix}$$

where $\mu_1 = \mu_2 = \mu_3 = 3\mu$, since the service time of each type follows Erlang distribution.

3. THE QUASI BIRTH-DEATH PROCESS

We now define a number of variables needed to describe the model under study.

- $N(t)$ indicates the number of customers in the system at time t .
- $S(t)$ indicates the state of the server at time t , where

$$S(t) = \begin{cases} 1, & \text{main server is busy} \\ 2, & \text{main server under repair and standby server on progress} \\ 3, & \text{main server under single working vacation} \end{cases}$$

- $B(t)$ indicates the phase of service process at time t .
- $J(t)$ indicates the type of service at time t .

Then $\{(N(t), S(t), B(t), J(t)), t \geq 0\}$, is a continuous-time Markov chain with state space Ω as follows:

$$\Omega = S(0) \cup S(i)$$

where,

$$S(0) = \{(0, s) : 1 \leq s \leq 3\}$$

$$\text{for } i \geq 1, \quad S(i) = \{(i, s, b, j) : 1 \leq s \leq 3; 1 \leq b \leq 3; 1 \leq j \leq 3\}.$$

The generator matrix is as below,

$$Q = \begin{bmatrix} B_{00} & B_{01} & \dots & & & & & & & \\ B_{10} & A_1 & A_0 & \dots & & & & & & \\ \vdots & A_2 & A_1 & A_0 & \dots & & & & & \\ & \vdots & A_2 & A_1 & A_0 & \dots & & & & \\ & & \vdots & \vdots & \vdots & & & & & \end{bmatrix}$$

where the block matrices in Q are as follows:

$$B_{00} = \begin{bmatrix} -(\lambda + \psi) & \psi & 0 \\ \epsilon & -(\lambda + 2\epsilon) & \epsilon \\ \alpha & \psi & -(\lambda + \alpha + \psi) \end{bmatrix},$$

$$B_{01} = \begin{bmatrix} \delta_1 \lambda \beta & \delta_2 \lambda \beta & \delta_3 \lambda \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_1 \lambda \beta & \delta_2 \lambda \beta & \delta_3 \lambda \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_1 \lambda \beta & \delta_2 \lambda \beta & \delta_3 \lambda \beta \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_1 V^0 & \gamma_2 V^0 & \gamma_3 V^0 & \zeta \gamma_1 V^0 & \zeta \gamma_2 V^0 & \zeta \gamma_3 V^0 & \tau \gamma_1 V^0 & \tau \gamma_2 V^0 & \tau \gamma_3 V^0 & 0 \end{bmatrix}^T,$$

$$A_1 = \begin{bmatrix} U_1 & 0 & 0 & \psi I_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & U_2 & 0 & \psi I_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & U_3 & 0 & 0 & \psi I_3 & 0 & 0 & 0 \\ \epsilon I_3 & 0 & 0 & U_4 & 0 & 0 & \epsilon I_3 & 0 & 0 \\ 0 & \epsilon I_3 & 0 & 0 & U_5 & 0 & 0 & \epsilon I_3 & 0 \\ 0 & 0 & \epsilon I_3 & 0 & 0 & U_6 & 0 & 0 & \epsilon I_3 \\ \alpha I_3 & 0 & 0 & \psi I_3 & 0 & 0 & U_7 & 0 & 0 \\ 0 & \alpha I_3 & 0 & 0 & \psi I_3 & 0 & 0 & U_8 & 0 \\ 0 & 0 & \alpha I_3 & 0 & 0 & \psi I_3 & 0 & 0 & U_9 \end{bmatrix}, A_0 = \begin{bmatrix} \lambda I_9 & 0 & 0 \\ 0 & \lambda I_9 & 0 \\ 0 & 0 & \lambda I_9 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \delta_1 \gamma_1 V^0 \beta & \delta_2 \gamma_1 V^0 \beta & \delta_3 \gamma_1 V^0 \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_1 \gamma_2 V^0 \beta & \delta_2 \gamma_2 V^0 \beta & \delta_3 \gamma_2 V^0 \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_1 \gamma_3 V^0 \beta & \delta_2 \gamma_3 V^0 \beta & \delta_3 \gamma_3 V^0 \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_1 \zeta \gamma_1 V^0 \beta & \delta_2 \zeta \gamma_1 V^0 \beta & \delta_3 \zeta \gamma_1 V^0 \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_1 \zeta \gamma_2 V^0 \beta & \delta_2 \zeta \gamma_2 V^0 \beta & \delta_3 \zeta \gamma_2 V^0 \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_1 \zeta \gamma_3 V^0 \beta & \delta_2 \zeta \gamma_3 V^0 \beta & \delta_3 \zeta \gamma_3 V^0 \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_1 \tau \gamma_1 V^0 \beta & \delta_2 \tau \gamma_1 V^0 \beta & \delta_3 \tau \gamma_1 V^0 \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_1 \tau \gamma_2 V^0 \beta & \delta_2 \tau \gamma_2 V^0 \beta & \delta_3 \tau \gamma_2 V^0 \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_1 \tau \gamma_3 V^0 \beta & \delta_2 \tau \gamma_3 V^0 \beta & \delta_3 \tau \gamma_3 V^0 \beta & 0 \end{bmatrix}$$

where

$$U_g = \begin{cases} \gamma_g V - (\psi + \lambda)I_3, & 1 \leq g \leq 3; \\ \zeta \gamma_{g-3} V - (2\epsilon + \lambda)I_3, & 4 \leq g \leq 6; \\ \tau \gamma_{g-6} V - (\alpha + \psi + \lambda)I_3, & 7 \leq g \leq 9; \end{cases}$$

4. THE STEADY-STATE ANALYSIS

In this section we will discuss the steady-state analysis of the model under study. We first derive the stability condition for the system to reach a steady state.

4.1. The Stability Condition. Let the stationary probability vector π be partitioned as

$$\pi = \{(\pi_{l1}, \pi_{l2}, \pi_{l3}), 1 \leq l \leq 9\}$$

of the irreducible generator $A = A_0 + A_1 + A_2$.

$$A = \begin{bmatrix} T_1 & \delta_2 \gamma_1 V^0 \beta & \delta_3 \gamma_1 V^0 \beta & \psi I_3 & 0 & 0 & 0 & 0 & 0 \\ \delta_1 \gamma_2 V^0 \beta & T_2 & \delta_3 \gamma_2 V^0 \beta & 0 & \psi I_3 & 0 & 0 & 0 & 0 \\ \delta_1 \gamma_3 V^0 \beta & \delta_2 \gamma_3 V^0 \beta & T_3 & 0 & 0 & \psi I_3 & 0 & 0 & 0 \\ \epsilon I_3 & 0 & 0 & T_4 & \delta_2 \zeta \gamma_1 V^0 \beta & \delta_3 \zeta \gamma_1 V^0 \beta & \epsilon I_3 & 0 & 0 \\ 0 & \epsilon I_3 & 0 & \delta_1 \zeta \gamma_2 V^0 \beta & T_5 & \delta_3 \zeta \gamma_2 V^0 \beta & 0 & \epsilon I_3 & 0 \\ 0 & 0 & \epsilon I_3 & \delta_1 \zeta \gamma_3 V^0 \beta & \delta_2 \zeta \gamma_3 V^0 \beta & T_6 & 0 & 0 & \epsilon I_3 \\ \alpha I_3 & 0 & 0 & \psi I_3 & 0 & 0 & T_7 & \delta_2 \tau \gamma_1 V^0 \beta & \delta_3 \tau \gamma_1 V^0 \beta \\ 0 & \alpha I_3 & 0 & 0 & \psi I_3 & 0 & \delta_1 \tau \gamma_2 V^0 \beta & T_8 & \delta_3 \tau \gamma_2 V^0 \beta \\ 0 & 0 & \alpha I_3 & 0 & 0 & \psi I_3 & \delta_1 \tau \gamma_3 V^0 \beta & \delta_2 \tau \gamma_3 V^0 \beta & T_9 \end{bmatrix}$$

where,

$$T_g = \begin{cases} \gamma_g (V + \delta_k V^0 \beta) - \psi I_3, & 1 \leq g \leq 3; \\ \gamma_{g-3} (\zeta V + \delta_k \zeta V^0 \beta) - 2\epsilon I_3, & 4 \leq g \leq 6; \\ \gamma_{g-6} (\tau V + \delta_k \tau V^0 \beta) - (\alpha + \psi) I_3, & 7 \leq g \leq 9; \end{cases}$$

The steady state equations should satisfy $\pi A = 0$, $\pi e = 1$ and get stability condition using $\pi A_0 e < \pi A_2 e$, from [21].

4.2. Steady-state probability vector. Let $P = (p_0, p_1, p_2, \dots)$ be the steady-state probability vector associated with Q .

Let

$$\begin{aligned} p_0 &= (p_{01}, p_{02}, p_{03}), \\ p_i &= (p_{i11}, p_{i12}, p_{i13}, p_{i21}, p_{i22}, p_{i23}, p_{i31}, p_{i32}, p_{i33}) \quad \text{for } i \geq 1. \end{aligned}$$

such that

$$PQ = 0, Pe = 1 \quad (4.1)$$

where e is a column vector of 1's of appropriate dimension.

Furthermore, after the model's stability requirement is met, the steady-state probability vector P could be calculated using the following equation:

$$p_i = p_1 R^{(i-1)} \quad \text{for } i \geq 2 \quad (4.2)$$

where R is the minimal non-negative solution of the matrix quadratic equation as

$$R^2 A_2 + R A_1 + A_0 = 0 \quad (4.3)$$

defined by [21]. The computation of the R matrix can be carried out iteratively using (4.3).

Substituting the equation (4.2) in (4.1), we have

$$p_0 B_{01} + p_1 (A_1 + R A_2) = 0 \quad (4.4)$$

and to uniquely determine the probabilities p_0 and p_1 , we further need the normalization equation. i.e.,

$$p_0 e + p_1 (I - R)^{-1} e = 1 \quad (4.5)$$

The vectors p_0, p_1, p_2, \dots are obtained by solving the equations (4.4) and (4.5).

5. MEASURES OF SYSTEM PERFORMANCE

In this section, to understand the qualitative aspects of the queueing model under study, we demonstrate the performance measure as follows,

- The probability that the main server is busy with no customers (P_0) = $p_0 e$
- The probability that the main server is busy (P_b) = $p_1 (I - R)^{-1} e_1$
- The probability that the standby server is busy while the main server is under repair (P_s) = $p_1 (I - R)^{-1} e_2$
- The probability that the main server is on working vacation (P_v) = $p_1 (I - R)^{-1} e_3$
- Expected system size (E_s) = $p_1 (I - R)^{-2} e$

6. APPLICATION ON AN IMAGE COMPRESSION SYSTEM

Our proposed queueing model can be effectively mapped to practical scenarios in the context of an image compression system. The single server is represented by the image compression server (main server), which provides three type of services such as high, medium and low quality compression. The server transition between three states: busy state (Regular Image Compression Operation), single working vacation (low demand and maintenance), and breakdown state (Server Failure and Standby Server Activation). During the busy state, the image compression system operates normally, processing images through three phases: transformation (using techniques like Discrete Cosine Transform (DCT) or wavelet transform), quantization (reducing the number of bits required to store transformed coefficients), and entropy coding (using methods such as Huffman coding or arithmetic coding).

When there is no demand for compression, the server enters a working vacation state, performing background tasks like optimizing compression parameters or conducting maintenance. This state ensures that the server remains productive even without handling incoming compression tasks. If data arrives during this state, the server processes image compression tasks as same as in busy state but at a slower speed.

In either the busy or working vacation state, if the main image compression server crashes due to a hardware or software issue, a standby server is activated to continue compressing images. Although the standby server may operate at a reduced speed, it ensures that service is still provided. The standby server begins with the transformation phase, picking up where the main server left off, continues with the quantization process to maintain data integrity, and completes the entropy coding phase, ensuring that the image compression process is not disrupted.

7. NUMERICAL EXAMPLES

In this section, we examine the performance of a single-server three phase queueing system with three types of service. We interpretate the queueing model with the application on the image compression system. We provide the fixed parameter numerical values that describe the queueing model under various scenarios by systematically varying its parameters. To ensure stability, we carefully fix the parameters according to the stability condition. Specifically, we set the parameters as follows: $\lambda = 1$, $\mu = 4$, $\epsilon = 3.5$, $\psi = 2.5$, $\zeta = 0.5$, $\tau = 0.7$, $\delta = (0.5, 0.3, 0.2)$, $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\gamma_3 = 0.5$ and $\alpha = 2$.

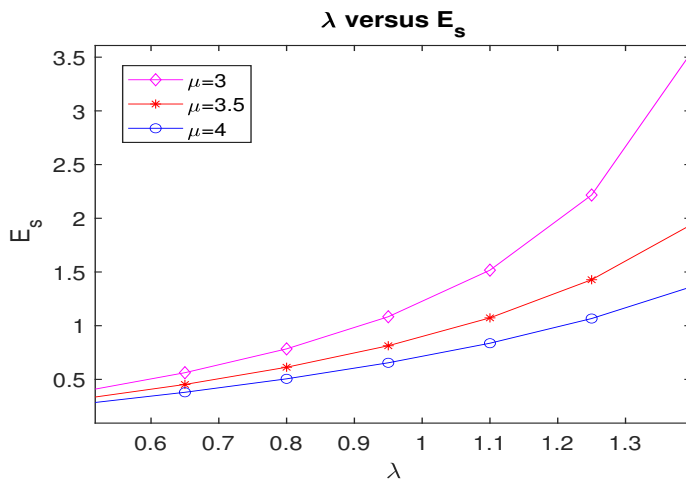


FIGURE 2. Varying λ on E_s

7.1. Example: 1. In figure 2, we investigate the behaviour of arrival rate of image data on expected system size. As the arrival rate increases, the expected system size increases. This is because, with a fixed service rate of image compression, increasing the arrival rate leads to the more image data in the system. The above statements holds for various service rates $\mu = 3, 3.5$ and 4.

7.2. Example: 2. In figure 3, we investigate the influence of service rate of image compression on expected system size. As the service rate increases, the expected system size decreases. This is because a higher service rate means a shorter service time duration for image compression, which leads to increased server availability. Consequently, Images are compressed more quickly, resulting in a decrease

in the expected system size. The above statements holds for various arrival rates of images $\lambda = 0.5, 0.8$ and 1.1 .

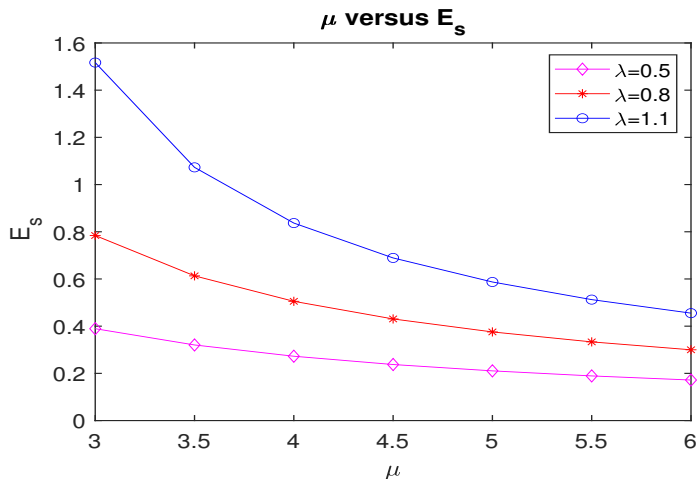


FIGURE 3. Varying μ on E_s

7.3. Example: 3. In figure 4, we picturise the influence of the breakdown rate and repair rate of the Image compression server on the expected system size. As the repair rate increases, the server returns to image compression operation more quickly, which reduces the expected system size. Conversely, as the breakdown (i.e., hardware or software issue) rate increases, the expected system size also increases because frequent breakdowns occur. When the repair rate decreases, the server remains unavailable for longer periods, further increasing the expected system size.

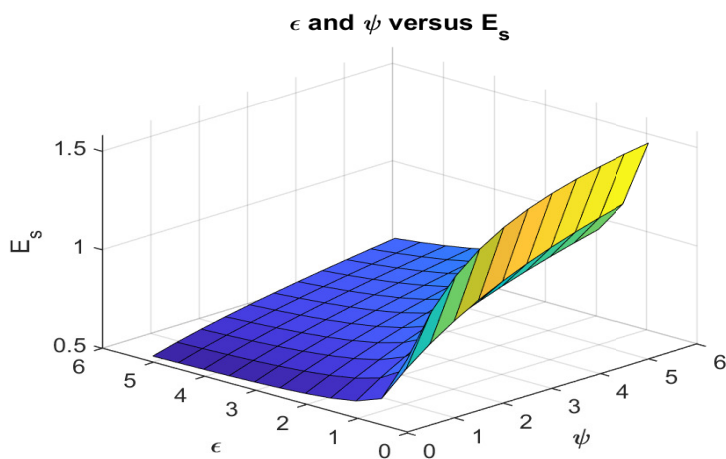


FIGURE 4. Varying ϵ and ψ on E_s

7.4. **Example: 4.** In figures 5, we investigate the influence of the working vacation rate on the expected size of the system. As the vacation rate increases, the expected system size decreases. This is due to an increase in the vacation rate which implies a decrease in the duration of vacation time of the image compression server for various arrival rate $\lambda = 0.5, 0.8$ and 1.1 .

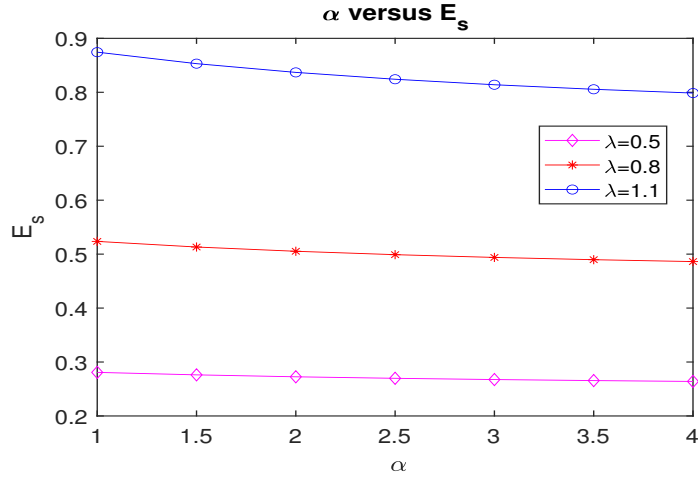


FIGURE 5. Varying α on E_s

7.5. **Example: 5.** In figures 6, we investigate the influence of the service rates $\gamma_1, \gamma_2, \gamma_3$ of high, medium and low image compression with respect to μ on the expected size of the system, respectively. We fix the value of the parameters for the relation as follows

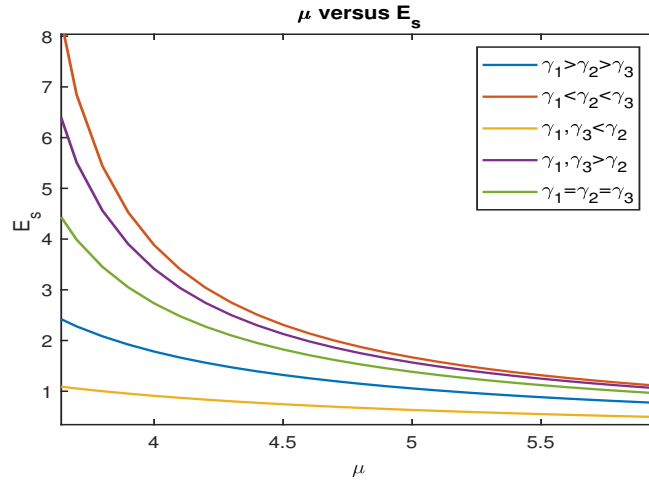


FIGURE 6. Effect of $\gamma_1, \gamma_2, \gamma_3$ on E_s

$$\left\{ \begin{array}{ll} \gamma_1 > \gamma_2 > \gamma_3, & \gamma_1 = 1, \gamma_2 = 0.7, \gamma_3 = 0.5 \\ \gamma_1 < \gamma_2 < \gamma_3, & \gamma_1 = 0.5, \gamma_2 = 0.7, \gamma_3 = 1 \\ \gamma_1, \gamma_3 < \gamma_2, & \gamma_1 = 0.5, \gamma_2 = 1, \gamma_3 = 0.7 \\ \gamma_1, \gamma_3 > \gamma_2, & \gamma_1 = 0.7, \gamma_2 = 0.5, \gamma_3 = 1 \\ \gamma_1 = \gamma_2 = \gamma_3, & \gamma_1 = \gamma_2 = \gamma_3 = 1 \end{array} \right.$$

We get the results from the figure 6 as the service rate of three types are equal and higher, then we get the lowest E_s than other relations.

8. COST ANALYSIS

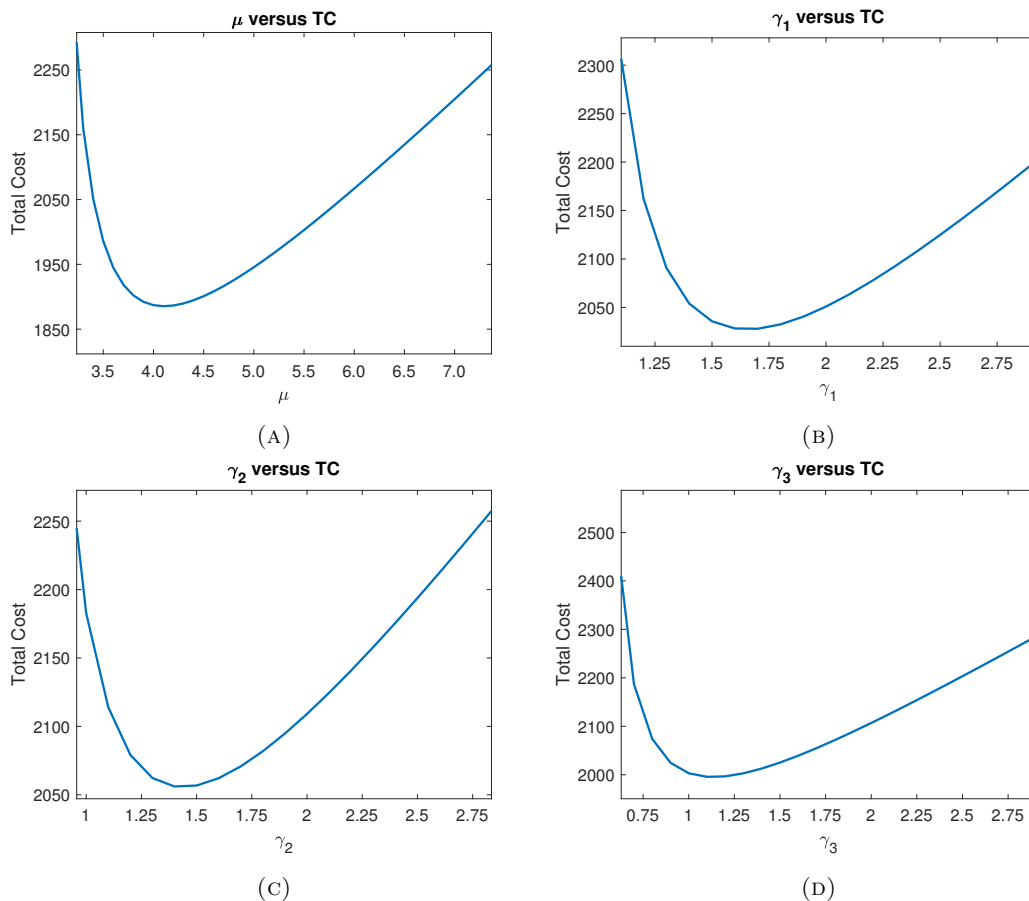
For a queueing system, it is crucial to determine a optimal cost. For an excellent survey, see Tadj and Choudhury [26]. In this model, the service rate μ , γ_1 , γ_2 , γ_3 are decision variables. The goal is to determine the optimal values of the service rates denoted as μ^* , γ_1^* , γ_2^* , γ_3^* , to minimize the expected total cost per unit time. To find the optimal values, it is necessary to show that the cost function is convex. We fix the parameters as follows: $\lambda = 2$, $\mu = 3$, $\psi = 3.5$, $\epsilon = 5$, $\zeta = 0.6$, $\tau = 0.8$, $\delta = (0.5, 0.3, 0.2)$, $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\gamma_3 = 0.5$ and $\alpha = 2$. Let us define the cost elements as follows:

- C_h holding cost per unit time for each customer present in the system.
- C_0 cost per unit time when the server is idle.
- C_b cost per unit time when the server is busy.
- C_s cost per unit time when the server is breakdown.
- C_v cost per unit time when the server is on vacation.
- C_1 cost per unit time of providing a service rate for busy period.
- C_2 cost per unit time of providing a service rate for breakdown period.
- C_3 cost per unit time of providing a service rate for SWV period.

Employing the definitions of each cost element listed above and their corresponding system characteristics, the expected total cost (TC) function is given by:

$$\begin{aligned} TC = & E_s C_h + P_b C_b + P_0 C_0 + P_v C_v + P_s C_s + \psi C_r + \alpha C_v \\ & + (\gamma_1 + \gamma_2 + \gamma_3)(\mu_1 + \mu_2 + \mu_3)(C_1 + \zeta C_2 + \tau C_3) \end{aligned}$$

Cost analysis can play a significant role in improving system design. For illustration, when compression server failures or faults are inevitable, the service rates are considered as decision variables to minimize costs for image compression. Here the service rate depends on μ , γ_1 , γ_2 and γ_3 . The range of the decision variable is configured between 3 to 8 for μ and 1 to 3 for remaining variables. The expected cost function plotted in 7 (A) - (D) demonstrates the impact of services on TC. We summarize the numerical results for the optimal decision variables μ^* , γ_1^* , γ_2^* , γ_3^* and the optimal cost TC^* . It is observed that the expected total cost initially decreases and then starts increasing with the growth of fixed values of other parameters. The convex nature of the cost function with respect to these decision variables indicates the trend for the optimum cost by increasing the service domain of the image data. The rate μ is common for all the type of service, the optimal cost for (μ^*, TC^*) obtained at (4.1, 1885.46) in 7 (A). Specifically, the optimal cost for high image compression service (γ_1^*, TC^*) obtained at (1.7, 2028.01), medium image compression service (γ_2^*, TC^*) obtained at (1.4, 2056.04) and low image compression service (γ_3^*, TC^*) obtained at (1.1, 1995.46) in 7 (B), 7 (C) and 7 (D), respectively.

FIGURE 7. $\mu, \gamma_1, \gamma_2, \gamma_3$ vs TC

9. CONCLUSION

This paper presents an investigation into the steady-state solution of a single server Markovian queueing model with three types of service, encompassing breakdown, repair, and standby server. Utilizing the matrix-geometric method, stability condition and significant performance measures were obtained. The study focuses on refining the model to support practical applications, such as Image Compression Systems, by analyzing the three types of service through phases. Impactness of parameters on image compression system were analysed in numerical results and conducting a comprehensive cost analysis. The findings provide valuable insights into the cost-effectiveness of system performance, offering a robust framework for image compression scenario. This research contributes to the development of more efficient and reliable queueing systems, advancing strategic decision-making.

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