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DYNAMICAL ANALYSIS OF DISPERSAL IMPACT ON A THREE-PATCH BASED PREDATOR-PREY SYSTEM WITH STRONG ALLEE EFFECTED PREYS

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ABSTRACT. This paper describes a three-patch based predator-prey interaction model where a strong Allee effect accomplishes each prey species in their respective patches. It is also assumed that only the prey species are movable among the patches following the balanced dispersal rule. The stability behavior at the inner equilibrium point has been studied in the presence and non-existence of dispersal. This study also observes a vital characteristic of the Allee edge regarding the stability of the equilibrium of the model when prey species progress freely in their patches. Also, the dispersal can destabilize the interior equilibrium of populations, i.e., if the interior equilibrium becomes unstable in the absence of dispersal under some conditions, it becomes stabilized under some conditions whenever the prey species move among their patches. The Hopf bifurcation behavior of the system has been studied and numerical simulations have been performed taking hypothetical parameter values using MATHEMATICA and MATLAB software.

1. Introduction

Conventionally, the prey-predator interaction models are formulated in homogeneous atmosphere. But often find that the environment is heterogeneous in an actual situation so that the prey-predator modelling can be developed by different patches related to immigration. The predator immigration effort on the stable behaviour of the prey-predator interaction has already been analysed in different studies [40]. They considered that the predator species remain in the present patch if the population size of the prey species is sufficiently large; otherwise, they leave the present patch. Researchers also discussed ([18],[30]) about population dispersal in a multi-patch environment. Amarasekare [2] studied a model of a single species involving two-patch and observed the effect of Allee effect and dispersal on the persistence ([6],[35]) of the system. Donahue et al. [22] presented a dispersal model and explained the dynamics between habitat patches. Padrón and Trevisan [45] discussed a single-species dispersal model consisting of some habitats linked with linear migration rates and having logistic growth. Cantrell et al. [18] proposed a dispersal model in heterogeneous patchy environment. Again, the metapopulation dynamics was similarly studied by researchers working in this field ([38]-[47]). Hassell et al. [27] studied a host-parasitoid interactions model where in each generation, the host and parasitoid can move to adjacent patches. Bascompte and Solé [12] studied the spatial degrees of freedom over the qualitative characteristics of single population maps. Ruxton [46] discussed a single population system adding density-dependent migration among nearest-neighbour populations. All of these models consist of the concept of fixed dispersal rate. In most cases, random dispersal among the patches is taken as static. The dispersal rate plays a vital role for the persistence of the species. Again, the dispersal rate also gives the evolutionarily stable patterns of the species.

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There are different ways for study the dispersal effect on species. One of the best paths is to replace weak fit phenotypes by better fit phenotypes due to the evolution process. In this process, the dispersal possession of the phenotypes is very important. Now, let us recall the idea of IFD (ideal free distribution). The basic idea of IFD is to consider the same fitness of individuals in all occupied areas. The concept of IFD was supported by many researchers through their works ([18],[36]). The leading factor of IFD is the balanced dispersal which is fixed at equilibrium in population dynamics. There is another way to examine the stability behaviour of dispersal; otherwise, an unstable behaviour of the population is obtained. To be precise, the prey-predator system can stabilize for passive animal dispersal at fixed rates ([41]). Also, passive dispersal can create a negative density dependence recruitment rate for obtaining a stable equilibrium state of the system. Therefore, the unstable nature of the population may become stable due to the heterogeneity of the dispersal rates in either patch at the metapopulation level. It is also observed that the dispersal speed cannot be taken very high, because it synchronizes the local behaviour of the system between the patches. There are some model structures in which the authors are considered to have fixed dispersal rates ([19],[37]). Also, these types of model are based on the assumption that the population densities change infinitely fast due to the dispersal approach. Krivan [37] found an interesting result for two competitive population where the fast adaptive animal dispersal destabilize the population equilibrium which does not disperse. Many other researchers observed that, population dynamics is hugely affected by different dispersal dynamics ([1]-[3], [45]-[51]).
On the other hand, recently many researchers have considered a prey-predator dispersal system with predator density dependent [11]. To produce the most accurate model, it is necessary to consider the revolting of predators on prey. When the predator species is augmented in one patch, prey dispersal speed steadily upsurges from the patch. For example, in the aquatic environment, there are various species of zooplankton that are moved vertically in each day for light and food. During the day, because of the sun's light, the predator risk is always higher than at night. For safety reasons, some species migrate downward into darkness to escape the prey. Again, at night, in the absence of sunlight those species come upward to devour the phytoplanktons that are created in the presence of sun light [3]. Slusarczyck et al. [48] made an experiment for finding the abilities of this type of prey to recognize the danger of predation. All the individuals are moved for the case of vertical dial migration which occurrs on each day.
The positive density dependence of population growth at low densities is known as the Allee effect ([4],[16],[49]). Mate finding, social dysfunction, inbreeding, food exploitation etc. [50] are the main causes of the Allee effect. Two types of Allee effect are there namely the strong Allee effect ([9], [44]-[56]) and the weak Allee effect ([25],[56]). This classification is based on the per capita enlargement rate of the population at low density. The growth equation due to the Allee effect takes the form: $f(x) = rx\left(1-\frac{x}{k}\right)\left(\frac{x}{k_0}-1\right)$, where x is the density of the population at time t, r is the per capita growth rate and k denotes the carrying capacity of the environment with $0 < k_0 << k$. The population growth rate decreaces ([8]-[33]) and goes to be extinct when $k_0 > 0$ and the population size remains below the threshold level k_0. This phenomenon is recognized as the strong Allee effect. On the other hand, if the growth rate of the population decreases, remaining positive at low population density, then the phenomenon is known as the weak Allee effect. There is no importance of the threshold level of the population size for the weak Allee effect. Different studies taking saturated recovery function, harvesting, disease in prey, intraspecific competition among predators, two-species competitive environment etc. have already been done together with the Allee effect. Stability behaviour of prey-predator model in presence and the absence of the Allee effect is enormously studied by Celik and Duman [14]. Javidi and Nyamorady [32] studied the effect of the Allee effect on a prey-predator harvesting system. Wang et al. [53] find that, the prey-predator system exhibit intricate population dynamics for the Allee effect.

Zhou et al. [57] discussed the population interaction of a prey-predator system with the Allee effect. Wang et al. [54] proposed competitive population model subject to the Allee effect. Kent et al. [34] used the Allee effect as prey outluxes in their model. Therefore, a large number of research works ([5]-[52]) on prey-predator interactions have already been analysed in the field of ecological modeling. In this work, we have formulated an ecological model combining population interaction, adaptive dispersal and Allee effect in a three-patch environment. To understand the situation clearly, this study presents three patches of predator-prey interactions in two parts, where the first part provides the standard prey-predator system and the other part gives the prey-predator interaction with dispersal. The present model has been divided into three patches namely Patch 1, Patch 2 and Patch 3, respectively. Each patch in our model consists of a pair of prey and predator and strong Allee effect is introduced in each patch for prey population growth only. The dispersal between the patches is formulated in such a way that individuals can move between the patches according to their higher fitness. Our main objective in the present paper is to study how dispersal and the Allee effect influence the dynamics of our anticipated system.

The remainder of the portion of our present study is sketched as follows: Formulation of the mathematical model is done in section 2. Positivity with boundedness of the system are discussed in section 3. Equilibria and stability analysis in the presence and absence of dispersal of our present model have been included in Section 4. Bifurcation analysis has been done in section 5. Numerical simulations using MATLAB are introduced in section 6. Lastly, section 7 provides some conclusions about the proposed model including the biological implications of our mathematical and analytical findings.

2. Formulation of mathematical model

The Allee effect is a natural phenomenon that tends to reduce population density. But the dispersion can increase or decrease the population density in any patch. So, a three patch based model is considered to study the effect of dispersal and Allee effect on the dynamics of a prey-predator relationship. This study considers a six species prey-predator system in a three patch ($i = 1, 2, 3$) environment for each prey and predator depending on the following notation and assumptions:

- x_i : Population density of prey species in patch i.
- y_i : Population density of predator species in patch i.
- r_i : Growth rate of prey in patch i.
- k_i : Environmental carrying capacity of the prey species in patch i in non-attendance of predation and dispersal.
- $\overline{k}_i$: Threshold value of the Allee effect in patch i without predation and dispersal.
- λ_i : Predation rate in patch i.
- δ : Speed of the dispersal between the patches.
- d_{ij} : Probability of dispersion from the i th patch to the j th patch ($i \neq j$).
- e_i : Conversion rate of prey biomass to predator biomass in patch i.
- m_i : Mortality rate of the predator species in patch i.

Statements of system behaviour:

(i) Growth rate of prey population is affected by Allee effect in all three patches.
(ii) All predator species are free from the Allee effect.
(iii) Only the prey species are movable to higher fitness patches.
(iv) All parameters are non-negative in nature.

(v) We consider balanced dispersal, mathematically $\frac{d_{12}}{d_{21}} = \frac{k_2}{k_1}$, $\frac{d_{13}}{d_{31}} = \frac{k_3}{k_1}$, $\frac{d_{23}}{d_{32}} = \frac{k_3}{k_2}$ i.e., there is no mesh progress between the patches.

The graphical view of our proposed system of the three-patch environment is presented in Figure 1.

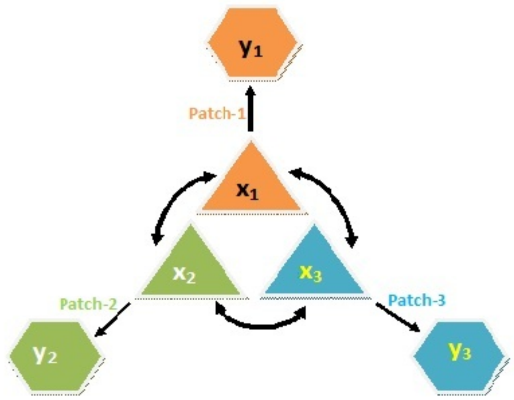

FIGURE 1. Prey-predator interaction in three patch environment.

Based on the above notation and assumptions, the proposed population dispersal dynamics can be described by the following set of differential equations:

$$\begin{cases} \dfrac{dx_1}{dt} = r_1 x_1 \left(1 - \dfrac{x_1}{k_1}\right)\left(\dfrac{x_1}{\overline{k}_1} - 1\right) - \lambda_1 x_1 y_1 + \delta\left(d_{21}x_2 + d_{31}x_3 - d_{12}x_1 - d_{13}x_1\right) \\ \dfrac{dx_2}{dt} = r_2 x_2 \left(1 - \dfrac{x_2}{k_2}\right)\left(\dfrac{x_2}{\overline{k}_2} - 1\right) - \lambda_2 x_2 y_2 + \delta\left(d_{12}x_1 + d_{32}x_3 - d_{21}x_2 - d_{23}x_2\right) \\ \dfrac{dx_3}{dt} = r_3 x_3 \left(1 - \dfrac{x_3}{k_3}\right)\left(\dfrac{x_3}{\overline{k}_3} - 1\right) - \lambda_3 x_3 y_3 + \delta\left(d_{13}x_1 + d_{23}x_2 - d_{31}x_3 - d_{32}x_3\right) \\ \dfrac{dy_1}{dt} = e_1 x_1 y_1 - m_1 y_1 \\ \dfrac{dy_2}{dt} = e_2 x_2 y_2 - m_2 y_2 \\ \dfrac{dy_3}{dt} = e_3 x_3 y_3 - m_3 y_3 \end{cases} \tag{2.1}$$

with the initial condition

$$x_i(0) > 0 \text{ and } y_i(0) > 0 \;\; (i = 1, 2, 3) \tag{2.2}$$

Dispersion sometimes acts as an alternative to avoid predation. In this paper, we are interested in analysing the balanced dispersal among preys. Here, we have considered the balanced dispersal. Hence, mathematically we can write $\frac{d_{12}}{d_{21}} = \frac{k_2}{k_1}$, $\frac{d_{13}}{d_{31}} = \frac{k_3}{k_1}$, $\frac{d_{23}}{d_{32}} = \frac{k_3}{k_2}$. Thus, the identical carrying capacities to the population abundances in all patches, always implies that, there is no mesh progress among the patches. Thus, this situation is exactly the same as without dispersal. Therefore, in absence of dispersal, our proposed model (2.1) reduces to the following system of equations:

$$\begin{cases} \dfrac{dx_1}{dt} = r_1 x_1 \left(1 - \dfrac{x_1}{k_1}\right)\left(\dfrac{x_1}{\overline{k}_1} - 1\right) - \lambda_1 x_1 y_1 \equiv f_1 \\ \dfrac{dx_2}{dt} = r_2 x_2 \left(1 - \dfrac{x_2}{k_2}\right)\left(\dfrac{x_2}{\overline{k}_2} - 1\right) - \lambda_2 x_2 y_2 \equiv f_2 \\ \dfrac{dx_3}{dt} = r_3 x_3 \left(1 - \dfrac{x_3}{k_3}\right)\left(\dfrac{x_3}{\overline{k}_3} - 1\right) - \lambda_3 x_3 y_3 \equiv f_3 \\ \dfrac{dy_1}{dt} = e_1 x_1 y_1 - m_1 y_1 \equiv f_4 \\ \dfrac{dy_2}{dt} = e_2 x_2 y_2 - m_2 y_2 \equiv f_5 \\ \dfrac{dy_3}{dt} = e_3 x_3 y_3 - m_3 y_3 \equiv f_6 \end{cases} \tag{2.3}$$

with the initial condition $x_i(0) > 0$ and $y_i(0) > 0$ for $i = 1, 2, 3$.

3. Positivity and Boundedness of the system

Theorem 3.1. *All the solutions of the system (2.1) with initial condition (2.2) exist in the interval $[0, \infty)$ and $x_i(t) > 0$, $y_i(t) > 0$ for $i = 1, 2, 3$ for all $t \geq 0$ at which the solution exists.*

Proof. As the right hand side of the system (2.1) is completely continuous and locally Lipschitzian in the space of continuous functions (C), the solution $(x_1(t), x_2(t), x_3(t), y_1(t), y_2(t), y_3(t))$ of (2.1) with the initial condition (2.2) exists and is unique on a maximal interval of existence $[0,\zeta)$ where $0 < \zeta < +\infty$. Since each equation in (2.3) is of Gaussian type, that is, it can be written as $u'(t) = ug(u)$, the positiveness of each variable of the solution remains positive for $t \in [0, \zeta)$ (by $u(t) = u(0)\exp(\int_0^t g(u(s))ds)$). Therefore, $x_i(t) > 0$ and $y_i(t) > 0$ $(i = 1, 2, 3)$ for all $t \in [0, \zeta)$. This completes the proof. □

Theorem 3.2. *All the solutions of the system (2.1) are bounded forever.*

Proof. Let us consider

$$P = \sum_{i=1}^{3} [Q x_i(t) + R_i y_i(t)] \text{ where } R_i = Q\frac{\lambda_i}{e_i}(i = 1, 2, 3).$$

Also consider

$$\epsilon_i = \frac{r_i(k_i + \overline{k}_i)}{3} + \frac{\sqrt{r_i^2(k_i^2 + \overline{k}_i^2 - k_i\overline{k}_i) + 3r_i k_i \overline{k}_i}}{3r_i} \quad \text{for } i = 1, 2, 3$$

and

$$Q = \frac{1}{\sum\limits_{i=1}^{3} \left[1 + (1/k_i + 1/\overline{k}_i) r_i \epsilon_i - r_i \epsilon_i^2 / k_i \overline{k}_i - r_i\right] \epsilon_i}.$$

Then,

$$\begin{aligned}\frac{dP}{dt} &= \sum_{i=1}^{3}(Q\frac{dx_i}{dt} + R_i\frac{dy_i}{dt})\\ &= \sum_{i=1}^{3}[Q\{r_ix_i\left(1-\frac{x_i}{k_i}\right)\left(\frac{x_i}{\overline{k}_i}-1\right) - \lambda_ix_iy_i\} + R_i\{e_ix_iy_i - m_iy_i\}]\\ &= \sum_{i=1}^{3}Q\{r_ix_i\left(1-\frac{x_i}{k_i}\right)\left(\frac{x_i}{\overline{k}_i}-1\right) + x_i\} - \sum_{i=1}^{3}(Qx_i - R_im_iy_i)\end{aligned}$$

Now, taking

$$S = \{r_ix_i\left(1-\frac{x_i}{k_i}\right)\left(\frac{x_i}{\overline{k}_i}-1\right) + x_i\},$$

we found that, S attains its maximum at $(\epsilon_1, \epsilon_2, \epsilon_3)$ and $S_{\max} = \frac{1}{Q}$. Therefore,

$$\frac{dP}{dt} \leq S_{\max} \times Q - \phi P \text{ where } \phi = \min\{1, m_1, m_2, m_3\}$$

This implies,

$$\frac{dP}{dt} \leq 1 - \phi P$$

Now, applying the theory of differential inequality, we get,

$$P(t) \leq \frac{1}{\phi}(1 - e^{-\phi t}) + P(x_1(0), x_2(0), x_3(0), y_1(0), y_2(0), y_3(0))e^{-\phi t}$$

Taking $t \to \infty$,

$$P(t) \leq \frac{1}{\phi}$$

Then all the solutions of (2.1) that initiates in $\mathbb{R}^6_+$ are confined in the region:

$$\Pi = \{(x_1, x_2, x_3, y_1, y_2, y_3) : x_i \in [0, k_i], i = 1, 2, 3; \quad 0 \leq P(t) \leq \frac{1}{\phi} + \varepsilon, \varepsilon > 0\}$$

Thus, the solutions of (2.1) are bounded forever. □

Remark 3.1. The boundedness established in Theorem 3.2 together with the extension theory of ODEs implies that the maximal interval of existence is actually $[0, \infty)$, meaning that the unique solution claimed in Theorem 3.1 indeed exists globally.

4. Equilibria and Stability of Proposed Model

In the absence of dispersal ($\delta = 0$), apart from the trivial equilibrium $E_0 = (0,0,0,0,0,0)$, there are other equilibrium points as follows:

$$E_1 = (k_1,0,0,0,0,0),\quad E_2 = (\overline{k}_1,0,0,0,0,0),$$
$$E_3 = (0,k_2,0,0,0,0),\quad E_4 = (0,\overline{k}_2,0,0,0,0),$$
$$E_5 = (0,0,k_3,0,0,0),\quad E_6 = (0,0,\overline{k}_3,0,0,0),$$
$$E_7 = \left(\frac{m_1}{e_1},0,0,\frac{r_1(e_1k_1-m_1)(m_1-e_1\overline{k}_1)}{\lambda_1 e_1^2 k_1\overline{k}_1},0,0\right),$$
$$E_8 = \left(0,\frac{m_2}{e_2},0,0,\frac{r_2(e_2k_2-m_2)(m_2-e_2\overline{k}_2)}{\lambda_2 e_2^2 k_2\overline{k}_2},0\right),$$
$$E_9 = \left(0,0,\frac{m_3}{e_3},0,0,\frac{r_3(e_3k_3-m_3)(m_3-e_3\overline{k}_3)}{\lambda_3 e_3^2 k_3\overline{k}_3}\right),$$
$$E_{10} = \left(\frac{m_1}{e_1},\frac{m_2}{e_2},0,\frac{r_1(e_1k_1-m_1)(m_1-e_1\overline{k}_1)}{\lambda_1 e_1^2 k_1\overline{k}_1},\frac{r_2(e_2k_2-m_2)(m_2-e_2\overline{k}_2)}{\lambda_2 e_2^2 k_2\overline{k}_2},0\right),$$
$$E_{11} = \left(\frac{m_1}{e_1},0,\frac{m_3}{e_3},\frac{r_1(e_1k_1-m_1)(m_1-e_1\overline{k}_1)}{\lambda_1 e_1^2 k_1\overline{k}_1},0,\frac{r_3(e_3k_3-m_3)(m_3-e_3\overline{k}_3)}{\lambda_3 e_3^2 k_3\overline{k}_3}\right),$$
$$E_{12} = \left(0,\frac{m_2}{e_2},\frac{m_3}{e_3},0,\frac{r_2(e_2k_2-m_2)(m_2-e_2\overline{k}_2)}{\lambda_2 e_2^2 k_2\overline{k}_2},\frac{r_3(e_3k_3-m_3)(m_3-e_3\overline{k}_3)}{\lambda_3 e_3^2 k_3\overline{k}_3}\right).$$

Again, in the presence of dispersal ($\delta \neq 0$), $E_i,\ i = 0,1,\cdots 6$, are still equilibrium points of (2.1); however, the equilibrium points $E_7 - E_{12}$ for (2.3) are now revised to the following equilibria for (2.1) which reflect the the impact of the dispersal rates:

$$\hat{E}_7 = \left(\frac{m_1}{e_1},0,0,(\frac{e_1}{\lambda_1 m_1}(\frac{r_1m_1(e_1k_1-m_1)(m_1-e_1\overline{k}_1)}{e_1^3k_1\overline{k}_1}-\frac{\delta m_1}{e_1}(d_{12}+d_{13})),0,0\right),$$
$$\hat{E}_8 = \left(0,\frac{m_2}{e_2},0,0,(\frac{e_2}{\lambda_2 m_2}(\frac{r_2m_2(e_2k_2-m_2)(m_2-e_2\overline{k}_2)}{e_2^3k_2\overline{k}_2}-\frac{\delta m_2}{e_2}(d_{21}+d_{23})),0\right),$$
$$\hat{E}_9 = \left(0,0,\frac{m_3}{e_3},0,0,(\frac{e_3}{\lambda_3 m_3}(\frac{r_3m_3(e_3k_3-m_3)(m_3-e_3\overline{k}_3)}{e_3^3k_3\overline{k}_3}-\frac{\delta m_3}{e_3}(d_{31}+d_{32}))\right),$$

$$\hat{E}_{10} = \left(\frac{m_1}{e_1},\frac{m_2}{e_2},0,\left(\frac{e_1}{\lambda_1 m_1}\left(\frac{r_1m_1(e_1k_1-m_1)(m_1-e_1\overline{k}_1)}{e_1^3k_1\overline{k}_1}+\delta\left(d_{21}\frac{m_2}{e_2}-d_{12}\frac{m_1}{e_1}-d_{13}\frac{m_1}{e_1}\right)\right)\right),\right.$$
$$\left.\left(\frac{e_2}{\lambda_2 m_2}\left(\frac{r_2m_2(e_2k_2-m_2)(m_2-e_2\overline{k}_2)}{e_2^3k_2\overline{k}_2}+\delta\left(d_{12}\frac{m_1}{e_1}-d_{21}\frac{m_2}{e_2}-d_{23}\frac{m_2}{e_2}\right)\right)\right),0\right),$$

$$\hat{E}_{11} = \left(\frac{m_1}{e_1},0,\frac{m_3}{e_3},\left(\frac{e_1}{\lambda_1 m_1}\left(\frac{r_1m_1(e_1k_1-m_1)(m_1-e_1\overline{k}_1)}{e_1^3k_1\overline{k}_1}+\delta\left(d_{21}\frac{m_2}{e_2}-d_{12}\frac{m_1}{e_1}-d_{13}\frac{m_1}{e_1}\right)\right)\right),\right.$$
$$\left.0,\left(\frac{e_3}{\lambda_3 m_3}\left(\frac{r_3m_3(e_3k_3-m_3)(m_3-e_3\overline{k}_3)}{e_3^3k_3\overline{k}_3}+\delta\left(d_{13}\frac{m_1}{e_1}-d_{31}\frac{m_3}{e_3}-d_{32}\frac{m_3}{e_3}\right)\right)\right)\right),$$
$$\hat{E}_{12} = \left(0,\frac{m_2}{e_2},\frac{m_3}{e_3},0,\left(\frac{e_2}{\lambda_2 m_2}\left(\frac{r_2m_2(e_2k_2-m_2)(m_2-e_2\overline{k}_2)}{e_2^3k_2\overline{k}_2}+\delta\left(d_{12}\frac{m_1}{e_1}-d_{21}\frac{m_2}{e_2}-d_{23}\frac{m_2}{e_2}\right)\right)\right),\right.$$
$$\left.\left(\frac{e_3}{\lambda_3 m_3}\left(\frac{r_3m_3(e_3k_3-m_3)(m_3-e_3\overline{k}_3)}{e_3^3k_3\overline{k}_3}+\delta\left(d_{13}\frac{m_1}{e_1}-d_{31}\frac{m_3}{e_3}-d_{32}\frac{m_3}{e_3}\right)\right)\right)\right).$$

For possible positive equilibrium point of the proposed model (2.1), we distinguish the case without dispersals ($\delta = 0$) and the case with dispersals ($\delta \neq 0$) to proceed respectively below.

Case 1. $\underline{\delta = 0}$.

A positive equilibrium $E^* = (x_{10}^*, x_{20}^*, x_{30}^*, y_{10}^*, y_{20}^*, y_{30}^*)$ can be found by solving the system of equations:

$$r_1\left(1-\frac{x_1}{k_1}\right)\left(\frac{x_1}{\overline{k}_1}-1\right)-\lambda_1 y_1=0, \tag{4.1}$$

$$r_2\left(1-\frac{x_2}{k_2}\right)\left(\frac{x_2}{\overline{k}_2}-1\right)-\lambda_2 y_2=0, \tag{4.2}$$

$$r_3\left(1-\frac{x_3}{k_3}\right)\left(\frac{x_3}{\overline{k}_3}-1\right)-\lambda_3 y_3=0, \tag{4.3}$$

$$e_1x_1-m_1=0, \tag{4.4}$$

$$e_2x_2-m_2=0, \tag{4.5}$$

$$e_3x_3-m_3=0. \tag{4.6}$$

Equations (4.4), (4.5) and (4.6) give $x_{i0}^* = \frac{m_i}{e_i}$ $(i = 1, 2, 3)$. Also from (4.1)-(4.3) we have:

$$\begin{aligned} y_{i0}^* &= \frac{r_i}{\lambda_i}\left\{-\frac{x_i^2}{k_i\overline{k}_i}+\left(\frac{1}{k_i}+\frac{1}{\overline{k}_i}\right)x_i-1\right\}_{x_i=\frac{m_i}{e_i}} \\ &= \frac{r_i}{\lambda_i k_i\overline{k}_i}\left\{-\frac{m_i^2}{e_i^2}+\left(k_i+\overline{k}_i\right)\frac{m_i}{e_i}-k_i\overline{k}_i\right\}, \quad i=1,2,3. \end{aligned}$$

Therefore, $y_{i0}^* > 0$ if

$$\left(k_i+\overline{k}_i\right)\frac{m_i}{e_i}-k_i\overline{k}_i>\frac{m_i^2}{e_i^2}, \quad i=1,2,3. \tag{4.7}$$

Assume

$$k_i+\overline{k}_i>\frac{2m_i}{e_i}, \quad i=1,2,3. \tag{4.8}$$

Let

$$\begin{aligned} k_1&=\frac{m_1}{e_1}+\eta_1, \quad \overline{k}_1=\frac{m_1}{e_1}-\eta_2, \quad k_2=\frac{m_2}{e_2}+\eta_3, \\ \overline{k}_2&=\frac{m_2}{e_2}-\eta_4, \quad k_3=\frac{m_3}{e_3}+\eta_5, \quad \overline{k}_3=\frac{m_3}{e_3}-\eta_6. \end{aligned}$$

where

$$\eta_1>\eta_2, \eta_3>\eta_4, \quad \eta_5>\eta_6; \quad \eta_2<\frac{m_1}{e_1}, \quad \eta_4<\frac{m_2}{e_2}, \quad \eta_6<\frac{m_3}{e_3}.$$

Therefore, (4.8) is satisfied. Now,

$$\left(k_1+\overline{k}_1\right)\frac{m_1}{e_1}-k_1\overline{k}_1-\frac{m_1^2}{e_1^2}=2\frac{m_1^2}{e_1^2}+(\eta_1-\eta_2)\frac{m_1}{e_1}-\left(\frac{m_1}{e_1}+\eta_1\right)\left(\frac{m_1}{e_1}-\eta_2\right)-\frac{m_1^2}{e_1^2}=\eta_1\eta_2>0,$$

$$\left(k_2+\overline{k}_2\right)\frac{m_2}{e_2}-k_2\overline{k}_2-\frac{m_2^2}{e_2^2}=2\frac{m_2^2}{e_2^2}+(\eta_3-\eta_4)\frac{m_2}{e_2}-\left(\frac{m_2}{e_2}+\eta_3\right)\left(\frac{m_2}{e_2}-\eta_4\right)-\frac{m_2^2}{e_2^2}=\eta_3\eta_4>0,$$

$$\left(k_3+\overline{k}_3\right)\frac{m_3}{e_3}-k_3\overline{k}_3-\frac{m_3^2}{e_3^2}=2\frac{m_3^2}{e_3^2}+(\eta_5-\eta_6)\frac{m_3}{e_3}-\left(\frac{m_3}{e_3}+\eta_5\right)\left(\frac{m_3}{e_3}-\eta_6\right)-\frac{m_3^2}{e_3^2}=\eta_5\eta_6>0.$$

Therefore, (4.7) is verified. Hence (4.7) or (4.8) implies $y_{i0}^* > 0, i = 1, 2, 3$. Thus

$$x_{i0}^* = \frac{m_i}{e_i} \tag{4.9}$$

$$y_{i0}^* = \frac{r_i}{\lambda_i k_i\overline{k}_i}\left\{-\frac{m_i^2}{e_i^2}+\left(k_i+\overline{k}_i\right)\frac{m_i}{e_i}-k_i\overline{k}_i\right\}, (i=1,2,3)\,. \tag{4.10}$$

4.1. **Local stability analysis without dispersal.** The Jacob matrix of the system (2.3) at E_0 is

$$J(E_0)=\begin{bmatrix} -r_1 & 0 & 0 & 0 & 0 & 0\\ 0 & -r_2 & 0 & 0 & 0 & 0\\ 0 & 0 & -r_3 & 0 & 0 & 0\\ 0 & 0 & 0 & -m_1 & 0 & 0\\ 0 & 0 & 0 & 0 & -m_2 & 0\\ 0 & 0 & 0 & 0 & 0 & -m_3 \end{bmatrix}$$

from which, we can conclude that E_0 is asymptotically stable.

The Jacob matrix of (2.3) at $E_1=(k_1,0,0,0,0,0)$ is

$$J(E_1)=\begin{bmatrix} -r_1(\frac{k_1}{\overline{k}_1}-1) & 0 & 0 & -\lambda_1 k_1 & 0 & 0\\ 0 & -r_2 & 0 & 0 & 0 & 0\\ 0 & 0 & -r_3 & 0 & 0 & 0\\ 0 & 0 & 0 & e_1k_1-m_1 & 0 & 0\\ 0 & 0 & 0 & 0 & -m_2 & 0\\ 0 & 0 & 0 & 0 & 0 & -m_3 \end{bmatrix}.$$

As $k_1>\overline{k}_1$, $J(E_1)$ is a stable matrix if $e_1k_1<m_1$ and hence E_1) is asymptotically stable for $e_1k_1<m_1$.

Similarly, $E_2=(0,k_2,0,0,0,0)$ and $E_3=(0,0,k_3,0,0,0)$ are asymptotically stable for $e_ik_i<m_i,\quad i=2,3$.

Furthermore, the characteristic equation for the variational matrix of (2.3) at $\big(\overline{k}_1,0,0,0,0,0\big)$ is

$$\begin{vmatrix} -r_1(1-\frac{\overline{k}_1}{k_1})-\lambda & 0 & 0 & -\lambda_1\overline{k}_1 & 0 & 0\\ 0 & -r_2-\lambda & 0 & 0 & 0 & 0\\ 0 & 0 & -r_3-\lambda & 0 & 0 & 0\\ 0 & 0 & 0 & e_1\overline{k}_1-m_1-\lambda & 0 & 0\\ 0 & 0 & 0 & 0 & -m_2-\lambda & 0\\ 0 & 0 & 0 & 0 & 0 & -m_3-\lambda \end{vmatrix}=\mathbf{0}.$$

As $\overline{k}_1 << k_1$, all the roots of the equation are negative for $e_1\overline{k}_1<m_1$ and hence $(k_1,0,0,0,0,0)$ is stable for $e_1\overline{k}_1<m_1$.

In a similar way $\big(0,\overline{k}_2,0,0,0,0\big)$ and $\big(0,0,\overline{k}_3,0,0,0\big)$ are stable for $e_i\overline{k}_i<m_i(i=2,3)$ as $\overline{k}_i<k_i(i=2,3)$.

Again, the characteristic equation of (2.3) at E_7, is

$$\begin{vmatrix} \frac{r_1m_1(e_1k_1+e_1\overline{k}_1-2m_1)}{e_1^2k_1\overline{k}_1}-\lambda & 0 & 0 & \frac{-\lambda_1m_1}{e_1} & 0 & 0\\ 0 & -r_2-\lambda & 0 & 0 & 0 & 0\\ 0 & 0 & -r_3-\lambda & 0 & 0 & 0\\ \frac{r_1(e_1k_1-m_1)(m_1-e_1\overline{k}_1)}{\lambda_1e_1k_1\overline{k}_1} & 0 & 0 & -\lambda & 0 & 0\\ 0 & 0 & 0 & 0 & -m_2-\lambda & 0\\ 0 & 0 & 0 & 0 & 0 & -m_3-\lambda \end{vmatrix}=\mathbf{0}.$$

Simplifying we get,

$$\lambda=-r_2,-r_3,-m_2,-m_3,$$

and

$$\lambda^2+\{\frac{r_1m_1(2m_1-e_1k_1-e_1\overline{k}_1)}{e_1^2k_1\overline{k}_1}\}\lambda+\frac{r_1m_1(e_1k_1-m_1)(m_1-e_1\overline{k}_1)}{e_1^2k_1\overline{k}_1}=0.$$

So, all the roots of the characteristic equation are negative for

$$2m_1 > e_1k_1 + e_1\overline{k}_1$$

and

$$(e_1k_1 - m_1)(m_1 - e_1\overline{k}_1) > 0.$$

Therefore, the equilibrium E_7 is stable if

$$2m_1 > e_1k_1 + e_1\overline{k}_1$$

and

$$(e_1k_1 - m_1)(m_1 - e_1\overline{k}_1) > 0.$$

Similarly, E_8 (resp.E_9) is also stable if

$$2m_i > e_ik_i + e_i\overline{k}_i$$

and

$$(e_ik_i - m_i)(m_i - e_i\overline{k}_i) > 0$$

for $i = 2$ (resp. $i = 3$).

Next, the characteristic equation of (2.3) at E_{10} is

$$\begin{vmatrix} \frac{r_1m_1(e_1k_1+e_1\overline{k}_1-2m_1)}{e_1^2k_1\overline{k}_1} - \lambda & 0 & 0 & \frac{-\lambda_1m_1}{e_1} & 0 & 0 \\ 0 & \frac{r_2m_2(e_2k_2+e_2\overline{k}_2-2m_2)}{e_2^2k_2\overline{k}_2} - \lambda & 0 & 0 & \frac{-\lambda_2m_2}{e_2} & 0 \\ 0 & 0 & -r_3 - \lambda & 0 & 0 & 0 \\ \frac{r_1(e_1k_1-m_1)(m_1-e_1\overline{k}_1)}{\lambda_1e_1k_1\overline{k}_1} & \frac{r_2(e_2k_2-m_2)(m_2-e_2\overline{k}_2)}{\lambda_2e_2k_2\overline{k}_2} & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & -m_3 - \lambda \end{vmatrix} = \mathbf{0}.$$

Simplifying we get, $\lambda = -r_3, -m_3$ and

$$\lambda^4 + T_1\lambda^3 + T_2\lambda^2 + T_3\lambda + T_4 = 0$$

where,

$$\begin{aligned} T_1 &= \frac{r_1m_1(2m_1 - e_1k_1 - e_1\overline{k}_1)}{e_1^2k_1\overline{k}_1} + \frac{r_2m_2(2m_2 - e_2k_2 - e_2\overline{k}_2)}{e_2^2k_2\overline{k}_2}, \\ T_2 &= \frac{r_1r_2m_1m_2(e_1k_1 + e_1\overline{k}_1 - 2m_1)(e_2k_2 + e_2\overline{k}_2 - 2m_2)}{e_1e_2k_1\overline{k}_1k_2\overline{k}_2} + \frac{m_1r_1(e_1k_1 - m_1)(m_1 - e_1\overline{k}_1)}{e_1^2k_1\overline{k}_1} \\ &\quad + \frac{m_2r_2(e_2k_2 - m_2)(m_2 - e_2\overline{k}_2)}{e_2^2k_2\overline{k}_2}, \\ T_3 &= \frac{r_1r_2m_1m_2}{e_1^2e_2^2k_1\overline{k}_1k_2\overline{k}_2}\left[(2m_1 - e_1k_1 - e_1\overline{k}_1)(e_2k_2 - m_2)(m_2 - e_2\overline{k}_2)\right. \\ &\quad \left.+(2m_2 - e_2k_2 - e_2\overline{k}_2)(e_1k_1 - m_1)(m_1 - e_1\overline{k}_1)\right], \\ T_4 &= \frac{r_1r_2m_1m_2}{e_1^2e_2^2k_1\overline{k}_1k_2\overline{k}_2}\left[(e_1k_1 - m_1)(m_1 - e_1\overline{k}_1)(e_2k_2 - m_2)(m_2 - e_2\overline{k}_2)\right]. \end{aligned}$$

The Hurwitz matrix for the biquadratic equation is

$$L = \begin{bmatrix} T_1 & 1 & 0 & 0 \\ T_3 & T_2 & T_1 & 1 \\ 0 & T_4 & T_3 & T_2 \\ 0 & 0 & 0 & T_4 \end{bmatrix}, \Delta_1 = T_1, \Delta_2 = \begin{vmatrix} T_1 & 1 \\ T_3 & T_2 \end{vmatrix}, \Delta_3 = \begin{vmatrix} T_1 & 1 & 0 \\ T_3 & T_2 & T_1 \\ 0 & T_4 & T_3 \end{vmatrix}.$$

The roots of the auxiliary equation have negative real parts if and only if all the principal diagonal minors of the Hurwitz matrix are positive provided $\Delta_i > 0, (i = 1, 2, 3)$ and $T_4 > 0$. Therefore, E_{10} is stable for $\Delta_i > 0, (i = 1, 2, 3)$ and $T_4 > 0$ where $T_i (i = 1, 2, 3, 4)$ are expressed as above.

Similarly, E_{11} is stable if $\Delta_i > 0, (i = 1, 2, 3)$ and $T_4 > 0$ where

$$\Delta_1 = T_1, \Delta_2 = \begin{vmatrix} T_1 & 1 \\ T_3 & T_2 \end{vmatrix}, \Delta_3 = \begin{vmatrix} T_1 & 1 & 0 \\ T_3 & T_2 & T_1 \\ 0 & T_4 & T_3 \end{vmatrix},$$

$$\begin{aligned}
T_1 &= \frac{r_1 m_1 (2m_1 - e_1 k_1 - e_1 \overline{k}_1)}{e_1^2 k_1 \overline{k}_1} + \frac{r_3 m_3 (2m_3 - e_3 k_3 - e_3 \overline{k}_3)}{e_3^2 k_3 \overline{k}_3}, \\
T_2 &= \frac{r_1 r_3 m_1 m_3 (e_1 k_1 + e_1 \overline{k}_1 - 2m_1)(e_3 k_3 + e_3 \overline{k}_3 - 2m_3)}{e_1 e_3 k_1 \overline{k}_1 k_3 \overline{k}_3} + \frac{m_1 r_1 (e_1 k_1 - m_1)(m_1 - e_1 \overline{k}_1)}{e_1^2 k_1 \overline{k}_1} \\
&\quad + \frac{m_3 r_3 (e_3 k_3 - m_3)(m_3 - e_3 \overline{k}_3)}{e_3^2 k_3 \overline{k}_3}. \\
T_3 &= \frac{r_1 r_3 m_1 m_3}{e_1^2 e_3^2 k_1 \overline{k}_1 k_3 \overline{k}_3} \left[(2m_1 - e_1 k_1 - e_1 \overline{k}_1)(e_3 k_3 - m_3)(m_3 - e_3 \overline{k}_3) \right. \\
&\quad \left. + (2m_3 - e_3 k_3 - e_3 \overline{k}_3)(e_1 k_1 - m_1)(m_1 - e_1 \overline{k}_1) \right], \\
T_4 &= \frac{r_1 r_3 m_1 m_3}{e_1^2 e_3^2 k_1 \overline{k}_1 k_3 \overline{k}_3} \left[(e_1 k_1 - m_1)(m_1 - e_1 \overline{k}_1)(e_3 k_3 - m_3)(m_3 - e_3 \overline{k}_3) \right].
\end{aligned}$$

Also E_{12} is stable for $\Delta_i > 0, (i = 1, 2, 3)$ and $T_4 > 0$ where,

$$\Delta_1 = T_1, \Delta_2 = \begin{vmatrix} T_1 & 1 \\ T_3 & T_2 \end{vmatrix}, \Delta_3 = \begin{vmatrix} T_1 & 1 & 0 \\ T_3 & T_2 & T_1 \\ 0 & T_4 & T_3 \end{vmatrix},$$

$$\begin{aligned}
T_1 &= \frac{r_2 m_2 (2m_2 - e_2 k_2 - e_2 \overline{k}_2)}{e_2^2 k_2 \overline{k}_2} + \frac{r_3 m_3 (2m_3 - e_3 k_3 - e_3 \overline{k}_3)}{e_3^2 k_3 \overline{k}_3}, \\
T_2 &= \frac{r_2 r_3 m_2 m_3 (e_2 k_2 + e_2 \overline{k}_2 - 2m_2)(e_3 k_3 + e_3 \overline{k}_3 - 2m_3)}{e_2 e_3 k_2 \overline{k}_2 k_3 \overline{k}_3} + \frac{m_2 r_2 (e_2 k_2 - m_2)(m_2 - e_2 \overline{k}_2)}{e_2^2 k_2 \overline{k}_2} \\
&\quad + \frac{m_3 r_3 (e_3 k_3 - m_3)(m_3 - e_3 \overline{k}_3)}{e_3^2 k_3 \overline{k}_3}, \\
T_3 &= \frac{r_2 r_3 m_2 m_3}{e_2^2 e_3^2 k_2 \overline{k}_2 k_3 \overline{k}_3} \left[(2m_2 - e_2 k_2 - e_2 \overline{k}_2)(e_3 k_3 - m_3)(m_3 - e_3 \overline{k}_3) \right. \\
&\quad \left. + (2m_3 - e_3 k_3 - e_3 \overline{k}_3)(e_2 k_2 - m_2)(m_2 - e_2 \overline{k}_2) \right], \\
T_4 &= \frac{r_2 r_3 m_2 m_3}{e_2^2 e_3^2 k_2 \overline{k}_2 k_3 \overline{k}_3} \left[(e_2 k_2 - m_2)(m_2 - e_2 \overline{k}_2)(e_3 k_3 - m_3)(m_3 - e_3 \overline{k}_3) \right].
\end{aligned}$$

For the stability of the interior (positive) equilibrium E^*, we have the following:

Theorem 4.1. *The interior equilibrium E^* of the system (2.1) without dispersal $(\delta = 0)$ is unstable under the condition $k_i + \overline{k}_i > \frac{2m_i}{e_i}, (i = 1, 2, 3)$.*

Proof. The characteristic equation of $V(E_1)$ is given by

$$\begin{vmatrix} \{-\frac{r_1}{k_1\bar{k}_1}x_{10}^*(2x_{10}^*-k_1-\bar{k}_1)\}-\lambda & 0 & 0 & -\lambda_1 x_{10}^* & 0 & 0 \\ 0 & \{-\frac{r_2}{k_2\bar{k}_2}x_{20}^*(2x_{20}^*-k_2-\bar{k}_2)\}-\lambda & 0 & 0 & -\lambda_2 x_{20}^* & 0 \\ 0 & 0 & \{-\frac{r_3}{k_3\bar{k}_3}x_{30}^*(2x_{30}^*-k_3-\bar{k}_3)\}-\lambda & 0 & 0 & -\lambda_3 x_{30}^* \\ e_1 y_{10}^* & 0 & 0 & -\lambda & 0 & 0 \\ 0 & e_2 y_{20}^* & 0 & 0 & -\lambda & 0 \\ 0 & 0 & e_3 y_{30}^* & 0 & 0 & -\lambda \end{vmatrix} = 0.$$

Expanding we get,

$$\begin{aligned}
&\lambda_3 x_{30}^* e_3 y_{30}^* \Big\{ \Big(\lambda^2 \lambda_2 x_{20}^* - \frac{\lambda r_1 x_{10}^* \lambda_2 x_{20}^*}{k_1} - \frac{\lambda r_1 x_{10}^* \lambda_2 x_{20}^*}{\bar{k}_1} + \frac{2\lambda r_1 x_{10}^{*2} \lambda_2 x_{20}^*}{k_1\bar{k}_1} + e_1 y_{10}^* \lambda_1 x_{10}^* \lambda_2 x_{20}^* \Big) e_2 y_{20}^* \\
&\quad - \lambda \Big(\lambda^2 - \frac{\lambda r_1 x_{10}^*}{k_1} - \frac{\lambda r_1 x_{10}^*}{\bar{k}_1} + \frac{2\lambda r_1 x_{10}^{*2}}{k_1\bar{k}_1} + e_1 y_{10}^* \lambda_1 x_{10}^* \Big) \Big(-\lambda + \frac{r_2 x_{20}^*}{k_2} + \frac{r_2 x_{20}^*}{\bar{k}_2} - \frac{2 r_2 x_{20}^{*2}}{k_2\bar{k}_2} \Big) \Big\} \\
&\quad - \lambda \Big\{ \Big(\lambda^2 \lambda_2 x_{20}^* - \frac{\lambda r_1 x_{10}^* \lambda_2 x_{20}^*}{k_1} - \frac{\lambda r_1 x_{10}^* \lambda_2 x_{20}^*}{\bar{k}_1} + \frac{2\lambda r_1 x_{10}^{*2} \lambda_2 x_{20}^*}{k_1\bar{k}_1} + e_1 y_{10}^* \lambda_1 x_{10}^* \lambda_2 x_{20}^* \Big) e_2 y_{20}^* \\
&\quad - \lambda \Big(\lambda^2 - \frac{\lambda r_1 x_{10}^*}{k_1} - \frac{\lambda r_1 x_{10}^*}{\bar{k}_1} + \frac{2\lambda r_1 x_{10}^{*2}}{k_1\bar{k}_1} + e_1 y_{10}^* \lambda_1 x_{10}^* \Big) \Big(-\lambda + \frac{r_2 x_{20}^*}{k_2} + \frac{r_2 x_{20}^*}{\bar{k}_2} - \frac{2 r_2 x_{20}^{*2}}{k_2\bar{k}_2} \Big) \Big\} \\
&\quad \times \Big(\frac{r_3 x_{30}^*}{k_3} - \frac{2 r_3 x_{30}^{*2}}{k_3\bar{k}_3} + \frac{r_3 x_{30}^*}{\bar{k}_3} - \lambda \Big) = 0
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \lambda_3 x_{30}^* e_3 y_{30}^* \Big\{ \Big(\lambda^2 \lambda_2 x_{20}^* - \frac{\lambda r_1 x_{10}^* \lambda_2 x_{20}^*}{k_1} - \frac{\lambda r_1 x_{10}^* \lambda_2 x_{20}^*}{\bar{k}_1} + \frac{2\lambda r_1 x_{10}^{*2} \lambda_2 x_{20}^*}{k_1\bar{k}_1} + e_1 y_{10}^* \lambda_1 x_{10}^* \lambda_2 x_{20}^* \Big) e_2 y_{20}^* \\
&\quad -\lambda \Big(\lambda^2 - \frac{\lambda r_1 x_{10}^*}{k_1} - \frac{\lambda r_1 x_{10}^*}{\bar{k}_1} + \frac{2\lambda r_1 x_{10}^{*2}}{k_1\bar{k}_1} + e_1 y_{10}^* \lambda_1 x_{10}^* \Big) \Big(-\lambda + \frac{r_2 x_{20}^*}{k_2} + \frac{r_2 x_{20}^*}{\bar{k}_2} - \frac{2 r_2 x_{20}^{*2}}{k_2\bar{k}_2} \Big) \Big\} \\
&\quad -\lambda \Big\{ \Big(\lambda^2 \lambda_2 x_{20}^* - \frac{\lambda r_1 x_{10}^* \lambda_2 x_{20}^*}{k_1} - \frac{\lambda r_1 x_{10}^* \lambda_2 x_{20}^*}{\bar{k}_1} + \frac{2\lambda r_1 x_{10}^{*2} \lambda_2 x_{20}^*}{k_1\bar{k}_1} + e_1 y_{10}^* \lambda_1 x_{10}^* \lambda_2 x_{20}^* \Big) e_2 y_{20}^* \\
&\quad -\lambda \Big(\lambda^2 - \frac{\lambda r_1 x_{10}^*}{k_1} - \frac{\lambda r_1 x_{10}^*}{\bar{k}_1} + \frac{2\lambda r_1 x_{10}^{*2}}{k_1\bar{k}_1} + e_1 y_{10}^* \lambda_1 x_{10}^* \Big) \Big(-\lambda + \frac{r_2 x_{20}^*}{k_2} + \frac{r_2 x_{20}^*}{\bar{k}_2} - \frac{2 r_2 x_{20}^{*2}}{k_2\bar{k}_2} \Big) \Big\} \\
&\quad \times \Big(\frac{r_3 x_{30}^*}{k_3} - \frac{2 r_3 x_{30}^{*2}}{k_3\bar{k}_3} + \frac{r_3 x_{30}^*}{\bar{k}_3} - \lambda \Big) = 0
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \Big\{ \Big(\lambda^2 \lambda_2 x_{20}^* - \frac{\lambda r_1 x_{10}^* \lambda_2 x_{20}^*}{k_1} - \frac{\lambda r_1 x_{10}^* \lambda_2 x_{20}^*}{\bar{k}_1} + \frac{2\lambda r_1 x_{10}^{*2} \lambda_2 x_{20}^*}{k_1\bar{k}_1} + e_1 y_{10}^* \lambda_1 x_{10}^* \lambda_2 x_{20}^* \Big) e_2 y_{20}^* \\
&\quad -\lambda \Big(\lambda^2 - \frac{\lambda r_1 x_{10}^*}{k_1} - \frac{\lambda r_1 x_{10}^*}{\bar{k}_1} + \frac{2\lambda r_1 x_{10}^{*2}}{k_1\bar{k}_1} + e_1 y_{10}^* \lambda_1 x_{10}^* \Big) \Big(-\lambda + \frac{r_2 x_{20}^*}{k_2} + \frac{r_2 x_{20}^*}{\bar{k}_2} - \frac{2 r_2 x_{20}^{*2}}{k_2\bar{k}_2} \Big) \Big\} \\
&\quad \times \Big\{ \lambda_3 x_{30}^* e_3 y_{30}^* - \lambda \Big(\frac{r_3 x_{30}^*}{k_3} - \frac{2 r_3 x_{30}^{*2}}{k_3\bar{k}_3} + \frac{r_3 x_{30}^*}{\bar{k}_3} - \lambda \Big) \Big\} = 0.
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \Big[\Big(\lambda^2 - \frac{\lambda r_1 x_{10}^*}{k_1} - \frac{\lambda r_1 x_{10}^*}{\bar{k}_1} + \frac{2\lambda r_1 x_{10}^{*2}}{k_1\bar{k}_1} + e_1 y_{10}^* \lambda_1 x_{10}^* \Big) \Big\{ \lambda_2 x_{20}^* e_2 y_{20}^* - \lambda \Big(-\lambda + \frac{r_2 x_{20}^*}{k_2} + \frac{r_2 x_{20}^*}{\bar{k}_2} - \frac{2 r_2 x_{20}^{*2}}{k_2\bar{k}_2} \Big) \Big\} \Big] \\
&\quad \times \Big\{ \lambda_3 x_{30}^* e_3 y_{30}^* - \lambda \Big(\frac{r_3 x_{30}^*}{k_3} - \frac{2 r_3 x_{30}^{*2}}{k_3\bar{k}_3} + \frac{r_3 x_{30}^*}{\bar{k}_3} - \lambda \Big) \Big\} = 0.
\end{aligned}$$

$$\Rightarrow \Big(\lambda^2 - \frac{\lambda r_1 x_{10}^*}{k_1} - \frac{\lambda r_1 x_{10}^*}{\bar{k}_1} + \frac{2\lambda r_1 x_{10}^{*2}}{k_1\bar{k}_1} + e_1 y_{10}^* \lambda_1 x_{10}^* \Big) \times \Big(\lambda_2 x_{20}^* e_2 y_{20}^* + \lambda^2 - \frac{\lambda r_2 x_{20}^*}{k_2} + \frac{2\lambda r_2 x_{20}^{*2}}{k_2\bar{k}_2} - \frac{\lambda r_2 x_{20}^*}{\bar{k}_2} \Big)$$

$$\times\Big(\lambda_3 x_{30}^* e_3 y_{30}^* - \frac{\lambda r_3 x_{30}^*}{k_3} + \frac{2\lambda r_3 x_{30}^{*2}}{k_3 \overline{k}_3} - \frac{\lambda r_3 x_{30}^*}{\overline{k}_3} + \lambda^2\Big) = 0.$$

Therefore,

$$\Big[\lambda^2 + \frac{r_1 x_{10}^*}{k_1 \overline{k}_1}(2x_{10}^* - k_1 - \overline{k}_1)\lambda + e_1\lambda_1 x_{10}^* y_{10}^*\Big] \times \Big[\lambda^2 + \frac{r_2 x_{20}^*}{k_2 \overline{k}_2}(2x_{20}^* - k_2 - \overline{k}_2)\lambda + e_2\lambda_2 x_{20}^* y_{20}^*\Big]$$

$$\times\Big[\lambda^2 + \frac{r_3 x_{30}^*}{k_3 \overline{k}_3}(2x_{30}^* - k_3 - \overline{k}_3)\lambda + e_3\lambda_3 x_{30}^* y_{30}^*\Big] = 0.$$

Simplifying, we obtain,

$$\Big[\lambda^2 + \frac{r_1 x_{10}^*}{k_1 \overline{k}_1}\left(2x_{10}^* - k_1 - \overline{k}_1\right)\lambda + e_1\lambda_1 x_{10}^* y_{10}^*\Big]\Big[\lambda^2 + \frac{r_2 x_{20}^*}{k_2 \overline{k}_2}\left(2x_{20}^* - k_2 - \overline{k}_2\right)\lambda + e_2\lambda_2 x_{20}^* y_{20}^*\Big] \tag{4.11}$$

$$\times\Big[\lambda^2 + \frac{r_3 x_{30}^*}{k_3 \overline{k}_3}\left(2x_{30}^* - k_3 - \overline{k}_3\right)\lambda + e_3\lambda_3 x_{30}^* y_{30}^*\Big] = 0. \tag{4.12}$$

Therefore, we must have,

$$\lambda^2 + \frac{r_1 x_{10}^*}{k_1 \overline{k}_1}\left(2x_{10}^* - k_1 - \overline{k}_1\right)\lambda + e_1\lambda_1 x_{10}^* y_{10}^* = 0, \tag{4.13}$$

$$\lambda^2 + \frac{r_2 x_{20}^*}{k_2 \overline{k}_2}\left(2x_{20}^* - k_2 - \overline{k}_2\right)\lambda + e_2\lambda_2 x_{20}^* y_{20}^* = 0, \tag{4.14}$$

$$\lambda^2 + \frac{r_3 x_{30}^*}{k_3 \overline{k}_3}\left(2x_{30}^* - k_3 - \overline{k}_3\right)\lambda + e_3\lambda_3 x_{30}^* y_{30}^* = 0. \tag{4.15}$$

Clearly, $e_i\lambda_i x_{i0}^* y_{i0}^* > 0$ and

$$\frac{r_i x_{i0}^*}{k_i \overline{k}_i}\left(2x_{i0}^* - k_i - \overline{k}_i\right) < 0$$

if $k_i + \overline{k}_i > 2m_i/$, $(i = 1, 2, 3)$. So, the roots of (4.13)-(4.15) are positive and hence the interior equilibrium E^* of (2.1) is unstable. This proves our theorem. □

Case 2. $\delta \neq 0$.

An interior (positive) equilibrium $\hat{E}^* = (x_{1\delta}^*, x_{2\delta}^*, x_{3\delta}^*, y_{1\delta}^*, y_{2\delta}^*, y_{3\delta}^*)$ is a positive solution of the system of equations:

$$\begin{aligned}
&r_1x_1\left(1-\tfrac{x_1}{k_1}\right)\left(\tfrac{x_1}{\overline{k}_1}-1\right)-\lambda_1x_1y_1+\delta\left(d_{21}x_2+d_{31}x_3-d_{12}x_1-d_{13}x_1\right)=0,\\
&r_2x_2\left(1-\tfrac{x_2}{k_2}\right)\left(\tfrac{x_2}{\overline{k}_2}-1\right)-\lambda_2x_2y_2+\delta\left(d_{12}x_1+d_{32}x_3-d_{21}x_2-d_{23}x_2\right)=0,\\
&r_3x_3\left(1-\tfrac{x_3}{k_3}\right)\left(\tfrac{x_3}{\overline{k}_3}-1\right)-\lambda_3x_3y_3+\delta\left(d_{13}x_1+d_{23}x_2-d_{31}x_3-d_{32}x_3\right)=0,\\
&e_1x_1-m_1=0,\\
&e_2x_2-m_2=0,\\
&e_3x_3-m_3=0.
\end{aligned} \tag{4.16}$$

Solving (4.16) we get,

$$\begin{aligned}
x_{i\delta}^* &= \tfrac{m_i}{e_i}\left(i=1,2,3\right),\\
y_{1\delta}^* &= \tfrac{r_1}{\lambda_1}\left(1-\tfrac{m_1}{e_1k_1}\right)\left(\tfrac{m_1}{e_1\overline{k}_1}-1\right)+\tfrac{\delta}{\lambda_1}\left(\tfrac{e_1d_{21}m_2}{e_2m_1}+\tfrac{e_1d_{31}m_3}{e_3m_1}-d_{12}-d_{13}\right),\\
y_{2\delta}^* &= \tfrac{r_2}{\lambda_2}\left(1-\tfrac{m_2}{e_2k_2}\right)\left(\tfrac{m_2}{e_2\overline{k}_2}-1\right)+\tfrac{\delta}{\lambda_2}\left(\tfrac{e_2d_{12}m_1}{e_1m_2}+\tfrac{e_2d_{32}m_3}{e_3m_2}-d_{21}-d_{23}\right),\\
y_{3\delta}^* &= \tfrac{r_3}{\lambda_3}\left(1-\tfrac{m_3}{e_3k_3}\right)\left(\tfrac{m_3}{e_3\overline{k}_3}-1\right)+\tfrac{\delta}{\lambda_3}\left(\tfrac{e_3d_{13}m_1}{e_1m_3}+\tfrac{e_3d_{23}m_2}{e_2m_3}-d_{31}-d_{32}\right),
\end{aligned} \tag{4.17}$$

where $x_{1\delta}^*$, $x_{2\delta}^*$, $x_{3\delta}^*$, $y_{1\delta}^*$, $y_{2\delta}^*$ and $y_{3\delta}^*$ are all assumed to be positive.

4.2. **Stability of equilibrium in the presence of dispersal.** The variational matrix of the system (2.1) at E_0 is

$$\begin{vmatrix} -r_1-\delta(d_{12}+d_{13}) & \delta d_{21} & \delta d_{31} & 0 & 0 & 0 \\ \delta d_{12} & -r_2-\delta(d_{21}+d_{23}) & \delta d_{32} & 0 & 0 & 0 \\ \delta d_{13} & \delta d_{23} & -r_3-\delta(d_{31}+\delta d_{32}) & 0 & 0 & 0 \\ 0 & 0 & 0 & -m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -m_3 \end{vmatrix}.$$

So, the characteristic equation is,

$$\begin{vmatrix} A-\lambda & \delta d_{21} & \delta d_{31} & 0 & 0 & 0 \\ \delta d_{12} & E-\lambda & \delta d_{32} & 0 & 0 & 0 \\ \delta d_{13} & \delta d_{23} & I-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -m_1-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -m_2-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & -m_3-\lambda \end{vmatrix} = \mathbf{0}.$$

Simplifying we get,

$$\lambda = -m_1, -m_2, -m_3$$

and

$$\lambda^3-(A+E+I)\lambda^2+(AE+AI+EI-BD-CG-HF)\lambda+AHF+BID+CGE-AEI-BGF-CDH=0$$

where

$$A=-r_1-\delta(d_{12}+d_{13}), B=\delta d_{21}, C=\delta d_{31}, D=\delta d_{12},$$

$$E=-r_2-\delta(d_{21}+d_{23}), F=\delta d_{32}, G=\delta d_{13}, H=\delta d_{23},$$

$$I=-r_3-\delta(d_{31}+\delta d_{32}).$$

Now, let the cubic equation is

$$\lambda^3+T_1\lambda^2+T_2\lambda+T_3=0$$

where

$$T_1=-(A+E+I),$$

$$T_2=AE+AI+EI-BD-CG-HF,$$

$$T_3=AHF+BID+CGE-AEI-BGF-CDH.$$

The Hurwitz matrix for the cubic equation is

$$L=\begin{bmatrix} T_1 & 1 & 0 \\ T_3 & T_2 & T_1 \\ 0 & 0 & T_3 \end{bmatrix}, \ \ \Delta_1=T_1, \ \ \Delta_2=\begin{vmatrix} T_1 & 1 \\ T_3 & T_2 \end{vmatrix}.$$

The roots of the auxiliary equation have negative real parts if and only if all the principal diagonal minors of the Hurwitz matrix are positive provided $\Delta_1>0, \Delta_2>0$ and $T_3>0$. Clearly, $\Delta_1=T_1>0$. Thus, $(0,0,0,0,0,0)$ is stable for $T_1T_2-T_3>0$ and $T_3>0$, where T_1,T_2,T_3 are expressed as above.

Again, the characteristic equation of the variational matrix of (2.1) at $E1 = (k_1, 0, 0, 0, 0, 0)$ is

$$\begin{vmatrix} A-\lambda & \delta d_{21} & \delta d_{31} & -\lambda_1 k_1 & 0 & 0 \\ \delta d_{12} & E-\lambda & \delta d_{32} & 0 & 0 & 0 \\ \delta d_{13} & \delta d_{23} & I-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1k_1 - m_1 - \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -m_2-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & -m_3-\lambda \end{vmatrix} = 0.$$

Simplifying we get,

$$\lambda = e_1k_1 - m_1, -m_2, -m_3$$

and

$$\lambda^3 - (A+E+I)\lambda^2 + (AE+AI+EI-BD-CG-HF)\lambda + AHF+BID+CGE-AEI-BGF-CDH = 0$$

where

$$A = -r_1(\frac{k_1}{\overline{k}_1} - 1) - \delta(d_{12} + d_{13}), B = \delta d_{21}, C = \delta d_{31}, D = \delta d_{12},$$
$$E = -r_2 - \delta(d_{21} + d_{23}), F = \delta d_{32}, G = \delta d_{13},$$
$$H = \delta d_{23}, I = -r_3 - \delta(d_{31} + \delta d_{32}).$$

Now, let the cubic equation is

$$\lambda^3 + T_1\lambda^2 + T_2\lambda + T_3 = 0$$

where

$$T_1 = -(A+E+I),$$
$$T_2 = AE + AI + EI - BD - CG - HF,$$
$$T_3 = AHF + BID + CGE - AEI - BGF - CDH.$$

The Hurwitz matrix for the cubic equation is

$$L = \begin{bmatrix} T_1 & 1 & 0 \\ T_3 & T_2 & T_1 \\ 0 & 0 & T_3 \end{bmatrix}, \ \ \Delta_1 = T_1, \ \ \Delta_2 = \begin{vmatrix} T_1 & 1 \\ T_3 & T_2 \end{vmatrix}.$$

The roots of the auxiliary equation have negative real parts if and only if all the principal diagonal minors of the Hurwitz matrix are positive provided $\Delta_1 > 0, \Delta_2 > 0$ and $T_3 > 0$. We have $k_1 >> \overline{k}_1$. Also $\Delta_1 = T_1 > 0$. Thus, $(k_1, 0, 0, 0, 0, 0)$ is stable for $e_1k_1 < m_1$, $T_1T_2 - T_3 > 0$ and $T_3 > 0$ where T_1, T_2, T_3 are expressed as above.

Similarly $E_2 = (0, k_2, 0, 0, 0, 0)$ is stable if $e_2k_2 < m_2$, $T_1T_2 - T_3 > 0$ and $T_3 > 0$. where

$$T_1 = -(A+E+I),$$
$$T_2 = AE + AI + EI - BD - CG - HF,$$
$$T_3 = AHF + BID + CGE - AEI - BGF - CDH,$$
$$A = -r_1 - \delta(d_{12} + d_{13}), B = \delta d_{21}, C = \delta d_{31}, D = \delta d_{12},$$
$$E = -r_2(\frac{k_2}{\overline{k}_2} - 1) - \delta(d_{21} + d_{23}), F = \delta d_{32}, G = \delta d_{13}, H = \delta d_{23}, I = -r_3 - \delta(d_{31} + \delta d_{32}).$$

And $E_3 = (0, 0, k_3, 0, 0, 0)$ is stable for $e_3k_3 < m_3$, $T_1T_2 - T_3 > 0$ and $T_3 > 0$ where

$$T_1 = -(A+E+I),$$
$$T_2 = AE + AI + EI - BD - CG - HF,$$
$$T_3 = AHF + BID + CGE - AEI - BGF - CDH,$$

$$A = -r_1 - \delta(d_{12} + d_{13}), B = \delta d_{21}, C = \delta d_{31},$$
$$D = \delta d_{12}, E = -r_2 - \delta(d_{21} + d_{23}), F = \delta d_{32},$$
$$G = \delta d_{13}, H = \delta d_{23}, I = -r_3(\frac{k_3}{\overline{k}_3} - 1) - \delta(d_{31} + \delta d_{32}).$$

Next, the characteristic equation is of the variational matrix of (2.1) at $\big(\overline{k}_1, 0, 0, 0, 0, 0\big)$ is

$$\begin{vmatrix} A-\lambda & \delta d_{21} & \delta d_{31} & -\lambda_1 \overline{k}_1 & 0 & 0 \\ \delta d_{12} & E-\lambda & \delta d_{32} & 0 & 0 & 0 \\ \delta d_{13} & \delta d_{23} & \begin{matrix} -r_3 \\ -\delta(d_{31}+\delta d_{32})-\lambda \end{matrix} & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1\overline{k}_1 - m_1 - \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -m_2-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & -m_3-\lambda \end{vmatrix} = 0.$$

Simplifying we get,

$$\lambda = e_1\overline{k}_1 - m_1, -m_2, -m_3$$

and

$$\lambda^3 - (A+E+I)\lambda^2 + (AE+AI+EI-BD-CG-HF)\lambda + AHF+BID+CGE-AEI-BGF-CDH = 0$$

where,

$$A = r_1(1 - \frac{\overline{k}_1}{k_1}) - \delta(d_{12} + d_{13}), B = \delta d_{21}, C = \delta d_{31},$$
$$D = \delta d_{12}, E = -r_2 - \delta(d_{21} + d_{23}),$$
$$F = \delta d_{32}, G = \delta d_{13}, H = \delta d_{23},$$
$$I = -r_3 - \delta(d_{31} + \delta d_{32}).$$

Now, consider the cubic equation is

$$\lambda^3 + T_1\lambda^2 + T_2\lambda + T_3 = 0$$

where

$$T_1 = -(A+E+I), T_2 = AE+AI+EI-BD-CG-HF, T_3 = AHF+BID+CGE-AEI-BGF-CDH.$$

The Hurwitz matrix for the cubic equation is

$$L = \begin{bmatrix} T_1 & 1 & 0 \\ T_3 & T_2 & T_1 \\ 0 & 0 & T_3 \end{bmatrix}, \ \ \Delta_1 = T_1, \ \ \Delta_2 = \begin{vmatrix} T_1 & 1 \\ T_3 & T_2 \end{vmatrix}.$$

The roots of the auxiliary equation have negative real parts if and only if all the principal diagonal minors of the Hurwitz matrix are positive provided $\Delta_1 > 0, \Delta_2 > 0$ and $T_3 > 0$. Hence $\big(\overline{k}_1, 0, 0, 0, 0, 0\big)$ is stable for $e_1\overline{k}_1 < m_1$, $T_1 > 0$, $T_1T_2 - T_3 > 0$ and $T_3 > 0$ where T_1, T_2, T_3 are expressed as above.

In a similar way we conclude that

$$\big(0, \overline{k}_2, 0, 0, 0, 0\big)$$

is stable for

$$e_2\overline{k}_2 < m_2, \ \ T_1 > 0, T_1T_2 - T_3 > 0$$

and

$$T_3 > 0$$

where

$T_1=-(A+E+I), T_2=AE+AI+EI-BD-CG-HF, T_3=AHF+BID+CGE-AEI-BGF-CDH,$

$$A=-r_1-\delta(d_{12}+d_{13}), B=\delta d_{21}, C=\delta d_{31},$$

$$D=\delta d_{12}, E=r_2(1-\frac{\overline{k}_2}{k_2})-\delta(d_{21}+d_{23}),$$

$$F=\delta d_{32}, G=\delta d_{13}, H=\delta d_{23}, I=-r_3-\delta(d_{31}+\delta d_{32}).$$

Similarly,

$$\left(0,0,\overline{k}_3,0,0,0\right)$$

is stable for

$$e_3\overline{k}_3<m_3,\ \ T_1>0, T_1T_2-T_3>0$$

and

$$T_3>0$$

where

$T_1=-(A+E+I), T_2=AE+AI+EI-BD-CG-HF, T_3=AHF+BID+CGE-AEI-BGF-CDH,$

$$A=-r_1-\delta(d_{12}+d_{13}), B=\delta d_{21}, C=\delta d_{31}, D=\delta d_{12},$$

$$E=-r_2-\delta(d_{21}+d_{23}), F=\delta d_{32}, G=\delta d_{13},$$

$$H=\delta d_{23}, I=r_3(1-\frac{\overline{k}_3}{k_3})-\delta(d_{31}+\delta d_{32}).$$

As for $\hat{E}_7$ the characteristic equation of (2.1) at E_7 is

$$\begin{vmatrix} A-\lambda & \delta d_{21} & \delta d_{31} & \frac{-\lambda_1 m_1}{e_1} & 0 & 0 \\ \delta d_{12} & \begin{matrix}-r_2\\ -\delta(d_{21}+d_{23})-\lambda\end{matrix} & \delta d_{32} & 0 & 0 & 0 \\ \delta d_{13} & \delta d_{23} & \begin{matrix}-r_3\\ -\delta(d_{31}+\delta d_{32})-\lambda\end{matrix} & 0 & 0 & 0 \\ J & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -m_2-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & -m_3-\lambda \end{vmatrix}=\mathbf{0}$$

Simplifying we get, $\lambda=-m_2,-m_3$ and $\lambda^4-(A+E+I)\lambda^3+(AE+AI+EI-HF-BD-CG-JL)\lambda^2+(AHF+BDI+CGE+JLE+JLI-BGF-CDH-AEI)\lambda+JLHF-JLEI=0$.
where,

$$A=\frac{r_1m_1(e_1k_1+e_1\overline{k}_1-2m_1)}{e_1^2k_1\overline{k}_1},$$

$$B=\delta d_{21}, C=\delta d_{31}, D=\delta d_{12},$$

$$E=-r_2-\delta(d_{21}+d_{23}), F=\delta d_{32},$$

$$G=\delta d_{13}, H=\delta d_{23}, I=-r_3-\delta(d_{31}+\delta d_{32}),$$

$$J=\frac{e_1^2}{\lambda_1 m_1}(\frac{r_1m_1(e_1k_1-m_1)(m_1-e_1\overline{k}_1)}{e_1^3k_1\overline{k}_1}-\frac{\delta m_1}{e_1}(d_{12}+d_{13})), L=\frac{-\lambda_1 m_1}{e_1}.$$

Let the biquadratic equation be

$$\lambda^4+T_1\lambda^3+T_2\lambda^2+T_3\lambda+T_4=0$$

where

$$T_1 = -(A+E+I),$$
$$T_2 = AE + AI + EI - HF - BD - CG - JL,$$
$$T_3 = AHF + BDI + CGE + JLI + JLE - BGF - CDH - AEI,$$
$$T_4 = JLHF - IJLE.$$

The Hurwitz matrix for the biquadratic equation is

$$L = \begin{bmatrix} T_1 & 1 & 0 & 0 \\ T_3 & T_2 & T_1 & 1 \\ 0 & T_4 & T_3 & T_2 \\ 0 & 0 & 0 & T_4 \end{bmatrix}, \Delta_1 = T_1, \Delta_2 = \begin{vmatrix} T_1 & 1 \\ T_3 & T_2 \end{vmatrix}, \Delta_3 = \begin{vmatrix} T_1 & 1 & 0 \\ T_3 & T_2 & T_1 \\ 0 & T_4 & T_3 \end{vmatrix}$$

The roots of the auxiliary equation have negative real parts if and only if all the principal diagonal minors of the Hurwitz matrix are positive provided $\Delta_i > 0, (i = 1, 2, 3)$ and $T_4 > 0$. Therefore, $\hat{E}_7$ is stable if $\Delta_i > 0, (i = 1, 2, 3)$and $T_4 > 0$ where $T_i (i = 1, 2, 3, 4)$ are expressed as above.

Similarly $\hat{E}_8$ is stable if $T_1 > 0, T_1T_2 - T_3 > 0, T_1T_2T_3 - T_1^2T_4 - T_3^2 > 0$and $T_4 > 0$. where

$$T_1 = -(A+E+I), T_2 = AE + AI + EI - HF - BD - CG - JL,$$
$$T_3 = AHF + BDI + CGE + AJL + JLI - BGF - CDH - AEI,$$
$$T_4 = CGJL - JLAI, A = -r_1 - \delta(d_{12} + d_{13}),$$
$$B = \delta d_{21}, C = \delta d_{31}, D = \delta d_{12}, E = \frac{r_2 m_2(e_2k_2 + e_2\overline{k}_2 - 2m_2)}{e_2^2 k_2 \overline{k}_2},$$
$$F = \delta d_{32}, G = \delta d_{13}, H = \delta d_{23}, I = -r_3 - \delta(d_{31} + \delta d_{32}),$$
$$J = \frac{e_2^2}{\lambda_2 m_2}\left(\frac{r_2 m_2(e_2k_2 - m_2)(m_2 - e_2\overline{k}_2)}{e_2^3 k_2 \overline{k}_2} - \frac{\delta m_2}{e_2}(d_{21} + d_{23}), L = \frac{-\lambda_2 m_2}{e_2}.\right.$$

As well, $\hat{E}_9$ is also stable if $T_1 > 0, T_1T_2 - T_3 > 0, T_1T_2T_3 - T_1^2T_4 - T_3^2 > 0$and $T_4 > 0$ where

$$T_1 = -(A+E+I), T_2 = AE + AI + EI - HF - BD - CG - JL,$$
$$T_3 = AHF + BDI + CGE + JLE + JLA - BGF - CDH - AEI, T_4 = BDJL - JLEA,$$
$$A = -r_1 - \delta(d_{12} + d_{13}), B = \delta d_{21}, C = \delta d_{31}, D = \delta d_{12}, E = -r_2 - \delta(d_{21} + d_{23}),$$
$$F = \delta d_{32}, G = \delta d_{13}, H = \delta d_{23}, I = \frac{r_3 m_3(e_3k_3 + e_3\overline{k}_3 - 2m_3)}{e_3^2 k_3 \overline{k}_3},$$
$$J = \frac{e_3^2}{\lambda_3 m_3}\left(\frac{r_3 m_3(e_3k_3 - m_3)(m_3 - e_3\overline{k}_3)}{e_3^3 k_3 \overline{k}_3} - \frac{\delta m_3}{e_3}(d_{31} + d_{32}), L = \frac{-\lambda_3 m_3}{e_3}.\right.$$

At the equilibrium $\hat{E}_{10}$, the characteristic equation of (2.1) is

$$\begin{vmatrix} A-\lambda & \delta d_{21} & \delta d_{31} & \frac{-\lambda_1 m_1}{e_1} & 0 & 0 \\ \delta d_{12} & E-\lambda & \delta d_{32} & 0 & \frac{-\lambda_2 m_2}{e_2} & 0 \\ \delta d_{13} & \delta d_{23} & \begin{matrix} -r_3 \\ -\delta(d_{31}+\delta d_{32}) - \lambda \end{matrix} & 0 & 0 & 0 \\ A\frac{e_1^2}{\lambda_1 m_1} & 0 & 0 & -\lambda & 0 & 0 \\ 0 & J\frac{e_2^2}{\lambda_2 m_2} & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & -m_3 - \lambda \end{vmatrix} = 0$$

Simplifying we get, $\lambda = -m_3$ and

$$\lambda^5 + T_1\lambda^4 + T_2\lambda^3 + T_3\lambda^2 + T_4\lambda + T_5 = 0$$

where

$$T_1 = -(A+E+I), T_2 = AE+AI+EI-HF-BD-CG-JL-MK,$$

$$T_3 = AHF+BDI+CGE+JLE+JLI+AMK+IMK-BGF-CDH-AEI,$$

$$T_4 = AIKM+KCMG+JLHF+JKLM-EIJL, T_5 = -JKLMI.$$

The Hurwitz matrix for the equation is

$$L = \begin{bmatrix} T_1 & 1 & 0 & 0 & 0 \\ T_3 & T_2 & T_1 & 1 & 0 \\ T_5 & T_4 & T_3 & T_2 & T_1 \\ 0 & 0 & T_5 & T_4 & T_3 \\ 0 & 0 & 0 & 0 & T_5 \end{bmatrix},$$

which has principal diagonal minors as

$$\Delta_1 = T_1, \;\; \Delta_2 = \begin{vmatrix} T_1 & 1 \\ T_3 & T_2 \end{vmatrix}, \; \Delta_3 = \begin{vmatrix} T_1 & 1 & 0 \\ T_3 & T_2 & T_1 \\ 0 & T_4 & T_3 \end{vmatrix}, \; \nabla_4 = \begin{vmatrix} T_1 & 1 & 0 & 0 \\ T_3 & T_2 & T_1 & 1 \\ T_5 & T_4 & T_3 & T_2 \\ 0 & 0 & T_5 & T_4 \end{vmatrix}.$$

The roots of the auxiliary equation have negative real parts if and only if all the principal diagonal minors of the Hurwitz matrix are positive provided $\Delta_i > 0, (i = 1,2,3,4)$ and $T_4 > 0$. Therefore, E_{10} is stable if $\Delta_i > 0, (i = 1,2,3,4)$ and $T_4 > 0$ where $T_i (i = 1,2,3,4,5)$ are expressed as above.

Similarly, $\hat{E}_{11}$ is stable when $\Delta_i > 0, (i = 1,2,3,4)$ and $T_4 > 0$ where

$$\begin{aligned}
&T_1 = -(A+E+I),\\
&T_2 = AE+AI+EI-HF-BD-CG-JL-MK,\\
&T_3 = AHF+BDI+CGE+JLE+JLI+AMK+IMK-BGF-CDH-AEI,\\
&T_4 = AIKM+KCMG+JLHF+JKLM-EIJL,\\
&T_5 = -JKLME,\\
&A = \frac{r_1 m_1 (e_1 k_1 + e_1 \overline{k}_1 - 2m_1)}{e_1^2 k_1 \overline{k}_1},\\
&B = \delta d_{21}, \quad C = \delta d_{31}, \quad D = \delta d_{12},\\
&E = -r_2 - \delta(d_{21} + d_{23}),\\
&F = \delta d_{32}, \quad G = \delta d_{13}, \quad H = \delta d_{23},\\
&I = \frac{r_3 m_3 (e_3 k_3 + e_3 \overline{k}_3 - 2m_3)}{e_3^2 k_3 \overline{k}_3},\\
&J = \frac{e_1^2}{\lambda_1 m_1}\left(\frac{r_1 m_1 (e_1 k_1 - m_1)(m_1 - e_1 \overline{k}_1)}{e_1^3 k_1 \overline{k}_1} - \frac{\delta m_1}{e_1}(d_{12} + d_{13})\right),\\
&K = \frac{e_3^2}{\lambda_3 m_3}\left(\frac{r_3 m_3 (e_3 k_3 - m_3)(m_3 - e_3 \overline{k}_3)}{e_3^3 k_3 \overline{k}_3} - \frac{\delta m_3}{e_3}(d_{31} + d_{32})\right),\\
&L = \frac{-\lambda_1 m_1}{e_1}, \quad M = \frac{-\lambda_3 m_3}{e_3}.
\end{aligned}$$

Also $\hat{E}_{12}$ is stable when $\Delta_i > 0, (i = 1,2,3,4)$ and $T_4 > 0$ where

$$T_1 = -(A+E+I), T_2 = AE+AI+EI-HF-BD-CG-JL-MK,$$

$$T_3 = AHF+BDI+CGE+JLA+JLI+AMK+IMK-BGF-CDH-AEI,$$

$$T_4 = AIKM + KCMG + JLAI + JKLM - EIJL, T_5 = -JKLMA,$$

$$A = -r_1 - \delta(d_{12} + d_{13}), B = \delta d_{21}, C = \delta d_{31}, D = \delta d_{12},$$

$$E = \frac{r_2 m_2(e_2 k_2 + e_2 \bar{k}_2 - 2m_2)}{e_2^2 k_2 \bar{k}_2},$$

$$F = \delta d_{32}, G = \delta d_{13}, H = \delta d_{23}, I = \frac{r_3 m_3(e_3 k_3 + e_3 \bar{k}_3 - 2m_3)}{e_3^2 k_3 \bar{k}_3},$$

$$J = \frac{e_1^2}{\lambda_1 m_1}\Big(\frac{r_1 m_1(e_1 k_1 - m_1)(m_1 - e_1 \bar{k}_1)}{e_1^3 k_1 \bar{k}_1} - \frac{\delta m_1}{e_1}(d_{12} + d_{13})\Big),$$

$$K = \frac{e_3^2}{\lambda_3 m_3}\Big(\frac{r_3 m_3(e_3 k_3 - m_3)(m_3 - e_3 \bar{k}_3)}{e_3^3 k_3 \bar{k}_3} - \frac{\delta m_3}{e_3}(d_{31} + d_{32}), L = \frac{-\lambda_2 m_2}{e_2}, M = \frac{-\lambda_3 m_3}{e_3}.$$

The following theorem ensures the stability of the equilibrium point $\hat{E}^*$.

Theorem 4.2. *Equilibrium point* $\hat{E}^* = (x_{1\delta}^*, x_{2\delta}^*, x_{3\delta}^*, y_{1\delta}^*, y_{2\delta}^*, y_{3\delta}^*)$ *of the system (2.1) is stable when* $\nabla_i > 0$, *(*$i = 1, 2, 3, 4, 5$*) and* $T_6 > 0$ *where*

$$\nabla_1 = T_1, \nabla_2 = \begin{vmatrix} T_1 & 1 \\ T_3 & T_2 \end{vmatrix} \nabla_3 = \begin{vmatrix} T_1 & 1 & 0 \\ T_3 & T_2 & T_1 \\ T_5 & T_4 & T_3 \end{vmatrix}$$

$$\nabla_4 = \begin{vmatrix} T_1 & 1 & 0 & 0 \\ T_3 & T_2 & T_1 & 1 \\ T_5 & T_4 & T_3 & T_2 \\ 0 & T_6 & T_5 & T_4 \end{vmatrix} \nabla_5 = \begin{vmatrix} T_1 & 1 & 0 & 0 & 0 \\ T_3 & T_2 & T_1 & 1 & 0 \\ T_5 & T_4 & T_3 & T_2 & T_1 \\ 0 & T_6 & T_5 & T_4 & T_3 \\ 0 & 0 & 0 & T_6 & T_5 \end{vmatrix}.$$

Here,

$$\begin{aligned}
T_1 =& AQ - N - Q - S - A - F - K,\\
T_2 =& NQ + SN + AN + NF + NK + SQ - AQF - AQK + QF + QK + SA + SF + SK + AF + AK\\
&+ FK + BE - CI - MD,\\
T_3 =& NFK - NQS - NQA - NQF - NQK - SAN - SNF - SNK - ANF - ANK - SQA - SQF\\
&- SQK + AQFK - QFK - SAF - SAK - SFK - AFK - NBE - QBE - SBE - KBE\\
&- BIG - CEJ + NCI + QCI + CIS - CIF + NRL - RBL + SPH + APH + PHN + PHK\\
&+ MDQ + MDS + MDF + MDK,\\
T_4 =& NQSA + NQSF + NQSK + NQAF + NQAK + NQFK + SANF + SANK + SNFK + ANFK\\
&+ SQAF + SQAK + SQFK + SAFK - JG + NQBE + NSBE + NKBE + QSBE + QKBE\\
&+ SKBE + QBIG + SBIG + NBIG + NCEJ + QCEJ + SCEJ - NQCI - NSCI - NFCI\\
&- QSCI - QFCI - CISF - NRL - QRLN - RLFN + NRBL + QRBL + RBLF - SAPH\\
&- SPHN - SPHK - APHN - APHK - PHNK + CPHI + PRHL - MDQS - MDQF\\
&- MDQK - MDSF - MDSK - MDFK + MDGJ - LRMD + MDHP,
\end{aligned}$$

$$\begin{aligned}T_5 =& NJG + QJG - NQSAF - NQSAK - NQSFK - NQAFK - SANFK - SAQFK - NQSBE\\ &- NQKBE - NSKBE - QSKBE - QSBIG - QNBIG - SNBIG - NQCEJ - NSCEJ\\ &- QSCEJ + NQSCI + NQFCI + NSFCI + QSKCI + QRLN + RLFN + QRLFN - NQRBL\\ &- NRBLF - QRBLF + SAPHN + SAPHK + SPHNK + APHNK - CPHIS - CPHIN\\ &- PRHLA - PRHLN + MDQSF + MDQSK + MDQFK + MDSFK - MDGJQ - MDGJS\\ &+ LDMRF + LDMRQ - MDHPS - MDHPK,\\ T_6 =& NQSAFK + NQSKBE - NQJG + QSNBIG + NQSCBJ - NQSFCI - QRLFN + NQRBLF\\ &- SAPHNK + SNCPHI + ANPHRL - MDQSFK + MDGJQS - LRFMDQ + MDHPSK\\ &- MDHLPR,\end{aligned}$$

with

$$\begin{aligned}&A = \frac{r_1}{k_1\bar{k}_1}x^*_{1\delta}(2k_1 + 2\bar{k}_1) - \frac{3r_1x^{*2}_{1\delta}}{k_1\bar{k}_1} - r_1 - \lambda_1 y^*_{1\delta} - \delta(d_{12} + d_{13}), B = \delta d_{21}, C = \delta d_{31}, D = -\lambda_1 x^*_{1\delta},\\ &E = \delta d_{12}, F = \frac{r_2}{k_2\bar{k}_2}x^*_{2\delta}(2k_2 + 2\bar{k}_2) - \frac{3r_2x^{*2}_{2\delta}}{k_2\bar{k}_2} - r_2 - \lambda_2 y^*_{2\delta} - \delta(d_{21} + d_{23}),\\ &G = \delta d_{32}, H = -\lambda_2 x^*_{2\delta}, I = \delta d_{13}, J = \delta d_{23},\\ &K = \frac{r_3}{k_3\bar{k}_3}x^*_{3\delta}(2k_3 + 2\bar{k}_3) - \frac{3r_3x^{*2}_{3\delta}}{k_2\bar{k}_2} - r_3 - \lambda_3 y^*_{3\delta} - \delta(d_{31} + d_{32}), L = -\lambda_3 x^*_{3\delta}, M = e_1 y^*_{1\delta},\\ &N = e_1x^*_{1\delta} - m_1, P = e_2 y^*_{2\delta}, Q = e_2 x^*_{2\delta} - m_2, R = e_3 y^*_{3\delta}, S = e_3 x^*_{3\delta} - m_3.\end{aligned}$$

Proof. The characteristic equation of the system (2.1) at the equilibrium point $\hat{E}^*$ is,

$$\begin{vmatrix} A-\lambda & \delta d_{21} & \delta d_{31} & -\lambda_1 x^*_{1\delta} & 0 & 0\\ \delta d_{12} & F-\lambda & \delta d_{32} & 0 & -\lambda_2 x^*_{2\delta} & 0\\ \delta d_{13} & \delta d_{23} & K-\lambda & 0 & 0 & -\lambda_3 x^*_{3\delta}\\ e_1 y^*_{1\delta} & 0 & 0 & N-\lambda & 0 & 0\\ 0 & e_2 y^*_{2\delta} & 0 & 0 & Q-\lambda & 0\\ 0 & 0 & e_3 y^*_{3\delta} & 0 & 0 & S-\lambda \end{vmatrix} = 0$$

Expanding and simplifying, we get,

$$\lambda^6 + T_1\lambda^5 + T_2\lambda^4 + T_3\lambda^3 + T_4\lambda^2 + T_5\lambda + T_6 = 0$$

where, $T_1, T_2, T_3, T_4, T_5, T_6$ are constants expressed as the statement. Now, the Hurwitz matrix

$$L = \begin{bmatrix} T_1 & 1 & 0 & 0 & 0 & 0\\ T_3 & T_2 & T_1 & 1 & 0 & 0\\ T_5 & T_4 & T_3 & T_2 & T_1 & 1\\ 0 & T_6 & T_5 & T_4 & T_3 & T_2\\ 0 & 0 & 0 & T_6 & T_5 & T_4\\ 0 & 0 & 0 & 0 & 0 & T_6 \end{bmatrix},$$

which has the principal diagonal minors as

$$\nabla_1 = T_1,\ \ \nabla_2 = \begin{vmatrix} T_1 & 1\\ T_3 & T_2 \end{vmatrix},\ \ \nabla_3 = \begin{vmatrix} T_1 & 1 & 0\\ T_3 & T_2 & T_1\\ T_5 & T_4 & T_3 \end{vmatrix},$$

$$\nabla_4 = \begin{vmatrix} T_1 & 1 & 0 & 0 \\ T_3 & T_2 & T_1 & 1 \\ T_5 & T_4 & T_3 & T_2 \\ 0 & T_6 & T_5 & T_4 \end{vmatrix}, \ \nabla_5 = \begin{vmatrix} T_1 & 1 & 0 & 0 & 0 \\ T_3 & T_2 & T_1 & 1 & 0 \\ T_5 & T_4 & T_3 & T_2 & T_1 \\ 0 & T_6 & T_5 & T_4 & T_3 \\ 0 & 0 & 0 & T_6 & T_5 \end{vmatrix}$$

The roots of the auxiliary equation have negative real parts if and only if all the principal diagonal minors of the Hurwitz matrix are positive provided that $\nabla_i > 0$, $(i = 1, 2, 3, 4, 5)$ and $T_6 > 0$. Therefore, the equilibrium point $\hat{E}^*$ is stable iff $\nabla_i > 0$, $(i = 1, 2, 3, 4, 5)$ and $T_6 > 0$ where $T_i(i = 1, 2, 3, 4, 5, 6)$ are expressed as the statement. □

5. Bifurcation Analysis

Hopf bifurcation behaviour occurs for non-hyperbolic and non-linear equations for two and more than two dimensions. We have studied the possible existence of a simple Hopf bifurcation near $E_2\left(x^*_{1\delta}, x^*_{2\delta}, x^*_{3\delta}, y^*_{1\delta}, y^*_{2\delta}, y^*_{3\delta}\right)$ taking δ as the bifurcation parameter. The following theorem gives the necessary and sufficient condition for the existence of a simple Hopf bifurcation of the proposed system (2.1):

Theorem 5.1. *The necessary and sufficient conditions for the system (2.1) undergoes simple Hopf bifurcation at $\delta = \delta_0$ near the interior equilibrium point $E_2\left(x^*_{1\delta}, x^*_{2\delta}, x^*_{3\delta}, y^*_{1\delta}, y^*_{2\delta}, y^*_{3\delta}\right)$ are that, $T_6 > 0$, $\nabla_i(\delta_0) > 0$, $(i = 1, 2, 3, 4)$, $\nabla_5(\delta_0) = 0$ and $\frac{d\nabla_5(\delta_0)}{d\delta} \neq 0$.*

Proof. The characteristic equation for $V(E_2)$ is

$$\lambda^6 + T_1(\delta)\lambda^5 + T_2(\delta)\lambda^4 + T_3(\delta)\lambda^3 + T_4(\delta)\lambda^2 + T_5(\delta)\lambda + T_6(\delta) = 0$$

where, $T_i(\delta)$ for $i = 1, 2, 3, 4, 5, 6$ is a smooth function of δ. Now, the Hurwitz matrix

$$L(\delta) = \begin{bmatrix} T_1(\delta) & 1 & 0 & 0 & 0 & 0 \\ T_3(\delta) & T_2(\delta) & T_1(\delta) & 1 & 0 & 0 \\ T_5(\delta) & T_4(\delta) & T_3(\delta) & T_2(\delta) & T_1(\delta) & 1 \\ 0 & T_6(\delta) & T_5(\delta) & T_4(\delta) & T_3(\delta) & T_2(\delta) \\ 0 & 0 & 0 & T_6(\delta) & T_5(\delta) & T_4(\delta) \\ 0 & 0 & 0 & 0 & 0 & T_6(\delta) \end{bmatrix},$$

which has the principal diagonal minors

$$\nabla_1(\delta) = T_1(\delta),\ \nabla_2(\delta) = \begin{vmatrix} T_1(\delta) & 1 \\ T_3(\delta) & T_2(\delta) \end{vmatrix},\ \nabla_3(\delta) = \begin{vmatrix} T_1(\delta) & 1 & 0 \\ T_3(\delta) & T_2(\delta) & T_1(\delta) \\ T_5(\delta) & T_4(\delta) & T_3(\delta) \end{vmatrix},$$

$$\nabla_4(\delta) = \begin{vmatrix} T_1(\delta) & 1 & 0 & 0 \\ T_3(\delta) & T_2(\delta) & T_1(\delta) & 1 \\ T_5(\delta) & T_4(\delta) & T_3(\delta) & T_2(\delta) \\ 0 & T_6(\delta) & T_5(\delta) & T_4(\delta) \end{vmatrix}, \ \nabla_5(\delta) = \begin{vmatrix} T_1(\delta) & 1 & 0 & 0 & 0 \\ T_3(\delta) & T_2(\delta) & T_1(\delta) & 1 & 0 \\ T_5(\delta) & T_4(\delta) & T_3(\delta) & T_2(\delta) & T_1(\delta) \\ 0 & T_6(\delta) & T_5(\delta) & T_4(\delta) & T_3(\delta) \\ 0 & 0 & 0 & T_6(\delta) & T_5(\delta) \end{vmatrix}$$

Using the condition for simple Hopf bifurcation [39] at $\delta = \delta_0$ around $E_2\left(x^*_{1\delta}, x^*_{2\delta}, x^*_{3\delta}, y^*_{1\delta}, y^*_{2\delta}, y^*_{3\delta}\right)$, we get $T_6 > 0, \nabla_i(\delta_0) > 0$, for $i = 1, 2, 3, 4$, $\nabla_5(\delta_0) = 0$ and $\frac{d\nabla_5(\delta_0)}{d\delta} \neq 0$. Hence, the theorem is proved. □

6. Numerical simulations

This section verified by numerical simulations of analytic observations of the study consists of three patch prey-predator interactions in which the dispersion facility is available to prey only; simultaneously, they are subject to the Allee effect. Therefore, it is essential to identify the changing impacts of carrying capacities, death rates, and dispersal on equilibrium. Analytical findings are verified via two cases where, in case I, the simulation in the absence of dispersal between patches has been analyzed, and in case II, the simulation in the presence of dispersal between the patches has been provided.

6.1. **Simulations in the absence of dispersal.** For the present case ($\delta = 0$), we have taken the parameter values given in Table 1.

Table 1: Parameter values for the system (2.1)

Parameter	Values	Parameter	Values	Parameter	Values
r_1	2.5	r_2	2	r_3	1.5
k_1	4.3	k_2	0.42	k_3	3.5
$\overline{k_1}$	0.45	$\overline{k_2}$	0.42	$\overline{k_3}$	0.35
e_1	0.32	e_2	0.35	e_3	0.8
λ_1	0.7	λ_2	0.6	λ_3	0.7
m_1	0.75	m_2	0.76	m_3	1.5
d_{12}	0.4	d_{21}	0.45	d_{13}	0.35
d_{31}	0.45	d_{23}	0.3	d_{32}	0.45

Using the parameter values of Table 1, we obtain $E_1\,(2.34375, 2.17143, 1.875, 6.83767, 6.35439, 4.33491)$. We have chosen $m_1 = 0.75$, $m_2 = 0.76$ and $m_3 = 1.5$ so that $k_1 + \overline{k_1} = 4.75 > 4.69 = 2\frac{m_1}{e_1}, k_2 + \overline{k_2} = 4.42 > 4.34 = 2\frac{m_2}{e_2}$ and $k_3+\overline{k_3} = 3.85 > 3.75 = 2\frac{m_3}{e_3}$. Therefore, $k_i+\overline{k_i} > 2\frac{m_i}{e_i}(i = 1, 2, 3)$ is satisfied and hence by Theorem 4.1, E_1 is unstable. The behaviour of the species with the progress of time and the corresponding phase plane portrait with the initial condition $(x_1\,(0)\,, x_2\,(0)\,, x_3\,(0)\,, y_1\,(0)\,, y_2\,(0)\,, y_3\,(0)) = (2, 2, 2, 6, 5, 4)$ are presented in Figure 2. Figure 2 depicts that, the inner equilibrium of the proposed

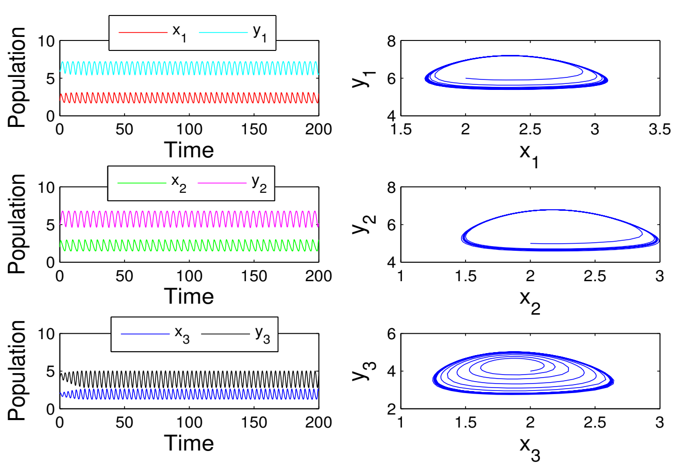

FIGURE 2. Time graph and phase portrait of the six species population without dispersal.

system becomes unstable with time progress without dispersal.

Now, for the assumed parameter values, we can observe the effect of the carrying capacities of the prey population on different species. We also observe the effect of taking same carrying capacity for all prey species. Figure 3 presents the trophic cascade of the six species population $(x_1, x_{2,}x_{3,}y_1, y_2, y_3)$ with respect to k_1, k_2, k_3 and $k_1 = k_2 = k_3 = k$.

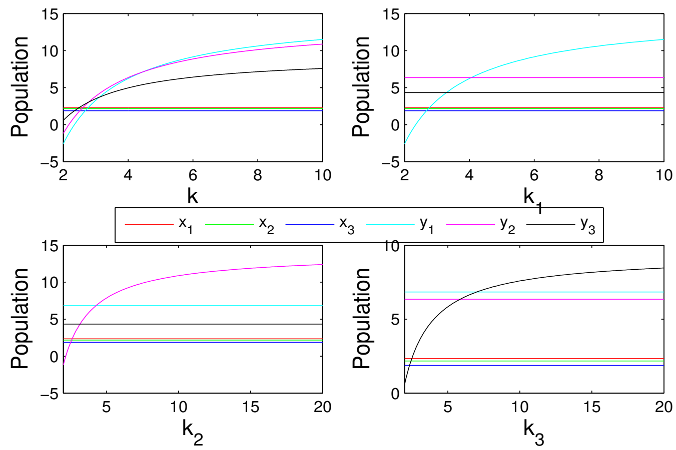

FIGURE 3. Effect of different carrying capacities on species.

From Figure 3, we observe that, with the increasing values of k_i, the i^{th} predator y_i $(i = 1; 2; 3)$ gradually increases and the remaining species remain constant throughout. Also, when k increases we observe that all predator species gradually increase but all prey species remain constant throughout. Therefore, we conclude that changing carrying capacity only affects a particular patch. Thus, this study shows that the environmental carrying capacity of any patch plays a vital role in the density of the predator population.

Next, we observe the effect of the death rate of predators on all species. We also consider the effect of the same death rate for all predator species. The behaviour of six species population with respect to $m_1, m_2, m_3, m_1 = m_2 = m_3 = m$ are depicted by Figure 4. From Figure 4, we find that, as m_i increases in the i^{th} patch for all $i = 1, 2, 3$, x_i species gradually increases and the y_i species follow the parabolic type path and all the other species remain constant throughout. Also, as m increases, three prey species x_i gradually increase, and predator species y_i follow the parabolic path. This fact biologically interprets and also verifies that, predator's death rate is directly related to the density of prey.

6.2. **Simulation in the presence of dispersal.** If we take $\delta = 5$ and the parameter values provided in Table 1, we obtain the interior equilibrium $E_2\,(2.34375, 2.17143, 1.875, 7.02991, 6.94033, 3.51297)$. The corresponding graphical presentation of the species with time progress is given in Figure 5 taking the initial condition as $(x_1\,(0)\,, x_2\,(0)\,, x_3\,(0)\,, y_1\,(0)\,, y_2\,(0)\,, y_3\,(0)) = (2, 2, 2, 6, 5, 4)$. From Figure 5, we observe that, the prey-predator system behaves like an unstable system as the time progresses. For the changes in the carrying capacity of environment, the behaviour of six species population are presented in Figure 6. From Figure 6, we observe that, as $k_i(i = 1, 2, 3)$ increases, $y_i(i = 1, 2, 3)$ species gradually increase, and all prey species remain constant throughout. Again, if k increases all predator species $y_i(i = 1, 2, 3)$ gradually increases, and all prey species $x_i(i = 1, 2, 3)$ remain constant throughout.

The tropic cascade of the six species population with respect to different death rates of predators is represented in Figure 7. In Figure 7, we observe that, as m_i $(i = 1, 2, 3)$ increases, $x_i(i = 1, 2, 3)$

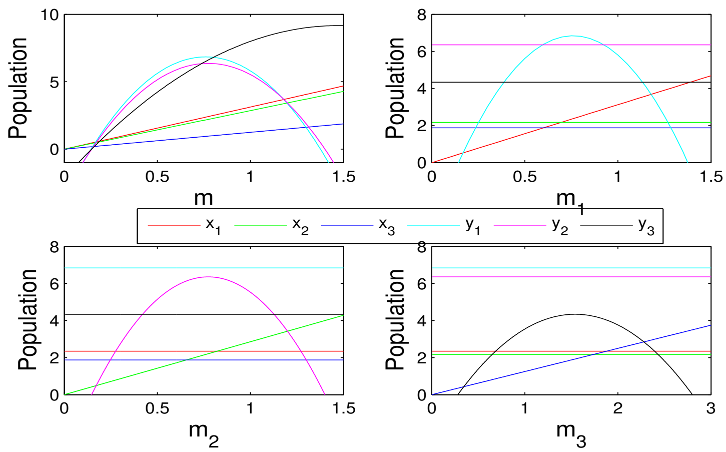

FIGURE 4. Effect of predators death rate on species.

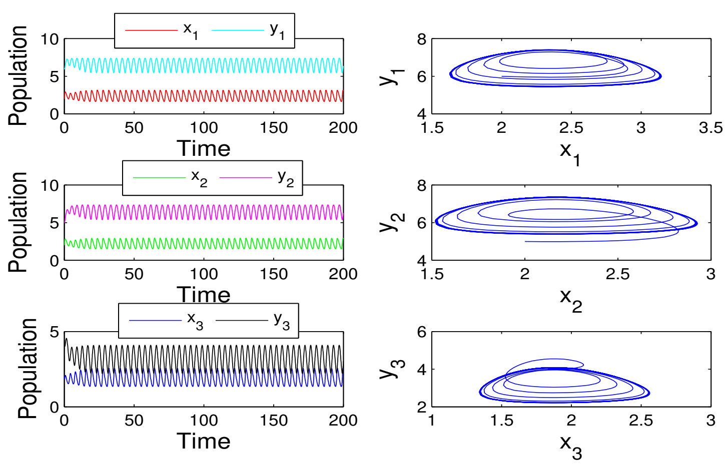

FIGURE 5. Time graph and phase portrait in the presence of dispersal.

gradually increases and two prey species remain constant. On the other hand, with the increasing values of m_i $(i = 1, 2, 3)$, the species $y_i(i = 1, 2, 3)$ gradually decreases and the remaining two predator species gradually increase. Also, when m increases, three predator species follow the parabolic path, and three prey species gradually increase.

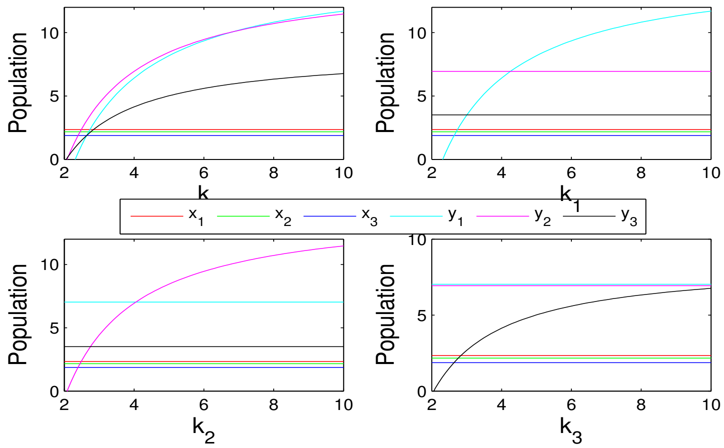

FIGURE 6. Changing effect of environmental carrying capacity on species.

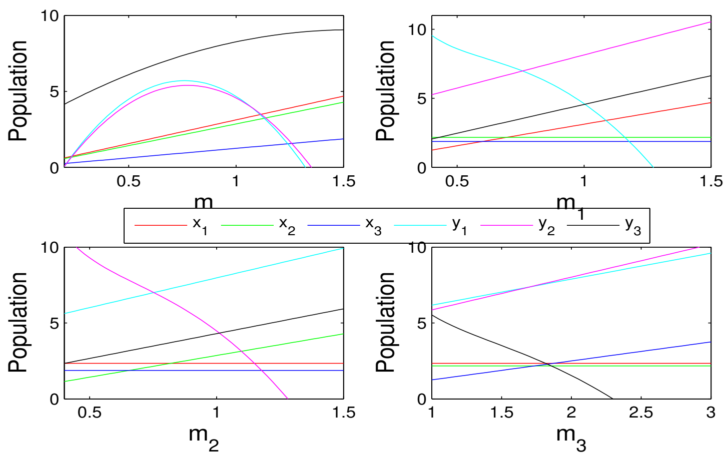

FIGURE 7. Changing behaviour of species for different death rate of predators.

Now, if we change the value of dispersal δ, then the equilibrium will change. Table 2 reflects the different equilibrium points of the proposed prey-predator system for different dispersal rates.

Table 2: Equilibrium points for different δ

δ	Equilibrium point (E_2)
0	(2.34375,2.17143,1.875,6.83767,6.35439,4.33491)
5	(2.34375,2.17143,1.875,7.02991,6.94033,3.51297)
10	(2.34375,2.17143,1.875,7.22216,7.52626,2.69103)
15	(2.34375,2.17143,1.875,7.4144,8.1122,1.8691)
20	(2.34375,2.17143,1.875,7.60665,8.69814,1.04716)
25	(2.34375,2.17143,1.875,7.79889,9.28408,0.225219)

From Table 2, we observe that, the population level of the prey species x_i $(i = 1, 2, 3)$ at the equilibrium level is independent of δ as we obtained $x_i = \frac{m_i}{e_i}(i = 1, 2, 3)$ in the theoretical section. On the other hand, we observe that, the equilibrium level of two predator species gradually increases, but the equilibrium of the third predator species gradually increases as δ increases. Therefore, equilibrium levels of the predator population are affected by the dispersal speed δ. Theoretical results can be verified by this conclusion. Figure 8 shows this fact graphically. From Table 2 and Figure 8, we observed that

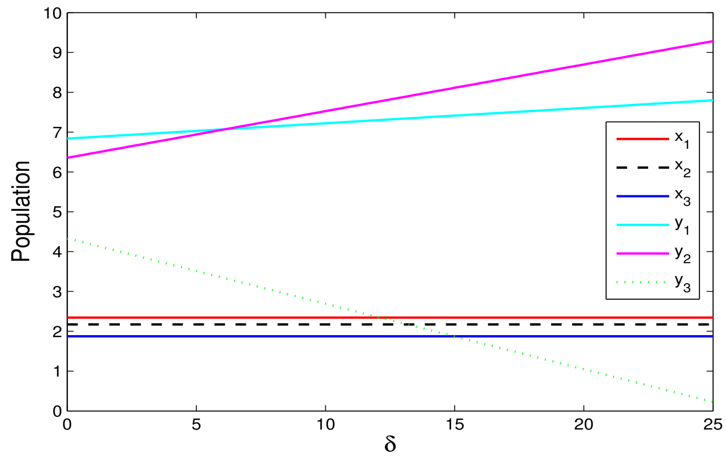

FIGURE 8. Effect of dispersal on equilibrium.

the predator species is very sensitive to the changing values of δ and therefore, the dispersal speed is very important for the persistence of the three patch-based system.

6.3. **Simulation for Hopf-bifurcation.** To explore the system geometry for the proposed model with changing dispersal rate between patches, we gradually increase the value of δ and find that, the interior equilibrium point switches towards periodic limit cycle oscillations from a stable steady state via the Hopf bifurcation. These shifting dynamics through the Hopf bifurcation are plotted in Figure 9. Taking δ as the bifurcation parameter, we assumed the other parameters as follows:

Table 3: Parameter values for Hopf-bifurcation

Parameter	Value	Parameter	Value
r_1	2.6	e_1	0.32
k_1	4.29	e_2	0.381
$\overline{k_1}$	0.43	e_3	0.825
λ_1	0.69	m_1	0.75
r_2	0.5	m_2	0.8
k_2	4.19	m_3	1.6
$\overline{k_2}$	0.47	d_{12}	0.419
λ_2	0.6	d_{21}	0.429
r_3	1.43	d_{13}	0.36
k_3	3.6	d_{31}	0.429
$\overline{k_3}$	0.45	d_{23}	0.36
λ_3	0.71	d_{32}	0.419

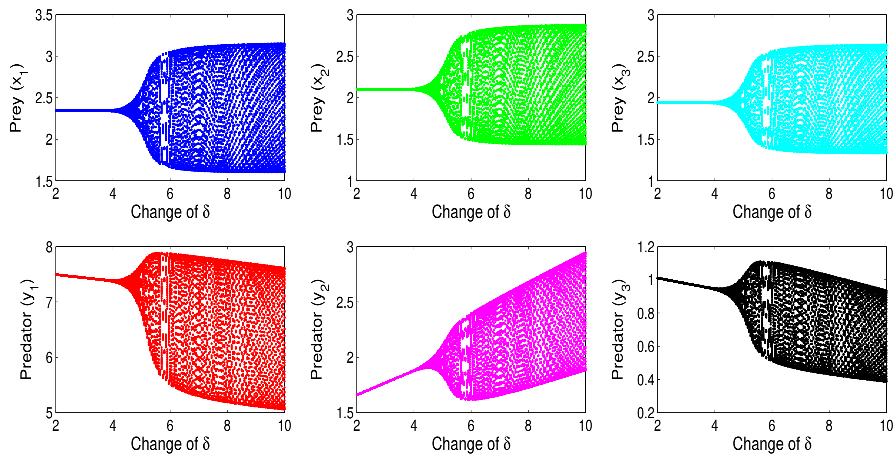

Figure 9. Hopf bifurcation diagrams.

Figure 9 shows that, a Hopf bifurcation value has been localized at $\delta = 4.245$. For $\delta > 4.245$, system (2.1) undergoes a limit cycle as in Figure 9. When $\delta < 4.245$, the equilibrium point becomes locally asymptotically stable as shown in Figure 9.

Here, a three-patch prey-predator system has incorporated dispersal among preys on each of the patches, and the preys are independent of occupying suitable patches. From these numerical studies, we can conclude that, the biological phenomenon of dispersal of the population is beneficial for the existence of species.

7. Discussion and conclusion

In the present work, we have tried to show the effect of the dispersal and Allee effect on the dynamical behaviour of the anticipated model in patchy environment. We observed that the dispersal rate decreases below a threshold value, creating the Hopf bifurcation. It is also found that the stability nature of the system is changing for different death rates of predators. We also observed that, the system is unstable in absence of dispersal depending on some restrictions. But the equilibrium of the model becomes stable under some restriction on parameters in the presence of dispersal. Therefore, stability behaviour of the equilibrium of the three patch-based system is dependent on the dispersal rate among the patches. The behavioural changes of the assumed prey-predator dynamics of the three patch based model is prominently shown by the graphical presentation. The hopf bifurcation analysis is also presented.

The present system shows a rich population dynamics. This analysis revealed that dispersal can act as a buffer against predatory attacks and other obstacles to prey. We also evaluated the importance of changing the carrying capacities of multi-patch prey-predator environments. These observations indicate that dispersal among predators can also help predator species to avoid the threats of existence. As we know, ingestion of prey species by predator species is not a continuous procedure. This process takes a certain time delay, which is termed gestation delay. The said concept may be included in the present model. In addition, we considered that, only prey species disperse among patches. But predator species can also be taken to move between patches. Therefore, the concept of predator dispersal can also be included to improve the present model. The modifications mentioned above in the current model are left for future work considerations.

8. Declarations

Authors' Contributions: S. Biswas was responsible for conceptualization, methodology, software, and investigation; D. Pal was responsible for methodology, software, investigation, and writing—drafting and editing; P.K. Santra was responsible for conceptualization, software, and visualization; G. S. Mahapatra was responsible for conceptualization, visualization, supervision, and writing—review and editing.

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