

THE IMPACT OF POLLUTION REDUCTION ON THE OPTIMAL HARVESTING STRATEGY IN A SEASONALLY CHANGING AND POLLUTED ENVIRONMENT

SIMON D. ZAWKA AND P. D. N. SRINIVASU

ABSTRACT. This paper deals with the optimal harvesting of single-species populations in a polluted and seasonally varying environment. The presence of pollution negatively impacts resource growth, and as a result, the economic rent from harvest suffers. Assuming that pollution is inevitable, the harvester, who has a fixed flow of funds for harvesting activity, intends to invest the unused funds left over after harvesting in pollution control, expecting some increment in the economic rent. Thus, optimal harvesting problems in two different scenarios have been studied, and the solutions are compared to evaluate the impact of investing in pollution reduction on revenues. Pontryagin's Maximum Principle and the concept of blocked intervals are used to solve the aforesaid problems. Findings of the study indicate that funding pollution control programs should be considered as an alternative to promote conservation of resources and enhance net economic rent for harvesting systems impacted by pollution and seasonal fluctuations.

1. INTRODUCTION

The continuous developmental activities to facilitate ever-growing human population has become the major cause for increase in environmental pollution. This inevitable evil has direct impact on sustainability of the renewable resources that support human population and has indirect impact on the revenues generated through these resources. For example, pollutants that are released into the water bodies from some external sources significantly impact the growth of renewable resource such as fish. The effect of pollutants on the survival of single species and interacting populations is extensively studied in the literature [18, 19, 14, 38, 16, 33, 25, 26, 8, 22, 31, 32]. On the other hand, the influence of seasonal variations on the dynamics of single-species populations has received considerable attention in the scientific community [13, 11, 2, 37, 21]. The authors in [38, 23, 12, 39, 35, 20] presented the effect of environmental pollution on the harvested single-species populations. Several studies discussed the optimal exploitation of renewable resources in the presence of seasonal changes in the environment [15, 17, 4, 6, 28].

Majority of the literature mentioned above emphasize that pollution, seasonal fluctuations and unregulated harvesting as the major threats to the sustenance of resources. Despite the alarming impacts, investigations on the optimal utilization of the stock (which suffers from all the three threats simultaneously) are rarely found in the existing literature. While the study [29] underlines the influence of pollution control on harvesting a single species (in the absence of seasonal variation), [6] presents the optimal harvesting of a single-species population in a seasonally varying environment that doesn't involve environmental pollution. The authors in [28] discussed optimal harvesting of a single-species

Received by the editors 31 January 2024; accepted 7 June 2024; published online 13 June 2024.

2020 *Mathematics Subject Classification.* 92D25, 34H05.

Key words and phrases. Seasonal change, polluted environment, optimal harvesting, pollution reduction, blocked intervals.

population in a seasonally varying environment and highlighted the crucial role played by restrictions on the harvesting effort to conserve resources.

The current study deals with the optimal harvesting of single-species populations in a polluted, seasonally changing environment involving restrictions on harvest effort and pollution reduction measures. To the best of our knowledge, there is no literature that addresses this issue. In this study, we focus on the comparison of optimal harvesting problems with and without pollution control measures. First, we study the problem by considering seasonality, environmental pollution, and restrictions on effort capacity. Then, the model is modified to accommodate pollution-reduction measures within limited financial resources. The optimal solutions are constructed for the two problems using the Pontryagin maximum principle and the concept of blocked intervals. This study helps the harvester to make an informed decision on the strategy to be followed to improve his revenues.

The paper is organized as follows: In Section 2, we introduce two optimal harvesting problems to be considered in this study. The singular solutions of these two problems are constructed in Section 3. Section 4 deals with the optimal solutions of the problems in two different situations, i.e., with and without binding constraints on the control variable. Section 5 presents several numerical examples to demonstrate the significant outcomes of the study. Finally, discussion and conclusions are presented in Section 6 and Section 7, respectively.

2. THE PROBLEMS

Let us consider harvesting a renewable resource, such as a fishery, under the influence of environmental pollution and seasonal variations. Without the effect of pollution, resource growth is defined by (ref. [6]),

$$\frac{dx}{dt} = x(a(t) - b(t)x) - qEx, \quad (2.1)$$

where a, b are continuously differentiable positive periodic functions of the same period (say ω), and q and E represent the catchability coefficient and the fishing effort (in vessels), respectively. The model (2.1) can be rewritten as

$$\frac{dx}{dt} = a(t)x \left(1 - \frac{x}{K(t)} \right) - qEx, \quad (2.2)$$

where $K(t) = \frac{a(t)}{b(t)}$. Here, $a(t)$ and $K(t)$, respectively, represent the intrinsic growth rate and environmental carrying capacity (with no pollution effects).

Now, consider the presence of pollution (due to various external sources such as industrial chemicals, agricultural runoff, sewage, and so on) in the ecosystem. We assume that pollution impacts both the intrinsic growth rate and the carrying capacity of the environment (ref. [29]). Thus, resource growth, with the effects of pollution and seasonal change, is defined by

$$\frac{dx}{dt} = a(t)x \left(1 - \frac{x}{K(t)} \right) - \gamma(t)x - qEx, \quad (2.3)$$

where γ is a twice continuously differentiable positive function of period ω that stands for the death rate of the species due to pollution. Setting $a_\gamma(t) = a(t) - \gamma(t)$ and $K_\gamma(t) = \frac{a(t) - \gamma(t)}{b(t)}$, (2.3) takes the following form:

$$\frac{dx}{dt} = a_\gamma(t)x \left(1 - \frac{x}{K_\gamma(t)} \right) - qEx. \quad (2.4)$$

Here, a_γ and K_γ , respectively, stand for the growth rate and environmental carrying capacity with pollution effects.

Let's assume that the owner has availability of \mathcal{M} amount of fund at each time t for the harvesting activity. The informed fund may be a salary or income from any source that is guaranteed at each

time t . The owner wishes to utilize the available continuous stream of funds for the harvesting activity, thereby maximizing the net economic rent. Suppose E represents the fishing effort (in vessels) with the natural constraint $0 \leq E^{min} \leq E \leq E^{max}$, i.e., the availability of vessels for fishing activity is bounded below and above by E^{min} and E^{max} , respectively. Let c be the cost per unit fishing effort, and it is assumed that the available fund accommodates hiring the maximum allowable vessels, i.e., $cE^{max} \leq \mathcal{M}$. If $Y(t)$ denotes the yield at each time t , we have

$$Y(t) = qE(t)x(t),$$

and the instantaneous net revenue is given by

$$\tau qE(t)x(t) - cE(t),$$

where τ is the price per unit harvest. Thus, the discounted net revenue over the infinite time horizon, denoted by PV , is given by

$$PV = \int_0^{\infty} e^{-\delta t} (\tau q x(t) - c) E(t) dt,$$

where δ is the discount rate. The aim is to maximize the discounted total net revenue subject to the given resource dynamic equation and restrictions on the size of fishing efforts. Formally expressed, the optimal harvesting problem is as follows:

$$\max_{\{E\}} \int_0^{\infty} e^{-\delta t} (\tau q x - c) E dt \quad (2.5a)$$

$$\text{s.t: } \frac{dx}{dt} = a_{\gamma}(t)x \left(1 - \frac{x}{K_{\gamma}(t)}\right) - qEx, \quad x(0) = x_0 > 0 \quad (2.5b)$$

$$E^{min} \leq E \leq E^{max}. \quad (2.5c)$$

Clearly, (2.5) is an optimal control problem with the control variable E and the state variable x . Solving (2.5) amounts to finding out the optimal pair (E^*, x^*) so that the integral (2.5a) is as large as possible.

The adverse effects of pollution on resource growth ultimately result in depleted revenue due to a fall in stock. Hence, any effort made, such as reducing the pollution effect on the resource to help improve the stock level, could result in enhanced economic revenue. Therefore, we consider investing in pollution reduction as an alternative to help improve the resource stock level and, hence, indirectly, the net revenue.

The problem (2.5) involves utilizing the amount $cE(t)$ out of the available fund \mathcal{M} for resource harvesting. Hence, we contemplate diverting the remaining fund $\mathcal{M} - cE(t)$ towards pollution reduction. Incorporating this modifies the problem (2.5) to the following:

$$\max_{\{E\}} \int_0^{\infty} e^{-\delta t} (\tau q EX - \mathcal{M}) dt \quad (2.6a)$$

$$\text{s.t: } \frac{dX}{dt} = \bar{a}_{\gamma}(t)X \left(1 - \frac{X}{\bar{K}_{\gamma}(t)}\right) - qEX, \quad X(0) = X_0 > 0 \quad (2.6b)$$

$$E^{min} \leq E \leq E^{max}, \quad (2.6c)$$

where

$$\begin{aligned} \bar{a}_{\gamma}(t) &= a(t) - \gamma(t) + \alpha(\mathcal{M} - cE)\gamma(t), \\ \bar{K}_{\gamma}(t) &= \frac{a(t)}{b(t)} - \frac{\gamma(t)}{b(t)} + \frac{\alpha(\mathcal{M} - cE)\gamma(t)}{b(t)}. \end{aligned}$$

Note that the overall cost in (2.6) is \mathcal{M} , which is the sum of harvesting cost (cE) and pollution reduction cost ($\mathcal{M} - cE$). A description of the parameters and constants involved in the problems (2.5) and (2.6)

is presented in Table 1. The term $\alpha(\mathcal{M} - cE)\gamma(t)$ represents enhancement in the growth rate of the species due to effort made to reduce the pollution in the environment utilising the fund $(\mathcal{M} - cE)$. Here we assume $0 < \alpha(\mathcal{M} - cE) < 1$ for $E \in [E^{min}, E^{max}]$. That is, pollution cannot be totally removed from the environment even if the maximum possible amount of money (i.e., $\mathcal{M} - cE^{min}$) is devoted to pollution reduction.

Before proceeding to solve the problems (2.5) and (2.6), we discuss the existence and global stability behavior of solutions for (2.5b) and (2.6b). Clearly, (2.5b)((2.6b)) always admits a trivial solution $x(t) \equiv 0$ ($X(t) \equiv 0$). To discuss about the non-trivial solution, let's consider the notation

$$\langle g \rangle = \frac{1}{\omega} \int_0^\omega g(t) dt.$$

The following lemma discusses the existence and global stability of the non-trivial solution of (2.5b), whose proof directly follows from [6].

Lemma 2.1. *Consider the differential equation (2.5b) with the given initial condition. Let $E(t)$ be ω -periodic.*

- a. *If $\langle a - \gamma - qE \rangle > 0$, then (2.5b) admits a positive globally asymptotically stable periodic orbit of period ω .*
- b. *If $\langle a - \gamma - qE \rangle < 0$, then all solutions of (2.5b) with the given initial condition approach the trivial solution $x(t) \equiv 0$ asymptotically (i.e., the trivial solution is globally asymptotically stable).*
- c. *If $0 \leq E_1(t) < E_2(t)$ be two ω -periodic functions such that*

$$\langle a - \gamma - qE_2 \rangle = \frac{1}{\omega} \int_0^\omega (a(s) - \gamma(s) - qE_2(s)) ds > 0$$

and $x_{E_1}(t)$ and $x_{E_2}(t)$ are the non-trivial globally asymptotically stable ω -periodic solutions of (2.5b) with $E(t) = E_1(t)$ and $E(t) = E_2(t)$, respectively, then $0 < x_{E_2}(t) < x_{E_1}(t)$ for all t .

A similar statement can be given for (2.6b) with the given initial condition. In the sections to follow, we discuss the construction of optimal singular solutions and the optimal solutions for the aforesaid problems (2.5) and (2.6).

3. CHARACTERIZATION OF SINGULAR SOLUTION

In this section, we construct the singular effort functions and the associated singular stock paths pertaining to (2.5) and (2.6). First, let's consider problem (2.5). The current value Hamiltonian (\mathcal{H}) is given by

$$\mathcal{H}(x, E, \lambda, t) = (\tau qx - c)E + \lambda[x(a - bx) - \gamma x - qEx]$$

and the adjoint differential equation is

$$\dot{\lambda} = \delta\lambda - \tau qE - \lambda(a - 2bx - \gamma - qE). \quad (3.1)$$

The partial derivate of \mathcal{H} with respect to E gives $\mathcal{H}_E = (\tau - \lambda)qx - c$, and hence the switching function, which we denote by $s(t)$, is given by

$$s(t) = (\tau - \lambda)qx - c. \quad (3.2)$$

In the case of singular solution, we must have $s(t) = 0$, i.e.,

$$(\tau - \lambda)qx - c = 0, \quad (3.3)$$

provided that $\tau - \lambda > 0$. Using (2.5b), (3.1), (3.3), and after simplification, the singular effort function (which we denote by $E_s(t)$), is

$$E_s(t) = \frac{1}{q} \left(a(t) - \gamma(t) - b(t)x_s - \frac{\tau q(a'(t) - \gamma'(t)) + b'(t)c - 2b'(t)\tau q x_s}{4b(t)\tau q x_s - b(t)c - \tau q(a(t) - \gamma(t) - \delta)} \right), \quad (3.4)$$

where $x_s(t)$ denotes the associated singular stock path which is the positive solution of the quadratic equation

$$2b(t)\tau q x^2 - (b(t)c + \tau q(a(t) - \gamma(t) - \delta))x - \delta c = 0, \quad (3.5)$$

given by

$$x_s(t) = \frac{b(t)c + \tau q(a(t) - \gamma(t) - \delta) + \sqrt{(b(t)c + \tau q(a(t) - \gamma(t) - \delta))^2 + 8b(t)\tau q \delta c}}{4b(t)\tau q}. \quad (3.6)$$

Similarly, one can verify that the singular effort function (which we denote by $\mathbf{E}_s(t)$) for (2.6) is given by

$$\mathbf{E}_s(t) = \frac{1}{q + \alpha c \gamma(t)} \left(a(t) - (1 - \alpha \mathcal{M})\gamma(t) - b(t)X_s - \frac{a'(t) - 2b'(t)X_s - \Gamma(t)}{2b(t)X_s} \right), \quad (3.7)$$

where

$$\Gamma(t) = (1 - \alpha \mathcal{M})\gamma'(t) + \frac{\alpha c \gamma''(t)}{q + \alpha c \gamma(t)} - \left(\frac{\alpha c \gamma'(t)}{q + \alpha c \gamma(t)} \right)^2$$

and $X_s(t)$ represents the associated singular stock path, given by

$$X_s(t) = \frac{1}{2b(t)} \left(a(t) - (1 - \alpha \mathcal{M})\gamma(t) - \frac{\alpha c \gamma'(t)}{q + \alpha c \gamma(t)} - \delta \right). \quad (3.8)$$

Note that the function $X_s(t)$ is positive provided that

$$a(t) - (1 - \alpha \mathcal{M})\gamma(t) - \frac{\alpha c \gamma'(t)}{q + \alpha c \gamma(t)} - \delta > 0. \quad (3.9)$$

Lemma 3.1. *The functions $E_s(t)$, $x_s(t)$, $\mathbf{E}_s(t)$ and $X_s(t)$ are periodic of period ω .*

Proof. We observe that the function $x_s(t)$ (in (3.6)) is well defined and positive since the constant term in the quadratic equation is negative. Moreover, all the coefficients in the quadratic equation (3.5) are ω -periodic. Hence, the function $x_s(t)$ is also periodic of period ω . The function $E_s(t)$ is well defined, provided that the denominator term in (3.4) is nonzero. Note that the denominator term is the derivative (w.r.t. x) of (3.5), and hence it is nonzero since it doesn't have a double zero for each t . The periodicity follows from the definition. In a parallel manner, we can establish that the functions $X_s(t)$ and $\mathbf{E}_s(t)$ are well-defined and ω -periodic. Note that the positivity of $X_s(t)$ follows from (3.9). \square

If the effort function $E_s(t)$ satisfies the control constraint (2.5c), then $E_s(t)$ becomes optimal singular control and $x_s(t)$ is the associated optimal singular stock path. Similarly, if the control constraint (2.6c) is satisfied, then $\mathbf{E}_s(t)$ becomes the optimal singular control and $X_s(t)$ is the associated optimal singular stock path [5].

Note that the effort functions $E_s(t)$ and $\mathbf{E}_s(t)$ become optimal singular controls only when these functions fall within the bounds prescribed for the control variables. However, the functions may exceed the upper bound or fall below the lower bound in some situations. In that case, the singular effort functions cannot be optimal, and hence, the optimal control must switch to an appropriate bang-bang control (E^{max} or E^{min}) on some interval of positive length [3]. K.J. Arrow in 1964 called such an interval a *blocked interval*. The reason for blocked intervals is the presence of binding constraints on the control variable. In the next section, we study the problems (2.5) and (2.6) for two scenarios where the control constraint is non-binding and binding.

4. CHARACTERIZATION OF OPTIMAL SOLUTION

In the previous section, we have constructed the singular solutions for problems (2.5) and (2.6). Here, we construct the optimal solutions, taking into consideration the initial state of the stock and constraints on the harvesting effort.

4.1. Problems with non-binding constraints. Recall the singular effort functions $E_s(t)$ and $\mathbf{E}_s(t)$ associated with (2.5) and (2.6), respectively. Suppose these singular effort functions satisfy the control constraints (2.5c) and (2.6c), respectively, i.e.,

$$E^{min} \leq E_s(t) \leq E^{max}$$

and

$$E^{min} \leq \mathbf{E}_s(t) \leq E^{max}$$

for all t . Since problems (2.5) and (2.6) are linear in the control variable, the optimal control shall be a combination of bang-bang and singular controls [10]. For the given initial conditions x_0 and X_0 , if $x_s(0) = x_0$ and $X_s(0) = X_0$, then the singular solutions become optimal solutions. Otherwise, the initial adjustment is needed so that the given initial condition is met. To do this, we use the most rapid approach strategy [10].

When $x_0 > x_s(0)$, the most rapid approach strategy is to employ the bang-bang control $E = E^{max}$ initially until the singular stock path is reached. On the other hand, if $x_0 < x_s(0)$, then the most rapid approach strategy is employing $E = E^{min}$ until the singular stock path is reached. If \tilde{E} denotes the most rapid approach strategy employed to reach the singular stock path with the minimum time $t = T_0$, then we have the following:

$$\tilde{E} = \begin{cases} E^{max}, & \text{for } 0 \leq t < T_0, \text{ when } x(0) > x_s(0), \\ E^{min}, & \text{for } 0 \leq t < T_0, \text{ when } x(0) < x_s(0). \end{cases} \quad (4.1)$$

Clearly, the bang-bang control policy given above reduces to the well-known policy given in terms of the switching function as follows:

$$\tilde{E} = \begin{cases} E^{max}, & \text{if } s(t) > 0, \\ E^{min}, & \text{if } s(t) < 0, \end{cases} \quad (4.2)$$

where $s(t)$ is the switching function defined in (3.2). Thus, the optimal pair (E^*, x^*) of the problem (2.5) is fully determined as follows. The optimal harvesting strategy is

$$E^*(t) = \begin{cases} \tilde{E}(t), & \text{for } 0 \leq t < T_0, \\ E_s(t), & \text{for } t \geq T_0, \end{cases} \quad (4.3)$$

and the associated optimal stock path is

$$x^*(t) = \begin{cases} \tilde{x}(t), & \text{for } 0 \leq t \leq T_0, \\ x_s(t), & \text{for } t \geq T_0, \end{cases} \quad (4.4)$$

where $\tilde{x}(t)$ denotes the optimal approach path which is given by

$$\tilde{x}(t) = \begin{cases} x_M(t), & \text{for } 0 \leq t \leq T_0, \text{ when } x(0) > x_s(0), \\ x_m(t), & \text{for } 0 \leq t \leq T_0, \text{ when } x(0) < x_s(0). \end{cases} \quad (4.5)$$

Here, $x_m(t)$ and $x_M(t)$ represent the solutions of (2.5b) with $E = E^{min}$ and $E = E^{max}$, respectively, which are explicitly given below:

$$x_m(t) = \frac{e^{\int_0^t (a(s) - \gamma(s) - qE^{min}) ds}}{\frac{1}{x(0)} + \int_0^t e^{\int_0^s (a(\eta) - \gamma(\eta) - qE^{min}) d\eta} b(s) ds}, \quad (4.6a)$$

$$x_M(t) = \frac{e^{\int_0^t (a(s) - \gamma(s) - qE^{max}) ds}}{\frac{1}{x(0)} + \int_0^t e^{\int_0^s (a(\eta) - \gamma(\eta) - qE^{max}) d\eta} b(s) ds}. \quad (4.6b)$$

Note that if the singular control $E_s(t)$ is employed right from the initial state, the corresponding stock path will approach the singular solution asymptotically due to the global asymptotic stability behavior of $x_s(t)$. Similarly, the optimal pair (\mathbf{E}^*, X^*) for (2.6) is given by:

$$\mathbf{E}^*(t) = \begin{cases} \tilde{\mathbf{E}}(t), & \text{for } 0 \leq t < T_0, \\ \mathbf{E}_s(t), & \text{for } t \geq T_0, \end{cases} \quad (4.7)$$

and the associated optimal stock path is

$$X^*(t) = \begin{cases} \tilde{X}(t), & \text{for } 0 \leq t \leq T_0, \\ X_s(t), & \text{for } t \geq T_0, \end{cases} \quad (4.8)$$

where

$$\tilde{\mathbf{E}} = \begin{cases} E^{max}, & \text{for } 0 \leq t < T_0, \text{ when } X(0) > X_s(0), \\ E^{min}, & \text{for } 0 \leq t < T_0, \text{ when } X(0) < X_s(0), \end{cases} \quad (4.9)$$

and

$$\tilde{X}(t) = \begin{cases} X_M(t), & \text{for } 0 \leq t \leq T_0, \text{ when } X(0) > X_s(0), \\ X_m(t), & \text{for } 0 \leq t \leq T_0, \text{ when } X(0) < X_s(0). \end{cases} \quad (4.10)$$

4.2. Problems with binding constraints. Here, we construct optimal solutions for (2.5) and (2.6) where the control constraint is binding. Recall the singular effort functions $E_s(t)$ and $\mathbf{E}_s(t)$ once again, which are given in Section 3, and consider the following notations:

$$E_s^m = \min_{t \in [t_0, t_0 + \omega]} E_s(t) \text{ and } E_s^M = \max_{t \in [t_0, t_0 + \omega]} E_s(t), \quad (4.11)$$

$$\mathbf{E}_s^m = \min_{t \in [t_0, t_0 + \omega]} \mathbf{E}_s(t) \text{ and } \mathbf{E}_s^M = \max_{t \in [t_0, t_0 + \omega]} \mathbf{E}_s(t). \quad (4.12)$$

For simplicity, we assume that the effort functions have a unique maximum and a unique minimum in one full cycle (ref. Frames A and B of Figure 1). As discussed earlier, whenever $E^{min} > E_s^m$ or $E^{max} < E_s^M$, the effort constraints become binding, and hence the optimal control cannot completely follow the singular control; instead, it has to switch to the appropriate extremal control. Due to the periodic nature of the binding constraints, the aforesaid situation occurs in a periodic manner. The main task is then determining the time, say $t = t_1^m$ (or $t = t_1^M$), when the optimal control switches from singular arc to E^{min} (or when the optimal control switches from singular curve to E^{max}) and $t = t_2^m$ (or $t = t_2^M$) the time at which it switches back to the singular arc, resulting in the blocked interval $[t_1^m, t_2^m]$ (or $[t_1^M, t_2^M]$). Due to the periodic nature of the singular curve, we first look for the optimal periodic solutions. Hence, we focus on the following modified optimal control problems:

$$\max_{E^{min} \leq E(t) \leq E^{max}} \int_{t_0}^{t_0 + \omega} e^{-\delta t} (\tau q x - c) E dt \quad (4.13a)$$

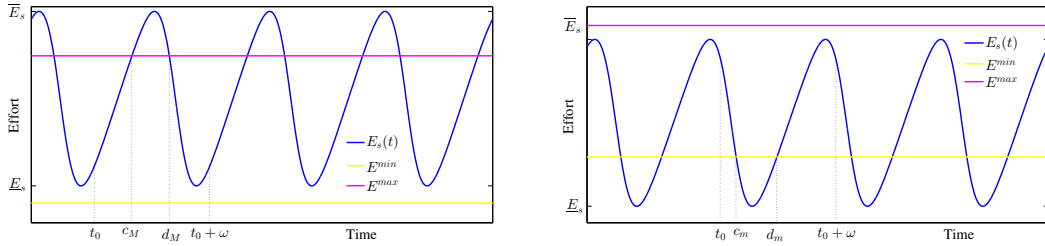
$$\text{s.t: } \frac{dx}{dt} = a_\gamma(t)x \left(1 - \frac{x}{K_\gamma(t)} \right) - qEx, \quad (4.13b)$$

and

$$\max_{E^{min} \leq E(t) \leq E^{max}} \int_{t_0}^{t_0+\omega} e^{-\delta t} (\tau q E X - \mathcal{M}) dt \quad (4.14a)$$

$$\text{s.t: } \frac{dX}{dt} = \bar{a}_\gamma(t) X \left(1 - \frac{X}{\bar{K}_\gamma(t)} \right) - q E X. \quad (4.14b)$$

Observe that solving (4.13) and (4.14) gives the solutions on $t_0 \leq t \leq t_0 + \omega$ only. The optimal periodic solutions on the entire planning horizon, which we denote by $E_{o\omega}(t)$ and $\mathbf{E}_{o\omega}(t)$ (for efforts) and $x_{o\omega}(t)$ and $X_{o\omega}(t)$ (for stocks), respectively, are obtained by extending the solutions for one full cycle onto the entire plan period in a periodic manner. Once the optimal periodic solution is identified, we apply the most rapid approach strategy to fulfill the given initial conditions.



(A) The constraint $E(t) \leq E^{max}$ is binding

(B) The constraint $E^{min} \leq E(t)$ is binding

FIGURE 1. The figure depicts cases where the control constraints are binding in a periodic manner.

Consider (4.13b) with the singular effort function $E_s(t)$ as shown in Frame A of Figure 1 where the constraint $E(t) \leq E^{max}$ is binding on the interval $[c_M, d_M]$. Here, $E^{max} < E_s^M$ on an open interval (c_M, d_M) and $E_s(c_M) = E^{max} = E_s(d_M)$. Consequently, the optimal control is not allowed to follow the singular curve on the interval (c_M, d_M) , and hence there is a need to switch from the singular effort curve to the bang-bang control $E = E^{max}$. Whenever the optimal control is forced to leave the singular path due to binding constraints, it switches at a time (say $t = t_1^M$) before the constraint becomes binding, i.e., $t_1^M < c_M$, and it switches back to the singular curve at a time, say $t = t_2^M$, such that $d_M < t_2^M < t_1^M + \omega$ [3]. The periodicity requirement on the optimal solution ensures t_2^M to satisfy $d_M < t_2^M < t_1^M + \omega$. If t_0 is any point of the t -axis such that $t_0 \in (t_2^M - \omega, t_1^M)$, then the optimal periodic harvest policy $E_{o\omega}(t)$ on the interval $[t_0, t_0 + \omega]$ becomes a combination of E^{max} and $E_s(t)$, which is given as follows:

$$E_{o\omega}(t) = \begin{cases} E_s(t), & \text{if } t_0 \leq t < t_1^M, \\ E^{max}, & \text{if } t_1^M \leq t < t_2^M, \\ E_s(t), & \text{if } t_2^M \leq t \leq t_0 + \omega, \end{cases} \quad (4.15)$$

and the optimal periodic stock path becomes

$$x_{o\omega}(t) = \begin{cases} x_s(t), & \text{if } t_0 \leq t \leq t_1^M, \\ x_M(t), & \text{if } t_1^M \leq t \leq t_2^M, \\ x_s(t), & \text{if } t_2^M \leq t \leq t_0 + \omega, \end{cases} \quad (4.16)$$

where $x_M(t)$ is given by

$$x_M(t) = \frac{e^{\int_{t_1^M}^t (a(s) - \gamma(s) - qE^{max}) ds}}{\frac{1}{x_s(t_1^M)} + \int_{t_1^M}^t e^{\int_{t_1^M}^s (a(\eta) - \gamma(\eta) - qE^{max}) d\eta} b(s) ds}. \quad (4.17)$$

Now, the discounted total net economic rent on one full cycle is

$$PV_{[t_0, t_0+\omega]} = \int_{t_0}^{t_1^M} e^{-\rho t} (pqx_s(t) - c)E_s(t)dt + \int_{t_1^M}^{t_2^M} e^{-\rho t} (pqx_M(t) - c)E^{max} dt + \int_{t_2^M}^{t_0+\omega} e^{-\rho t} (pqx_s(t) - c)E_s(t)dt. \quad (4.18)$$

From the optimality of $x_{\omega}(t)$ it follows that the switching time t_1^M maximizes (4.18). This critical value t_1^M can be identified as follows. By continuity of $x_{\omega}(t)$ we have

$$x_s(t_1^M) = x_M(t_1^M) \quad (4.19)$$

and

$$x_M(t_2^M) = x_s(t_2^M). \quad (4.20)$$

From (4.16)-(4.20) we observe that t_2^M can be expressed (implicitly) as a function of t_1^M . Thus, (4.18) can be viewed as a function of t_1^M alone. Now treating t_1^M as a parameter we solve the equation

$$\frac{dPV_{[t_0, t_0+T]}}{dt_1^M} = 0, \quad (4.21)$$

where

$$\begin{aligned} \frac{dPV_{[t_0, t_0+\omega]}}{dt_1^M} &= e^{-\rho t_1^M} (pqx_s(t_1^M) - c)E_s(t_1^M) - e^{-\rho t_1^M} (pqx_M(t_1^M) - c)E^{max} \\ &+ e^{-\rho t_2^M} (pqx_M(t_2^M) - c)E^{max} \frac{dt_2^M}{dt_1^M} + pqE^{max} \int_{t_1^M}^{t_2^M} e^{-\rho t} \left(\frac{\partial x_M(t, t_1^M)}{\partial t_1^M} \right) dt \\ &- e^{-\rho t_2^M} (pqx_s(t_2^M) - c)E_s(t_2^M) \frac{dt_2^M}{dt_1^M}. \end{aligned}$$

Simplifying (4.21) we obtain

$$\begin{aligned} e^{-\rho t_1^M} (pqx_s(t_1^M) - c)(E^{max} - E_s(t_1^M)) &= pqE^{max} \int_{t_1^M}^{t_2^M} e^{-\rho t} \left(\frac{\partial x_M(t, t_1^M)}{\partial t_1^M} \right) dt \\ &+ e^{-\rho t_2^M} (pqx_M(t_2^M) - c)(E^{max} - E_s(t_2^M)) \frac{dt_2^M}{dt_1^M}. \end{aligned} \quad (4.22)$$

The term $\frac{dt_2^M}{dt_1^M}$ in (4.22) can be evaluated using (4.19) and (4.20). Thus, (4.19), (4.20), and (4.22) can be used to identify the values of t_1^M and t_2^M . Similarly, one can construct the optimal periodic solution for (2.6) with the binding constraint $E(t) \leq E^{max}$ as shown in Frame A of Figure 1. If the constraint $E^{min} \leq E(t)$ is binding in (2.5) or (2.6) (ref. Frame B of Figure 1), the optimal periodic solutions can be constructed in the same manner as what is presented above, except that E^{max} is replaced by E^{min} in the later case.

Note that $[t_1^m, t_2^m]$ and $[t_1^M, t_2^M]$ represent blocked intervals over which the optimal control follows E^{min} and E^{max} , respectively, where t_1^m and t_1^M are the switching times when the optimal control leaves the singular arc, and t_2^m and t_2^M are times when it gets back to the singular arc. While $E_{\omega}(t)$ and $x_{\omega}(t)$ represent the optimal periodic effort and the associated optimal periodic stock path for (2.5), $\mathbf{E}_{\omega}(t)$ and $\mathbf{X}_{\omega}(t)$ represent that of (2.6).

5. NUMERICAL SIMULATIONS

Here, we present numerical examples to illustrate the significant findings of the study. For this purpose, we have chosen certain function forms associated with the coefficients. The numerical values assigned to the parameters and constants are related to the actual values one might have in a fishery (ref. [1]). Recall problems (2.5) and (2.6) with the coefficient functions given below and the numerical

TABLE 1. Numerical values that are assigned to the associated parameters and constants.

Symbol	Discription	Values	Unit
r	Intrinsic growth rate	1	1/month
K	Carrying capacity	3.7×10^6	ton
σ_K	Amplitude of K -fluctuation	1.55×10^6	ton
γ	Death rate due to pollution	0.45	1/month
q	Catability coefficient	0.00027	1/vessel/month
\mathcal{M}	Available flow of fund	4×10^6	\$/month
c	Cost per unit effort	2000	\$/vessel/month
τ	Price per unit catch	500	\$/ ton
ω	Environmental cycle	12	months
δ	Discount rate	0.05	1/month
α	Conversion factor	2.25×10^{-7}	1/\$/month

values in Table 1:

$$a(t) = r, \quad b(t) = \frac{r}{K - \sigma_K \sin\left(\frac{2\pi t}{\omega}\right)}, \quad \gamma(t) = \gamma,$$

where r, K, σ_K, ω and γ are all constants, and their numerical values can be found in Table 1. Clearly, the functions $a(t), b(t)$, and $\gamma(t)$ are ω -periodic. Here, the time t is measured on a monthly basis, and seasonal change in the environment occurs every twelve months, or one year. We consider $x_0 = 1.5 \times 10^5 = X_0$ (in tons), and $E^{\min} = 200, E^{\max} = 2000$ (in vessels).

Example 5.1 (Singular solutions when $\gamma = 0.20$ and $\gamma = 0.45$). Consider the given coefficient functions and the numerical data present in Table 1. It can be verified that $\langle a - \gamma - qE \rangle > 0$ for each $E \in [E^{\min}, E^{\max}]$, and hence (2.5b) admits a globally asymptotically stable ω -periodic solution $(x_s(t))$. Similarly, $\langle a - (1 - \alpha(\mathcal{M} - cE))\gamma - qE \rangle$ is positive for each $E \in [E^{\min}, E^{\max}]$, and hence (2.6b) also admits a globally asymptotically stable ω -periodic solution $(X_s(t))$. The singular effort curves, the associated singular stock path, the yield curves, and the revenue curves for the two problems are shown in Figure 2. From the figure, one can observe that as the influence of pollution increases, the fishing effort, stock, yield, and net economic rent decrease in both scenarios. It is also evident from the figure that the portion of effort devoted to pollution reduction increases with the effect of pollution (ref. Frame A). The figure further highlights that the stock, yield, and net revenue are superior in the case of pollution reduction.

Example 5.2 (Optimal solutions of (2.5) and (2.6) when $\gamma = 0.45$). Consider problems (2.5) and (2.6) with the numerical data given in Table 2. Then we have both $E_s(t)$ and $\mathbf{E}_s(t)$ within bounds

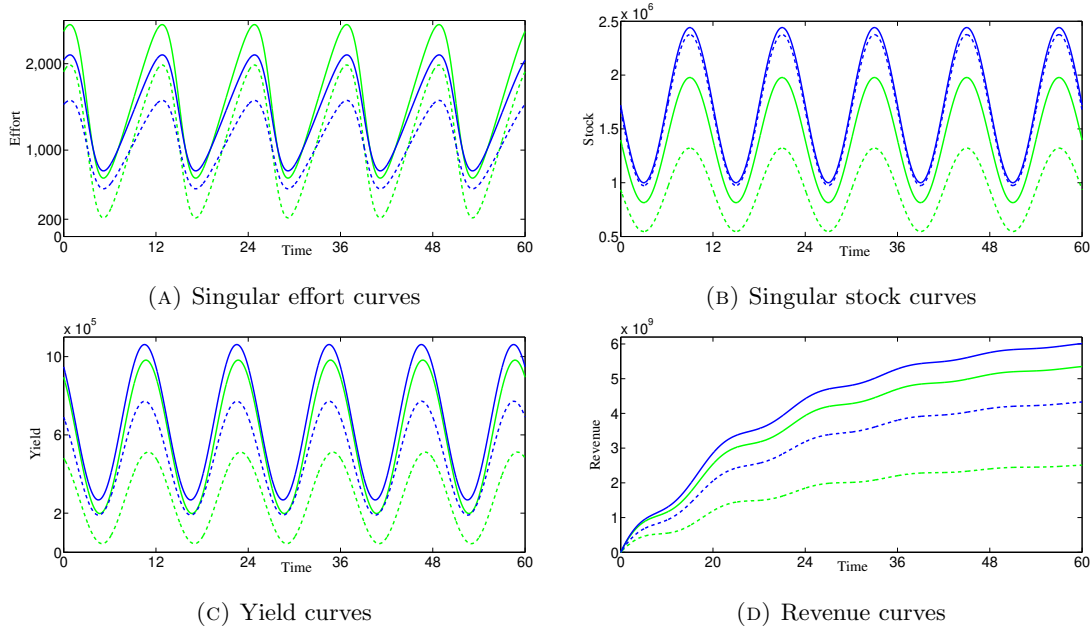


FIGURE 2. This figure depicts the singular effort curves (Frame A), associated stock paths (Frame B), yield curves (Frame C), and revenue curves (Frame D) pertaining to problems (2.5) and (2.6). In each frame, while the solid and dashed curves correspond to $\gamma = 0.20$ and $\gamma = 0.45$, respectively, the green and blue curves represent problems (2.5) and (2.6), respectively.

of the control constraints (see the dashed curves in Figure 2(A) for $\gamma = 0.45$). However, $x_0 < x_s(0)$ and $X_0 < X_s(0)$ (see the dashed curves in Frame B of Figure 2). Therefore, an initial adjustment is needed. Following (4.1) and (4.3), the optimal harvest strategy is the combination of E^{min} and E_s , where $E = E^{min}$ is employed initially, and then the optimal control switches to $E = E_s$ at time $T_0 = 3.88$. Here, the most rapid approach path x_m (starting from x_0) reaches the singular path x_s with the minimum time $T_0 = 3.88$. Similarly, employing $\mathbf{E} = E^{min}$ initially, the optimal control switches to $\mathbf{E} = \mathbf{E}_s$ at time $T_0 = 2.82$. Here, the most rapid approach path X_m (starting from X_0) reaches the singular path X_s with the minimum time $T_0 = 2.82$. Hence, we have the following:

$$E^*(t) = \begin{cases} E^{min}, & \text{for } 0 \leq t < 3.88, \\ E_s(t), & \text{for } t \geq 3.88; \end{cases}, \quad x^*(t) = \begin{cases} x_m(t), & \text{for } 0 \leq t \leq 3.88, \\ x_s(t), & \text{for } t \geq 3.88; \end{cases}$$

$$\mathbf{E}^*(t) = \begin{cases} E^{min}, & \text{for } 0 \leq t < 2.82, \\ \mathbf{E}_s(t), & \text{for } t \geq 2.82; \end{cases}, \quad X^*(t) = \begin{cases} X_m(t), & \text{for } 0 \leq t \leq 2.82, \\ X_s(t), & \text{for } t \geq 2.82. \end{cases}$$

Here, it is worth mentioning that the minimum time (T_0) required to reach the singular path for (2.6) is smaller than that for (2.5). This is because in the former case, not only the minimum effort $E = E^{min}$ is utilized for fishing, but also the maximum allowable amount $\mathcal{M} - cE^{min}$ is invested in pollution reduction, which makes the most rapid approach path grow faster. The singular effort curves, singular stock paths, optimal effort curves, and optimal stock paths of the two problems are shown in Figures 3 and 4, respectively. From the figures, it is evident that, while the singular curves are periodic, the optimal curves become periodic only after a finite time. The present value of the total net revenues for the first 60 months computed for the two problems below highlights that the net revenue is superior in

the case of (2.6), where pollution reduction is implemented.

$$PV_{[0,60]} = \int_0^{3.88} e^{-\delta t} (\tau q x_m(t) - c) E^{min} dt + \int_{3.88}^{60} e^{-\delta t} (\tau q x_s(t) - c) E_s(t) dt = 2.0389 \times 10^9 \text{ (\$US)},$$

$$PV_{[0,60]} = \int_0^{2.82} e^{-\delta t} (\tau q E^{min} X_m(t) - \mathcal{M}) dt + \int_{2.82}^{60} e^{-\delta t} (\tau q \mathbf{E}_s(t) X_s(t) - \mathcal{M}) dt = 3.7008 \times 10^9 \text{ (\$US)}.$$

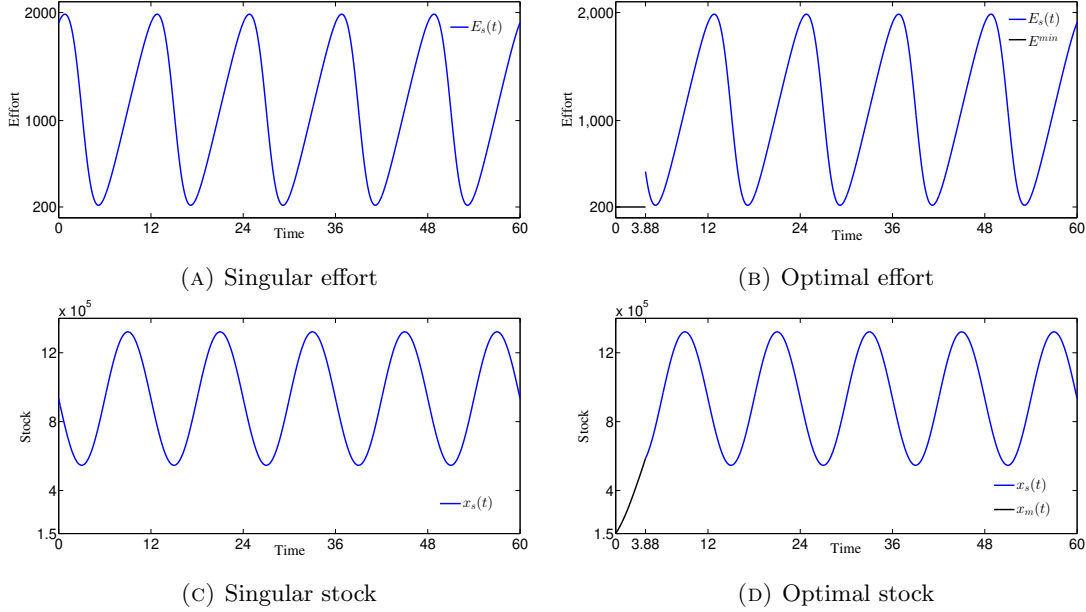


FIGURE 3. The figure presents the singular effort curve (Frame A), singular stock path (Frame C), optimal effort (Frame B), and optimal stock path (Frame D) pertaining to problem (2.5). The black arc (in Frame D) given for the time period $[0, 3.88]$ represents the most rapid approach path.

Example 5.3 (Optimal solution of (2.5) when $\gamma = 0.20$). Consider the optimal harvesting problem (2.5) and the numerical data in Table 1 except for γ , which is replaced by $\gamma = 0.20$ in the present case. Here, the effort function $E_s(t)$ fails to satisfy the control constraint $E(t) \leq E^{max}$ (see the solid green curve in Frame A of Figure 2). Consequently, the control constraint becomes binding on the interval $[c_M, d_M] = [10.47, 14.36]$, resulting in the blocked interval $[t_1^M, t_2^M] = [10.46, 15.47]$. The optimal periodic effort ($E_{o\omega}(t)$) and the associated optimal periodic stock path ($x_{o\omega}(t)$) are given as follows:

$$E_{o\omega}(t) = \begin{cases} E^{max}, & 0 \leq t < t_2^M - \omega \\ E_s(t), & t_2^M - \omega \leq t < t_1^M \\ E^{max}, & t_1^M \leq t < t_2^M \\ E_s(t), & t_2^M \leq t < t_1^M + \omega \\ E^{max}, & t_1^M + \omega \leq t < t_2^M + \omega \\ E_s(t), & t_2^M + \omega \leq t < t_1^M + 2\omega \\ E^{max}, & t_1^M + 2\omega \leq t < t_2^M + 2\omega \\ E_s(t), & t_2^M + 2\omega \leq t < t_1^M + 3\omega \\ E^{max}, & t_1^M + 3\omega \leq t < t_2^M + 3\omega \\ E_s(t), & t_2^M + 3\omega \leq t < t_1^M + 4\omega \\ E^{max}, & t_1^M + 4\omega \leq t \leq 60 \end{cases}, \quad x_{o\omega}(t) = \begin{cases} x_M(t), & \text{if } 0 \leq t \leq t_2^M - \omega \\ x_s(t), & t_2^M - \omega \leq t \leq t_1^M \\ x_M(t), & t_1^M \leq t \leq t_2^M \\ x_s(t), & t_2^M \leq t \leq t_1^M + \omega \\ x_M(t), & t_1^M + \omega \leq t \leq t_2^M + \omega \\ x_s(t), & t_2^M + \omega \leq t \leq t_1^M + 2\omega \\ x_M(t), & t_1^M + 2\omega \leq t \leq t_2^M + 2\omega \\ x_s(t), & t_2^M + 2\omega \leq t \leq t_1^M + 3\omega \\ x_M(t), & t_1^M + 3\omega \leq t \leq t_2^M + 3\omega \\ x_s(t), & t_2^M + 3\omega \leq t \leq t_1^M + 4\omega \\ x_M(t), & t_1^M + 4\omega \leq t \leq 60 \end{cases}.$$

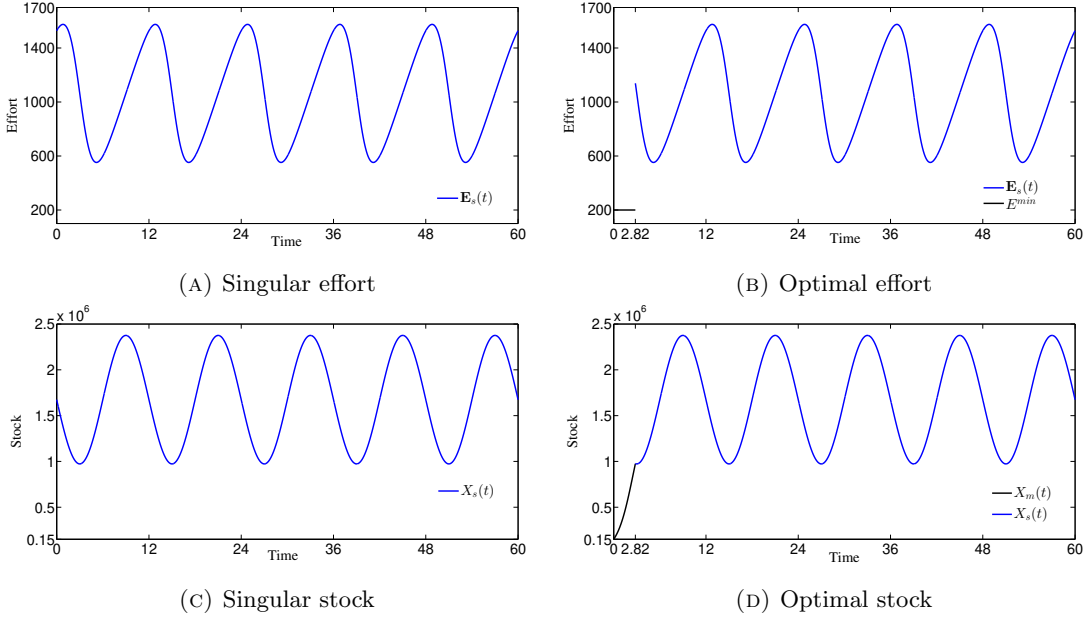


FIGURE 4. The figure depicts the singular effort curve (Frame A), singular stock path (Frame C), optimal effort (Frame B), and optimal stock path (Frame D) pertaining to problem (2.6). The black arc (in Frame D) given for the time period $[0, 2.82]$ represents the most rapid approach path.

The graphs of $E_{ow}(t)$ and $x_{ow}(t)$ are shown in Frames A and C of Figure 5, respectively. Since $x_0 < x_{ow}(0)$, an initial adjustment is needed to fully determine the solution to the problem (2.5). Following (4.1) and (4.3), the optimal harvesting strategy first follows $E = E^{min}$ and then switches to $E = E_s$ at $T_0 = 3.14$. The optimal harvesting strategy and the associated optimal stock path are given as follows:

$$E^*(t) = \begin{cases} E^{min}, & \text{if } 0 \leq t < 3.14, \\ E_{ow}(t), & \text{if } 3.14 \leq t \leq 60; \end{cases}, \quad x^*(t) = \begin{cases} x_m(t), & \text{if } 0 \leq t \leq 3.14, \\ x_{ow}(t), & \text{if } 3.14 \leq t \leq 60. \end{cases}$$

The graphs for $E^*(t)$ and $x^*(t)$ are shown in Frames B and D of Figure 5. Note that while $E_{ow}(t)$ and $x_{ow}(t)$ are periodic, $E^*(t)$ and $x^*(t)$ become periodic only after a finite time. Here, the present value for the first two full cycles is given by

$$PV_{[t_0, 60]} = \int_0^{3.14} e^{-\delta t} (\tau q x_m(t) - c) E^{min} dt + \int_{3.14}^{60} e^{-\delta t} (\tau q x_{ow}(t) - c) E_{ow}(t) dt = 4.5005 \times 10^9 \text{ (\$US)}.$$

Example 5.4 (Optimal solution for (2.6) when $\gamma = 0.20$). Consider problem (2.6) with the set of parameter values given in Table 1 except for γ , which is given by $\gamma = 0.20$ in the present case. The singular effort function $\mathbf{E}_s(t)$ doesn't satisfy the control constraint $E(t) \leq E^{max}$ (see the solid blue curve in Frame A of Figure 2). Consequently, the control constraint becomes binding on the interval $[c_M, d_M] = [11.73, 13.69]$, resulting in the blocked interval $[t_1^M, t_2^M] = [11.72, 14.35]$. The optimal periodic

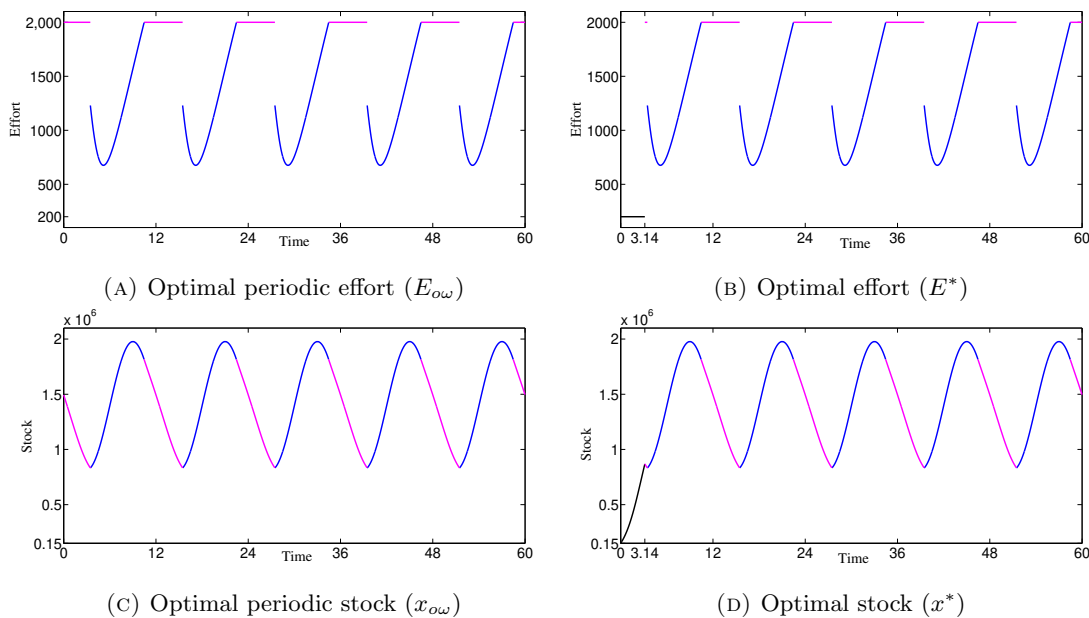


FIGURE 5. The figure depicts the optimal periodic effort, the associated optimal periodic stock, the optimal effort, and the associated optimal stock pertaining to (2.5) where $\gamma = 0.2$. The black arc in Frame (D) stands for the most rapid approach path of the initial adjustment.

effort (\mathbf{E}_{ow}) and the associated stock path (X_{ow}) for the first five years are given below.

$$\mathbf{E}_{ow}(t) = \begin{cases} E^{max}, & \text{if } 0 \leq t < t_2^M - \omega \\ \mathbf{E}_s(t), & t_2^M - \omega \leq t < t_1^M \\ E^{max}, & t_1^M \leq t < t_2^M \\ \mathbf{E}_s(t), & t_2^M \leq t < t_1^M + \omega \\ E^{max}, & t_1^M + \omega \leq t < t_2^M + \omega \\ \mathbf{E}_s(t), & t_2^M + \omega \leq t < t_1^M + 2\omega \\ E^{max}, & t_1^M + 2\omega \leq t < t_2^M + 2\omega \\ \mathbf{E}_s(t), & t_2^M + 2\omega \leq t < t_1^M + 3\omega \\ E^{max}, & t_1^M + 3\omega \leq t < t_2^M + 3\omega \\ \mathbf{E}_s(t), & t_2^M + 3\omega \leq t < t_1^M + 4\omega \\ E^{max}, & t_1^M + 4\omega \leq t \leq 60 \end{cases}, \quad X_{ow}(t) = \begin{cases} X_M(t), & \text{if } 0 \leq t \leq t_2^M - \omega \\ X_s(t), & t_2^M - \omega \leq t \leq t_1^M \\ X_M(t), & t_1^M \leq t \leq t_2^M \\ X_s(t), & t_2^M \leq t \leq t_1^M + \omega \\ X_M(t), & t_1^M + \omega \leq t \leq t_2^M + \omega \\ X_s(t), & t_2^M + \omega \leq t \leq t_1^M + 2\omega \\ X_M(t), & t_1^M + 2\omega \leq t \leq t_2^M + 2\omega \\ X_s(t), & t_2^M + 2\omega \leq t \leq t_1^M + 3\omega \\ X_M(t), & t_1^M + 3\omega \leq t \leq t_2^M + 3\omega \\ X_s(t), & t_2^M + 3\omega \leq t \leq t_1^M + 4\omega \\ X_M(t), & t_1^M + 4\omega \leq t \leq 60 \end{cases}.$$

The graphs of \mathbf{E}_{ow} and X_{ow} are shown in Frames A and C of Figure 6, respectively. Since $X_0 < X_{ow}(0)$ (see Frame C of Figure 6), an initial adjustment is needed to fully determine the optimal solution to the problem (2.6). Following (4.1) and (4.3), the optimal harvesting strategy first follows $E = E^{min}$ and then switches to $E = \mathbf{E}_s$ at $T_0 = 2.69$. Hence, the optimal harvesting strategy and the associated optimal stock path are given as follows:

$$\mathbf{E}^*(t) = \begin{cases} E^{min}, & \text{if } 0 \leq t < 2.69, \\ \mathbf{E}_{ow}(t), & \text{if } 2.69 \leq t \leq 60; \end{cases}, \quad X^*(t) = \begin{cases} X_m(t), & \text{if } 0 \leq t \leq 2.69, \\ X_{ow}(t), & \text{if } 2.69 \leq t \leq 60. \end{cases}$$

Graphically, the optimal harvesting strategy and the associated stock path are shown in Frames B and D of Figure 6, respectively. The total discounted net revenue for the first five years is given by,

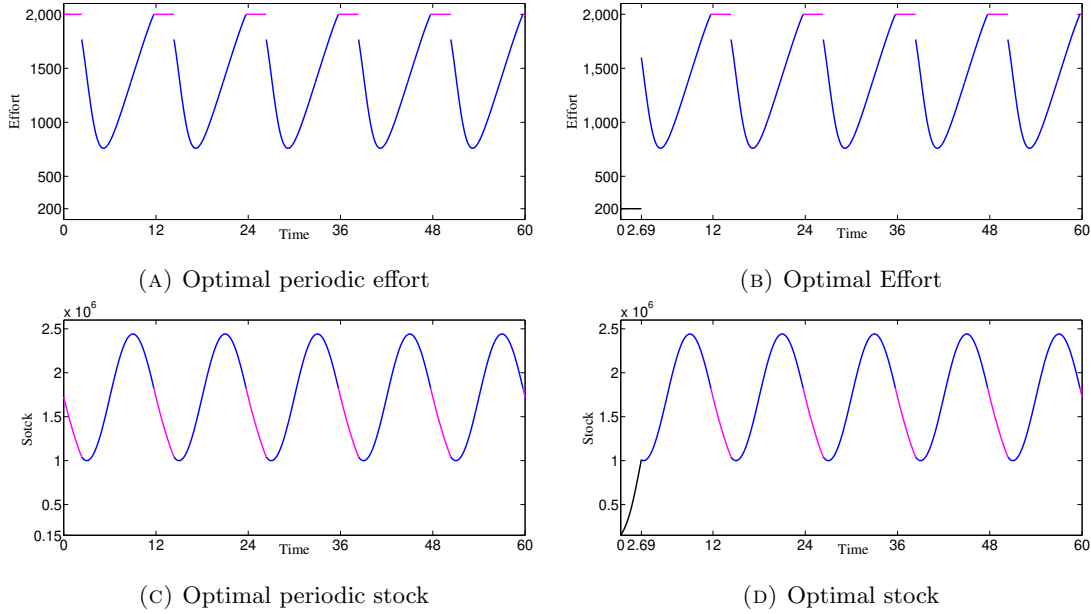


FIGURE 6. The figure depicts the optimal periodic solution and the optimal solution pertaining to (2.6) with binding control constraints. While frames (A) and (B) represent the optimal periodic effort and optimal effort, frames (C) and (D) represent the associated optimal periodic stock and optimal stock paths, respectively.

$$PV_{[0,60]} = \int_0^{2.69} e^{-\delta t} (\tau q E^{min} X_m(t) - \mathcal{M}) dt + \int_{2.69}^{60} e^{-\delta t} (\tau q \mathbf{E}_{ow}(t) X_{ow}(t) - \mathcal{M}) dt = 5.1604 \times 10^9 \text{ (\$US)}.$$

Example 5.5 (Optimal solution of (2.5) and (2.6) when $\gamma = 0.55$).

First, let's consider problem (2.6) with the set of parameter values in Table 1 except for γ , which is taken as $\gamma = 0.55$ in the present case. It can be seen that $\mathbf{E}_s(t)$ satisfies the control constraint (ref. Frame A of Figure 7). However, $X_0 < X_s(0)$ (ref. Frame B of Figure 7). Therefore, an initial adjustment is required. Using the most rapid approach technique, the optimal solution is given as follows:

$$\mathbf{E}^*(t) = \begin{cases} E^{min}, & \text{for } 0 \leq t < 2.88, \\ \mathbf{E}_s(t), & \text{for } t \geq 2.88; \end{cases}, \quad X^*(t) = \begin{cases} X_m(t), & \text{for } 0 \leq t \leq 2.88, \\ X_s(t), & \text{for } t \geq 2.88. \end{cases}$$

Graphically, the optimal harvesting strategy and the associated stock path are shown in Frames A and B of Figure 7, respectively. The discounted total net revenue is given by

$$PV_{[0,60]} = \int_0^{2.88} e^{-\delta t} (\tau q E^{min} X_m(t) - \mathcal{M}) dt + \int_{2.88}^{60} e^{-\delta t} (\tau q \mathbf{E}_s(t) X_s(t) - \mathcal{M}) dt = 3.2959 \times 10^9 \text{ (\$US)}.$$

Now, let's consider problem (2.5) with the numerical data in Table 1 except for γ , which is replaced by $\gamma = 0.55$. We observe that the effort function E_s fails to satisfy the control constraint $E^{min} \leq E(t)$. Consequently, the constraint is binding on the interval $[c_m, d_m] = [4.21, 6.23]$, resulting in the blocked interval $[t_1^m, t_2^m] = [4.20, 7.57]$. The optimal periodic effort (E_{ow}) and the associated stock path (x_{ow})

for the first five years are given as follows:

$$E_{ow}(t) = \begin{cases} E_s(t), & 0 \leq t < t_1^m \\ E^{min}, & t_1^m \leq t < t_2^m \\ E_s(t), & t_2^m \leq t < t_1^m + \omega \\ E^{min}, & t_1^m + \omega \leq t < t_2^m + \omega \\ E_s(t), & t_2^m + \omega \leq t < t_1^m + 2\omega \\ E^{min}, & t_1^m + 2\omega \leq t < t_2^m + 2\omega \\ E_s(t), & t_2^m + 2\omega \leq t < t_1^m + 3\omega \\ E^{min}, & t_1^m + 3\omega \leq t < t_2^m + 3\omega \\ E_s(t), & t_2^m + 3\omega \leq t < t_1^m + 4\omega \\ E^{min}, & t_1^m + 4\omega \leq t < t_2^m + 4\omega \\ E_s(t), & t_2^m + 4\omega \leq t \leq 60 \end{cases}, \quad x_{ow}(t) = \begin{cases} x_s(t), & 0 \leq t \leq t_1^m \\ x_m(t), & t_1^m \leq t \leq t_2^m \\ x_s(t), & t_2^m \leq t \leq t_1^m + \omega \\ x_m(t), & t_1^m + \omega \leq t \leq t_2^m + \omega \\ x_s(t), & t_2^m + \omega \leq t \leq t_1^m + 2\omega \\ x_m(t), & t_1^m + 2\omega \leq t \leq t_2^m + 2\omega \\ x_s(t), & t_2^m + 2\omega \leq t \leq t_1^m + 3\omega \\ x_m(t), & t_1^m + 3\omega \leq t \leq t_2^m + 3\omega \\ x_s(t), & t_2^m + 3\omega \leq t \leq t_1^m + 4\omega \\ x_m(t), & t_1^m + 4\omega \leq t \leq t_2^m + 4\omega \\ x_s(t), & t_2^m + 4\omega \leq t \leq 60 \end{cases}.$$

Since $x(0) < x_{ow}(0)$, the initial adjustment is needed to determine the optimal solution. With the most

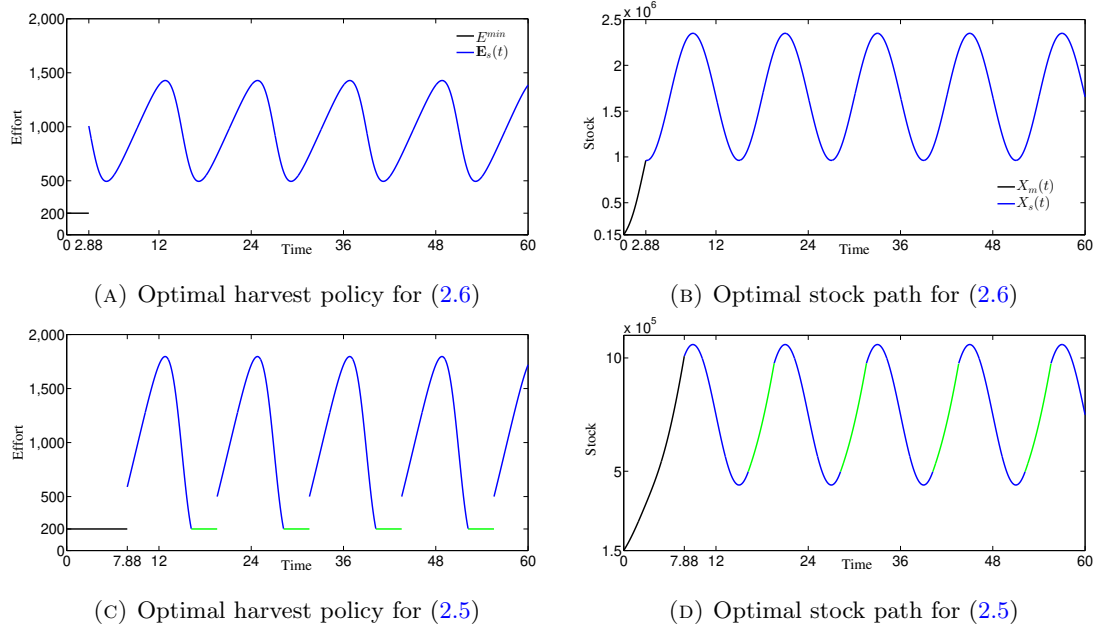


FIGURE 7. The figure depicts optimal solutions pertaining to (2.5) and (2.6) when $\gamma = 0.55$.

rapid approach strategy $E = E^{min}$, the path $x_{ow}(t)$ is reached at the minimum time $T_0 = 7.88$. Hence, the optimal effort and the associated path are given by:

$$E^*(t) = \begin{cases} E^{min}, & \text{if } 0 \leq t < 7.88 \\ E_{ow}(t), & \text{if } 7.88 \leq t \leq 60 \end{cases}, \quad x^*(t) = \begin{cases} x_m(t), & \text{if } 0 \leq t \leq 7.88 \\ x_{ow}(t), & \text{if } 7.88 \leq t \leq 60 \end{cases}.$$

The optimal effort and the associated stock path are shown in Frames C and D of Figure 7, respectively. The present value for the first five years is

$$PV_{[0,60]} = \int_0^{7.88} e^{-\delta t} (\tau q x_m(t) - c) E^{min} dt + \int_{7.88}^{60} e^{-\delta t} (\tau q x_{ow}(t) - c) E_{ow}(t) dt = 1.3272 \times 10^9 \text{ (\$US)}.$$

TABLE 2. A summary of the maximum and minimum levels of the singular effort functions and the associated stock paths pertaining to problems (2.5) and (2.6) for different values of γ , where $x_s^m = \min_{t \in [t_0, t_0 + \omega]} x_s(t)$, $x_s^M = \max_{t \in [t_0, t_0 + \omega]} x_s(t)$, $X_s^m = \min_{t \in [t_0, t_0 + \omega]} X_s(t)$, $X_s^M = \max_{t \in [t_0, t_0 + \omega]} X_s(t)$, and (P) stands for the given optimal harvesting problem.

γ	(P)	E^{min}	$E_s^m(\mathbf{E}_s^m)$	$E_s^M(\mathbf{E}_s^M)$	E^{max}	$x_s^m(X_s^m)$	$x_s^M(X_s^M)$
0.20	(2.5)	200	676	2452	2000	8.1464×10^5	1.9771×10^6
	(2.6)	200	760	2102	2000	9.9978×10^5	2.4412×10^6
0.45	(2.5)	200	216	1985	2000	5.4636×10^5	1.3214×10^6
	(2.6)	200	552	1575	2000	9.7288×10^5	2.3756×10^6
0.55	(2.5)	200	33	1796	2000	4.3922×10^5	1.0592×10^6
	(2.6)	200	495	1428	2000	9.6213×10^5	2.3493×10^6

TABLE 3. A summary of the optimal harvesting strategies and the occurrence of binding constraints pertaining to problems (2.5) and (2.6) for different values of γ where (P) stands for the given optimal harvesting problem.

γ	(P)	$E^{min} \leq E(t)$	$E(t) \leq E^{max}$	Optimal harvesting strategy
0.20	(2.5)	Non-binding	Binding	E^{min} , $0 \leq t < 3.14$, $E_{\omega}(t)$, $t \geq 3.14$
	(2.6)	Non-binding	Binding	E^{min} , $0 \leq t < 2.69$, $\mathbf{E}_{\omega}(t)$, $t \geq 2.69$
0.45	(2.5)	Non-binding	Non-binding	E^{min} , $0 \leq t < 3.88$, $E_s(t)$, $t \geq 3.88$
	(2.6)	Non-binding	Non-binding	E^{min} , $0 \leq t < 2.82$, $\mathbf{E}_s(t)$, $t \geq 2.82$
0.55	(2.5)	Binding	Non-binding	E^{min} , $0 \leq t < 7.88$, $E_{\omega}(t)$, $t \geq 7.88$
	(2.6)	Non-binding	Non-binding	E^{min} , $0 \leq t < 2.88$, $\mathbf{E}_s(t)$, $t \geq 2.88$

TABLE 4. A summary of the discounted total net revenues pertaining to (2.5) and (2.6) for different values of γ . Here, (P) stands for the optimal harvesting problem, $PV_s(\mathbf{PV}_s)$, $PV_{\omega}(\mathbf{PV}_{\omega})$, and $PV_o(\mathbf{PV}_o)$ represent the net revenue computed for singular solution, optimal periodic solution, and optimal solution, respectively.

γ	(P)	$PV_s(\mathbf{PV}_s)$	$PV_{\omega}(\mathbf{PV}_{\omega})$	$PV_o(\mathbf{PV}_o)$	$\mathbf{PV}_s - PV_s$	$\mathbf{PV}_o - PV_o$
0.20	(2.5)	5.3510×10^9	5.3337×10^9	4.5005×10^9	6.6150×10^8	6.5990×10^8
	(2.6)	6.0125×10^9	6.0120×10^9	5.1604×10^9		
0.45	(2.5)	2.5113×10^9	N/A	2.0389×10^9	1.8177×10^9	1.6619×10^9
	(2.6)	4.3290×10^9	N/A	3.7008×10^9		
0.55	(2.5)	1.6709×10^9	1.6704×10^9	1.3272×10^9	2.1912×10^9	1.9687×10^9
	(2.6)	3.8621×10^9	N/A	3.2959×10^9		

6. DISCUSSION

A summary of the maximum and minimum levels of the singular fishing efforts and the associated singular stock paths computed for the two problems is given in Table 2. When the effect of pollution increases, the maximum and minimum effort levels fall, and hence it is more likely that the constraint $E^{min} \leq E(t)$ becomes binding. Comparing the maximum (minimum) effort levels of the two problems, one can observe that, for all pollution levels, the maximum effort level (\mathbf{E}_s^M) of (2.6) is smaller than E_s^M of (2.5), and the minimum effort level (\mathbf{E}_s^m) of (2.6) is larger than E_s^m of (2.5). This highlights that pollution reduction efforts may turn the problem involving binding constraints into one with nonbinding constraints. From the table, the comparison of E_s^m and E^{min} and E_s^M and E^{max} clearly highlights the occurrence of binding constraints. For both problems, the maximum (minimum) stock level decreases as the effect of pollution increases. We also observe that the maximum (minimum) stock level is superior when pollution reduction is undertaken.

A summary of the optimal harvesting strategies for the two problems is presented in Table 3. It also highlights the occurrence of binding constraints for the varying effects of pollution. We can observe that the optimal approach path takes less time to reach the optimal singular solution (or optimal periodic solution) in (2.6) than (2.5). The reason is that, while it is only the fishing effort that is reduced to the minimum level E^{min} in (2.5), the maximum possible effort $\mathcal{M} - cE^{min}$ is also devoted to pollution reduction in addition to reducing the fishing effort to E^{min} in (2.6). Moreover, the minimum time the optimal approach path takes to reach the optimal singular solution (or optimal periodic solution) increases with the effect of pollution in both (2.6) and (2.5). The relative position of the initial state to the singular solutions and optimal periodic solution is the deriving force for the employment of E^{min} for the most rapid approach strategy.

A summary of the discounted total net revenues computed for the two problems over the time interval $[0, 60]$ is presented in Table 4. In both problems, the revenues are computed along the optimal singular solution (when the control constraint is nonbinding), the optimal periodic solution (when the control constraint is binding), and the values appear in the decreasing order of their appearance, which was expected (ref. Table 3). It can also be observed that, in all cases, the revenue is superior when pollution reduction is undertaken. Moreover, the difference increases with the effect of pollution, which underlines the importance of pollution reduction efforts when the effect of pollution is severe.

7. CONCLUSIONS

In this paper, we have investigated the impact of pollution control on the optimal harvesting of a single species population in a seasonally changing and polluted environment. Considering a constant amount (\mathcal{M}) of funding for the harvesting activity at each time, we have proposed two distinct ways to use this fund to maximize the net economic rent: the first involves allocating the available funds for harvesting exclusively with the effort cE (such that $0 < cE \leq \mathcal{M}$), with the remaining sum $\mathcal{M} - cE$ not being used for the harvesting activity. The second involves allocating the excess $\mathcal{M} - cE$ to pollution reduction (rather than holding the money) in the hopes of enhancing the net economic revenue indirectly. Both seasonal change and environmental pollution affect the intrinsic growth rate and carrying capacity of the environment.

We have constructed the optimal singular solutions for each of the two scenarios and noted that the fishing effort, stock level, yield, and revenue decrease with the effect of pollution. Through numerical simulation, we observed that, while strong restrictions on the control variable for different reasons are a major cause, a fall in the fishing effort due to the effect of pollution may also cause binding constraints on the control variable. We have also observed that, while the maximum level of singular fishing effort

is reduced and the minimum level rises after the implementation of pollution reduction, the associated stock level and yield remain superior all the time in the case of pollution reduction. The main reason for the rise in net revenue is the improved stock level due to pollution reduction efforts.

Following the determination of singular solutions, we have constructed the optimal solutions in two different situations: when the control constraint is binding and when it is nonbinding. When the problem involves binding constraints, first we have constructed the optimal periodic solution (a solution satisfying control constraints), and then the optimal solution (which fulfills the initial condition) is obtained using the most rapid approach strategy. By considering the varying effects of pollution, we have computed the net revenues along singular solutions, optimal periodic solutions, and optimal solutions. In all cases, the net revenue improves when investment in pollution reduction efforts is considered. Thus, based on the results of this study, we strongly recommend investing in pollution management as an alternative to benefit resource conservation and enhance the net economic rent for the harvesting system in a seasonally changing and polluted environment.

DISCLOSURE STATEMENT

No potential conflict of interest was reported by the authors.

FUNDING

The authors received no direct funding for this research.

REFERENCES

- [1] L. G. Anderson, J. C. Seijo, *Bio-economics of Fisheries Management*, Wiley-Blackwell Publishing, New York, 2010.
- [2] A. Ardito, P. Ricciardi and A. Schiaffino, *The effects of seasonal fluctuations on an open access fishery problem*, J. Math. Biol. **31**(1993), 879-890.
- [3] K. J. Arrow, *Optimal capital policy with irreversible investment*. In *Value, capital and growth*. Routledge, 2017, 1-20.
- [4] L. Bai and K. Wang, *Optimal impulsive harvest policy for time-dependent logistic equation with periodic coefficients*, Electr. J. Diff. Equations, **121**(2003), 1-9.
- [5] D. J. Bell and D. H. Jacobson, *Singular Optimal Control Problems*, Academic Press, New York, 1975.
- [6] César Castilho and P. D. N. Srinivasu, *Bio-economics of a renewable resource in a seasonally varying environment*, Math. Biosci. **205**(2007), 1-18.
- [7] L. Cesari, *Optimization-theory and applications: problems with ordinary differential equations*, Vol. 17, Springer Science and Business Media, 2012.
- [8] S. Chauhan, S. K. Bhatia and P. Chaudhary, *Effect of pollution on prey-predator system with infected predator*, Commun. Math. Biol. Neurosci. **2017** (2017): 14.
- [9] C. W. Clark, *Mathematical Bioeconomics: The Optimal Management of Renewable Resources*, Second Edition, Wiley, New York, 1990.
- [10] C. W. Clark, *Mathematical bioeconomics: The mathematics of conservation* Vol. 91, John Wiley & Sons, 2010.
- [11] J. M. Cushing, *Oscillatory population growth in periodic environments*, Theor. Popul. Biol. **30**(1986), 289-308.
- [12] T. Das, R. N. Mukherjee and K. S. Chaudhuri, *Harvesting of a prey-predator fishery in the presence of toxicity*, Appl. Math. Model. **33**(2009), 2282-2292.
- [13] M. de Gee and J. Grassman, *Sustainable yields from seasonally fluctuating biological populations*, Ecol. Model. **109**(1998), 203-212.
- [14] J. T. De Luna and T. G. Hallam, *Effects of toxicants on populations: A qualitative approach IV. Resource-Consumer-Toxicant models*, Ecol. Model. **35**(1987), 249-273.
- [15] L. Dong, L. Chen and L. Sun, *Optimal harvesting policies for periodic Gompertz systems*, Nonlinear Anal. Real World Appl. **8**(2007), 572-578.
- [16] B. Dubey and J. Hussain, *Modelling the survival of species dependent on a resource in a polluted environment*, Nonlinear Anal. Real World Appl. **7**(2006), 187-210.
- [17] M. Fan and K. Wang, *Optimal harvesting policy for single population with periodic coefficients*, Math. Biosci. **152**(1998), 165-178.

- [18] T. G. Hallam, C. E. Clark and G. S. Jordan, *Effects of toxicants on populations: a qualitative approach II. First order kinetics*, J. Math. Biol. **18**(1983), 25-37.
- [19] T. G. Hallam and J. T. De Luna, *Effects of toxicants on populations: a qualitative Approach III. Environmental and Food Chain Pathways*, J. Theor. Biol. **109**(1984), 411-429.
- [20] M. Haque and S. Sarwardi, *Effect of toxicity on a harvested fishery model*, Model. Earth Syst. Environ. **2**(2016): 122.
- [21] M. Hasanbulli, S. P. Rogovchenko and Y. V. Rogovchenko, *Dynamics of a Single Species in a Fluctuating Environment under Periodic Yield Harvesting*, J. Appl. Math. **2013**(2013): 167671.
- [22] Q. Huang, H. Wang and M. A. Lewis, *The impact of environmental toxins on predator-prey dynamics*, J. Theor. Biol. **378**(2015), 12-30.
- [23] T. K. Kar and K. S. Chaudhuri, *On non-selective harvesting of two competing fish species in the presence of toxicity*, Ecol. Modell. **161**(2003), 125-137.
- [24] E. Keeler, M. Spence and R. Zeckhauser, *The optimal control of pollution*, J. Econ. Theory **4**(1972), 19-34.
- [25] D. E. Kime, *The effects of pollution on reproduction in fish*, Rev. Fish Biol. Fish. **5**(1995), 52-96.
- [26] L. D. Mee and M. Bloxham, *Assessing the impact of pollution on aquatic systems at a global and regional level*, Mar. Environ. Res. **54**(2002), 223-228.
- [27] J. R. Olsen and J. S. Shortle, *The Optimal Control of Emissions and Renewable Resource Harvesting Under Uncertainty*, Environ. Resource Econ. **7**(1996), 97-115.
- [28] S. D. N. Pichika and S. D. Zawka, *Renewable resource management in a seasonally fluctuating environment with restricted harvesting effort*, Math. Biosci. **301**(2018), 1-9.
- [29] S. D. N. Pichika and S. D. Zawka, *Optimal harvesting of a renewable resource in a polluted environment: An allocation problem of the sole owner*, Nat. Resour. Model. **32**(2019): e12206.
- [30] L. S. Pontryagin, V. G. Bol'tyankii, R. V. Gamkrelidze and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, Pergamon Press, New York, 1964.
- [31] G. P. Samanta, *A two-species competitive system under the influence of toxic substances*, Appl. Math. Comput. **216**(2010), 291-299.
- [32] M. Schaefer, *Renewable resources and pollution*, Math. Comput. Modelling, **14**(1990), 1177-1181.
- [33] J. B. Shukla, S. Sharma, B. Dubey and P. Sinha, *Modeling the survival of a resource dependent population: Effects of toxicants (pollutants) emitted from external sources as well as formed by its precursors*, Nonlinear Anal. Real World Appl. **10**(2009), 54-70.
- [34] P. D. N. Srinivasu and S. D. Zawka, *Influence of Investing in Treating a Polluted Environment on the Harvest: A Problem of Optimal Allocation*, J. Biol. Syst. **27**(2019), 257-279.
- [35] P. D. N. Srinivasu and S. D. Zawka, *Bio-Economics of a Renewable Resource in the Presence of Pollution: The Problem of Optimal Effort Allocation*, Nonlinear Dyn. Syst. Theory **20**(2020), 552-567.
- [36] O. Tahvonen, *On the dynamics of renewable resource harvesting and pollution control*, Environ. Resour. Econ., **1**(1991), 97-117.
- [37] L. S. L. Teh, D. Zeller, A. Cabanban, L. C. L. Teh and U. R. Sumaila, *Seasonality and historic trends in the reef fisheries of Pulau Banggi, Sabah, Malaysia*, Coral Reefs, **26**(2007), 251-263.
- [38] H. Wu and F. Chen, *Harvesting of a single-species system incorporating stage structure and toxicity*, Discrete Dyn. Nat. Soc. **2009**(2009), 1-16.
- [39] S. D. Zawka and D. N. S. Pichika, *Existence and optimal harvesting of two competing species in a polluted environment with pollution reduction effect*, J. Math. Model. **9**(2021), 517-536.

SIMON D. ZAWKA, CORRESPONDING AUTHOR, DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES, ARBA MINCH UNIVERSITY, ARBA MINCH, ETHIOPIA

Email address: simon.derke@amu.edu.et

P. D. N. SRINIVASU, DEPARTMENT OF MATHEMATICS, ANDHRA UNIVERSITY, VISAKHAPATNAM-530003, ANDHRA PRADESH, INDIA

Email address: pdnsrini@gmail.com