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DYNAMICAL PROPERTIES OF AN SIRI EPIDEMIC MODEL WITH RELAPSE AND FREE BOUNDARIES

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Abstract. In this paper, we study an SIRI epidemic model with relapse and free boundaries, which can describe the spreading process of diseases well. At first, the existence and uniqueness of the global solution is proven. And then we obtain some sufficient conditions for the disease spreading and vanishing. We can find an $\mathcal{R}_0$ such that the disease will vanish for $\mathcal{R}_0 \le 1$; otherwise, whether the disease will spread or not is determined by the initial infected region h_0 and spreading capacity μ of the spreading front. Moreover, the longtime behavior of the solution is given. Finally, numerical simulations are provided to illustrate the results of the analysis.

1. Introduction

Infectious diseases have been a major threat to human health all the time. A large number of mathematical models have been proposed to study various infectious disease. In 1927, Kermack and McKendrick [12] considered the following SIR model:

$$\begin{cases} \frac{dS}{dt} = -\beta SI, \\ \frac{dI}{dt} = \beta SI - \gamma I, \\ \frac{dR}{dt} = \gamma I, \end{cases}$$

where S, I and R stand for susceptible, infectious, and recovered individuals respectively; the positive constant β represents the effective contact rate, and the positive constant γ is the recovery rate. If we denote the total constant population by N, then $N = S + I + R$.

However, for some diseases, the recovered individuals has a temporary period of immunity and may relapse. Namely, the recovered individuals will once again become infectious individuals after losing immunity. For example, herpes, human and bovine tuberculosis, and so on. Therefore, we need to consider the relapse of the disease. In view of that herpes viral infections are the susceptible-infective-latent-infective type, Tudor [18] proposed the following SIRI epidemic model:

$$\begin{cases} S'(t) = b - kS - \beta SI, \\ I'(t) = \beta SI - \gamma I + \delta R - kI, \\ R'(t) = \gamma I - \delta R - kR, \end{cases} \tag{1.1}$$

where b is the constant birth rate, k is the natural death rate, δ is the relapse rate, β represents the effective contact rate, γ is the recovery rate, these parameters are positive constants. Then, Moreira and Wang [17] modified this model by replacing βSI with a nonlinear incidence rate $\varphi(S)I$, they constructed

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the sufficient conditions for the persistence and extinction of disease, then gave the global asymptotic stability of the disease-free and endemic equilibria. After these, there are many researchers devoting into studying the model with relapse, please refer to [1, 5, 7, 30, 31] and references therein.

All of above models ignore the spatial diffusion. It is necessary to consider the following model with diffusion:

$$\begin{cases} S_t(t,x) - d\Delta S(t,x) = b - kS(t,x) - \beta S(t,x)I(t,x), & t>0,\ x\in\Omega, \\ I_t(t,x) - d\Delta I(t,x) = \beta S(t,x)I(t,x) - \gamma I(t,x) & \\ \qquad\qquad\qquad +\delta R(t,x) - kI(t,x), & t>0,\ x\in\Omega, \\ R_t(t,x) - d\Delta R(t,x) = \gamma I(t,x) - \delta R(t,x) - kR(t,x), & t>0,\ x\in\Omega, \\ \partial_n S(t,x) = \partial_n I(t,x) = \partial_n R(t,x), & t>0,\ x\in\partial\Omega, \\ S(0,x) = S_0(x), I(0,x) = I_0(x), R(0,x) = R_0(x), & x\in\overline{\Omega}, \end{cases} \tag{1.2}$$

where $S(t,x)$, $I(t,x)$ and $R(t,x)$ stand for the population densities of susceptible, infective and recovered individuals at $t>0$ and $x\in\Omega$ respectively. Assume that the diffusion rate of S, I and R are the same, and denoted by the positive constant d. There are many researchers studying this model. For example, [27] considered the dynamics of a diffusive SIRI model with nonlinear incidence rate, and [26] studied traveling waves for a spatial SIRI epidemic model. Besides, [19] investigated the dynamics of epidemiological model with replace and disease-induced deaths, which is extended by Ding et al. [4] to a diffusive epidemic model with replace and bilinear incidence rate later.

However, the above proposed models can not describe the spreading front well, which is important during studying the spreading of disease and species. To overcome this shortcoming, free boundary conditions are first introduced by Du and Lin [3] to study the spreading of invasive species. Later, many researchers used this free boundary problems to consider the spreading of diseases. For example, Kim et al. [11] considered an SIR epidemic model with free boundary; Huang and Wang [9] studied the reaction-diffusion system for an SIR epidemic model with a free boundary; Cao et al. [2] studied dynamics of a nonlocal SIS epidemic model with free boundary. For other problems related to free boundary, we can refer to [20, 16, 8] and references therein.

Inspired by the above works, we investigate the following SIRI epidemic model with relapse and free boundaries:

$$\begin{cases} S_t - dS_{xx} = b - kS - \beta SI, & t>0,\ x\in\mathbb{R}, \\ I_t - dI_{xx} = \beta SI - \gamma I + \delta R - kI, & t>0,\ x\in(g(t),h(t)), \\ R_t - dR_{xx} = \gamma I - \delta R - kR, & t>0,\ x\in(g(t),h(t)), \\ I(t,x) = R(t,x) = 0, & t\geq 0,\ x\in\mathbb{R}\backslash(g(t),h(t)), \\ g'(t) = -\mu I_x(t,g(t)), & t>0, \\ h'(t) = -\mu I_x(t,h(t)), & t>0, \\ S(0,x) = S_0(x), & x\in\mathbb{R}, \\ -g(0) = h(0) = h_0,\ I(0,x) = I_0(x), R(0,x) = R_0(x), & x\in[-h_0,h_0], \end{cases} \tag{1.3}$$

where $x=g(t)$ and $x=h(t)$ are the moving boundary to the determined, μ is the moving rate of the free boundary. Assume that the initial functions $S_0(x)$, $I_0(x)$ and $R_0(x)$ satisfy

$$\begin{cases} S_0\in C^2(\mathbb{R})\cap L^\infty(\mathbb{R}),\ I_0, R_0\in C^2([-h_0,h_0]), \\ I_0(x) = R_0(x) = 0,\ x\in\mathbb{R}\backslash(-h_0,h_0), \\ S_0(x)>0, x\in\mathbb{R},\ I_0(x)>0, R_0(x)>0, x\in(-h_0,h_0). \end{cases} \tag{1.4}$$

We should mention that this model was considered by Ding et al. in [4]. Unfortunately, there are some cases remaining open in [4]. In this paper, we aim to obtain the results for all the cases.

Denote

$$\mathcal{R}_0 = \frac{\beta b}{k(k+\gamma)} + \frac{\delta\gamma}{(k+\delta)(k+\gamma)}. \tag{1.5}$$

The rest of this paper is arranged as follows. The existence and uniqueness of the global solution of problem (1.3) will be proved in Section 2. In Section 3, we will establish the criteria for spreading and vanishing. Then, we will show the long time behavior of solution (S, I, R, g, h) for problem (1.3) in Section 4. In Section 5, we give some numerical simulations and brief discussions on the spreading and vanishing of diseases. The last section is a brief discussion.

2. EXISTENCE AND UNIQUENESS

In this section, we first prove the local existence and uniqueness result. Then we use some suitable estimates to show that the solution is defined for all $t > 0$.

Theorem 2.1. *For any given (S_0, I_0, R_0) satisfying (1.4) and any given $\alpha \in (0,1)$, there exists $T > 0$ such that problem (1.3) admits a unique solution*

$$(S, I, R, g, h) \in \mathcal{C}_T \times [C^{1+\frac{\alpha}{2}, 2+\alpha}(\overline{D}_T)]^2 \times [C^{1+\frac{\alpha}{2}}([0,T])]^2,$$

and

$$\|I\|_{C^{1+\frac{\alpha}{2},2+\alpha}(\overline{D}_T)} + \|R\|_{C^{1+\frac{\alpha}{2},2+\alpha}(\overline{D}_T)} + \|g\|_{C^{1+\frac{\alpha}{2}}([0,T])} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C,$$

where

$$\mathcal{C}_T = L^\infty([0,T]\times\mathbb{R}) \cap C^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\mathbb{R}), \;\; D_T = \{(t,x) \in \mathbb{R}^2 : t \in [0,T], x \in (g(t), h(t))\}.$$

Here C and T depend on h_0, α, $\|S_0\|_{C^2(\mathbb{R})}$, $\|S_0\|_{L^\infty(\mathbb{R})}$, $\|I_0\|_{C^2([-h_0,h_0])}$ and $\|R_0\|_{C^2([-h_0,h_0])}$.

Proof. The proof of the local existence and uniqueness of the solution is similar to that in [10, Theorem 2.1], [24, Theorem 2.1] and [25, Theorem 2.1]. For completeness, we give the details as follows.

Step 1: We first straighten the free boundaries. Let $\zeta(y)$ be a function in $C^3(\mathbb{R})$ satisfying

$$\zeta(y) = 1 \text{ if } |y - h_0| < \frac{h_0}{4},\;\; \zeta(y) = 0 \text{ if } |y-h_0| > \frac{h_0}{2},\;\; |\zeta'(y)| < \frac{h_0}{6},$$

and let

$$\xi(y) = \zeta(-y) \text{ for any } y \in \mathbb{R}.$$

Consider the transformation

$$(t,y) \to (t,x), \text{ where } x = y + \zeta(y)(h(t)-h_0) + \xi(y)(g(t)+h_0),\;\; y \in \mathbb{R}.$$

If

$$|h(t)-h_0| + |g(t)+h_0| \le \frac{h_0}{16},$$

then the above transformation is a diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Moreover, it changes the free boundaries $x = g(t)$ and $x = h(t)$ to the lines $y = -h_0$ and $y = h_0$. Direct calculations give that

$$\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{1}{1+\zeta'(y)(h(t)-h_0)+\xi'(y)(g(t)+h_0)} =: \sqrt{A(g(t),h(t),y)},\\
\frac{\partial^2 y}{\partial x^2} &= -\frac{\zeta''(y)(h(t)-h_0)+\xi''(y)(g(t)+h_0)}{[1+\zeta'(y)(h(t)-h_0)+\xi'(y)(g(t)+h_0)]^3} =: B(g(t),h(t),y),\\
\frac{\partial y}{\partial t} &= -\frac{\zeta(y)h'(t)+\xi(y)g'(t)}{1+\zeta'(y)(h(t)-h_0)+\xi'(y)(g(t)+h_0)} =: C(g(t),g'(t),h(t),h'(t),y).
\end{aligned}$$

Let

$$\begin{aligned}
&S(t,x)=S(t,y+\zeta(y)(h(t)-h_0)+\xi(y)(g(t)+h_0))=:u(t,y),\\
&I(t,x)=I(t,y+\zeta(y)(h(t)-h_0)+\xi(y)(g(t)+h_0))=:v(t,y),\\
&R(t,x)=R(t,y+\zeta(y)(h(t)-h_0)+\xi(y)(g(t)+h_0))=:w(t,y),
\end{aligned}$$

we have

$$\begin{aligned}
&S_t=u_t+C(g(t),g'(t),h(t),h'(t),y)u_y, \qquad S_x=\sqrt{A(g(t),h(t),y)}u_y,\\
&I_t=v_t+C(g(t),g'(t),h(t),h'(t),y)v_y, \qquad I_x=\sqrt{A(g(t),h(t),y)}v_y,\\
&R_t=w_t+C(g(t),g'(t),h(t),h'(t),y)w_y, \qquad R_x=\sqrt{A(g(t),h(t),y)}w_y,\\
&S_{xx}=A(g(t),h(t),y)u_{yy}+B(g(t),h(t),y)u_y,\\
&I_{xx}=A(g(t),h(t),y)v_{yy}+B(g(t),h(t),y)v_y,\\
&R_{xx}=A(g(t),h(t),y)w_{yy}+B(g(t),h(t),y)w_y,
\end{aligned}$$

then problem (1.3) becomes

$$\begin{cases}
u_t-Adu_{yy}-(dB-C)u_y=b-ku-\beta uv, & t>0,\ y\in\mathbb{R},\\
v_t-Adv_{yy}-(dB-C)v_y=(\beta u-\gamma-k)v+\delta w, & t>0,\ |y|<h_0,\\
w_t-Adw_{yy}-(dB-C)w_y=\gamma v-(\delta+k)w, & t>0,\ |y|<h_0,\\
g'(t)=-\mu v_y(t,-h_0), g(0)=-h_0, & t>0,\\
h'(t)=-\mu v_y(t,h_0), h(0)=h_0, & t>0,\\
v(t,y)=w(t,y)=0, & t\ge 0,\ |y|\ge h_0,\\
u(0,y)=u_0(y):=S_0(y), & y\in\mathbb{R},\\
v(0,y)=v_0(x):=I_0(y),\ w(0,y)=w_0(y):=R_0(y), & y\in[-h_0,h_0],
\end{cases} \tag{2.1}$$

where $A=A(g(t),h(t),y)$, $B=B(g(t),h(t),y)$, $C=C(g(t),g'(t),h(t),h'(t),y)$.

Let

$$T_1=\min\left\{\frac{h_0}{16(1+g^*)},\frac{h_0}{16(1+h^*)}\right\}.$$

For $0<T<T_1$, we define $\Delta_T=[0,T]\times[-h_0,h_0]$ and

$$\begin{aligned}
&X_{1T}=\{v\in C(\Delta_T):v(t,\pm h_0)=0,v(0,y)=v_0,\|v-v_0\|_{C(\Delta_T)}\le 1\},\\
&X_{2T}=\{w\in C(\Delta_T):w(t,\pm h_0)=0,w(0,y)=w_0,\|w-w_0\|_{C(\Delta_T)}\le 1\},\\
&X_{3T}=\{g\in C^1([0,T]):g(0)=-h_0,g'(0)=g^*,\|g'-g^*\|_{C([0,T])}\le 1\},\\
&X_{4T}=\{h\in C^1([0,T]):h(0)=h_0,h'(0)=h^*,\|h'-h^*\|_{C([0,T])}\le 1\},
\end{aligned}$$

where $g^*=-\mu v_0'(-h_0)$, $h^*=-\mu v_0'(h_0)$. It is easy to see that $X_T:=X_{1T}\times X_{2T}\times X_{3T}\times X_{4T}$ is a complete metric space with the metric

$$\begin{aligned}
&d((v_1,w_1,g_1,h_1),(v_2,w_2,g_2,h_2))\\
&=\|v_1-v_2\|_{C(\Delta_T)}+\|w_1-w_2\|_{C(\Delta_T)}+\|g_1'-g_2'\|_{C([0,T])}+\|h_1'-h_2'\|_{C([0,T])}.
\end{aligned}$$

For any given $(v,w,g,h)\in X_T$ with $0<T<T_1$, we consider

$$\begin{cases}
u_t-Adu_{yy}-(dB-C)u_y=b-\beta uv-ku, & 0<t\le T,\ y\in\mathbb{R},\\
u(0,y)=u_0(y), & y\in\mathbb{R}.
\end{cases}$$

Applying standard partial differential equation theory [6], this problem has a unique solution $u \in L^\infty([0,T]\times\mathbb{R}) \cap C^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\mathbb{R})$. For above (u,v,w,g,h), we consider

$$\begin{cases} \tilde{v}_t - Ad\tilde{v}_{yy} - (dB-C)\tilde{v}_y = v(\beta u - \gamma - k) + \delta w, & 0 < t \le T,\ |y| < h_0, \\ \tilde{w}_t - Ad\tilde{w}_{yy} - (dB-C)\tilde{w}_y = \gamma v - (k+\delta)w, & 0 < t \le T,\ |y| < h_0, \\ \tilde{v}(t,\pm h_0) = \tilde{w}(t,\pm h_0) = 0, & 0 < t \le T, \\ \tilde{v}(0,y) = v_0(y), \tilde{w}(0,y) = w_0(y), & y \in [-h_0,h_0]. \end{cases}$$

It follows from the standard L^p theory and Sobolev's embedding theorem that this problem admits a unique solution $(\tilde{v},\tilde{w}) \in [C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)]^2$. Just like the arguments in [21], we have, for $T \in (0,T_1]$,

$$\|\tilde{v}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)} \le \|\tilde{v}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_{T_1})} \le C(T_1^{-1})\|\tilde{v}\|_{W_p^{1,2}(\Delta_{T_1})} \le K_1(T_1^{-1},T_1), \tag{2.2}$$

$$\|\tilde{w}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)} \le \|\tilde{w}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_{T_1})} \le C(T_1^{-1})\|\tilde{w}\|_{W_p^{1,2}(\Delta_{T_1})} \le K_1(T_1^{-1},T_1), \tag{2.3}$$

where K_1 is a constant depending on T_1, Δ_{T_1}, α, h_0, p, β, b, γ, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|v_0\|_{C^2([-h_0,h_0])}$ and $\|w_0\|_{C^2([-h_0,h_0])}$.

For $0 < T < T_1$, defining

$$\tilde{g}(t) = -h_0 - \int_0^t \mu\tilde{v}_y(\tau,-h_0)d\tau,\ \tilde{h}(t) = h_0 - \int_0^t \mu\tilde{v}_y(\tau,h_0)d\tau,\ 0 \le t \le T.$$

We have

$$\tilde{g}'(t) = -\mu\tilde{v}_y(t,-h_0),\ \tilde{h}'(t) = -\mu\tilde{v}_y(t,h_0),\ -\tilde{g}(0) = \tilde{h}(0) = h_0,$$

and then $\tilde{g}'(t), \tilde{h}'(t) \in C^{\frac{\alpha}{2}}([0,T])$,

$$\|\tilde{g}'\|_{C^{\frac{\alpha}{2}}([0,T])}, \|\tilde{h}'\|_{C^{\frac{\alpha}{2}}([0,T])} \le \mu K_1 := K_2, \tag{2.4}$$

where K_2 depends on μ, h_0, K_1.

Now, we define $\mathcal{F} : X_T \to [C(\Delta_T)]^2 \times [C^1([0,T])]^2$ by

$$\mathcal{F}(v,w,g,h) = (\tilde{v},\tilde{w},\tilde{g},\tilde{h}).$$

It is clear that $(v,w,g,h) \in X_T$ is a fixed point of $\mathcal{F}$ if and only if it solves (2.1).

At first, we prove that $\mathcal{F}$ is a self-mapping on X_T for $T > 0$ sufficiently small. By (2.2), (2.3) and (2.4), we know that $\mathcal{F}$ is continuous and compact, and

$$\begin{aligned} &\|\tilde{g}' - g^*\|_{C([0,T])} \le \|\tilde{g}'\|_{C^{\frac{\alpha}{2}}([0,T])} T^{\frac{\alpha}{2}} \le K_2 T^{\frac{\alpha}{2}}, \\ &\|\tilde{h}' - h^*\|_{C([0,T])} \le \|\tilde{h}'\|_{C^{\frac{\alpha}{2}}([0,T])} T^{\frac{\alpha}{2}} \le K_2 T^{\frac{\alpha}{2}}, \\ &\|\tilde{v} - v_0\|_{C(\Delta_T)} \le \|\tilde{v}\|_{C^{\frac{1+\alpha}{2},0}(\Delta_T)} T^{\frac{1+\alpha}{2}} \le K_1 T^{\frac{1+\alpha}{2}}, \\ &\|\tilde{w} - w_0\|_{C(\Delta_T)} \le \|\tilde{w}\|_{C^{\frac{1+\alpha}{2},0}(\Delta_T)} T^{\frac{1+\alpha}{2}} \le K_1 T^{\frac{1+\alpha}{2}}. \end{aligned}$$

If we take

$$0 < T \le \min\left\{T_1, K_2^{-\frac{2}{\alpha}}, K_1^{-\frac{2}{1+\alpha}}\right\},$$

then $\mathcal{F}$ maps X_T into itself. It follows from the Schauder fixed point theorem that $\mathcal{F}$ has at least one fixed point $(v,w,g,h) \in X_T$. Therefore, (2.1) has at least one solution (u,v,w,g,h) and

$$u \in L^\infty([0,T]\times\mathbb{R}) \cap C^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\mathbb{R}),\ v,w \in C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T),$$

$$g,h \in C^{1+\frac{\alpha}{2}}([0,T]),\ g'(t) < 0,\ h'(t) > 0 \text{ in } [0,T].$$

Hence, problem (1.3) has a solution

$$(S, I, R, g, h) \in \mathcal{C}_T \times [C^{\frac{1+\alpha}{2}, 1+\alpha}(\overline{D}_T)]^2 \times [C^{1+\frac{\alpha}{2}}([0, T])]^2.$$

Step 2: Let $(S_i, I_i, R_i, g_i, h_i)(i = 1, 2)$ be the solution of (1.3), which is defined for $t \in [0, T]$ with $0 < T \ll 1$. Making the same transformation as in Step 1, we have

$$\begin{aligned}
S_i(t, x) &= S_i(t, y + \zeta(y)(h_i(t) - h_0) + \xi(y)(g_i(t) + h_0)) =: u_i(t, y),\ i = 1, 2,\\
I_i(t, x) &= I_i(t, y + \zeta(y)(h_i(t) - h_0) + \xi(y)(g_i(t) + h_0)) =: v_i(t, y),\ i = 1, 2,\\
R_i(t, x) &= R_i(t, y + \zeta(y)(h_i(t) - h_0) + \xi(y)(g_i(t) + h_0)) =: w_i(t, y),\ i = 1, 2.
\end{aligned}$$

Letting

$$U = u_1 - u_2,\ V = v_1 - v_2,\ W = w_1 - w_2,\ G = g_1 - g_2 \text{ and } H = h_1 - h_2,$$

then $U(t, y)$, $V(t, y)$ and $W(t, y)$ satisfy

$$\begin{cases}
U_t - A_1 dU_{yy} - (dB_1 - C_1)U_y + [\beta v_1 + k]U = (A_1 - A_2)du_{2yy} & \\
\qquad\qquad + d(B_1 - B_2)u_{2y} + (C_2 - C_1)u_{2y} - \beta u_2 V, & 0 < t < T,\ y \in \mathbb{R},\\
V_t - A_1 dV_{yy} - (dB_1 - C_1)V_y - (\beta u_1 - \gamma - k)V = (A_1 - A_2)dv_{2yy} & \\
\qquad\qquad + d(B_1 - B_2)v_{2y} + (C_2 - C_1)v_{2y} + \beta v_2 U + \delta W, & 0 < t < T,\ y < |h_0|,\\
W_t - A_1 dW_{yy} - (dB_1 - C_1)W_y + (k + \delta)W = (A_1 - A_2)dw_{2yy} & \\
\qquad\qquad + d(B_1 - B_2)w_{2y} + (C_2 - C_1)w_{2y} + \gamma V, & 0 < t < T,\ y < |h_0|,\\
U(0, y) = 0, & y \in \mathbb{R},\\
V(t, \pm h_0) = 0,\ W(t, \pm h_0) = 0, & 0 \le t \le T,\\
V(0, y) = 0,\ W(0, y) = 0, & y \le |h_0|,
\end{cases}$$

and

$$\begin{cases}
G'(t) = -\mu V_y(t, -h_0), & 0 < t < T,\\
H'(t) = -\mu V_y(t, h_0), & 0 < t < T,\\
G(0) = H(0) = 0,
\end{cases}$$

where $A_i = A(g_i, h_i, y)$, $B_i = B(g_i, h_i, y)$, $C_i = C(g_i, g_i', h_i, h_i', y)$ $(i = 1, 2)$.

Just like the arguments in [21], we have, for $T \in (0, T_1]$,

$$\begin{aligned}
\|u_1 - u_2\|_{L^\infty([0,T]\times\mathbb{R})} &\le K_3(T_1)(\|v_1 - v_2\|_{C(\Delta_T)} + \|g_1 - g_2\|_{C^1([0,T])}\\
&\qquad + \|h_1 - h_2\|_{C^1([0,T])}),\\
\|v_1 - v_2\|_{C^{\frac{1+\alpha}{2}, 1+\alpha}(\Delta_T)} &\le K_4(T_1)\left(\|u_1 - u_2\|_{L^\infty([0,T]\times\mathbb{R})} + \|w_1 - w_2\|_{C(\Delta_T)}\right.\\
&\qquad \left. + \|g_1 - g_2\|_{C^1([0,T_1])} + \|h_1 - h_2\|_{C^1([0,T_1])}\right)\\
&\le K_5(T_1)\left(\|v_1 - v_2\|_{C(\Delta_T)} + \|w_1 - w_2\|_{C(\Delta_T)}\right.\\
&\qquad \left. + \|g_1 - g_2\|_{C^1([0,T_1])} + \|h_1 - h_2\|_{C^1([0,T_1])}\right),\\
\|w_1 - w_2\|_{C^{\frac{1+\alpha}{2}, 1+\alpha}(\Delta_T)} &\le K_6(T_1)(\|v_1 - v_2\|_{C(\Delta_T)} + \|g_1 - g_2\|_{C^1([0,T_1])}\\
&\qquad + \|h_1 - h_2\|_{C^1([0,T_1])}),
\end{aligned} \tag{2.5}$$

where K_3, K_4 and K_6 depend on K_1, K_2 and the functions A_i, B_i, C_i in the definition of the transformation $(t, y) \to (t, x)$, K_5 depends on K_3 and K_4.

By similar arguments in the proof of [21, Theorem 1.1], we can obtain that

$$[v_{1y} - v_{2y}]_{C^{\frac{\alpha}{2}, \alpha}(\Delta_T)} \le K_7 \|v_1 - v_2\|_{W_p^{1,2}(\Delta_T)} \le K_8 \|v_1 - v_2\|_{C^{\frac{1+\alpha}{2}, 1+\alpha}(\Delta_T)},$$

where K_7 and K_8 are independent of T^{-1}. Then

$$\begin{aligned}\|g_1'-g_2'\|_{C^{\frac{\alpha}{2}}([0,T])}+\|h_1'-h_2'\|_{C^{\frac{\alpha}{2}}([0,T])} &\le 2\mu\|v_{1y}-v_{2y}\|_{C^{\frac{\alpha}{2},0}(\Delta_T)}\\ \le K_9(\|(v_1-v_2,w_1-w_2)\|_{C(\Delta_T)}&+\|(g_1-g_2,h_1-h_2)\|_{C^1([0,T])}),\end{aligned}$$

where K_9 depends on K_5, K_8 and μ. Recalling that $G(0)=G'(0)=0$ and $H(0)=H'(0)=0$, we have for $T\ll 1$,

$$\begin{aligned}&\|g_1-g_2\|_{C^1([0,T])}+\|h_1-h_2\|_{C^1([0,T])}\\ \le\ & T^{\frac{\alpha}{2}}(\|g_1'-g_2'\|_{C^{\frac{\alpha}{2}}([0,T])}+\|h_1'-h_2'\|_{C^{\frac{\alpha}{2}}([0,T])})\\ \le\ & K_{10}T^{\frac{\alpha}{2}}\left(\|v_1-v_2\|_{C(\Delta_T)}+\|w_1-w_2\|_{C(\Delta_T)}+\|g_1-g_2\|_{C^1([0,T_1])}+\|h_1-h_2\|_{C^1([0,T_1])}\right).\end{aligned}$$

Combining this fact and (2.5), we have, if $T<1$,

$$\begin{aligned}&\|v_1-v_2\|_{C(\Delta_T)}+\|w_1-w_2\|_{C(\Delta_T)}+\|g_1-g_2\|_{C^1([0,T])}+\|h_1-h_2\|_{C^1([0,T])}\\ \le\ & T^{\frac{1+\alpha}{2}}\|v_1-v_2\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)}+T^{\frac{1+\alpha}{2}}\|w_1-w_2\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Delta_T)}\\ &+T^{\frac{\alpha}{2}}\|g_1'-g_2'\|_{C^{\frac{\alpha}{2}}([0,T])}+T^{\frac{\alpha}{2}}\|h_1'-h_2'\|_{C^{\frac{\alpha}{2}}([0,T])}\\ \le\ & K_{11}T^{\frac{\alpha}{2}}(\|v_1-v_2\|_{C(\Delta_T)}+\|w_1-w_2\|_{C(\Delta_T)}+\|g_1-g_2\|_{C^1([0,T])}+\|h_1-h_2\|_{C^1([0,T])}).\end{aligned}$$

Therefore, for $0<T\ll 1$, we have $v_1=v_2$, $w_1=w_2$, $g_1=g_2$ and $h_1=h_2$. Consequently, $u_1=u_2$.

Step 3: By the Schauder estimates, we have additional regularity for (u,v,w,g,h) as a solution of (2.1), namely,

$$u\in L^\infty([0,T]\times\mathbb{R})\cap C_{loc}^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\mathbb{R});\ \ v,w\in C^{1+\frac{\alpha}{2},2+\alpha}(\Delta_T)\ \text{ and }\ g,h\in C^{1+\frac{\alpha}{2}}([0,T]),$$

and (2.2), (2.3) and (2.4) hold. Hence, (u,v,w,g,h) is a unique local classical solution of the problem (2.1), and then (S,I,R,g,h) is a unique local classical solution of the problem (1.3). □

To show that the local solution obtained in Theorem 2.1 can be defined for all $t>0$, we need the following estimate.

Lemma 2.2. *Let (S,I,R,g,h) be a solution to problem (1.3) defined for $t\in(0,T_0)$ for some $T_0\in(0,+\infty)$, and satisfies*

$$\begin{aligned}&0<S(t,x)\le M_1,\ 0<t\le T_0,\ x\in\mathbb{R},\\ &0<I(t,x),R(t,x)\le M_1,\ 0<t\le T_0,\ g(t)<x<h(t),\\ &-M_2\le g'(t)<0, 0<h'(t)<M_2,\ 0<t\le T_0,\end{aligned}$$

where $M_i(i=1,2)$ is independent of T_0.

Proof. Using the strong maximum principle, we are easy to see that $S(t,x)>0$ in $(0,T_0]\times\mathbb{R}$ and $I(t,x),R(t,x)>0$ in $(0,T_0]\times(g(t),h(t))$. By direct calculations, we have that $N(t,x)$ satisfies

$$\begin{cases}N_t-dN_{xx}=b-kN, & 0<t\le T_0,\ x\in\mathbb{R}\setminus\{h(t),g(t)\},\\ N_x(t,g(t)-0)\le N_x(t,g(t)+0), & 0<t\le T_0,\\ N_x(t,h(t)-0)\le N_x(t,h(t)+0), & 0<t\le T_0,\\ N(0,x)=N_0(x), & x\in\mathbb{R},\end{cases}$$

where $N_0(x)=S_0(x)+I_0(x)+R_0(x)$. Let $\overline{N}(t,x)$ be the solution of

$$\begin{cases}\overline{N}_t-d\overline{N}_{xx}=b-k\overline{N}, & 0<t\le T_0,\ x\in\mathbb{R},\\ \overline{N}(0,x)=N_0(x), & x\in\mathbb{R}.\end{cases}$$

It follows from the comparison principle that we have

$$N(t,x) \le \overline{N}(t,x) \le \max\left\{\frac{b}{k}, \|N_0\|_\infty\right\} =: M_1.$$

Therefore, we have

$$\begin{aligned} &0 < S(t,x) \le M_1,\ 0 < t \le T_0,\ x \in \mathbb{R},\\ &0 < I(t,x), R(t,x) \le M_1,\ 0 < t \le T_0,\ g(t) < x < h(t). \end{aligned}$$

By the Hopf boundary lemma, we have $g'(t) < 0,\ h'(t) > 0,\ t \in (0,T_0)$. In the following, we will prove that $h'(t) \le M_2$ for all $t \in (0,T_0)$ with some M_2 independent of T_0. By the similar arguments in [15], we define

$$\begin{aligned} &\Omega_M := \{(t,x) : 0 < t < T_0,\ h(t) - M^{-1} < x < h(t)\},\\ &z(t,x) = M_1[2M(h(t)-x) - M^2(h(t)-x)^2],\ (t,x) \in \Omega_M, \end{aligned}$$

where M will be determined later. Direct calculations show that, for $(t,x) \in \Omega_M$,

$$\begin{aligned} &z_t - dz_{xx} - \beta SI + \gamma I - \delta R + kI\\ =\ &2M_1Mh'(t)(1 - M(h(t)-x)) + 2dM_1M^2 - \beta SI - \delta R + (\gamma + k)I\\ \ge\ &2dM_1M^2 - \beta M_1^2 - \delta M_1. \end{aligned}$$

If we choose $M \ge \sqrt{(\beta M_1 + \delta)/2d}$, then

$$z_t - dz_{xx} - \beta SI + \gamma I - \delta R + kI \ge 0.$$

It is easy to check that

$$z(t, h(t) - M^{-1}) = M_1 \ge I(t, h(t) - M^{-1}),\ z(t,h(t)) = 0 = I(t,h(t)).$$

If we can choose suitable M such that $I_0(x) \le z(0,x)$ for $x \in [h_0 - M^{-1}, h_0]$, then we can apply the maximum principle to $z - I$ over Ω_M to deduce that $I(t,x) \le z(t,x)$ for $(t,x) \in \Omega_M$. It follows that

$$\begin{aligned} &I_x(t,h(t)) \ge z_x(t,h(t)) = -2M_1M,\\ &h'(t) = -\mu I_x(t,h(t)) \le 2\mu MM_1 =: M_2. \end{aligned}$$

In the following, we find some M independent of T_0 such that $I_0(x) \le z(0,x)$ for $x \in [h_0 - M^{-1}, h_0]$. Let

$$M := \max\left\{\sqrt{\frac{\beta M_1 + \delta}{2d}}, \frac{4\|I_0\|_{C^1([-h_0,h_0])}}{3M_1}\right\},$$

we have

$$z_x(0,x) = -2M_1M[1 - M(h_0 - x)] \le -MM_1 \le -\frac{4}{3}\|I_0\|_{C^1([-h_0,h_0])} \le I_0'(x),$$

for $x \in [h_0 - (2M)^{-1}, h_0]$. Noting that $z(0,h_0) = I_0(h_0) = 0$, we obtain

$$z(0,x) \ge I_0(x),\ x \in [h_0 - (2M)^{-1}, h_0].$$

Moreover,

$$I_0(x) \le \|I_0\|_{C^1([-h_0,h_0])}M^{-1} \le \frac{3}{4}M_1 \le z(0,x),$$

for $x \in [h_0 - (M)^{-1}, h_0 - (2M)^{-1}]$. Hence $I_0(x) \le z(0,x)$ for $x \in [h_0 - M^{-1}, h_0]$.

Similarly, we have $g'(t) \ge -M_2$. This proof is completed. □

Theorem 2.3. *The solution of problem (1.3) exists and is unique for all* $t \in (0,\infty)$.

Proof. This proof is similar to Theorem 2.4 in [11], we give the details of the proof process below. Let $[0, T_{max})$ be the maximal time interval in which the solution exists. By Theorem 2.1, $T_{max} > 0$. Next, we shall prove that $T_{max} = \infty$. By the contradiction argument, we assume that $T_{max} < \infty$. By Lemma 2.2, there are M_1 and M_2 and independent of T_{max} such that

$$\begin{aligned}
&0 < S(t,x) \leq M_1,\ (t,x) \in [0, T_{max}) \times \mathbb{R},\\
&0 < I(t,x), R(t,x) \leq M_1,\ (t,x) \in [0, T_{max}) \times [g(t), h(t)],\\
&-M_2 \leq g'(t) < 0, -h_0 - M_2 t \leq g(t) \leq -h_0,\ t \in [0, T_{max}),\\
&0 < h'(t) < M_2, h_0 \leq h(t) \leq h_0 + M_2 t,\ t \in [0, T_{max}).
\end{aligned}$$

We now fix $\delta_0 \in (0, T_{max})$ and $M > T_{max}$. By standard L^p estimates, the Sobolev embedding theorem, and the Hölder estimates for parabolic equations, we can find $M_3 > 0$ depending only on δ_0, M, M_1, M_2 such that

$$\|S(\cdot,t)\|_{C^2[0,\infty)}, \|I(\cdot,t)\|_{C^2[g(t),h(t)]}, \|R(\cdot,t)\|_{C^2[g(t),h(t)]} \leq M_3 \text{ for } t \in [\delta_0, T_{max}).$$

It then follows from the proof of Theorem 2.1 that there exists a $\tau > 0$ depending only on $M_i (i = 1, 2, 3)$ such that the solution of problem (1.3) with initial time $T_{max} - \frac{\tau}{2}$ can be extended uniquely to the time $T_{max} - \frac{\tau}{2} + \tau = T_{max} + \frac{\tau}{2}$. But this contradicts the assumption, thus $T_{max} = \infty$. This proof is completed. □

3. CRITERIA FOR SPREADING AND VANISHING

It follows from Lemma 2.2 that $g(t)$ and $h(t)$ are monotonically decreasing and increasing respectively. Then there exist h_∞ and $g_\infty \in (0, +\infty]$ such that $\lim_{t\to\infty} h(t) = h_\infty$, $\lim_{t\to\infty} g(t) = g_\infty$. We call that **spreading happens** if $\lim_{t\to\infty}(h(t) - g(t)) = \infty$, and **vanishing happens** if $\lim_{t\to\infty}(h(t) - g(t)) < \infty$. In this section, we will divide our discussion into two case: $\mathcal{R}_0 \leq 1$ and $\mathcal{R}_0 > 1$.

3.1. The case $\mathcal{R}_0 \leq 1$.

Lemma 3.1. *If $\mathcal{R}_0 \leq 1$ and $\|S_0\|_\infty \leq \frac{b}{k}$, then $h_\infty - g_\infty < \infty$.*

Proof. By $\|S_0\|_\infty \leq \frac{b}{k}$, we can use the comparison principle to obtain that

$$S(t,x) \leq \frac{b}{k} \text{ for } t > 0 \text{ and } x \in \mathbb{R}.$$

In view of $\mathcal{R}_0 \leq 1$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{g(t)}^{h(t)} \left[I(t,x) + \frac{\delta}{k+\delta}R(t,x)\right] dx\\
= &\int_{g(t)}^{h(t)} \left[I_t(t,x) + \frac{\delta}{k+\delta}R_t(t,x)\right] dx + h'(t)\left[I(t,h(t)) + \frac{\delta}{k+\delta}R(t,h(t))\right]\\
&- g'(t)\left[I(t,g(t)) + \frac{\delta}{k+\delta}R(t,g(t))\right]\\
= &\int_{g(t)}^{h(t)} \left[dI_{xx} + \beta SI - \gamma I - kI + \delta R + \frac{\delta}{k+\delta}(dR_{xx} + \gamma I - kR - \delta R)\right] dx\\
= &\int_{g(t)}^{h(t)} \left[(\beta S - \gamma - k + \frac{\delta\gamma}{k+\delta})I\right] dx + \int_{g(t)}^{h(t)} \left[d(I_{xx} + \frac{\delta}{k+\delta}R_{xx})dx\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{g(t)}^{h(t)} \left[(\beta\frac{b}{k} - \gamma - k + \frac{\delta\gamma}{k+\delta})I\right] dx + \int_{g(t)}^{h(t)} \left[d(I_{xx} + \frac{\delta}{k+\delta}R_{xx})dx\right] \\
&\leq \int_{g(t)}^{h(t)} (\gamma + k)(\mathcal{R}_0 - 1)I dx + \int_{g(t)}^{h(t)} \left[d(I_{xx} + \frac{\delta}{k+\delta}R_{xx})dx\right] \\
&\leq d\int_{g(t)}^{h(t)} I_{xx} dx \\
&= \ -\frac{d}{\mu}[h'(t) - g'(t)] \leq 0,
\end{aligned}$$

which implies

$$h(t) - g(t) \leq 2h_0 + \frac{d}{\mu}\int_{-h_0}^{h_0} \left[I_0(x) + \frac{\delta}{k+\delta}R_0(x)\right] dx < \infty \text{ for } t > 0.$$

Thus we finish the proof. □

Lemma 3.2. *If $h_\infty - g_\infty < \infty$, then $\lim_{t\to\infty} S(t,x) = b/k$ in $C_{loc}(\mathbb{R})$, and*

$$\lim_{t\to\infty} \|I(t,x) + R(t,x)\|_{C([g(t),h(t)])} = 0. \tag{3.1}$$

Proof. By [22, Proposition 3.1], it follows from the equations I satisfied that

$$\lim_{t\to\infty} \|I(t,x)\|_{C([g(t),h(t)])} = 0.$$

Then, for any given $\varepsilon_1 > 0$, there exists $T_1 > 0$ such that $I(t,x) < \varepsilon_1$ for $t \geq T_1$ and $x \in [g(t), h(t)]$. For above ε_1, we can use the comparison principle to obtain that

$$R(t,x) \leq \frac{\gamma\varepsilon_1}{k+\delta} \text{ for } t \in [T_1,\infty) \text{ and } x \in [g(t), h(t)].$$

By the arbitrariness of ε_1, we have

$$\lim_{t\to\infty} \|R(t,x)\|_{C([g(t),h(t)])} = 0.$$

Therefore, (3.1) is proved.

By the comparison principle, we can have

$$\lim_{t\to\infty} S(t,x) \leq \frac{b}{k} \text{ uniformly in } \mathbb{R}. \tag{3.2}$$

Noting that $I(t,x) = R(t,x) = 0$ for $t > 0$ and $x \in \mathbb{R}\backslash[g(t), h(t)]$, it follows from (3.1) that, for any small $\varepsilon_2 > 0$, there exists $T_2 \gg 1$ such that $I(t,x) + R(t,x) \leq \varepsilon_2$ for $t \geq T_2$ and $x \in \mathbb{R}$. Hence, S satisfies

$$\begin{cases} S_t - dS_{xx} \geq b - kS - \beta\varepsilon_2 S, & t > T_2,\ x \in \mathbb{R}, \\ S(T_2, x) > 0, & x \in \mathbb{R}. \end{cases}$$

Let $\underline{S}$ be the unique solution of

$$\begin{cases} \underline{S}_t - d\underline{S}_{xx} = b - k\underline{S} - \beta\varepsilon_2\underline{S}, & t > T_2,\ x \in \mathbb{R}, \\ \underline{S}(T_2, x) = S(T_2, x), & x \in \mathbb{R}. \end{cases}$$

It is obvious that $\lim_{t\to\infty} \underline{S}(t,x) = b/(k+\beta\varepsilon_2)$ in $C_{loc}(\mathbb{R})$. By a comparison argument and the arbitrariness of ε_2, we have $\liminf_{t\to\infty} S(t,x) \geq b/k$ in $C_{loc}(\mathbb{R})$. Combined with (3.2), we have $\lim_{t\to\infty} S(t,x) = b/k$ in $C_{loc}(\mathbb{R})$. The proof is completed. □

3.2. The case $\mathcal{R}_0 > 1$.

Consider the following eigenvalue problem:

$$\begin{cases} -d\phi'' = -a_{11}\phi + a_{12}\psi + \lambda\phi, & x \in (-L, L), \\ -d\psi'' = a_{21}\phi - a_{22}\psi + \lambda\psi, & x \in (-L, L), \\ \phi(\pm L) = \psi(\pm L) = 0, \end{cases} \tag{3.3}$$

where $a_{12}, a_{21}, a_{22} > 0$ and $a_{11} \in \mathbb{R}$ are all the constants. We denote its first eigenpair by $(\lambda, \phi(x), \psi(x))$. It follows from [25, Lemma 3.3] that

$$\lambda = \frac{1}{2}\left[\frac{d\pi^2}{2L^2} + a_{11} + a_{22} - \sqrt{\left(\frac{d\pi^2}{2L^2} + a_{11} + a_{22}\right)^2 - 4(\frac{d\pi^2}{4L^2} + a_{11})(\frac{d\pi^2}{4L^2} + a_{22}) + 4a_{12}a_{21}}\right],$$

and there exists some $\theta > 0$ such that $\phi = \theta\psi$ and

$$(\phi(x), \psi(x)) = (\theta\cos(\frac{\pi}{2L}x), \cos(\frac{\pi}{2L}x)).$$

Let $(\lambda_1, \phi(x), \varphi(x))$ be first eigenpair of (3.3) with $a_{11} = \gamma + k - \beta b/k$, $a_{12} = \delta$, $a_{21} = \gamma$ and $a_{22} = k + \delta$. By $\mathcal{R}_0 > 1$, we have

$$a_{12}a_{21} - a_{11}a_{22} = \delta\gamma - (\gamma + k - \beta\frac{b}{k})(k + \delta) > 0.$$

Then it follows from [29, Lemma 3.3] that there exists some $h^* > 0$ such that $\lambda_1(L) > 0$ for $L < h^*$, $\lambda_1(L) = 0$ for $L = h^*$ and $\lambda_1(L) < 0$ for $L > h^*$, where

$$h^* = L^*(a_{11}, a_{12}, a_{21}, a_{22}) := \frac{\pi}{2}\sqrt{\frac{2d^2}{\sqrt{d^2(a_{22} - a_{11})^2 + 4d^2a_{12}a_{21}} - d(a_{22} + a_{11})}}.$$

Lemma 3.3. *If $h_\infty - g_\infty < \infty$, then $h_\infty - g_\infty \leq 2h^*$.*

Proof. Assume on the contrary that $2h^* < h_\infty - g_\infty < \infty$. Then there exists some small $\varepsilon > 0$ such that $h_\infty - g_\infty > 2h^*_\varepsilon =: 2L^*(-\beta(\frac{b}{k} - \varepsilon) + \gamma + k, \delta, \gamma, k + \delta)$. By Lemma 3.2, we have, for above $\varepsilon > 0$, there exists some $T \gg 1$ such that

$$S(t, x) \geq \frac{b}{k} - \varepsilon \text{ for } t \geq T \text{ and } x \in [g_\infty, h_\infty],$$

and

$$h(t) - g(t) > 2h^*_\varepsilon \text{ for } t \geq T. \tag{3.4}$$

Hence

$$\begin{cases} I_t - dI_{xx} \geq (\beta\frac{b}{k} - \gamma - k - \beta\varepsilon)I + \delta R, & t > T,\ x \in (g(T), h(T)), \\ R_t - dR_{xx} = \gamma I - (k + \delta)R, & t > T,\ x \in (g(T), h(T)), \\ I(t, g(T)) > 0, I(t, h(T)) > 0, & t > T, \\ R(t, g(T)) > 0, R(t, h(T)) > 0, & t > T, \\ I(T, x) \geq 0, R(T, x) \geq 0, & x \in [g(T), h(T)]. \end{cases}$$

Let $(\lambda_1, \phi(x), \psi(x))$ be the eigenpair of (3.3) with $L = [h(T) - g(T)]/2$, $a_{11} = \gamma + k - b/k + \beta\varepsilon$, $a_{12} = \delta$, $a_{21} = \gamma$ and $a_{22} = k + \delta$, then $\lambda_1(L) < 0$. We define

$$\underline{I}(t, x) = m\phi\left(x - \frac{g(T) + h(T)}{2}\right),\ t \geq 0,\ x \in [g(T), h(T)],$$

$$\underline{R}(t, x) = m\psi\left(x - \frac{g(T) + h(T)}{2}\right),\ t \geq 0,\ x \in [g(T), h(T)],$$

where m will be determined later.

Noting that $\phi = \theta\psi$, we have

$$\begin{aligned}
&\underline{I}_t - d\underline{I}_{xx} - (\beta\frac{b}{k} - \gamma - k - \beta\varepsilon)\underline{I} - \delta\underline{R}\\
=& -dm\phi'' - q(k\frac{b}{k} - \gamma - k - \beta\varepsilon)\phi - \delta q\psi\\
=& \, q\left[(k\frac{b}{k} - \gamma - k - \beta\varepsilon)\phi + \delta\psi + \lambda_1\phi\right] - m(k\frac{b}{k} - \gamma - k - \beta\varepsilon)\phi - \delta m\psi\\
=& \, m\lambda_1\phi < 0,
\end{aligned}$$

and

$$\begin{aligned}
&\underline{R}_t - d\underline{R}_{xx} - \gamma\underline{I} + (k+\delta)\underline{R}\\
=& -dm\psi'' - m\gamma\phi - m(k+\delta)\psi\\
=& \, m\lambda_1\psi < 0.
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
&\underline{I}(t, g(T)) = \underline{I}(t, h(T)) = 0,\\
&\underline{R}(t, g(T)) = \underline{R}(t, h(T)) = 0.
\end{aligned}$$

If we choose some sufficiently small m such that

$$I(0,x) \geq \underline{I}(0,x) \text{ and } R(0,x) \geq \underline{R}(0,x),$$

then we can apply the comparison principle to get that

$$I(t,x) \geq \underline{I}(t,x),\ R(t,x) \geq \underline{R}(t,x) \text{ for } t \geq T \text{ and } x \in [g(T), h(T)].$$

Hence,

$$\lim_{t\to\infty} \|I(t,x) + R(t,x)\|_{C([g(t),h(t)])} > 0,$$

which is a contradiction to (3.1). We complete the proof. □

Corollary 3.4. *If $h_0 \geq h^*$, then spreading always happens.*

Lemma 3.5. *If $h_0 < h^*$, then there exists $\mu^0 > 0$ such that spreading occurs if $\mu > \mu^0$.*

Proof. This lemma can be proved by similar arguments in [23, Lemma 3.2]. We give the proof in detail.

Consider the following auxiliary free boundary problem

$$\begin{cases}
U_t - dU_{xx} = -(\gamma + k)U, & t > 0,\ r(t) < x < s(t),\\
V_t - dV_{xx} = -(k+\delta)V, & t > 0,\ r(t) < x < s(t),\\
U(t,x) = V(t,x) = 0, & t \geq 0,\ x = r(t) \text{ or } s(t),\\
r'(t) = -\mu U_x(t, r(t)), r(0) = -h_0, & t > 0,\\
s'(t) = -\mu U_x(t, s(t)), s(0) = h_0, & t > 0,\\
U(0,x) = I_0(x), V(0,x) = R_0(x), & x \in [-h_0, h_0].
\end{cases} \tag{3.5}$$

The proof of the existence and uniqueness of problem (3.5) is similar to that of problem (1.3), we can easily show that (3.5) admits a unique global solution (U, V, r, s) and $s'(t) > 0$ and $r'(t) < 0$ for $t > 0$. We write $(I^\mu, R^\mu, g^\mu, h^\mu)$ and $(U^\mu, V^\mu, r^\mu, s^\mu)$ in place of (I, R, g, h) and (U, V, r, s) to clarify the dependence of the solutions on the parameter μ. By using the comparison principle, we have

$$\begin{aligned}
&I^\mu(t,x) \geq U^\mu(t,x), R^\mu(t,x) \geq V^\mu(t,x),\\
&h^\mu \geq s^\mu(t), g^\mu \leq r^\mu(t) \text{ for } t \geq 0 \text{ and } x \in [r^\mu(t), s^\mu(t)].
\end{aligned} \tag{3.6}$$

In the following, we will prove that for all large μ,

$$s^\mu(2) - r^\mu(2) \geq 2h^*. \tag{3.7}$$

We first choose the smooth functions $\underline{s}(t)$ and $\underline{r}(t)$ such that $\underline{s}(0) = -\underline{r}(0) = h_0$, $\underline{s}'(t) > 0$ and $\underline{r}'(t) > 0$ and $\underline{s}(2) - \underline{r}(2) = 2h^*$. We next consider the following initial-boundary value problem

$$\begin{cases} \underline{U}_t - d\underline{U}_{xx} = -(\gamma + k)\underline{U}, & t > 0,\ \underline{r}(t) < x < \underline{s}(t), \\ \underline{V}_t - d\underline{V}_{xx} = -(k + \delta)\underline{V}, & t > 0,\ \underline{r}(t) < x < \underline{s}(t), \\ \underline{U}(t, \underline{r}(t)) = 0,\ \underline{U}(t, \underline{s}(t)) = 0, & t \geq 0, \\ \underline{V}(t, \underline{r}(t)) = 0,\ \underline{V}(t, \underline{s}(t)) = 0, & t \geq 0, \\ \underline{U}(0, x) = \underline{U}_0(x),\ \underline{V}(0, x) = \underline{V}_0(x), & x \in [-h_0, h_0], \end{cases} \tag{3.8}$$

where the functions $\underline{U}_0$ and $\underline{V}_0$ belonging to $C^1([-h_0, h_0])$ satisfy

$$\begin{cases} 0 < \underline{U}_0(x) \leq I_0(x) \text{ for all } x \in [-h_0, h_0],\ \underline{U}_0(-h_0) = \underline{U}_0(h_0) = 0, \\ 0 < \underline{V}_0(x) \leq R_0(x) \text{ for all } x \in [-h_0, h_0],\ \underline{V}_0(-h_0) = \underline{V}_0(h_0) = 0. \end{cases} \tag{3.9}$$

It follows from the standard theory for parabolic equations that problem (3.8) has a unique positive solution $(\underline{U}, \underline{V})$. By the Hopf lemma, we have $\underline{U}_x(t, \underline{s}(t)) < 0$ and $\underline{U}_x(t, \underline{r}(t)) > 0$ for all $t \in [0, 2]$. According to our choice of $\underline{s}(t)$, $\underline{r}(t)$, $\underline{U}_0(x)$ and $\underline{V}_0(x)$, there exist some $\mu^0 > 0$ such that, for all $\mu > \mu^0$,

$$\underline{s}'(t) \leq -\mu\underline{U}_x(t, \underline{s}(t)),\ \underline{r}'(t) \geq -\mu\underline{U}_x(t, \underline{r}(t)) \text{ for all } t \in [0, 2]. \tag{3.10}$$

Noting that $\underline{s}(0) = h_0 < s^\mu(0)$, $\underline{r}(0) = -h_0 > r^\mu(0)$, it follows from (3.5),(3.8),(3.9) and (3.10) that

$$U^\mu(t, x) \geq \underline{U}(t, x), V^\mu(t, x) \geq \underline{V}(t, x), s^\mu(t) \geq \underline{s}(t), r^\mu(t) \leq \underline{r}(t) \text{ for } t \in [0, 2] \text{ and } x \in [\underline{r}(t), \underline{s}(t)],$$

which implies that $s^\mu(2) - r^\mu(2) \geq \underline{s}(2) - \underline{r}(2) = 2h^*$. Thus, (3.7) holds. It follows that

$$h_\infty - g_\infty = \lim_{t \to \infty} [s^\mu(t) - r^\mu(t)] > s^\mu(2) - r^\mu(2) \geq 2h^*.$$

Hence, we obtain the desired result by Corollary 3.4. We complete the proof. □

Lemma 3.6. *If $\|S_0\|_{L^\infty(\mathbb{R})} \leq \frac{b}{k}$ and $h_0 < h^*$, there exists $\mu_0 > 0$ such that vanishing happens when $\mu < \mu_0$.*

Proof. By $\|S_0\|_{L^\infty(\mathbb{R})} \leq \frac{b}{k}$, it is easy to see that $0 \leq S(t, x) \leq \frac{b}{k}$ for $t \geq 0$ and $x \in \mathbb{R}$. Then I and R satisfy

$$\begin{cases} I_t - dI_{xx} \leq (\beta\frac{b}{k} - \gamma - k)I + \delta R, & t > 0,\ x \in (g(t), h(t)), \\ R_t - dR_{xx} = \gamma I - (k + \delta)R, & t > 0,\ x \in (g(t), h(t)), \\ g'(t) = -\mu I_x(t, g(t)), g(0) = -h_0, & t > 0, \\ h'(t) = -\mu I_x(t, h(t)), h(0) = h_0, & t > 0, \\ I(t, x) = R(t, x) = 0, & t \geq 0,\ x \in \mathbb{R}\backslash(g(t), h(t)), \\ I(0, x) = I_0(x), R(0, x) = R_0(x), & x \in [-h_0, h_0]. \end{cases} \tag{3.11}$$

In the following, we construct an upper solution of (3.11). Let $(\lambda_1, \phi(x), \psi(x))$ be the eigenpair of (3.3) with $L = h_0$, $a_{11} = \gamma + k - \beta\frac{b}{k}$, $a_{12} = \delta$, $a_{21} = \gamma$, $a_{22} = k + \delta$, then $\lambda_1(L) > 0$. Inspired by [25, Lemma 3.8], we define

$$\sigma(t) = h_0(1 + 2m - me^{-mt}),\ t \geq 0,$$

$$\overline{I}(t, x) = Ke^{-mt}\phi\left(\frac{h_0 x}{\sigma(t)}\right),\ t \geq 0,\ x \in [-\sigma(t), \sigma(t)],$$

$$\overline{R}(t,x) = Ke^{-mt}\psi\left(\frac{h_0 x}{\sigma(t)}\right),\ t \geq 0,\ x \in [-\sigma(t), \sigma(t)],$$

where $\phi = \theta\psi$, the positive constants K and m will be determined later.

Direct calculations yield

$$\begin{aligned}
&\overline{I}_t - d\overline{I}_{xx} - (\beta\frac{b}{k} - \gamma - k)\overline{I} - \delta\overline{R} \\
={}& Ke^{-mt}\left[-m\phi - \frac{h_0 x\sigma'}{\sigma^2(t)}\phi' - d\phi''\frac{h_0^2}{\sigma^2(t)} - (\beta\frac{b}{k} - \gamma - k)\phi - \delta\psi\right] \\
={}& Ke^{-mt}\left[-m\phi - m\phi''\frac{h_0^2}{\sigma^2(t)} - (\beta\frac{b}{k} - \gamma - k)\phi - \delta\psi\right] - Ke^{-mt}\frac{h_0 x\sigma'}{\sigma^2(t)}\phi' \\
={}& Ke^{-mt}\left[-m\phi + \frac{h_0^2}{\sigma^2(t)}\left((\beta\frac{b}{k} - \gamma - k)\phi + \delta\psi + \lambda_1\phi\right) - (\beta\frac{b}{k} - \gamma - k)\phi - \delta\psi\right] \\
&- Ke^{-mt}\frac{h_0 x\sigma'}{\sigma^2(t)}\phi' \\
={}& Ke^{-mt}\left[-m\phi + \left((\beta\frac{b}{k} - \gamma - k)\phi + \delta\psi\right)\left(\frac{h_0^2}{\sigma^2(t)} - 1\right) + \lambda_1\frac{h_0^2}{\sigma^2(t)}\phi\right] - Ke^{-mt}\frac{h_0 x\sigma'}{\sigma^2(t)}\phi' \\
={}& Ke^{-mt}\phi\left[-m + \left(\beta\frac{b}{k} - \gamma - k + \frac{\delta}{\theta}\right)\left(\frac{h_0^2}{\sigma^2(t)} - 1\right) + \lambda_1\frac{h_0^2}{\sigma^2(t)}\right] - Ke^{-mt}\frac{h_0 x\sigma'}{\sigma^2(t)}\phi' \\
\geq{}& Ke^{-mt}\phi\left[-m + \left(\beta\frac{b}{k} - \gamma - k + \frac{\delta}{\theta}\right)\left(\frac{h_0^2}{\sigma^2(t)} - 1\right) + \lambda_1\frac{h_0^2}{\sigma^2(t)}\right] =: \Delta_1 \geq 0
\end{aligned}$$

and

$$\begin{aligned}
&\overline{R}_t - d\overline{R}_{xx} - \gamma\overline{I} + (k+\delta)\overline{R} \\
={}& Ke^{-mt}\left[-m\psi - \frac{h_0 x\sigma'}{\sigma^2(t)}\psi' - d\psi''\frac{h_0^2}{\sigma^2(t)} - \gamma\phi + (k+\delta)\psi\right] \\
={}& Ke^{-mt}\left[-m\psi + \frac{h_0^2}{\sigma^2(t)}\left(\gamma\phi - (k+\delta)\psi + \lambda_1\psi\right) - \gamma\phi + (k+\delta)\psi\right] - Ke^{-mt}\frac{h_0 x\sigma'}{\sigma^2(t)}\psi' \\
={}& Ke^{-mt}\psi\left[-m + \left(\frac{h_0^2}{\sigma^2(t)} - 1\right)(\theta\gamma - k - \delta) + \lambda_1\frac{h_0^2}{\sigma^2(t)}\right] - Ke^{-mt}\frac{h_0 x\sigma'}{\sigma^2(t)}\psi' \\
\geq{}& Ke^{-mt}\psi\left[-m + \left(\frac{h_0^2}{\sigma^2(t)} - 1\right)(\theta\gamma - k - \delta) + \lambda_1\frac{h_0^2}{\sigma^2(t)}\right] =: \Delta_2 \geq 0.
\end{aligned}$$

If we choose small enough m such that

$$\begin{aligned}
&-m + \left(\beta\frac{b}{k} - \gamma - k + \frac{\delta}{\theta}\right)\left(\frac{h_0^2}{\sigma^2(t)} - 1\right) + \lambda_1\frac{h_0^2}{\sigma^2(t)} > 0, \\
&-m + \left(\frac{h_0^2}{\sigma^2(t)} - 1\right)(\theta\gamma - k - \delta) + \lambda_1\frac{h_0^2}{\sigma^2(t)} > 0,
\end{aligned}$$

then $\Delta_1 \geq 0$ and $\Delta_2 \geq 0$. Now we choose K sufficiently large such that

$$\overline{I}(0,x) = K\phi\left(\frac{h_0 x}{h_0(1+2m)}\right) \geq I_0(x),\ \overline{R}(0,x) = K\psi\left(\frac{h_0 x}{h_0(1+2m)}\right) \geq R_0(x) \text{ for } x \in [-h_0, h_0].$$

Let $\mu_0 = -\frac{h_0 m^2}{K\phi'(h_0)}$. If $0 < \mu \leq \mu_0$, then

$$-\mu\overline{I}_x(t, \sigma(t)) = -\frac{\mu h_0}{\sigma(t)}Ke^{-mt}\phi'(h_0) \leq -\mu Ke^{-mt}\phi'(h_0) \leq h_0 m^2 e^{-mt} = \sigma'(t)$$

for $t>0$. Similarly, we can obtain $-\sigma'(t) \le -\mu \overline{I}_x(t,-\sigma(t))$ for $t>0$. By the comparison principle, we have

$$\begin{aligned} &g(t) \ge -\sigma(t),\ \ h(t) \le \sigma(t) \text{ for } t>0,\\ &I(t,x) \le \overline{I}(t,x),\ \ R(t,x) \le \overline{R}(t,x) \text{ for } t>0 \text{ and } x \in [g(t),h(t)]. \end{aligned}$$

It follows that $h_\infty - g_\infty \le 2\lim_{t\to\infty}\sigma(t) = 2h_0(1+2m) < \infty$. The proof is completed. □

Theorem 3.7. *If $\|S_0\|_{L^\infty(\mathbb{R})} \le \frac{b}{k}$ and $h_0 < h^*$, then there exists $\mu^* \ge \mu_* > 0$ such that spreading happens if $\mu > \mu^*$, and vanishing occurs if $\mu \le \mu_*$ and $\mu = \mu^*$.*

Proof. As in [24, Theorem 5.2], we give detailed proof as follows.

We write $(S_\mu, T_\mu, R_\mu, g_\mu, h_\mu)$ instead of (S,I,R,g,h) to stress the dependence of the solution (S,I,R,g,h) of (1.3) on μ. Define $\Sigma^* := \{\mu > 0 : h_{\mu,\infty} - g_{\mu,\infty} \le 2h^*\}$. By Lemmas 3.6 and 3.3, we have $\Sigma^* \supset (0,\mu_0]$. Using Lemma 3.5, we can obtain that $\Sigma^* \cap [\mu^0,+\infty) = \emptyset$. Therefore, define $\mu^* := \sup\Sigma^* \in [\mu_0,\mu^0]$. By this definition and Corollary 3.4, we find that $h_{\mu,\infty} - g_{\mu,\infty} = +\infty$ when $\mu > \mu^*$. Hence, $\Sigma^* \subset (0,\mu^*]$.

Next, we will show that $\mu^* \in \Sigma^*$. Otherwise, $h_{\mu^*,\infty} - g_{\mu^*,\infty} = +\infty$. Hence we can find $T>0$ such that $h_{\mu^*}(T) - g_{\mu^*}(T) > 2h^*$. By the continuous dependence of $(S_\mu, T_\mu, R_\mu, g_\mu, h_\mu)$ on μ, we can find small enough $\varepsilon > 0$ such that $h_\mu(T) - g_\mu(T) > 2h^*$ for all $\mu \in [\mu^*-\varepsilon, \mu^*+\varepsilon]$. It follows that for all such μ,

$$\lim_{t\to\infty}[h_\mu(t) - g_\mu(t)] \ge h_\mu(T) - g_\mu(T) > 2h^*.$$

This implies that $[\mu^*-\varepsilon,\mu^*+\varepsilon] \cap \Sigma^* = \emptyset$, and $\sup\Sigma^* \le \mu^*-\varepsilon$. Contradicting the definition of μ^*. Thus we prove that $\mu^* \in \Sigma^*$.

Similarly, define

$$\Sigma_* = \{\varsigma : \varsigma \ge \mu_0 \text{ such that } h_{\mu,\infty} - g_{\mu,\infty} \le 2h^* \text{ for all } \mu \le \varsigma\},$$

where μ_0 is given by Lemma 3.6. Then $\mu_* := \sup\Sigma_* \le \mu^*$ and $(0,\mu_*) \subset \Sigma_*$. By the similar arguments as above, we can show that $\mu_* \in \Sigma_*$. This proof is completed. □

4. LONG-TIME BEHAVIORS

Lemma 4.1. *Let (S,I,R,g,h) be the unique solution of problem (1.3). If vanishing happens, then*

$$\lim_{t\to\infty} S(t,x) = \frac{b}{k} \text{ in } C_{loc}(\mathbb{R}),\ \ \lim_{t\to\infty}\|I(t,x)+R(t,x)\|_{C([g(t),h(t)])} = 0.$$

Proof. The proof of this theorem has already been done in Lemma 3.2. □

Lemma 4.2. *If $h_\infty - g_\infty = \infty$, then $h_\infty = \infty$ and $g_\infty = -\infty$.*

Proof. The proof of this Lemma is similar to [14, Lemma 3.10]. Assume on the contrary that $g_\infty = -\infty$ and $h_\infty < \infty$.

Step 1: By using [28, Lemma 2.5], we can follow the arguments in the Step 1 of the proof in [14, Lemma 3.7] to obtain that, for a given constant $L > 0$ and small $\varepsilon > 0$, there exist $T_1 > 0$ and $l_1 < l_2 < 0$ satisfying $l_2 - l_1 = L$ such that

$$N(t,x) \ge \frac{b}{k} - \varepsilon = N_* - \varepsilon \text{ for all } t \ge T_1 \text{ and } x \in [l_1,l_2].$$

Step 2: By using [13, Lemma 3.3], we have

$$\lim_{t\to\infty}\|I(t,\cdot)\|_{C([-L,h(t)])} = 0 \text{ for any given } L>0.$$

Step 3: By applying the argument in the Step 3 of the proof in [14, Lemma 3.10], we can have that for any given constant $L > 0$ and small $\varepsilon > 0$, there exist $T_1 > 0$ and $l_1 < l_2 < 0$ satisfying $l_2 - l_1 = L$ such that

$$S(t,x) \geq \frac{b}{k} - \varepsilon \text{ for all } t \geq T_1 \text{ and } x \in [l_1, l_2].$$

Step 4: We choose l_1 and l_2 as in Step 3 satisfying $l_2 - l_1 \geq 2h^*$, for small $\varepsilon > 0$ and large $T > 0$, we have

$$\begin{cases} I_t - dI_{xx} \geq (\beta(\frac{b}{k} - \varepsilon) - \gamma - k)I + \delta R, & t > T,\ x \in [l_1, l_2], \\ R_t - dR_{xx} = \gamma I - (k+\delta)R, & t > T,\ x \in [l_1, l_2], \\ I(t,x) > 0, R(t,x) > 0, & t > T,\ x = l_1 \text{ or } l_2, \\ I(T,x) \geq 0, R(T,x) \geq 0, & x \in [l_1, l_2]. \end{cases}$$

By $l_2 - l_1 \geq 2h^*$, we can argue as in the proof of Lemma 3.3 to obtain that

$$\liminf_{t\to\infty} I(t,x) > 0 \text{ for } x \in [l_1, l_2],$$

which contradicts to the conclusion of the Step 2.

Therefore, if $h_\infty - g_\infty = \infty$, then $h_\infty = \infty$. Similarly, we can prove $g_\infty = -\infty$. This proof is completed. ☐

Lemma 4.3. *Let (S, I, R, g, h) be the unique solution of problem (1.3). If spreading happens and $k + \delta > \gamma$, then*

$$\lim_{t\to\infty} (S(t,x), I(t,x), R(t,x)) = (S_*, I_*, R_*) \text{ locally uniformly in } \mathbb{R},$$

where

$$(S_*, I_*, R_*) = \left(\frac{\gamma + k}{\beta} - \frac{\delta\gamma}{\beta(k+\delta)}, \frac{b}{\gamma + k - \delta\gamma/(k+\delta)} - \frac{k}{\beta}, \frac{b\gamma}{(\gamma+k)(k+\delta) - \delta\gamma} - \frac{k\gamma}{\beta(k+\delta)}\right).$$

Proof. This Lemma can be proved by the similar arguments in [14, Lemma 3.11] and [29, Lemma 4.2]. If spreading happens, then $h_\infty = \infty$ and $g_\infty = -\infty$ by Lemma 4.2 .

Step 1: Let $N(t,x) = S(t,x) + I(t,x) + R(t,x)$. By using [28, Lemma 2.5], we can follow the arguments in the Step 1 of the proof in [14, Lemma 3.11] to obtain that

$$\lim_{t\to\infty} N(t,x) = \frac{b}{k} =: N_* \text{ locally uniformly in } \mathbb{R}.$$

Step 2: For any small $\varepsilon > 0$ and large $l > 0$, there exists $T_1 \gg 1$ such that

$$N(t,x) \leq N_* + \varepsilon \text{ for } t \geq T_1 \text{ and } x \in [-l, l],$$

and $g(t) < -l$ and $h(t) > l$ for $t > T_1$ by $h_\infty = -g_\infty = \infty$. Thus (I, R) satisfies

$$\begin{cases} I_t - dI_{xx} \leq [\beta(N_* + \varepsilon) - \gamma - k - \beta I]I + \delta R, & t > T_1,\ x \in [-l, l], \\ R_t - dR_{xx} = \gamma I - (k+\delta)R, & t > T_1,\ x \in [-l, l], \\ I(t,x) = R(t,x) = 0, & t > T_1,\ x \leq -l \text{ or } x \geq l, \\ I(T_1,x) \geq 0, R(T_1,x) \geq 0, & x \in [-l, l]. \end{cases}$$

Consider the following problem

$$\begin{cases} \overline{I}'(t) = [\beta(N_* + \varepsilon) - \gamma - k - \beta\overline{I}]\overline{I} + \delta\overline{R}, & t > T_1, \\ \overline{R}'(t) = \gamma\overline{I} - (k+\delta)\overline{R}, & t > T_1, \\ \overline{I}(T_1) \geq \|I(T_1,\cdot)\|_\infty, \overline{R}(T_1) \geq \|R(T_1,\cdot)\|_\infty. \end{cases} \tag{4.1}$$

By the comparison principle, we have $I(t,x) \le \overline{I}(t)$ and $R(t,x) \le \overline{R}(t)$ for $t \ge T_1$ and $x \in [-l,l]$. Let $\widetilde{\mathcal{R}}_0$ be the basic reproduction number of (4.1), where

$$\widetilde{\mathcal{R}}_0 = \frac{\beta b(k+\delta)}{k[(k+\delta)(k+\gamma)-\delta\gamma]} + \frac{\varepsilon(k+\delta)}{(k+\delta)(k+\gamma)-\delta\gamma}.$$

Thanks to the fact $\mathcal{R}_0 > 1$, we have $\widetilde{\mathcal{R}}_0 > \mathcal{R}_0 > 1$, and then $\lim\limits_{t\to\infty}(\overline{I}(t), \overline{R}(t)) = (\overline{I}_1^{\varepsilon}, \overline{R}_1^{\varepsilon})$, where

$$(\overline{I}_1^{\varepsilon}, \overline{R}_1^{\varepsilon}) = \left(\frac{\beta(N_* + \varepsilon) + \frac{\delta\gamma}{k+\delta} - \gamma - k}{\beta}, \frac{\gamma[\beta(N_*+\varepsilon) + \frac{\delta\gamma}{k+\delta} - \gamma - k]}{\beta(k+\delta)} \right)$$

is the unique positive equilibrium of (4.1). By the arbitrariness of ε, we have

$$\limsup_{t\to\infty} I(t,x) \le \overline{I}_1 \text{ locally uniformly in } \mathbb{R}, \tag{4.2}$$

$$\limsup_{t\to\infty} R(t,x) \le \overline{R}_1 \text{ locally uniformly in } \mathbb{R}, \tag{4.3}$$

where $(\overline{I}_1, \overline{R}_1)$ is the unique positive solution of

$$\begin{cases} (\beta N_* - \gamma - k - \beta\overline{I})\overline{I} + \delta\overline{R} = 0, \\ \gamma\overline{I} - (k+\delta)\overline{R} = 0. \end{cases}$$

Since $\mathcal{R}_0 > 1$, then $\overline{I}_1$ and $\overline{R}_1$ are positive.

By (4.2), for any small $\varepsilon_1 > 0$ and large l, there exists $T_2 > T_1$ such that

$$I(t,x) \le \overline{I}_1 + \varepsilon_1 \text{ for } t > T_2 \text{ and } x \in [-l,l].$$

Then S satisfies

$$\begin{cases} S_t - dS_{xx} \ge b - kS - \beta(\overline{I}_1 + \varepsilon_1)S, & t > T_2,\ x \in [-l,l], \\ S(T_2,x) > 0, & x \in \mathbb{R}. \end{cases}$$

Applying the arguments in the step 2 of the proof in [14, Lemma 3.11], we can show that

$$\liminf_{t\to\infty} S(t,x) \ge \frac{b}{k + \beta(\overline{I}_1 + \varepsilon_1)} \text{ locally uniformly in } \mathbb{R}.$$

By the arbitrariness of ε_1, we have

$$\liminf_{t\to\infty} S(t,x) \ge \frac{b}{k+\beta\overline{I}_1} =: \underline{S}_1 \text{ locally uniformly in } \mathbb{R},$$

where $\underline{S}_1$ is the unique positive solution of

$$b - kS - \beta\overline{I}_1 S = 0.$$

For any small $\varepsilon > 0$ and large $l > 0$, there exists $T_3 \gg 1$ such that

$$N(t,x) \ge N_* - \varepsilon \text{ for } t \ge T_3 \text{ and } x \in [-l,l],$$

and $g(t) < -l$ and $h(t) > l$ for $t > T_3$ by $h_\infty = -g_\infty = \infty$. In view of (4.3), for any small ε_2, there exists $T_4 > T_3$ such that

$$R(t,x) \le \overline{R}_1 + \varepsilon_2 \text{ for } t > T_4 \text{ and } x \in [-l,l].$$

Thus (I,R) satisfies

$$\begin{cases} I_t - dI_{xx} \ge [\beta(N_* - \varepsilon) - \beta(\overline{R}_1 + \varepsilon_2) - \gamma - k - \beta I]I + \delta R, & t > T_4,\ x \in [-l,l], \\ R_t - dR_{xx} = \gamma I - (k+\delta)R, & t > T_4,\ x \in [-l,l], \\ I(t,x) \ge 0, R(t,x) \ge 0, & t > T_4,\ x = \pm l, \\ I(T_4,x) \ge 0, R(T_4,x) \ge 0, & x \in [-l,l]. \end{cases}$$

Using the arguments in the step 2 of the proof in [29, Lemma 4.2], we can show that

$$\liminf_{t\to\infty} I(t,x) \geq \frac{\beta N_* + \frac{\delta\gamma}{k+\delta} - \beta\overline{R}_1 - \gamma - k}{\beta} = \overline{I}_1 - \overline{R}_1 =: \underline{I}_1 \text{ locally uniformly in } \mathbb{R}, \tag{4.4}$$

$$\liminf_{t\to\infty} R(t,x) \geq \frac{\gamma(\beta N_* + \frac{\delta\gamma}{k+\delta} - \beta\overline{R}_1 - \gamma - k)}{\beta(k+\delta)} = \frac{\gamma}{k+\delta}\underline{I}_1 =: \underline{R}_1 \text{ locally uniformly in } \mathbb{R}. \tag{4.5}$$

By $k+\delta > \gamma$ and $\mathcal{R}_0 > 1$, we have $\underline{I}_1$ and $\underline{R}_1$ are positive.

By (4.4), for any small $\varepsilon_3 > 0$ and large $l > 0$, there exists $T_5 > T_4$ such that

$$I(t,x) \geq \underline{I}_1 - \varepsilon_3 \text{ for } t > T_5 \text{ and } x \in [-l, l].$$

Then S satisfies

$$S_t - dS_{xx} \leq b - kS - \beta(\underline{I}_1 - \varepsilon_3)S \text{ for } t > T_5 \text{ and } x \in [-l, l].$$

Applying the arguments in the step 2 of the proof in [14, Lemma 3.11] again, we can show that

$$\limsup_{t\to\infty} S(t,x) \leq \frac{b}{k + \beta(\underline{I}_1 - \varepsilon_3)} \text{ locally uniformly in } \mathbb{R}.$$

By the arbitrariness of ε_3, we have

$$\limsup_{t\to\infty} S(t,x) \leq \frac{b}{k + \beta\underline{I}_1} =: \overline{S}_1 \text{ locally uniformly in } \mathbb{R}.$$

Step 3: By the similar arguments in Step 2, we have

$$\limsup_{t\to\infty} I(t,x) \leq \frac{\beta N_* + \frac{\delta\gamma}{k+\delta} - \gamma - k - \beta\underline{R}_1}{\beta} = \overline{I}_1 - \underline{R}_1 =: \overline{I}_2 \text{ locally uniformly in } \mathbb{R},$$

$$\limsup_{t\to\infty} R(t,x) \leq \frac{\gamma}{k+\delta}\overline{I}_2 =: \overline{R}_2 \text{ locally uniformly in } \mathbb{R}.$$

By $k+\delta > \gamma$ and $\mathcal{R}_0 > 1$, we have $\overline{I}_2$ and $\overline{R}_2$ are positive.

Moreover, we can obtain that

$$\liminf_{t\to\infty} S(t,x) \geq \frac{b}{k + \beta\overline{I}_2} =: \underline{S}_2 \text{ locally uniformly in } \mathbb{R}.$$

Step 4: Repeating above arguments, we can obtain six sequences $\{\overline{S}_n\}$,$\{\underline{S}_n\}$,$\{\overline{I}_n\}$,$\{\underline{I}_n\}$,$\{\overline{R}_n\}$, and $\{\underline{R}_n\}$ satisfying

$$\begin{aligned}
&\underline{S}_n \leq \liminf_{t\to\infty} S(t,x) \leq \limsup_{t\to\infty} S(t,x) \leq \overline{S}_n \text{ locally uniformly in } \mathbb{R},\\
&\underline{I}_n \leq \liminf_{t\to\infty} I(t,x) \leq \limsup_{t\to\infty} I(t,x) \leq \overline{I}_n \text{ locally uniformly in } \mathbb{R},\\
&\underline{R}_n \leq \liminf_{t\to\infty} R(t,x) \leq \limsup_{t\to\infty} R(t,x) \leq \overline{R}_n \text{ locally uniformly in } \mathbb{R},
\end{aligned}$$

where

$$\begin{aligned}
&\overline{S}_n = \frac{b}{k + \beta\underline{I}_n},\ \overline{I}_n = \frac{\beta N_* + \frac{\delta\gamma}{k+\delta} - \gamma - k - \beta\underline{R}_{n-1}}{\beta},\ \overline{R}_n = \frac{\gamma}{k+\delta}\bar{I}_n,\\
&\underline{I}_n = \frac{\beta N_* + \frac{\delta\gamma}{k+\delta} - \gamma - k - \beta\bar{R}_n}{\beta},\ \underline{R}_n = \frac{\gamma}{k+\delta}\underline{I}_n,\ \underline{S}_n = \frac{b}{k + \beta\bar{I}_n}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\underline{S}_1 < \underline{S}_2 < \cdots < \underline{S}_n < \cdots < \overline{S}_n < \cdots < \overline{S}_2 < \overline{S}_1,\\
&\underline{I}_1 < \underline{I}_2 < \cdots < \underline{I}_n < \cdots < \overline{I}_n < \cdots < \overline{I}_2 < \overline{I}_1,\\
&\underline{R}_1 < \underline{R}_2 < \cdots < \underline{R}_n < \cdots < \overline{R}_n < \cdots < \overline{R}_2 < \overline{R}_1.
\end{aligned}$$

Denote

$$\lim_{n\to\infty}(\overline{S}_n,\overline{I}_n,\overline{R}_n)=(\overline{S}_\infty,\overline{I}_\infty,\overline{R}_\infty) \text{ and } \lim_{n\to\infty}(\underline{S}_n,\underline{I}_n,\underline{R}_n)=(\underline{S}_\infty,\underline{I}_\infty,\underline{R}_\infty),$$

then we have

$$\begin{aligned}\overline{I}_\infty&=\frac{\beta N_*+\frac{\delta\gamma}{k+\delta}-\gamma-k-\beta\underline{R}_\infty}{\beta},\ \overline{R}_\infty=\frac{\gamma}{k+\delta}\overline{I}_\infty,\ \overline{S}_\infty=\frac{b}{k+\beta\underline{I}_\infty},\\ \underline{I}_\infty&=\frac{\beta N_*+\frac{\delta\gamma}{k+\delta}-\gamma-k-\beta\overline{R}_\infty}{\beta},\ \underline{R}_\infty=\frac{\gamma}{k+\delta}\underline{I}_\infty,\ \underline{S}_\infty=\frac{b}{k+\beta\overline{I}_\infty}.\end{aligned}$$

By direct computations, we have $(\underline{S}_\infty,\underline{I}_\infty,\underline{R}_\infty)=(\overline{S}_\infty,\overline{I}_\infty,\overline{R}_\infty)=(S_*,I_*,R_*)$. Therefore, this lemma is proved. □

5. Numerical simulation and discussion

In this section, we will use MATLAB to carry out some numerical simulations to illustrate the spreading and vanishing of diseases.

5.1. The case of $\mathcal{R}_0\le 1$.

Fix the coefficients and initial functions in (1.3) as follows:

$$\begin{aligned}&b=1,\ \beta=0.5,\ k=0.6,\ \gamma=0.5,\ \delta=0.2,\ d=1,\ h_0=1,\\ &S_0(x)=1+\frac{1}{2}\sin x,\ I_0(x)=\cos(\frac{\pi}{2}x).\end{aligned}$$

Then, we can have that $\mathcal{R}_0=0.8712<1$ by (1.5). By the following figure 1, it is easy to see that the disease will die out if $\mathcal{R}_0\le 1$.

Figure 1. The profiles of I and R.

5.2. The case of $\mathcal{R}_0>1$.

Firstly, we consider the case of $h_0\ge h_*$. Fix the coefficients and initial functions in (1.3) as follows:

$$\begin{aligned}&b=1,\ \beta=1,\ k=0.5,\ \gamma=0.5,\ \delta=0.2,\ d=1,\ h_0=3,\\ &S_0(x)=1+\frac{1}{2}\sin x,\ I_0(x)=\cos(\frac{\pi}{2}x).\end{aligned}$$

Then we have $\mathcal{R}_0=2.1429>1$ by (1.5). By direct calculations, we can obtain that $h_*=1.5279$ and $(S_*,I_*,R_*)=(0.8571,0.6667,0.4762)$, and then $h_0>h_*$. From figure 2, we can easily see that the solution (I,R) keeps positive and tends to an equilibrium (0.6667,0.4762) if $\mathcal{R}_0>1$ and $h_0\ge h_*$.

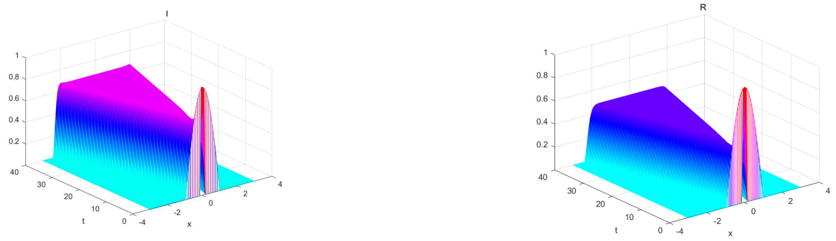

FIGURE 2. The profiles of I and R.

Next, we will discuss the impact of expending capability μ on disease transmission for the case of $h_0 < h_*$. Fix the coefficients and initial functions in (1.3) as follows:

$$b = 1,\ \beta = 1,\ k = 0.5,\ \gamma = 0.5,\ \delta = 0.2,\ d = 1,\ h_0 = 1.5,$$
$$S_0(x) = 1 + \frac{1}{2}\sin x,\ I_0(x) = \cos(\frac{\pi}{2}x).$$

Then we have that $\mathcal{R}_0 = 2.1429 > 1$ by (1.5). By direct calculations, we can obtain that $h_* = 1.5279 > h_0$ and $(S_*, I_*, R_*) = (0.8571, 0.6667, 0.4762)$. From figure 3, we can find the solution (I, R) keeps positive and tends to an equilibrium (0.6667,0.4762) for some large $\mu = 10$ if $\mathcal{R}_0 > 1$ and $h_0 < h_*$. From figure 4, it is easy to see that the disease will die out for some small $\mu = 0.5$ if $\mathcal{R}_0 > 1$ and $h_0 < h_*$. This means that whether the disease will spread or not for $\mathcal{R}_0 > 1$ and $h_0 < h_*$ depends on the expending capability μ.

FIGURE 3. The profiles of I and R with $\mu = 10$.

FIGURE 4. The profiles of I and R with $\mu = 0.5$.

6. DISCUSSION

This paper considers an SIRI epidemic model with relapse and free boundaries. At first, the existence and uniqueness of the global solution is proved. And then we provide the criteria for the disease spreading and vanishing. Roughly speaking, the disease will vanish if one of the following conditions holds:

(i) The basic reproduction number $\mathcal{R}_0 < 1$;
(ii) $\mathcal{R}_0 > 1$ and the initial infected region $h_0 < h^*$ and the boundary moving rate $\mu < \mu_0$;

and the disease will spread if one of the following conditions holds:

(i) The basic reproduction number $\mathcal{R}_0 > 1$ and $h_0 \geq h^*$;
(ii) $\mathcal{R}_0 > 1$ and the initial infected region $h_0 < h^*$ and the boundary moving rate $\mu > \mu^0$.

Moreover, we give the longtime behavior of the solution, which is given by the following spreading-vanishing dichotomy:

(i) Vanishing: if $h_\infty - g_\infty < \infty$, then

$$\lim_{t\to\infty} S(t,x) = \frac{b}{k} \text{ in } C_{loc}(\mathbb{R}),\ \lim_{t\to\infty} \|I(t,x) + R(t,x)\|_{C([g(t),h(t)])} = 0.$$

(ii) Spreading: if $h_\infty - g_\infty = \infty$ and $k + \delta > \gamma$, then

$$\lim_{t\to\infty} (S(t,x), I(t,x), R(t,x)) = (S_*, I_*, R_*) \text{ locally uniformly in } \mathbb{R}.$$

Finally, some numerical simulations are provided to illustrate the results of our analysis.

This paper mainly studies the effect of relapse. These results can be regarded as complement and extension of [4]. However, compared with [4], we establish the sharp criteria when considering the factor of relapse. The main results in this paper indicate that relapse makes $\mathcal{R}_0$ become lager, which means that relapse can increase the chance of epidemic spreading. This tell us that it is necessary to consider the effect of relapse when we propose the free boundary model to describe the spreading diseases.

References

[1] B.B. Anvarjonov, *On the SIRI free boundary model with relapse and transfer*, Uzbek. Math. J. **66** (2022), 21–26.

[2] J.F. Cao, W.T. Li, F.Y. Yang, *Dynamics of a nonlocal SIS epidemic model with free boundary*, Discrete Contin. Dyn. Syst. Ser. B **22** (2017), 247–266.

[3] Y. Du, Z. Lin, *Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary*, SIAM J. Math. Anal. **42** (2010), 377–405. Erratum: SIAM J. Math. Anal. **45** (2013), 1995–1996.

[4] Q. Ding, Y. Liu, Y. Chen, Z. Guo, *Dynamics of a reaction-diffusion SIRI model with relapse and free boundary*, Math. Biosci. Eng. **17** (2019), 1659–1676.

[5] P. van den Driessche, X. Zou, *Modeling relapse in infectious diseases*, Math. Biosci. **207** (2007), 89–103.

[6] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N.J., 1964.

[7] P. Georgescu, H. Zhang, *A Lyapnuov functional for a SIRI model with nonlinear incidence of infection and relapse*, Appl. Math. Comput. **219** (2013), 8496–8507.

[8] Y. Hu, X. Hao, X. Song, Y. Du, *A free boundary problem for spreading under shifting climate*, J. Differential Equations **269** (2020), 5931–5958.

[9] H. Huang, M. Wang, *The reaction-diffusion system for an SIR epidemic model with a free boundary*, Discrete Contin. Dyn. Syst. Ser. B **20** (2015), 2039–2050.

[10] H. Huang, M. Wang, *A nonlocal SIS epidemic problem with double free boundaries*, Z. Angew. Math. Phys. **70** (2019): 109.

[11] K.I. Kim, Z. Lin, Q. Zhang, *An SIR epidemic model with free boundary*, Nonlinear Anal. Real World Appl. **14** (2013), 1992–2001.

[12] W.O. Kermack, A.G.Mckendrick, *A contribution to the mathematical theory of epidemics*, Proc. Roy. Soc. London A **115** (1927), 700–721.

[13] L. Li, S. Liu, M. Wang, *A viral propagation model with a nonlinear infection rate and free boundaries*, Sci. China Math. **64** (2021), 1971–1992.

[14] L. Li, W. Ni, M. Wang, *Dynamical properties of a new SIR epidemic model*, Discrete Contin. Dyn. Syst. Ser. S **17** (2023), 690–707.

[15] Z. Lin, *A free boundary problem for a predator-prey model*, Nonlinearity **20** (2007), 1883–1892.

[16] Z. Lin, H. Zhu, *Spatial spreading model and dynamics of West Nile virus in birds and mosquitoes with free boundary*, J. Math. Biol. **75** (2017), 1381–1409.

[17] H.N. Moreira, Y. Wang, *Global stability in an $S \to I \to R \to I$ model*, SIMA Rev. **39** (1997), 496–502.

[18] D. Tudor, *A deterministic model for herps infectious in human and animal populations*, SIAM Rev. **32** (1990), 136–139.

[19] C. Vargas-De-León, *On the global stability of infectious diseases model with relapse*, Abstraction Application. **9** (2013), 50–61.

[20] M. Wang, *On some free boundary problems of the prey-predator model*, J. Differential Equations **256** (2014), 3365–3394.

[21] M. Wang, *Existence and uniqueness of solutions of free boundary problems in heterogeneous environments*, Discrete Contin. Dyn. Syst. Ser. B **24** (2019), 415–421. Erratum: Discrete Cont. Dyn. Syst. B **27** (2022), 5179–5180.

[22] M. Wang, Q. Zhang, *Dynamics for the diffusive Leslie-Gower model with double free boundaries*, Discrete Contin. Dyn. Syst. **38** (2018), 2591–2607.

[23] M. Wang, J. Zhao, *Free boundary problems for a Lotka-Volterra competition system*, J. Dyn. Diff. Equat. **26** (2014), 655–672.

[24] M. Wang, J. Zhao, *A free boundary problem for the predator-prey model with double free boundaries*, J. Dyn. Diff. Equat. **29** (2017), 957–979.

[25] R. Wang, Y. Du, *Long-time dynamics of a diffusive epidemic model with free boundaries*, Discrete Contin. Dyn. Syst. Ser. B **26** (2021), 2201–2238.

[26] Z. Xu, Y. Xu, Y. Huang, *Traveling Waves for a Spatial SIRI Epidemic Model*, Taiwan. J. Math. **23** (2019), 1435–1460.

[27] Y. Yang, J. Zhou, C.H. Hsu, *Threshold dynamics of a diffusive SIRI model with nonlinear incidence rate*, J. Math. Anal. Appl. **478** (2019), 874–896.

[28] M. Zhao, *The longtime behavior of an SIR epidemic model with free boundaries*, Journal of Nonlinear Modeling and Analysis, **6** (2024), 476–484.

[29] M. Zhao, *Dynamics of a reaction-diffusion waterborne pathogen model with free boundaries*, Nonlinear Anal. Real World Appl. **77** (2024): 104043.

[30] C.C. Zhu, W.T. Li, F.Y. Yang, *Traveling waves of a reaction-diffusion SIRQ epidemic model with relapse*, J. Appl. Anal. Comput. **7** (2017), 147–171.

[31] C.C. Zhu, J. Zhu, X.L. Liu, *Influence of spatial heterogeneous environment on long-term dynamics of a reaction-diffusion SVIR epidemic model with relapse*, Math. Biosci. Eng. **16** (2019), 5897–5922.

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