Mathematics in Applied Sciences and Engineering https://ojs.lib.uwo.ca/mase
Volume **5**, Number **3**, **2024**, pp.**254-275** https://doi.org/10.5206/mase/18954

TIME-DEPENDENT BEHAVIOUR OF AN $M/M/3$ HETEROGENEOUS SERVER QUEUEING SYSTEM WITH MULTIPLE VACATIONS SUBJECT TO SYSTEM DISASTER

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Abstract. This research paper investigates the dynamic behaviour of a heterogeneous three-server queuing model with multiple vacation. The model also accounts for the possibility that a disaster could occur during busy periods or on vacation, in which case the restoration process would begin immediately. The time-dependent probabilities of system size are presented in the work with explicit equations computed in terms of modified Bessel functions, employing generating functions. To support the theoretical conclusions, numerical examples are given.

1. Introduction

It is a common observation in queuing systems that the server leaves from the system for a certain amount of time while it is in operation; this is known as the vacation period. This is the time that the server can rest or finish up other tasks. The phrase "vacation queuing model" was originally coined in 1975 by Levy and Yechiali [13] to refer to a queueing approach that takes server vacations into consideration. Numerous scholars have thoroughly examined queueing models using vacations. Doshi [3], Takagi [16], Upadhyaya [20], Tian and Zhang [18], and Ke et al. [10] have all conducted extensive research on vacation queueing models that consider a wide range of scenarios.

Multiple vacations are among the many types of vacations available here. If there are no customers in the system when the server returns from a vacation, the server may take another one. Multiple vacations is the name of this type of vacation. Two server multiple vacation queueing model is studied by Krishnakumar and madheshwari [12],rencenly Karthick and Suvitha [9] are studied about three server multiple vacations queueing models. Researchers in various fields have Researchers in various fields have studied multi-server queues with vacations. Levy [13] and Vinod [21] do initial investigations on the $M/M/c$ queue with exponentially distributed vacation times. Tian [17] discusses the $M/M/c$ queueing system with vacation in detail and gives conditional stochastic decomposition results for waiting time and queue length. Several studies on multiserver queueing systems with vacations have been carried out by a number of researchers, including Kao and Narayanan [8], Igaki [6], Chao and Zhang [2], Zhang and Tian [22],[23], and Houalef et al. [4]. It is commonly assumed that servers in multiserver queueing systems are homogeneous, i.e., that each server has the same service rate. This presumption might only be accurate, though, if the servicing procedure is mechanically or electrically controlled. It is typical to witness servers charging varying prices for the same tasks in human-manned queuing systems. Because of this disparity in service rates, multiserver queueing systems with heterogeneous servers where service time distributions can fluctuate throughout servers need to be modelled. As was already mentioned, secondary occupations are taken into account by vacation queueing systems. Studying how secondary

Received by the editors 31 May 2024; accepted 1 September 2024; published online 11 September 2024.
2020 *Mathematics Subject Classification.* 60K25, 60K30, 90B22.
Key words and phrases. Transient analysis, three-server, multiple vacation, disaster.

jobs affect system performance is made possible by analysing queuing systems with heterogeneous servers and server vacations.

Queuing theory and network analysis are widely used to forecast congestion and blocking in computer systems, telecommunication networks, and other queueing systems. Unforeseen events can result in disasters, and these systems are commonly prone to failure. When a disaster strikes, all jobs in the buffer including the one the server is now processing are lost because the system becomes unusable. The first people to examine disaster-prone queuing systems were Towsley and Tripathi[19]. Artaljo and Gomez-Coral [1] investigated the idea of "stochastic clearing" for queueing systems with catastrophes. The following studies have offered transient analyses of single server queueing systems with disasters: Sudhesh et al.[14], Kumar et al. [11], and Jain and Singh [7]. Furthermore, Sudhesh et al. [15] investigated the transient analysis of two-server queueing models with system diaster.

This model is motivated by the following real-life situation: Consider a wireless network in a shopping mall where there are three Access Points (APs) that provide internet connectivity to users' devices. If there are no users in the vicinity of the mall with devices looking to connect to the internet, then all three APs remain Inactive. After the inactive time, if again there are no devices waiting for connections, then again the AP's go to inactive. If at least one user enters the mall and wants to connect to the internet, any one of the three APs can handle the connection. The other two APs remain at rest until more users arrive. If two users arrive and wish to connect to the internet simultaneously, any two of the three APs can handle their connections, while the third AP remains inactive. If three users arrive and want to connect to the internet at the same time, all three APs become active, providing connectivity to each user. It's also important to take into account the potential for a Denial of Service (DoS) attack against the system, which would initiate an urgent repair procedure.

2. Model description

In the present study, a $M/M/3$ heterogeneous servers queueing model with multiple vacations and the potential for disastrous breakdown and repair is taken into account. These are the main presumptions that underlie this model:

(i) Customer Arrivals: Customers arrive at a rate of λ, following a Poisson process. Customers are sorted into one queue according to their arrival order. The system's capacity is based on an infinite number of potential customers.

(ii) Service Process: For servers 1, 2, and 3, the service rates are indicated as μ_1, μ_2, and μ_3, respectively and the service follows exponential distribution. First-come, first-served is the order in which customers are serviced.

(iii) Vacation Policy: When there are no customers in queue, every server is allowed to take a self-imposed vacation. If a customer has been entered into the system at the end of a vacation, service will begin. If not, the server takes another vacation right away, and so on, until finally returning from vacation to find that at least a single customer is still waiting.

(iv) The duration of vacations taken by the servers follows exponential distributions with rates θ_1, θ_2, θ_3.

(v) Disastrous Breakdown and Repair: When the servers are either on vacation or busy, there is a possibility of a disastrous breakdown occurring. The occurrence of a breakdown follows an exponential distribution with a rate α. In the event of a breakdown, the repair process starts immediately, which follows exponential distribution with a rate β, allowing for the servers to resume their operation as soon as possible.

At the time t the number of customers in the systems is consider as $H(t)$ and let $I(t)$ be the servers state, where

$$I(t) = \begin{cases} 0, & \text{when all the servers are on vacation,} \\ 1, & \text{when the server 1 is alone busy,} \\ 2, & \text{when the server 2 alone is busy,} \\ 3, & \text{when the server 3 alone is busy,} \\ 4, & \text{when the servers 1 and 2 are busy,} \\ 5, & \text{when the servers 2 and 3 are busy,} \\ 6, & \text{when the servers 1 and 3 are busy,} \\ 7, & \text{when all the servers are busy,} \\ 8, & \text{when all the servers are disaster.} \end{cases}$$

Let $X(t) = \{H(t), I(t)\}$, then {X(t): t≥0} is a Continuous time Markov chain with a state space denoted by Ω as follows: $\Omega = \{\{(n,j), n \geq 0, j = 0,1,8\} \cup \{(n,j), n \geq 1, j = 2,3\} \cup \{(n,j), n \geq 2, j = 4,5,6\} \cup \{(n,j), n \geq 3, j = 7\}\}$. Let $P_{n,j}(t)$ be the time-dependent probability for the system to be in state j with n customers at time t.

$$P'_{0,0}(t) = -(\lambda+\alpha)P_{0,0}(t) + \mu_1 P_{1,1}(t) + \mu_2 P_{1,2}(t) + \mu_{1,3}(t), \tag{2.1}$$

$$P'_{n,0}(t) = -(\lambda+\alpha+\theta_1+\theta_2+\theta_3)P_{n,0}(t) + \lambda P_{n-1,0}(t) \text{for } n \geq 1, \tag{2.2}$$

$$P'_{0,1}(t) = -(\lambda+\alpha)P_{0,1}(t) + \beta P_{0,8}(t), \tag{2.3}$$

$$\begin{aligned} P'_{1,1}(t) = {} & -(\lambda+\alpha+\mu_1)P_{1,1}(t) + \lambda P_{0,1}(t) + \mu_2 P_{2,1}(t) + \theta_1 P_{1,0}(t) + \beta P_{1,8}(t) \\ & + \mu_2 P_{2,4}(t) + \mu_3 P_{2,6}(t), \end{aligned} \tag{2.4}$$

$$\begin{aligned} P'_{n,1}(t) = {} & -(\lambda+\alpha+\mu_1+\theta_2+\theta_3)P_{n,1}(t) + \lambda P_{n-1,1}(t) + \mu_1 P_{n+1,1}(t) \\ & + \theta_1 P_{n,0}(t) + \beta P_{n,8}(t) \quad \text{for } n \geq 2, \end{aligned} \tag{2.5}$$

$$P'_{1,2}(t) = -(\lambda+\alpha+\mu_2)P_{1,2}(t) + \mu_2 P_{2,2}(t) + \mu_1 P_{2,4}(t) + \mu_3 P_{2,5}(t) + \theta_2 P_{1,0}(t), \tag{2.6}$$

$$P'_{n,2}(t) = -(\lambda+\alpha+\mu_2+\theta_1+\theta_3)P_{n,2}(t) + \lambda P_{n-1,2}(t) + \mu_2 P_{n+1,2}(t) + \theta_2 P_{n,0}(t) \quad \text{for } n \geq 2, \tag{2.7}$$

$$P'_{1,3}(t) = -(\lambda+\alpha+\mu_3)P_{1,3}(t) + \mu_3 P_{2,3}(t) + \mu_2 P_{2,5}(t) + \mu_1 P_{2,6}(t) + \theta_3 P_{1,0}(t), \tag{2.8}$$

$$P'_{n,3}(t) = -(\lambda+\alpha+\mu_2+\theta_1+\theta_2)P_{n,3}(t) + \lambda P_{n-1,3}(t) + \mu_3 P_{n+1,3}(t) + \theta_3 P_{n,0}(t) \quad \text{for } n \geq 2, \tag{2.9}$$

$$P'_{2,4}(t) = -(\lambda+\alpha+\mu_1+\mu_2)P_{2,4}(t) + (\mu_1+\mu_2)P_{3,4}(t) + \theta_1 P_{2,2}(t) + \theta_2 P_{2,1}(t) + \mu_3 P_{3,7}(t) \tag{2.10}$$

$$\begin{aligned} P'_{n,4}(t) = {} & -(\lambda+\alpha+\mu_1+\mu_2+\theta_3)P_{n,4}(t) \\ & + (\mu_1+\mu_2)P_{n+1,4}(t) + \lambda P_{n-1,4}(t) + \theta_1 P_{n,2}(t) + \theta_2 P_{n,1}(t) + \mu_3 P_{3,7}(t) \end{aligned} \tag{2.11}$$

$$P'_{2,5}(t) = -(\lambda+\alpha+\mu_2+\mu_3)P_{2,5}(t) + (\mu_2+\mu_3)P_{3,5}(t) + \theta_2 P_{2,3}(t) + \theta_3 P_{2,2}(t) + \mu_3 P_{3,7}(t) \tag{2.12}$$

$$\begin{aligned} P'_{n,5}(t) = {} & -(\lambda+\alpha+\mu_2+\mu_3+\theta_1)P_{n,5}(t) + (\mu_2+\mu_3)P_{n+1,5}(t) + \lambda P_{n-1,5}(t) \\ & + \theta_2 P_{n,3}(t) + \theta_3 P_{n,2}(t) + \mu_3 P_{3,7}(t) \end{aligned} \tag{2.13}$$

$$P'_{2,6}(t) = -(\lambda+\alpha+\mu_1+\mu_3)P_{2,6}(t) + (\mu_1+\mu_3)P_{3,6}(t) + \theta_1 P_{2,3}(t) + \theta_3 P_{2,1}(t) + \mu_3 P_{3,7}(t) \tag{2.14}$$

$$\begin{aligned} P'_{n,6}(t) = {} & -(\lambda+\alpha+\mu_1+\mu_3+\theta_2)P_{n,6}(t) + (\mu_1+\mu_3)P_{n+1,6}(t) + \lambda P_{n-1,6}(t) \\ & + \theta_1 P_{n,3}(t) + \theta_3 P_{n,1}(t) + \mu_3 P_{3,7}(t), \end{aligned} \tag{2.15}$$

$$P'_{3,7}(t) = -(\lambda+\alpha+\mu_1+\mu_2+\mu_3)P_{3,7}(t) + (\mu_1+\mu_2+\mu_3)P_{4,7}(t) + \theta_1 P_{3,5}(t) + \theta_2 P_{3,6}(t) + \theta_3 P_{3,4}(t) + \mu_3 P_{3,7}(t), \tag{2.16}$$

$$P'_{n,7}(t) = -(\lambda+\alpha+\mu_1+\mu_2+\mu_3)P_{n,7}(t) + (\mu_1+\mu_2+\mu_3)P_{n+1,7}(t) + \theta_1 P_{n,5}(t) + \theta_2 P_{n,6}(t) + \theta_3 P_{n,4}(t), \tag{2.17}$$

$$P'_{0,8}(t) = -(\lambda+\beta)P_{0,8}(t) + \alpha\big(1 - \sum_{n=0}^{\infty} P_{n,8}(t)\big), \tag{2.18}$$

$$P'_{n,8}(t) = -(\lambda+\beta)P_{n,8}(t) + \lambda P_{n-1,8}(t) \ \text{ for } n \geq 1. \tag{2.19}$$

We assume the initial condition as

$$\begin{cases} P_{0,0}(0) = 1,\ P_{0,8}(0) = 0, P_{n,j}(0) = 0 \ \text{ for } \ n \geq 1,\ j = 0,1,2,3,8; \\ P_{n,j}(0) = 0 \ \text{ for } \ n \geq 2,\ j = 4,5,6 \ \text{ and } \ P_{n,j}(0) = 0 \ \text{ for } \ n \geq 3,\ j = 7. \end{cases}$$

3. Transient Probabilities

This section provides the time dependent system size probabilities for the system under consideration using Laplace transformation and Modified Bessels functions.

In Theorem 1 below, we prove the transient probabilities $P_{n,1}(t)$, $P_{n,2}(t)$, $P_{n,3}(t)$, for $n \geq 1$, $P_{n,4}(t)$, $P_{n,5}(t)$, $P_{n,6}(t)$, for $n \geq 2$, and $P_{n,7}(t)$, $n \geq 3$; and in Theorem 2, we give the transient probabilities $P_{n,0}(t), P_{n,8}(t)$ for $n \geq 0$.

Theorem 3.1. *The probabilities $P_{n,1}(t)$, for $n \geq 0$, $P_{n,2}(t)$, $P_{n,3}(t)$, for $n \geq 1$, $P_{n,4}(t)$, $P_{n,5}(t)$, $P_{n,6}(t)$, for $n \geq 2$, and $P_{n,7}(t)$, $n \geq 3$ are obtained from equations (2.4)- (2.17) in terms of Modified Bessels functions as*

$$P_{n,1}(t) = \int_0^t \bigg(\sum_{m=1}^{\infty} \big(\theta_1 P_{m,0}(t) + \beta P_{m,8}(t)\big) b_1^{n-m} \big[I_{n-m}(.) - I_{n+m}(.)\big] e^{-k_1(t-y)} \bigg) dy + \int_0^t \bigg(\big(\mu_2 P_{2,4}(t) + \mu_3 P_{2,6}(t) + \lambda P_{0,1}(t) + (\theta_2+\theta_3)P_{1,1}(t)\big) b_1^{n-1} \cdot \cdot \big[I_{n-1}(.) - I_{n+1}(.)\big] e^{-k_1(t-y)} \bigg) dy, \tag{3.1}$$

$$P_{n,2}(t) = \int_0^t \bigg(\sum_{m=1}^{\infty} \big(\theta_1 P_{m,0}(t) + \beta P_{m,8}(t)\big) b_2^{n-m} \big[I_{n-m}(.) - I_{n+m}(.)\big] e^{-k_1(t-y)} \bigg) dy + \int_0^t \bigg(\big(\mu_1 P_{2,5}(t) + \mu_3 P_{2,6}(t) + \lambda P_{0,2}(t) + (\theta_1+\theta_3)P_{1,2}(t)\big) b_2^{n-1} \cdot \cdot \big[I_{n-1}(.) - I_{n+1}(.)\big] e^{-k_2(t-y)} \bigg) dy, \tag{3.2}$$

$$P_{n,3}(t) = \int_0^t \bigg(\sum_{m=1}^{\infty} \big(\theta_1 P_{m,0}(t) + \beta P_{m,8}(t)\big) b_3^{n-m} \big[I_{n-m}(.) - I_{n+m}(.)\big] e^{-k_3(t-y)} \bigg) dy + \int_0^t \bigg(\big(\mu_1 P_{2,5}(t) + \mu_2 P_{2,5}(t) + \lambda P_{0,3}(t) + (\theta_1+\theta_2)P_{1,3}(t)\big) b_3^{n-1} \cdot \cdot \big[I_{n-1}(.) - I_{n+1}(.)\big] e^{-k_3(t-y)} \bigg) dy, \tag{3.3}$$

$$
\begin{aligned}
P_{n,4}(t) = &\int_0^t \Bigg(\sum_{m=1}^{\infty} \big(\theta_1 P_{m,1}(t) + \theta_2 P_{m,1}(t)\big) b_4^{n-m} \big[I_{n-m}(.) - I_{n+m}(.)\big] e^{-k_4(t-y)} \Bigg) dy \\
&+ \int_0^t \Bigg(\big(2(\mu_1+\mu_2) P_{2,4}(t) + 2\mu_3 P_{3,7}(t) + 2\theta_3 P_{2,4}(t)\big) b_4^{n-1} \cdot \\
&\cdot \big[I_{n-1}(.) - I_{n+1}(.)\big] e^{-k_4(t-y)} \Bigg) dy,
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
P_{n,5}(t) = &\int_0^t \Bigg(\sum_{m=1}^{\infty} \big(\theta_2 P_{m,3}(t) + \theta_3 P_{m,2}(t)\big) b_5^{n-m} \big[I_{n-m}(.) - I_{n+m}(.)\big] e^{-k_5(t-y)} \Bigg) dy \\
&+ \int_0^t \Bigg(\big(2(\mu_2+\mu_3) P_{2,5}(t) + 2\mu_1 P_{3,7}(t) + 2\theta_1 P_{2,5}(t)\big) b_5^{n-1} \cdot \\
&\cdot \big[I_{n-1}(.) - I_{n+1}(.)\big] e^{-k_5(t-y)} \Bigg) dy,
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
P_{n,6}(t) = &\int_0^t \Bigg(\sum_{m=1}^{\infty} \big(\theta_1 P_{m,3}(t) + \theta_3 P_{m,2}(t)\big) b_6^{n-m} \big[I_{n-m}(.) - I_{n+m}(.)\big] e^{-k_6(t-y)} \Bigg) dy \\
&+ \int_0^t \Bigg(\big(2(\mu_1+\mu_3) P_{2,6}(t) + 2\mu_2 P_{3,7}(t) + 2\theta_1 P_{2,5}(t)\big) b_6^{n-1} \cdot \\
&\cdot \big[I_{n-1}(.) - I_{n+1}(.)\big] e^{-k_6(t-y)} \Bigg) dy,
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
P_{n,7}(t) = &\int_0^t \Bigg(\sum_{m=1}^{\infty} \big(\theta_1 P_{m,5}(t) + \theta_2 P_{m,6}(t) + \theta_3 P_{m,7}(t)\big) b_7^{n-m} \big[I_{n-m}(.) - I_{n+m}(.)\big] e^{-k_7(t-y)} \Bigg) dy \\
&+ \int_0^t \Bigg(\big(2(\mu_1+\mu_2+\mu_3) P_{3,7}(t) + \times \big[I_{n-1}(.) - I_{n+1}(.)\big] e^{-k_7(t-y)} \Bigg) dy,
\end{aligned} \tag{3.7}
$$

and $P_{0,1}(t)$ is obtained from equation (2.3) as

$$
P_{0,1}(t) = \beta \int_0^t P_{0,8}(t) e^{-(\lambda+\alpha)(t-\mu)} d\mu. \tag{3.8}
$$

The proof of Theorem 3.1 is given in Appendix A.

Theorem 3.2. *The Probabilities $P_{n,0}(t)$ and $P_{n,8}(t)$ for $n \geq 0$ are obtained from equations (2.1), (2.2), (2.18), (2.19) using Laplace transform as*

$$
\begin{aligned}
P_{0,0}(t) = &e^{-(\lambda+\alpha)t} \cdot \Bigg(\delta(t) + \mu_1 G_{11}(t) + \mu_2 G_{21}(t) + \mu_3 G_{31}(t) + \big(1 + \mu_1 G_{12}(t) + \mu_2 G_{22}(t) \\
&+ \mu_3 G_{32}(t)\big) P_{0,8}(t) \Bigg),
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
P_{n,0}(t) = &\frac{\lambda^n t^{n-1}}{(n-1)!} e^{-(\lambda+\alpha+\theta_1+\theta_2+\theta_3)t} \cdot e^{-(\lambda+\alpha)t} \cdot \Bigg(\delta(t) + \mu_1 G_{11}(t) + \mu_2 G_{21}(t) + \mu_3 G_{31}(t) \\
&+ \big(1 + \mu_1 G_{12}(t) + \mu_2 G_{22}(t) + \mu_3 G_{32}(t)\big) P_{0,8}(t) \Bigg),
\end{aligned} \tag{3.10}
$$

$$\hat{P}_{n,1}(t) = \sum_{m=1}^{\infty} \theta_1 l_3 X_1(t) + \left[2b_1^{n-1}\frac{\psi(t)}{a_1} \cdot \left(\mu_2 \hat{G}_{41}(t) + \mu_3 \hat{G}_{61}(t) + (\theta_2+\theta_3)\hat{G}_{11}(t)\right)\right]$$
$$+ \left[2b_1^{n-1}\frac{\hat{\psi}(t)}{a_1}\left(\mu_2 \hat{G}_{42}(t) + \mu_3 \hat{G}_{62}(t) + (\theta_2+\theta_3)\hat{G}_{12}(t)\right)\right.$$
$$\left. + \sum_{j=0}^{\infty} \frac{\beta\lambda^m t^{m-1}}{(m-1)!^m} e^{-(\lambda+\beta)t} X_1(t) + 2\lambda\beta e^{-(\lambda+\beta)t} b_1^{n-1}\frac{\psi^n(t)}{a_1}\right]\hat{P}_{0,8}(t), \tag{3.11}$$

$$P_{0,8}(t) = \left[\sum_{j=0}^{\infty} - \left(\alpha e^{-(\lambda+\alpha+\beta)t} \cdot \sum_{n=1}^{\infty} \frac{\lambda^n t^{n-1} e^{-(\lambda+\beta)t}}{(n-1)!}\right)^j\right] \cdot \left[\frac{\alpha}{\lambda+\alpha+\beta}\left(1 - e^{-(\lambda+\alpha+\beta)t}\right)\right], \tag{3.12}$$

$$P_{n,8}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{(\lambda+\beta)^n} P_{0,8}(t) \text{ for } n \geq 1. \tag{3.13}$$

The proof of Theorem 3.2 is given in Appendix B.

Remarks:

(i) When there is only one server (assuming $\mu_2 = \mu_3 = \theta_2 == \theta_3$=0) and considering there is no disaster then the proposed model is reduced the model derived in Kalidas and Ramnath [5].

(ii) When there is only one server (assuming $\mu_2 = \mu_3 = \theta_2 = \theta_3$=0) then the proposed model is reduced single server queueing model with system disaster.

(iii) When there are 2 servers (assuming $\mu_3 = \theta_3$=0) then the proposed model is reduced to heterogeneous two server queueing model with multiple vacation with system disaster.

(iv) When there are 2 servers (assuming $\mu_3 = \theta_3$=0) and considering there is no disaster then the proposed model is reduced to the model proposed by Kumar and Madheswari [12].

4. Numerical Analysis

In the following section, graphs showing the system's transient probabilities in different circumstances, including single-server busy, two servers busy, all the three-server busy, servers on vacation, and failure states, are shown. The graphs have been generated using the following parameter values: $\lambda = 2$, $\mu_1 = 3$, $\mu_2 = 2.5$, $\mu_3 = 2.0$, $\theta_1 = 0.9$, $\theta_2 = 0.7$, $\theta_3 = 0.5$, $\alpha = 0.5$, $\beta = 1.5$. The behaviour of the transient probability of a single server-only busy state against time t for various values of n is depicted in Fig. 1. This graph depicts the probability curves starting at 0 and converging to a steady state over time. We saw in that graph that the transient probability values go down as the number of customers goes up. For a certain n value, the probability value of server 1 being busy by itself is higher than those of servers 2 and 3. Since the vacation rate of server 1 is greater than that of server 2 and server 3. The evolution of transient probabilities during server 1 and server 2 busy states, server 2 and server 3 busy states, and server 1 and server 3 busy states is shown in Fig. 2. From that figure, we notice that if the number of customers increases, then the transient probability of all the states decreases. Also, we notice that all probability curves start at 0 and gradually rise to some amount as t rises until stabilising. Fig. 3 exhibits the graph of all the servers vacation state, all the servers busy state, and failure state transient probability over time t. From that figure, we observed that if the number of customers increases, then the transient probability of all three states mentioned above decreases, and it is also evident that probability curves start at 0 and gradually rise to a particular value until they reach a steady state.

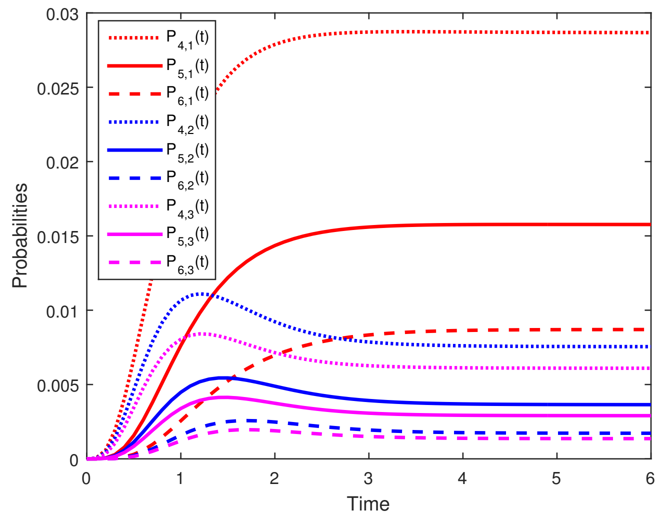

Figure 1. Probabilities Vs Time

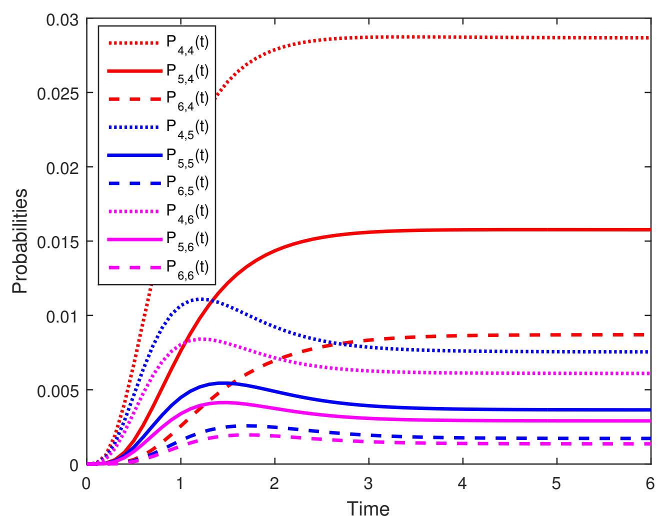

Figure 2. Probabilities Vs Time

5. Conclusion and Future work

This study explored a heterogeneous three-server queueing system with multiple vacation, disaster, and repair. The modified Bessel function of the first kind was used to derive the time-dependent probability of the system size. The proposed model's numerical results show that the time-dependent probabilities converge their respective steady-state probability.

Future work can expand upon this research by considering multi-server multiple vacation queueing systems with disaster and repair. Analysing such systems would provide us a more complete understanding of their performance and behaviour. It would also be advantageous to investigate stochastic

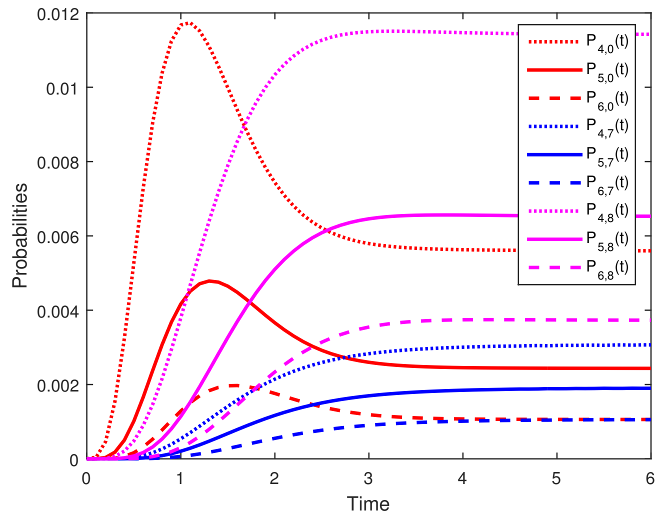

FIGURE 3. Probabilities Vs Time

decomposition for this model, as it can provide a deeper understanding of the dynamics of the system and aid in improving performance.

APPENDICES: PROOFS OF THEOREMS 3.1 AND 3.2

Appendix A : Proof of Theorem 3.1.

Define

$$Q_1(z,t) = \sum_{n=1}^{\infty} P_{n,1}(t)z^n.$$

Then

$$\frac{\partial Q_1(z,t)}{\partial t} = \sum_{n=1}^{\infty} P'_{n,1}(t)z^n.$$

Multiply (2.4) by z and (2.5) by z^n and summing over all possible values of n then we get,

$$\sum_{n=1}^{\infty} P'_{n,1}(t)z^n = -(\lambda+\alpha+\mu_1)Q_1(z,t) - (\theta_2+\theta_3)\sum_{n=2}^{\infty} P_{n,1}(t)z^n + \lambda z \sum_{n=1}^{\infty} P_{n-1,1}(t)z^{n-1}$$
$$+ \frac{\mu_1}{z}\sum_{n=1}^{\infty} P_{n+1,1}(t)z^{n+1} + \theta_1 \sum_{n=1}^{\infty} P_{n,0}(t)z^n + \beta \sum_{n=1}^{\infty} P_{n,8}(t)z^n + \mu_2 z P_{2,4}(t) + \mu_3 z P_{2,6}(t),$$

$$\frac{\partial Q_1(z,t)}{\partial t} + \left((\lambda+\alpha+\mu_1+\theta_2+\theta_3) - \left(\frac{\mu_1}{z} + \lambda z\right)\right)Q_1(z,t) = (\theta_2+\theta_3)P_{1,1}(t)z + \lambda z P_{0,1}(t)$$
$$- \frac{\mu_1}{z}P_{1,1}(t)z + \theta_1 \sum_{n=1}^{\infty} P_{n,0}(t)z^n + \beta \sum_{n=1}^{\infty} P_{n,8}(t)z^n$$
$$+ \mu_2 P_{2,4}(t) + \mu_3 P_{2,6}(t).$$

Upon integrating the above linear differential equation with respect to t, we get

$$Q_1(z,t) = \int_0^t \Big(\theta_1 \sum_{n=1}^{\infty} P_{n,0}(t)z^n + \beta \sum_{n=1}^{\infty} P_{n,8}(t)z^n - \mu_1 P_{1,1}(t) + (\theta_2+\theta_3)P_{1,1}(t)z + \mu_2 z P_{2,4}(t) + \mu_3 z P_{,6}(t)z + \lambda z P_{0,1}(t)\Big)(e^{-(\lambda+\alpha+\mu_1+\theta_2+\theta_3)(t-y)} e^{(\mu_1/z+\lambda z)(t-y)})dy.$$

If $a_i = 2\sqrt{\lambda\mu_i}$ and $b_i = \sqrt{\lambda/\mu_i}$, then

$$e^{(\mu_i/z+\lambda z)t} = \sum_{-\infty}^{\infty} (b_i z)^n I_n(a_i t)$$

where $I_n(a_i t)$ is a bessel funtion of order n. Using that fact in the above equation and comparing the terms coeffients of z^n for n=1,2,3,...

$$P_{n,1}(t) = \int_0^t \Big[\big(\theta_1 \sum_{m=1}^{\infty} P_{m,0}(t) + \beta \sum_{m=1}^{\infty} P_{m,8}(t)\big) b_1^{n-m} I_{n-m}(.) e^{-k_1(t-y)}\Big]dy + \int_0^t \Big[\big(\mu_2 P_{2,4}(t) + \mu_3 P_{2,6}(t) + \lambda P_{0,1}(t) + (\theta_2+\theta_3)P_{1,1}(t)\big) b_1^{n-1} I_{n-1}(.) e^{-k_1(t-y)}\Big]dy - \int_0^t \mu_1 P_{1,1}(t) b_1^n I_n(.) e^{-k_1(t-y)} dy. \tag{5.1}$$

Equating the coefficients of z^{-n} for n=1,2,... and applying $I_{-n}(.) = I_n(.)$ we get

$$0 = \int_0^t \Big[\big(\theta_1 \sum_{m=1}^{\infty} P_{m,0}(t) + \beta \sum_{m=1}^{\infty} P_{m,8}(t)\big) b_1^{-n-m} I_{n+m}(.) e^{-k_1(t-y)}\Big]dy + \int_0^t \Big[\big(\mu_2 P_{2,4}(t) + \mu_3 P_{2,6}(t) + \lambda P_{0,1}(t) + (\theta_2+\theta_3)P_{1,1}(t)\big) b_1^{-n-1} I_{n+1}(.) e^{-k_1(t-y)}\Big]dy - \int_0^t \Big[\mu_1 P_{1,1}(t) b_1^{-n} I_n(.) e^{-k_1(t-y)}\Big]dy \tag{5.2}$$

where $k_1 = \lambda + \alpha + \mu_1 + \theta_2 + \theta_3$ and $I_n(.) = I_n(a(t-y))$. Multiply equation (5.2) by b_1^{2n} and subtract from equation (5.1)

$$P_{n,1}(t) = \int_0^t \Big[\sum_{m=1}^{\infty} \big(\theta_1 P_{m,1}(t)z^n + \beta P_{m,4}(t)\big) b_1^{n-m} \big[I_{n-m}(.) - I_{n+m}(.)\big] e^{-k_1(t-y)}\Big]dy + \int_0^t \Big[\big(\mu_2 P_{2,4}(t) + \mu_3 P_{2,6}(t) + \lambda P_{0,1}(t) + (\theta_2+\theta_3)P_{1,1}(t)\big) b_1^{n-1} \big[I_{n-1}(.) - I_{n+1}(.)\big] e^{-k_1(t-y)}\Big]dy.$$

In similar way using the equations (2.6)-(2.17) we get (3.1)-(3.7)

Appendix B: Proof of Theorem 3.2.

Taking Laplace transformation of the above equations (2.1)-(2.3), (2.18), (2.19), one obtains,

$$\hat{P}_{0,0}(s) = \frac{1}{s+\lambda+\alpha}\big(1 + \mu_1 \hat{P}_{1,1}(s) + \mu_2 \hat{P}_{1,2}(s) + \mu_3 \hat{P}_{1,3}(s)\big), \tag{5.3}$$

$$\hat{P}_{n,0}(s) = \frac{\lambda}{(s+\lambda+\alpha+\theta_1+\theta_2+\theta_3)}\hat{P}_{n-1,0},(s) \tag{5.4}$$

$$\hat{P}_{0,1}(s) = \frac{\beta}{s+\lambda+\alpha}\hat{P}_{0,8}(s), \tag{5.5}$$

$$\hat{P}_{0,8}(s) = \frac{\alpha}{s(s+\lambda+\beta)} - \frac{\alpha}{s+\lambda+\beta}\sum_{n=0}^{\infty}\hat{P}_{n,8}(s), \tag{5.6}$$

$$\hat{P}_{n,8}(s) = \frac{\lambda}{(s+\lambda+\beta)}\hat{P}_{n-1,4}(s). \tag{5.7}$$

The above equations (5.4) and (5.7) recursively yield

$$\hat{P}_{n,0}(s) = \frac{\lambda^n}{(s+\lambda+\alpha+\theta_1+\theta_2+\theta_3)^n}\hat{P}_{0,0}(s), \tag{5.8}$$

$$\hat{P}_{n,8}(s) = \frac{\lambda^n}{(s+\lambda+\beta)^n}\hat{P}_{0,8}(s). \tag{5.9}$$

Taking The Laplace transforms of (3.1)-(3.7), we get

$$\hat{P}_{n,1}(s) = \frac{1}{\sqrt{\omega_1^2-a_1^2}}\sum_{m=1}^{\infty}\Big[\big(\theta_1\hat{P}_{m,0}(s)+\beta\hat{P}_{m,8}(s)\big)b_1^{n-m}\big(\hat{\psi}(s)^{n-m}-\hat{\psi}(s)^{n+m}\big)\Big]$$
$$+\big(2\mu_2\hat{P}_{2,4}(s)+2\mu_3\hat{P}_{2,6}(s)+2\lambda\hat{P}_{0,1}(s)+2(\theta_2+\theta_3)\hat{P}_{1,1}(s)\big)\frac{b_1^{n-1}\hat{\psi}(s)^n}{a_1}\ \text{ for } n\geq 1, \tag{5.10}$$

$$\hat{P}_{n,2}(s) = \frac{1}{\sqrt{\omega_2^2-a_2^2}}\sum_{m=1}^{\infty}\Big[\theta_2\hat{P}_{m,0}(s)b_2^{n-m}\big(\hat{\psi}(s)^{n-m}-\hat{\psi}(s)^{n+m}\big)\Big]$$
$$+\big(2\mu_1\hat{P}_{2,4}(s)+2\mu_3\hat{P}_{2,5}(s)+2(\theta_1+\theta_3)\hat{P}_{1,2}(s)\big)\frac{b_2^{n-1}\hat{\psi}(s)^n}{a_1}\ \text{ for } n\geq 1, \tag{5.11}$$

$$\hat{P}_{n,3}(s) = \frac{1}{\sqrt{\omega_3^2-a_3^2}}\sum_{m=1}^{\infty}\Big[\theta_3\hat{P}_{m,0}(s)b_3^{n-m}\big(\hat{\psi}(s)^{n-m}-\hat{\psi}(s)^{n+m}\big)\Big]$$
$$+\big(2\mu_2\hat{P}_{2,5}(s)+2\mu_1\hat{P}_{2,6}(s)+2(\theta_1+\theta_2)\hat{P}_{1,3}(s)\big)\frac{b_3^{n-1}\hat{\psi}(s)^n}{a_1}\ \text{ for } n\geq 1, \tag{5.12}$$

$$\hat{P}_{n,4}(s) = \frac{1}{\sqrt{\omega_4^2-a_4^2}}\sum_{m=2}^{\infty}\Big[\big(\theta_1\hat{P}_{m,2}(s)+\theta_2\hat{P}_{m,1}(s)\big)b_4^{n-m}\times\big(\hat{\psi}(s)^{n-m}-\hat{\psi}(s)^{n+m}\big)\Big]$$
$$+\big(2(\mu_1+\mu_2)\hat{P}_{2,4}(s)+2\mu_3\hat{P}_{3,7}(s)\big)\frac{b_4^{n-1}\hat{\psi}(s)^n}{a_4}+2\theta_3\hat{P}_{2,4}(s)\frac{b_4^{n-2}\hat{\psi}(s)^n}{a_4}\ \text{ for } n\geq 2, \tag{5.13}$$

$$\hat{P}_{n,5}(s) = \frac{1}{\sqrt{\omega_5^2-a_5^2}}\sum_{m=2}^{\infty}\Big[\big(\theta_2\hat{P}_{m,3}(s)+\theta_3\hat{P}_{m,2}(s)\big)b_5^{n-m}\big(\hat{\psi}(s)^{n-m}-\hat{\psi}(s)^{n+m}\big)\Big]$$
$$+\big(2(\mu_2+\mu_3)\hat{P}_{2,5}(s)+2\mu_1\hat{P}_{3,7}(s)\big)\frac{b_5^{n-1}\hat{\psi}(s)^n}{a_5}+2\theta_1\hat{P}_{2,5}(s)\frac{b_5^{n-2}\hat{\psi}(s)^n}{a_5}\ \text{ for } n\geq 2, \tag{5.14}$$

$$\hat{P}_{n,6}(s) = \frac{1}{\sqrt{\omega_6^2-a_6^2}}\sum_{m=2}^{\infty}\Big[\big(\theta_1\hat{P}_{m,3}(s)+\theta_3\hat{P}_{m,1}(s)\big)b_6^{n-m}\big(\hat{\psi}(s)^{n-m}-\hat{\psi}(s)^{n+m}\big)\Big]$$
$$+\big(2(\mu_1+\mu_3)\hat{P}_{2,6}(s)+2\mu_2\hat{P}_{3,7}(s)\big)\frac{b_6^{n-1}\hat{\psi}(s)^n}{a_6}+2\theta_2\hat{P}_{2,6}(s)\frac{b_6^{n-2}\hat{\psi}(s)^n}{a_6}\ \text{ for } n\geq 2, \tag{5.15}$$

$$\hat{P}_{n,7}(s) = \frac{1}{\sqrt{\omega_7^2 - a_7^2}} \sum_{m=3}^{\infty} \left[\big(\theta_1 \hat{P}_{m,5}(s) + \theta_2 \hat{P}_{m,6}(s) + \theta_3 \hat{P}_{m,4}(s)\big) b_7^{n-m} \big(\hat{\psi}(s)^{n-m} - \hat{\psi}(s)^{n+m}\big) \right]$$
$$+ \big(2(\mu_1 + \mu_2 + \mu_3) \frac{b_7^{n-1} \hat{\psi}(s)^n}{a_7} \hat{P}_{3,7}(s) \text{ for } n \geq 3. \tag{5.16}$$

Substituting equations (5.8), (5.9) in (5.10), (5.11) and (5.12) yields

$$\hat{P}_{n,1}(s) = \sum_{m=1}^{\infty} \left[\theta_1 l_1 + \frac{\beta \lambda^m}{(s+\lambda+\beta)^m} \hat{P}_{0,8}(s) \right] \hat{X}_1(s) + 2\mu_2 b_1^{n-1} \frac{\hat{\psi}(s)^n}{a_1} \hat{P}_{2,4}(s)$$
$$+ 2\mu_3 b_1^{n-1} \frac{\hat{\psi}(s)^n}{a_1} \hat{P}_{2,6}(s) + 2(\theta_2 + \theta_3) b_1^{n-1} \frac{\hat{\psi}(s)^n}{a_1} \hat{P}_{1,1}(s) + \frac{2\lambda\beta}{s+\lambda+\beta} \frac{b_1^{n-1} \hat{\psi}(s)^n}{a_1} \hat{P}_{0,8}(s), \tag{5.17}$$

where

$$l_1 = \frac{\lambda^m}{(s+\lambda+\alpha)(s+\lambda+\alpha+\theta_1+\theta_2+\theta_3)^m} \bigg(1 + \mu_1 \hat{P}_{1,1}(s) + \mu_2 \hat{P}_{1,2}(s) + \mu_3 \hat{P}_{1,3}(s)\bigg),$$

$$\hat{P}_{n,2}(s) = \sum_{m=1}^{\infty} \theta_2 l_1 \hat{X}_2(s) + 2\mu_1 b_2^{n-1} \frac{\hat{\psi}(s)^n}{a_2} \hat{P}_{2,4}(s) + 2\mu_3 \times b_2^{n-1} \frac{\hat{\psi}(s)^n}{a_2} \hat{P}_{2,5}(s)$$
$$+ 2(\theta_1 + \theta_3) b_2^{n-1} \frac{\hat{\psi}(s)^n}{a_2} \hat{P}_{1,2}(s) \tag{5.18}$$
$$\hat{P}_{n,3}(s) = \sum_{m=1}^{\infty} \theta_3 l_1 \hat{X}_3(s) + 2\mu_2 b_3^{n-1} \frac{\hat{\psi}(s)^n}{a_3} \hat{P}_{2,5}(s) + 2\mu_3 b_3^{n-1} \frac{\hat{\psi}(s)^n}{a_3} \hat{P}_{2,6}(s)$$
$$+ 2(\theta_1 + \theta_2) b_3^{n-1} \frac{\hat{\psi}(s)^n}{a_3} \hat{P}_{1,3}(s). \tag{5.19}$$

Substituting (5.17)-(5.19) in (5.13)-(5.15) leads to

$$\hat{P}_{n,4}(s) = \sum_{m=1}^{\infty} \Bigg[\theta_1 \theta_2 l_1 \bigg(\hat{X}_1(s) + \hat{X}_2(s)\bigg) + \bigg(\frac{\beta\lambda^m}{(s+\lambda+\beta)^m} + \frac{2\lambda\beta\theta_2}{s+\lambda+\beta} \frac{b_1^{n-1}\hat{\psi}(s)^n}{a_1}\bigg) \hat{P}_{0,8}(s)$$
$$+ 2\big(\mu_1 + \mu_2\big) b_1^{n-1} \cdot \frac{\hat{\psi}(s)^n}{a_1} \hat{P}_{2,4}(s) + 2\mu_3 b_2^{n-1} \frac{\hat{\psi}(s)^n}{a_2} \hat{P}_{2,5}(s) + 2(\theta_1 + \theta_3) b_2^{n-1} \frac{\hat{\psi}(s)^n}{a_2} \hat{P}_{1,2}(s)$$
$$+ 2\mu_3 b_1^{n-1} \frac{\hat{\psi}(s)^n}{a_1} \hat{P}_{2,6}(s) + 2(\theta_2 + \theta_3) b_1^{n-1} \frac{\hat{\psi}(s)^n}{a_1} \hat{P}_{1,1}(s) \Bigg] \hat{X}_4(s)$$
$$+ \big(2(\mu_1 + \mu_2) \hat{P}_{2,4}(s) + 2\mu_3 \hat{P}_{3,7}(s) + 2\theta_3 \hat{P}_{2,4}(s)\big) \frac{b_4^{n-1} \hat{\psi}(s)^n}{a_4} \quad \text{for } n \geq 2, \tag{5.20}$$
$$\hat{P}_{n,5}(s) = \sum_{m=1}^{\infty} \Bigg[\theta_2 \theta_3 l_1 \bigg(\hat{X}_3(s) + \hat{X}_2(s)\bigg) + 2\bigg(\mu_3 + \mu_2\bigg) b_2^{n-1} \frac{\hat{\psi}(s)^n}{a_2} \hat{P}_{2,5}(s) + 2\mu_1 b_2^{n-1} \frac{\hat{\psi}(s)^n}{a_2} \hat{P}_{2,4}(s)$$
$$+ 2(\theta_1 + \theta_3) b_2^{n-1} \frac{\hat{\psi}(s)^n}{a_2} \hat{P}_{1,2}(s) + 2\mu_1 b_3^{n-1} \frac{\hat{\psi}(s)^n}{a_3} \hat{P}_{2,6}(s) + 2(\theta_1 + \theta_2) b_3^{n-1} \frac{\hat{\psi}(s)^n}{a_3} \hat{P}_{1,3}(s) \Bigg] \hat{X}_5(s)$$
$$+ \big(2(\mu_2 + \mu_3) \hat{P}_{2,5}(s) + 2\mu_1 \hat{P}_{3,7}(s) + 2\theta_1 \hat{P}_{2,5}(s)\big) \frac{b_5^{n-1} \hat{\psi}(s)^n}{a_5} \text{ for } n \geq 2, \tag{5.21}$$

$$\hat{P}_{n,6}(s) = \sum_{m=1}^{\infty}\Big[\theta_1\theta_3 l_1\Big(\hat{X}_1(s)+\hat{X}_3(s)\Big) + \Big(\frac{\beta\lambda^m}{(s+\lambda+\beta)^m} + \frac{2\lambda\beta}{s+\lambda+\beta}\frac{b_1^{n-1}\hat{\psi}(s)^n}{a_1}\Big)\hat{P}_{0,8}(s)$$
$$+ 2\Big(\mu_1+\mu_3\Big)b_3^{n-1}\cdot\frac{\hat{\psi}(s)^n}{a_3}\hat{P}_{2,6}(s) + 2\mu_2 b_1^{n-1}\frac{\hat{\psi}(s)^n}{a_1}\hat{P}_{2,4}(s) + 2(\theta_1+\theta_2)b_3^{n-1}\frac{\hat{\psi}(s)^n}{a_3}\hat{P}_{1,3}(s)$$
$$+ 2\mu_2 b_3^{n-1}\frac{\hat{\psi}(s)^n}{a_3}\hat{P}_{2,5}(s) + 2(\theta_2+\theta_3)b_1^{n-1}\frac{\hat{\psi}(s)^n}{a_1}\hat{P}_{1,1}(s)\Big]\hat{X}_6(s) + \big(2(\mu_1+\mu_3)\hat{P}_{2,6}(s)$$
$$+ 2\mu_3\hat{P}_{3,7}(s) + 2\theta_2\hat{P}_{2,4}(s)\big)\frac{b_6^{n-1}\hat{\psi}(s)^n}{a_6} \quad \text{for } n \geq 2. \tag{5.22}$$

Substitute $n = 1$ in (5.17)-(5.19), $n = 2$ in (5.20)-(5.22) and $n = 3$ in (5.17) and after some simplifications, we obtain

$$\hat{P}_{1,i}(s) = \hat{G}_{i1}(s) + \hat{G}_{i2}(s)\hat{P}_{08}(s) \;\text{ for } i = 1,2,3,$$
$$\hat{P}_{2,i}(s) = \hat{G}_{i1}(s) + \hat{G}_{i2}(s)\hat{P}_{08}(s) \text{ for } \; i = 4,5,6,$$
$$\hat{P}_{3,7}(s) = \hat{F}_{11}(s) + \hat{F}_{12}(s)\hat{P}_{08}(s).$$

where

$$\hat{A}_{10}(s) = \sum_{j=0}^{\infty}\Big[\sum_{m=1}^{\infty}\Big(\theta_1\mu_1 l_0\hat{X}_1(s) + 2(\theta_2+\theta_3)\frac{\hat{\psi}(s)}{a_1}\Big)\Big]^j,$$
$$\hat{A}_{i1}(s) = \hat{A}_{i0}(s)\Big\{\sum_{m=1}^{\infty}\theta_i l_0\hat{X}_i(s)\Big\} \text{ for } i = 1,2,3,$$
$$\hat{A}_{12}(s) = \hat{A}_{10}(s)\Big\{\sum_{m=1}^{\infty}\Big(\frac{\beta\lambda^m}{(s+\lambda+\beta)^m}\Big)\hat{X}_1(s) + \frac{2\lambda\beta}{s+\lambda+\beta}\frac{\hat{\psi}(s)}{a_1}\Big\},$$
$$\hat{A}_{20}(s) = \sum_{j=0}^{\infty}\Big[\sum_{m=1}^{\infty}\Big(\theta_2\mu_2 l_0\hat{X}_1(s) + 2(\theta_1+\theta_3)\frac{\hat{\psi}(s)}{a_2}\Big)\Big]^j,$$
$$\hat{A}_{30}(s) = \sum_{j=0}^{\infty}\Big[\sum_{m=1}^{\infty}\Big(\theta_3\mu_3 l_0\hat{X}_3(s) + 2(\theta_1+\theta_2)\frac{\hat{\psi}(s)}{a_3}\Big)\Big]^j,$$
$$\hat{B}_{10}(s) = \Big\{\sum_{m=1}^{\infty}\hat{X}_4(s)2\Big(\mu_1+\mu_2\Big)b_1^{n-1}\frac{\hat{\psi}(s)^n}{a_1} + 2(\mu_1+\mu_2+\theta_3)\frac{b_4^{n-1}\hat{\psi}(s)^n}{a_4}\Big\}^j,$$
$$\hat{B}_{11}(s) = \hat{B}_{10}(s)\Big\{\sum_{m=1}^{\infty}\Big[\Big(\theta_1\theta_2 l_0\Big(\hat{X}_1(s)+\hat{X}_2(s)\Big)\Big]\hat{X}_4(s)\Big\},$$
$$\hat{B}_{12}(s) = \hat{B}_{10}(s)\Big\{\sum_{m=1}^{\infty}\Big[\Big(\theta_1\theta_2 l_0\mu_1\Big(\hat{X}_1(s)+\hat{X}_2(s)\Big) + 2(\theta_2+\theta_3)b_1^{n-1}\frac{\hat{\psi}(s)^n}{a_1}\Big]\hat{X}_4(s)\Big\},$$
$$\hat{B}_{13}(s) = \hat{B}_{10}(s)\Big\{\sum_{m=1}^{\infty}2\mu_3 b_2^{n-1}\frac{\hat{\psi}(s)^n}{a_2}\hat{X}_4(s)\Big\},$$
$$\hat{B}_{14}(s) = \hat{B}_{10}(s)\Big\{\sum_{m=1}^{\infty}2\mu_3 b_1^{n-1}\frac{\hat{\psi}(s)^n}{a_1}\hat{X}_4(s)\Big\},$$
$$\hat{B}_{15}(s) = \hat{B}_{10}(s)\sum_{m=1}^{\infty}\Big[\Big(\frac{\beta\lambda^m}{(s+\lambda+\beta)^m} + \frac{2\lambda\beta\theta_2}{s+\lambda+\beta}b_1^{n-1}\frac{\hat{\psi}(s)^n}{a_1}\Big)\Big]\hat{X}_4(s),$$

$$\hat{B}_{20}(s) =\left\{\sum_{m=1}^{\infty} \hat{X}_5(s)2\big(\mu_2+\mu_2\big)b_2^{n-1}\frac{\hat{\psi}(s)^n}{a_2}+2(\mu_2+\mu_3+\theta_1)\frac{b_5^{2-m}\hat{\psi}(s)^2}{a_5}\right\}^j,$$

$$l_0 =\frac{\lambda^m}{(s+\lambda+\alpha)(s+\lambda+\alpha+\theta_1+\theta_2)^m},$$

$$\hat{B}_{21}(s) =\hat{B}_{20}(s)\left\{\sum_{m=1}^{\infty}\left[\left(\theta_2\theta_3 l_0\left(\hat{X}_3(s)+\hat{X}_2(s)\right)\right]\hat{X}_5(s)\right\},$$

$$\hat{B}_{22}(s) =\hat{B}_{20}(s)\left\{\sum_{m=1}^{\infty}\left[\left(\theta_2\theta_3 l_0\mu_2\left(\hat{X}_3(s)+\hat{X}_2(s)\right)+2(\theta_1+\theta_3)b_2^{n-1}\frac{\hat{\psi}(s)^n}{a_2}\right]\hat{X}_5(s)\right\},$$

$$\hat{B}_{23}(s) =\hat{B}_{20}(s)\left\{\sum_{m=1}^{\infty} 2\mu_1 b_2^{n-1}\frac{\hat{\psi}(s)^n}{a_2}\hat{X}_5(s)\right\},$$

$$\hat{B}_{24}(s) =\hat{B}_{20}(s)\left\{\sum_{m=1}^{\infty} 2\mu_1 b_3^{n-1}\frac{\hat{\psi}(s)^n}{a_3}\hat{X}_5(s)\right\},$$

$$\hat{B}_{30}(s) =\left\{\sum_{m=1}^{\infty} \hat{X}_6(s)2\big(\mu_1+\mu_3\big)b_3^{n-1}\frac{\hat{\psi}(s)^n}{a_3}+2(\mu_1+\mu_3+\theta_1)\frac{b_6^{2-m}\hat{\psi}(s)^2}{a_6}\right\}^j,$$

$$\hat{B}_{31}(s) =\hat{B}_{30}(s)\left\{\sum_{m=1}^{\infty}\left[\left(\theta_1\theta_3 l_0\left(\hat{X}_1(s)+\hat{X}_3(s)\right)\right]\hat{X}_6(s)\right\},$$

$$\hat{B}_{32}(s) =\hat{B}_{30}(s)\left\{\sum_{m=1}^{\infty}\left[\left(\theta_1\theta_3 l_0\mu_3\left(\hat{X}_3(s)+\hat{X}_2(s)\right)+2(\theta_2+\theta_3)b_1^{1}\frac{\hat{\psi}(s)^2}{a_1}\right]\hat{X}_6(s)\right\},$$

$$\hat{B}_{33}(s) =\hat{B}_{30}(s)\left\{\sum_{m=1}^{\infty} 2\mu_2 b_1^{1}\frac{\hat{\psi}(s)^2}{a_1}\hat{X}_6(s)\right\},$$

$$\hat{B}_{34}(s) =\hat{B}_{30}(s)\left\{\sum_{m=1}^{\infty} 2\mu_2 b_3^{1}\frac{\hat{\psi}(s)^2}{a_3}\right\},$$

$$\hat{C}_{1,0}(s) =\sum_{j=0}^{\infty}\left[\left[2\theta_1\mu_1 b_5^2\frac{\hat{\psi(s)}^3}{a_5}+2\theta_2\mu_3 b_6^2\frac{\hat{\psi(s)}^3}{a_6}+2\theta_3\mu_3 b_4^2\frac{\hat{\psi(s)}^3}{a_7}\right]\hat{X}_7(s)+2\big(\mu_1+\mu_2+\mu_3\big)b_7^2\frac{\hat{\psi(s)}^3}{a_7}\right]^j,$$

$$\hat{C}_{11}(s) =\hat{C}_{1,0}(s)\left\{\sum_{m=1}^{\infty}\left[\theta_1\theta_2\theta_3 l_0\left[\left(\hat{X}_3(s)+\hat{X}_5(s)+\left(\hat{X}_1(s)+\hat{X}_3(s)\right)\hat{X}_6(s)+\left(\hat{X}_1(s)+\hat{X}_2(s)\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\times\hat{X}_4(s)\right]\right]\hat{X}_7(s)\right\},$$

$$\hat{C}_{12}(s) =\hat{C}_{1,0}(s)\left\{\hat{C}_{11}(s)+\left[\theta_2(\theta_1+\theta_3)b_1^2\frac{\hat{\psi}(s)^3}{a_1}\hat{X}_6(s)2\theta_2(\theta_2+\theta_3)b_1^2\frac{\hat{\psi}(s)^3}{a_1}\hat{X}_4\right]\right]\hat{X}_7(s)\right\},$$

$$\hat{C}_{13}(s) =\hat{C}_{1,0}(s)\left\{\hat{C}_{11}(s)+\left[\frac{1}{\sqrt{\omega_5^2-a_5^2}}2\theta_1(\theta_1+\theta_3)b_2^2\frac{\hat{\psi}(s)^3}{a_2}\hat{X}_5(s)+2\theta_3(\theta_1+\theta_3)b_2^2\frac{\hat{\psi}(s)^3}{a_2}\hat{X}_4\right]\hat{X}_7(s)\right\},$$

$$\hat{C}_{14}(s) =\hat{C}_{1,0}(s)\left\{\hat{C}_{11}(s)+\left[2\theta_1(\theta_1+\theta_2)b_3^2\frac{\hat{\psi}(s)^3}{a_3}\hat{X}_5(s+2\theta_2(\theta_1+\theta_2)b_3^2\frac{\hat{\psi}(s)^3}{a_3}\hat{X}_6\right]\right]\hat{X}_7(s)\right\},$$

$$\hat{C}_{1,5}(s) =\hat{C}_{1,0}(s)\left\{\sum_{m=1}^{\infty}\left[2\theta_1\mu_1 b_2^2\frac{\hat{\psi}(s)}{a_2}\hat{X}_5(s)+2\theta_2\mu_2 b_1^2\frac{\hat{\psi}(s)}{a_1}\hat{X}_6(s)+2\theta_3(\mu_1+\mu_2)\frac{b_1^2}{a_1}\hat{\psi}(s)\hat{X}_4(s)\right.\right.$$
$$\left.\left.+2\theta_3(\mu_1+\mu_2)b_4^2\frac{\hat{\psi}(s)}{a_4}+2\theta_3^2 b_4\frac{\hat{\psi}(s)}{a_4}\right]\hat{X}_7(s)\right\},$$

$$\hat{C}_{1,6}(s) =\hat{C}_{1,0}(s)\Big\{\sum_{m=1}^{\infty}\Big[2\theta_3\mu_3 b_2^{n-1}\frac{\hat{\psi}(s)}{a_2}\hat{X}_4(s)+2\theta_2\mu_2 b_3^2\frac{\hat{\psi}(s)}{a_3}\hat{X}_6(s)+2\theta_2(\mu_3+\mu_3)\frac{b_2^2}{a_2}\hat{\psi}(s)\hat{X}_5(s)$$

$$+2\theta_1(\mu_2+\mu_3)b_5^2\frac{\hat{\psi}(s)}{a_5}+2\theta_1^2 b_5\frac{\hat{\psi}(s)}{a_5}\Big]\hat{X}_7(s)\Big\},$$

$$\hat{C}_{1,7}(s) =\hat{C}_{1,0}(s)\Big\{\sum_{m=1}^{\infty}\Big[2\theta_1\mu_3 b_1^2\frac{\hat{\psi}^3(s)}{a_1}\hat{X}_4(s)+2\theta_1\mu_1 b_3^2\frac{\psi(\hat{s})^{n^3}}{a_3}\hat{X}_5(s)+2\theta_2(\mu_1+\mu_3)\frac{b_3^2}{a_3}\hat{\psi}(s)\hat{X}_6(s)$$

$$+2\theta_2(\mu_1+\mu_2)b_6^2\frac{\hat{\psi}(s)}{a_6}+2\theta_2^2 b_6\frac{\hat{\psi}(s)}{a_6}\Big]\hat{X}_7(s)\Big\},$$

$$\hat{C}_{1,8}(s) =\hat{C}_{1,0}(s)\Big\{\sum_{m=3}^{\infty}\Big[\Big(\frac{\beta\lambda^m}{(s+\lambda+\beta)^m}+\frac{2\lambda\beta}{s+\lambda+\beta}b_1^2\frac{\psi(\hat{s})^3}{a_1}\Big)\Big(\theta_2\hat{X}_6(s)+\theta_3\hat{X}_4(s)\Big)\Big]\hat{X}_7(s)\Big\},$$

$$\hat{D}_{10}(s)=\sum_{j=0}^{\infty}\Big[\hat{A}_{31}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\big(\hat{A}_{11}(s)\mu_1\hat{A}_{21}(s)\mu_1+\hat{A}_{11}(s)\mu_3\big)+\hat{A}_{31}(s)\mu_2\big(\hat{A}_{21}(s)\mu_1$$

$$\times\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_3+\hat{A}_{11}(s)\mu_3\big)+\hat{A}_{21}(s)\mu_1\big)\Big]^j,$$

$$\hat{D}_{11}(s) =\hat{D}_{10}(s)\Big[\hat{A}_{31}(s)+\hat{A}_{31}(s)\mu_2\big(\hat{A}_{21}+\hat{A}_{22}\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\big(\hat{A}_{11}(s)+\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\big)\big)\Big],$$

$$\hat{D}_{12}(s) =\hat{D}_{10}(s)\Big[\hat{A}_{31}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{20}(s)\Big\{2\mu_1\frac{\hat{\psi}(s)}{a_2}\Big\}+\hat{A}_{10}(s)\Big\{2\mu_2\frac{\hat{\psi}(s)}{a_1}\Big\}\Big)$$

$$+\hat{A}_{31}(s)\mu_2\big(\hat{A}_{21}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j bigg(\hat{A}_{11}(s)\mu_2\hat{A}_{20}(s)\Big\{2\mu_1\frac{\hat{\psi}(s)}{a_2}\Big\}$$

$$+\hat{A}_{10}(s)\Big\{2\mu_2\frac{\hat{\psi}(s)}{a_1}\Big\}\Big)+\hat{A}_{20}(s)\Big\{2\mu_1\frac{\hat{\psi}(s)}{a_2}\Big\})\Big],$$

$$\hat{D}_{13}(s) =\hat{D}_{10}(s)\Big[\hat{A}_{31}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\Big(\hat{A}_{11}(s\mu_2\hat{A}_{20}(s)\Big\{2\mu_3\frac{\hat{\psi}(s)}{a_2}\Big\}\Big)$$

$$+\hat{A}_{31}(s)\mu_2\big(\hat{A}_{21}(s)\mu_1\cdot\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\big(\hat{A}_{11}(s)\mu_2\hat{A}_{20}(s)\Big\{2\mu_3\frac{\hat{\psi}(s)}{a_2}\Big\}$$

$$+\hat{A}_{20}(s)\Big\{2\mu_3\frac{\hat{\psi}(s)}{a_2}\Big\})+\hat{A}_{30}(s)\Big\{2\mu_1\frac{\hat{\psi}(s)}{a_3}\Big\})\Big],$$

$$\hat{D}_{14}(s) =\hat{D}_{10}(s)\Big[\hat{A}_{31}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\hat{A}_{10}(s)\Big\{2\mu_3\frac{\hat{\psi}(s)}{a_1}\Big\}+\hat{A}_{31}(s)\mu_2\hat{A}_{21}(s)$$

$$\times\mu_1\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\hat{A}_{10}(s)\Big\{2\mu_3\frac{\hat{\psi}(s)}{a_1}\Big\}+\hat{A}_{30}(s)\Big\{2\mu_2\frac{\hat{\psi}(s)}{a_3}\Big\}\Big],$$

$$\hat{D}_{15}(s) =\hat{D}_{10}(s)\Big[\hat{A}_{31}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\times\hat{A}_{12}(s)+\hat{A}_{31}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big(\hat{A}_{11}(s)$$

$$\times\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\hat{A}_{12}(s)\Big],$$

$$\hat{D}_{21}(s)=\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\big[\hat{A}_{11}(s)+A_{12}(s)\hat{A}_{21}(s)+\big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_3+\hat{A}_{11}(s)\mu_3\big)\hat{D}_{11}(s)\big],$$

$$\hat{D}_{22}(s) =\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\Big[\big(\hat{A}_{11}(s)\mu_2A_{23}(s)+\hat{A}_{11}(s)\mu_3\big)\hat{D}_{12}(s)+\hat{A}_{11}(s)\mu_2\hat{A}_{20}(s)\Big\{2\mu_1\frac{\hat{\psi}(s)}{a_2}\Big\}$$
$$+\hat{A}_{10}(s)\Big\{2\mu_2\frac{\hat{\psi}(s)}{a_1}\Big\}\Big],$$
$$\hat{D}_{23}(s) =\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\Big\{\big[\big(\hat{A}_{11}(s)\mu_2A_{23}(s)+\hat{A}_{11}(s)\mu_3\big)\hat{D}_{13}(s)+\hat{A}_{11}(s)\mu_2\hat{A}_{20}(s)\Big\{2\mu_3\frac{\hat{\psi}(s)}{a_2}\Big\}\Big\},$$
$$\hat{D}_{24}(s) =\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\Big\{\big(\hat{A}_{11}(s)\mu_2A_{23}(s)+\hat{A}_{11}(s)\mu_3\big)\hat{D}_{14}(s)+\hat{A}_{10}(s)\Big\{2\mu_3\frac{\hat{\psi}(s)}{a_1}\Big\}\Big\},$$
$$\hat{D}_{25}(s) =\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\hat{A}_{21}(s)\mu_1,$$
$$\hat{D}_{31}(s) =\hat{A}_{21}(s)+\hat{A}_{21}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\big(\hat{A}_{11}(s)+\hat{A}_{12}\hat{A}_{21}(s)\big)+\Big(\hat{A}_{21}(s)\mu_1$$
$$\times\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\big(\hat{A}_{11}(s)\mu_2+\hat{A}_{21}(s)\mu_3\hat{A}_{11}(s)\mu_3\big)+\hat{A}_{21}(s)\mu_3\Big)\hat{D}_{11}(s),$$
$$\hat{D}_{32}(s) =\Big(\hat{A}_{21}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_3+\hat{A}_{11}(s)\mu_3\big)+\hat{A}_{21}(s)$$
$$\times\mu_3\Big)\hat{D}_{12}(s)+\Big(\hat{A}_{21}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\times\hat{A}_{21}(s)\mu_1\Big)^j\big(\hat{A}_{11}(s)\mu_2\hat{A}_{20}(s)\Big\{2\mu_1\frac{\hat{\psi}(s)}{a_2}\Big\}$$
$$+\hat{A}_{10}(s)\Big\{2\mu_2\frac{\hat{\psi}(s)}{a_1}\Big\})+\hat{A}_{20}(s)\Big\{2\mu_1\frac{\hat{\psi}(s)}{a_2}\Big\}\Big),$$
$$\hat{D}_{33}(s) =\Big(\hat{A}_{21}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\big(\hat{A}_{11}(s)\mu_2\times\hat{A}_{21}(s)\mu_3+\hat{A}_{11}(s)\mu_3\big)+\hat{A}_{21}(s)\mu_3\Big)\hat{D}_{13}(s)$$
$$+\Big(\hat{A}_{21}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\big(\hat{A}_{11}(s)\mu_2\hat{A}_{20}(s)\Big\{2\mu_3\frac{\hat{\psi}(s)}{a_2}\Big\}+\hat{A}_{20}(s)\Big\{2\mu_3\frac{\hat{\psi}(s)}{a_2}\Big\})\Big),$$
$$\hat{D}_{34}(s) =\Big(\hat{A}_{21}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_3+\hat{A}_{11}(s)\mu_3\big)+\hat{A}_{21}(s)\mu_3\Big)\hat{D}_{14}(s)$$
$$+\Big(\hat{A}_{21}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\hat{A}_{10}(s)\Big\{2\mu_3\frac{\hat{\psi}(s)}{a_1}\Big\}\Big),$$
$$\hat{D}_{35}(s) =\Big(\hat{A}_{21}(s)\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_3+\hat{A}_{11}(s)\mu_3\big)+\hat{A}_{21}(s)\mu_3\Big)$$
$$+\Big(\hat{A}_{21}(s)\times\mu_1\sum_{j=0}^{\infty}\Big(\hat{A}_{11}(s)\mu_2\hat{A}_{21}(s)\mu_1\Big)^j\hat{A}_{12}(s)\Big),$$
$$\hat{E}_{10}(s) =\Big[\sum_{j=0}^{\infty}\Big(\hat{B}_{12}(s)\hat{D}_{22}(s)+\hat{B}_{11}(s)\mu_2\hat{D}_{32}(s)+\hat{B}_{11}(s)\mu_3\hat{D}_{12}(s)\Big)^j\Big],$$
$$\hat{E}_{1i}(s) =\hat{E}_{10}(s)\Big[\hat{B}_{1i}(s)+\hat{B}_{12}(s)\hat{D}_{2i}(s)+\hat{B}_{11}(s)\mu_2\hat{D}_{3i}(s)+\hat{B}_{11}(s)\mu_3\hat{D}_{1i}(s)\Big]\ \text{for } i=1,2,3,4,$$
$$\hat{E}_{15}(s) =\hat{E}_{10}(s)\hat{B}_{10}(s)2\mu_3\frac{b_4^{n-1}\hat{\psi}(s)^n}{a_4},$$

$$
\begin{aligned}
\hat{F}_{10}(s) =&\sum_{j=0}^{\infty}\bigg(\big(\hat{C}_{12}(s)+\hat{D}_{22}(s)+\hat{C}_{13}(s)\hat{D}_{32}(s)+\hat{C}_{14}(s)\hat{D}_{12}(s)+\hat{C}_{15}(s)\big)\big(\hat{E}_{12}(s)\hat{E}_{22}(s)\hat{E}_{33}(s)\\
&+\hat{E}_{12}(s)\hat{E}_{24}(s)+\hat{E}_{13}(s)\hat{E}_{33}(s)+\hat{E}_{15}(s)\big)+\big(\hat{C}_{12}(s)\hat{D}_{23}(s)+\hat{C}_{13}(s)\hat{D}_{33}(s)+\hat{C}_{14}(s)\\
&\times\hat{D}_{13}(s)+\hat{C}_{16}(s)\big)\big(\hat{E}_{22}(s)\hat{E}_{33}(s)+\hat{E}_{24}(s)\big)+\big(\hat{C}_{12}(s)\hat{D}_{24}+\hat{C}_{13}(s)\hat{D}_{34}(s)+\hat{C}_{14}(s)\hat{D}_{14}(s)\\
&+\hat{C}_{17}(s)\big)\hat{E}_{33}(s)\bigg)^{j},\\
\hat{F}_{11}(s) =&\hat{F}_{10}(s)\bigg(\big(\hat{C}_{11}(s)+\hat{D}_{22}(s)\hat{D}_{22}(s)+\hat{C}_{13}(s)\hat{D}_{32}(s)+\hat{C}_{14}(s)\hat{D}_{12}(s)+\hat{C}_{15}(s)\big)\big(\hat{E}_{12}(s)\hat{E}_{22}(s)\hat{E}_{33}(s)\\
&+\hat{E}_{12}(s)\hat{E}_{24}(s)+\hat{E}_{13}(s)\hat{E}_{33}(s)+\hat{E}_{15}(s)\big)+\big(\hat{C}_{12}(s)\hat{D}_{23}(s)+\hat{C}_{13}(s)\hat{D}_{33}(s)+\hat{C}_{14}(s)\hat{D}_{13}(s)\\
&+\hat{C}_{16}(s)\big)\big(\hat{E}_{22}(s)\hat{E}_{33}(s)+\hat{E}_{24}(s)\big)+\big(\hat{C}_{12}(s)\hat{D}_{24}+\hat{C}_{13}(s)\hat{D}_{34}(s)+\hat{C}_{14}(s)\hat{D}_{14}(s)+\hat{C}_{17}(s)\big)\hat{E}_{33}(s)\bigg),\\
\hat{F}_{12}(s) =&\hat{F}_{10}(s)\bigg(\big(\hat{C}_{12}(s)\hat{D}_{22}(s)+\hat{C}_{13}(s)\hat{D}_{32}(s)+\hat{C}_{14}(s)\hat{D}_{12}(s)+\hat{C}_{15}(s)\big)\big(\hat{E}_{12}(s)\hat{E}_{23}(s)+\hat{E}_{12}(s)\hat{E}_{22}(s)\\
&+\hat{E}_{14}(s)\big)+\big(\hat{C}_{12}(s)\hat{D}_{23}(s)+\hat{C}_{13}(s)\hat{D}_{33}(s)+\hat{C}_{14}(s)\hat{D}_{13}(s)+\hat{C}_{16}(s)\big)\big(\hat{E}_{22}(s)\hat{E}_{32}(s)+\hat{E}_{23}(s)\big)\\
&+\big(\hat{C}_{12}(s)\hat{D}_{24}+\hat{C}_{13}(s)\hat{D}_{34}(s)+\hat{C}_{14}(s)\hat{D}_{14}(s)+\hat{C}_{17}(s)\big)\hat{E}_{32}(s)\bigg),\\
\hat{G}_{i1}(s) =&\hat{D}_{i1}(s)+\hat{D}_{i2}(s)\big(\hat{E}_{11}(s)+\hat{E}_{12}(s)\hat{E}_{23}(s)+\hat{E}_{13}(s)\hat{E}_{31}(s)+\hat{E}_{12}(s)\hat{E}_{22}(s)\hat{E}_{31}(s)+\big(\hat{E}_{12}(s)\hat{E}_{22}(s)\\
&\times\hat{E}_{33}(s)+\hat{E}_{12}(s)\hat{E}_{24}(s)+\hat{E}_{13}(s)\hat{E}_{33}(s)+\hat{E}_{15}(s)\big)\hat{F}_{11}(s)\big)\hat{D}_{i3}(s)\big(\hat{E}_{21}(s)\\
&+\hat{E}_{22}(s)\hat{E}_{31}(s)+\big(\hat{E}_{22}(s)\hat{E}_{33}(s)+\hat{E}_{24}(s)\big)\hat{F}_{11}(s)\big)+\hat{D}_{i4}\big(\hat{E}_{31}(s)\\
&+\hat{E}_{33}(s)\hat{F}_{11}(s)\big)\quad\text{for}\quad i=1,2,3,\\
\hat{G}_{i2}(s) =&\hat{D}_{i5}(s)+\hat{D}_{i2}(s)\big(\hat{E}_{12}(s)\hat{E}_{23}(s)+\hat{E}_{12}(s)\hat{E}_{22}(s)\hat{E}_{32}(s)+\hat{E}_{14}(s)\big)\big(\hat{E}_{12}(s)\hat{E}_{22}(s)\hat{E}_{33}(s)\\
&+\big(\hat{E}_{12}(s)\hat{E}_{24}(s)+\hat{E}_{13}(s)\hat{E}_{33}(s)+\hat{E}_{15}(s)\big)\hat{F}_{12}(s)\big)+\hat{D}_{i3}\big(\hat{E}_{22}(s)\hat{E}_{32}(s)+\hat{F}_{23}(s)\\
&+\big(\hat{E}_{22}(s)\hat{E}_{33}(s)\big)+\hat{E}_{24}(s)\big)\hat{F}_{12}(s)+\hat{D}_{i4}(s)\big(\hat{E}_{32}(s)+\hat{E}_{33}(s)\hat{F}_{12}(s)\big)\quad\text{for}\quad i=1,2,3,\\
\hat{G}_{41}(s) =&\hat{E}_{11}(s)+\hat{E}_{12}(s)\hat{E}_{21}(s)+\hat{E}_{13}(s)\hat{E}_{31}(s)+\hat{E}_{12}(s)\hat{E}_{22}(s)\hat{E}_{31}(s)+\big(\hat{E}_{12}(s)\hat{E}_{22}(s)\hat{E}_{33}(s)\\
&+\hat{E}_{12}(s)\hat{E}_{24}(s)+\hat{E}_{13}(s)\hat{E}_{33}(s)+\hat{E}_{15}(s)\big)\hat{F}_{11},\\
\hat{G}_{42}(s) =&\hat{E}_{14}(s)+\hat{E}_{12}(s)\hat{E}_{23}(s)+\hat{E}_{12}(s)\hat{E}_{22}(s)\hat{E}_{32}(s)+\big(\hat{E}_{12}(s)\hat{E}_{22}(s)\hat{E}_{33}(s)+\hat{E}_{12}(s)\hat{E}_{24}(s)\\
&+\hat{E}_{13}(s)\hat{E}_{33}(s)+\hat{E}_{15}(s)\big)\hat{F}_{12}(s),\\
\hat{G}_{51}(s) =&\hat{E}_{21}(s)+\hat{E}_{22}(s)\hat{E}_{31}(s)+\hat{E}_{13}(s)+\big(\hat{E}_{22}(s)\hat{E}_{33}(s)+\hat{E}_{24}(s)\big)\hat{F}_{11},\\
\hat{G}_{52}(s) =&\hat{E}_{22}(s)\hat{E}_{32}(s)+\hat{E}_{23}(s)+\big(\hat{E}_{22}(s)\hat{E}_{33}(s)\hat{E}_{24}(s)\big)\hat{F}_{12}(s),\\
\hat{G}_{61}(s) =&\hat{E}_{31}(s)+\hat{E}_{33}(s)\hat{F}_{11}(s),\\
\hat{G}_{62}(s) =&\hat{E}_{32}(s)+\hat{E}_{33}(s)\hat{F}_{12}(s).
\end{aligned}
$$

Substituting the values of $\hat{P}_{11}(s)$, $\hat{P}_{12}(s)$, $\hat{P}_{13}(s)$, $\hat{P}_{24}(s)$, $\hat{P}_{25}(s)$, $\hat{P}_{26}(s)$, $\hat{P}_{37}(s)$ in (5.3), (5.8) and (5.16)-(5.22) yieds

$$\hat{P}_{0,0}(s) = \frac{1}{(s+\lambda+\alpha)}\Big[\big(1+\mu_1\hat{G}_{11}(s)+\mu_2\hat{G}_{21}(s)+\mu_3\hat{G}_{31}(s)\big)+\big(1+\mu_1\hat{G}_{12}(s)+\mu_2\hat{G}_{22}(s)+\mu_3 \times \hat{G}_{32}(s)\big)\hat{P}_{0,8}(s)\Big], \tag{5.23}$$

$$\hat{P}_{n,0}(s) = \frac{\lambda^n}{(s+\lambda+\alpha)(s+\lambda+\alpha+\theta_1+\theta_2+\theta_3)^n}\Big[\big(1+\mu_1\hat{G}_{11}(s)+\mu_2\hat{G}_{21}(s)+\mu_3\hat{G}_{31}(s)\big) + \big(1+\mu_1\hat{G}_{12}(s)+\mu_2\hat{G}_{22}(s)+\mu_3\hat{G}_{32}(s)\big)\hat{P}_{0,8}(s)\Big], \tag{5.24}$$

$$\hat{P}_{n,1}(s) = \sum_{m=1}^{\infty}\theta_1 l_2\hat{X}_1(s) + \Big[2b_1^{n-1}\frac{\hat{\psi}(s)}{a_1}\Big(\mu_2\hat{G}_{41}(s)+\mu_3\hat{G}_{61}(s)+(\theta_2+\theta_3)\hat{G}_{11}(s)\Big)\Big] + \Big[2b_1^{n-1}\frac{\hat{\psi}(s)}{a_1}\cdot\Big(\mu_2\hat{G}_{42}(s)+\mu_3\hat{G}_{62}(s)+(\theta_2+\theta_3)\hat{G}_{12}(s)\Big) + \sum_{j=0}^{\infty}\frac{\beta\lambda^m}{(s+\lambda+\beta)^m}\hat{X}_1(s)+\frac{2\lambda\beta}{s+\lambda+\beta}\cdot b_1^{n-1}\frac{\hat{\psi}^n(s)}{a_1}\Big]\hat{P}_{0,8}(s), \tag{5.25}$$

$$\hat{P}_{n,2}(s) = \sum_{m=1}^{\infty}\theta_2 l_2\hat{X}_2(s) + \Big[2b_2^{n-1}\frac{\hat{\psi}(s)}{a_2}\Big(\mu_1\hat{G}_{41}(s)+\mu_3\hat{G}_{51}(s)+(\theta_1+\theta_3)\hat{G}_{21}(s)\Big)\Big] + \Big[2b_2^{n-1}\frac{\hat{\psi}(s)^n}{a_1}\Big(\mu_1\cdot\hat{G}_{42}(s)+\mu_3\hat{G}_{52}(s)+(\theta_1+\theta_3)\hat{G}_{22}(s)\Big)\Big]\hat{P}_{0,8}(s), \tag{5.26}$$

$$\hat{P}_{n,3}(s) = \sum_{m=1}^{\infty}\theta_3 l_2\hat{X}_3(s) + \Big[2b_3^{n-1}\frac{\hat{\psi}(s)}{a_2}\Big(\mu_1\hat{G}_{51}(s)+\mu_2\hat{G}_{61}(s)+(\theta_1+\theta_2)\hat{G}_{31}(s)\Big)\Big] + \Big[2b_2^{n-1}\frac{\hat{\psi}(s)^n}{a_1}\cdot\Big(\mu_1\hat{G}_{52}(s)+\mu_2\hat{G}_{62}(s)+(\theta_1+\theta_2)\hat{G}_{32}(s)\Big)\Big]\hat{P}_{0,8}(s), \tag{5.27}$$

$$l_2 = \frac{\lambda^m}{(s+\lambda+\alpha)(s+\lambda+\alpha+\theta_1+\theta_2+\theta_3)^m}\big(1+\mu_1\hat{G}_{11}(s)+\mu_2\hat{G}_{21}(s)+\mu_3\hat{G}_{31}(s)\big) + \big(1+\mu_1\hat{G}_{12}(s)+\mu_2\hat{G}_{22}(s)+\mu_3\hat{G}_{32}(s)\big), \tag{5.28}$$

$$\hat{P}_{n,4}(s) = \sum_{m=2}^{\infty}\Big[\theta_1\theta_2 l_2\Big(\hat{X}_1(s)+\hat{X}_2(s)\Big)\Big]\hat{X}_4(s) + \sum_{m=1}^{\infty}\Big[\Big(\mu_3\hat{G}_{51}(s)+(\theta_1+\theta_3)\hat{G}_{21}(s)\Big)b_2^{n-1}\frac{\hat{\psi}^n(s)}{a_2} + \Big((\mu_1+\mu_2)\hat{G}_{41}+\mu_3\hat{G}_{61}(s)+(\theta_1+\theta_3)\hat{G}_{11}(s)\Big)b_1^{n-1}\frac{\hat{\psi}^n}{a_1}(s)\Big]b_4^{n-1}\frac{\hat{\psi}^n}{a_4}(s) + \Big[\sum_{m=1}^{\infty}\Big[\Big(\mu_3\hat{G}_{52}(s)+(\theta_1+\theta_3)\hat{G}_{22}(s)\Big)b_2^{n-1}\frac{\hat{\psi}^n(s)}{a_2} + \Big((\mu_1+\mu_2)\hat{G}_{42}+\mu_3\hat{G}_{62}(s)+(\theta_1+\theta_3)\hat{G}_{12}(s)\Big)b_1^{n-1}\frac{\hat{\psi}^n}{a_1}(s)\Big]b_4^{n-1}\frac{\hat{\psi}^n}{a_4}(s) + \Big(\frac{\beta\lambda^m}{(s+\lambda+\beta)^m}+\frac{2\lambda\beta\theta_2}{s+\lambda+\beta}b_1^{n-1}\frac{\hat{\psi^n}(s)}{a_1}\Big)\Big]\hat{P}_{0,8}(s), \tag{5.29}$$

$$\hat{P}_{n,5}(s) = \sum_{m=2}^{\infty}\Big[\theta_2\theta_3 l_2\Big(\hat{X}_2(s)+\hat{X}_3(s)\Big)\Big]\hat{X}_5(s) + \sum_{m=1}^{\infty}\Big[\Big(\mu_1\hat{G}_{61}(s)+(\theta_1+\theta_2)\hat{G}_{31}(s)\Big)b_3^{n-1}\frac{\hat{\psi}^n(s)}{a_3} + \Big((\mu_2+\mu_3)\hat{G}_{51}+\mu_1\hat{G}_{41}(s)+(\theta_1+\theta_2)\hat{G}_{21}(s)\Big)b_2^{n-1}\frac{\hat{\psi}^n}{a_2}(s)\Big]b_5^{n-1}\frac{\hat{\psi}^n}{a_5}(s), \tag{5.30}$$

$$+\sum_{m=1}^{\infty}\bigg[\Big(\mu_1\hat{G}_{62}(s)+(\theta_1+\theta_2)\hat{G}_{32}(s)\Big)b_3^{n-1}\frac{\hat{\psi}^n(s)}{a_3}$$

$$+\Big((\mu_2+\mu_3)\hat{G}_{42}+\mu_1\hat{G}_{42}(s)+(\theta_1+\theta_2)\hat{G}_{22}(s)\Big)\cdot b_2^{n-1}\frac{\hat{\psi}^n}{a_2}(s)\bigg]b_5^{n-1}\frac{\hat{\psi}^n}{a_5}(s)\hat{P}_{0,8}(s), \tag{5.31}$$

$$\hat{P}_{n,6}(s)=\sum_{m=2}^{\infty}\bigg[\theta_1\theta_3 l_2\Big(\hat{X}_1(s)+\hat{X}_3(s)\Big)\bigg]\hat{X}_6(s)+\sum_{m=1}^{\infty}\bigg[\Big(\mu_2\hat{G}_{41}(s)+(\theta_2+\theta_3)\hat{G}_{11}(s)\Big)b_1^{n-1}\frac{\hat{\psi}^n(s)}{a_1}$$

$$+\Big((\mu_1+\mu_3)\hat{G}_{61}+\mu_2\hat{G}_{51}(s)+(\theta_2+\theta_3)\hat{G}_{31}(s)\Big)b_3^{n-1}\frac{\hat{\psi}^n}{a_3}(s)\bigg]b_6^{n-1}\frac{\hat{\psi}^n}{a_6}(s)$$

$$+\bigg[\sum_{m=1}^{\infty}\bigg[\Big(\mu_2\hat{G}_{42}(s)+(\theta_2+\theta_3)\hat{G}_{12}(s)\Big)b_1^{n-1}\frac{\hat{\psi}^n(s)}{a_1}$$

$$+\Big((\mu_1+\mu_3)\hat{G}_{62}+\mu_2\hat{G}_{52}(s)+(\theta_2+\theta_3)\hat{G}_{32}(s)\Big)b_3^{n-1}\frac{\hat{\psi}^n}{a_3}(s)\bigg]b_6^{n-1}\frac{\hat{\psi}^n}{a_6}(s),$$

$$+\bigg(\frac{\beta\lambda^m}{(s+\lambda+\beta)^m}+\frac{2\lambda\beta\theta_2}{s+\lambda+\beta}b_1^{n-1}\frac{\hat{\psi^n(s)}}{a_1}\bigg)\bigg]\hat{P}_{0,8}(s) \tag{5.32}$$

$$\hat{P}_{n,7}(s)=\sum_{m=2}^{\infty}\bigg[\theta_1\theta_2\theta_3 l_2\Big(\big(\hat{X}_1(s)+\hat{X}_2(s)\big)\hat{X}_4(s)+\big(\hat{X}_2(s)+\hat{X}_3(s)\big)\hat{X}_5(s)\big(\hat{X}_1(s)+\hat{X}_3(s)\big)\hat{X}_6(s)\Big)\bigg]\hat{X}_7(s)$$

$$+\sum_{m=1}^{\infty}\bigg[\theta_3\bigg[\Big(\mu_3\hat{G}_{51}(s)+(\theta_1+\theta_3)\hat{G}_{21}(s)\Big)b_2^{n-1}\frac{\hat{\psi}^n(s)}{a_2}$$

$$+\Big((\mu_1+\mu_2)\hat{G}_{41}+\mu_3\hat{G}_{61}(s)+(\theta_1+\theta_3)\hat{G}_{11}(s)\Big)\cdot b_1^{n-1}\frac{\hat{\psi}^n}{a_1}(s)\bigg]b_4^{n-1}\frac{\hat{\psi}^n}{a_4}(s)$$

$$+\theta_1\bigg[\Big(\mu_1\hat{G}_{61}(s)+(\theta_1+\theta_2)\hat{G}_{31}(s)\Big)b_3^{n-1}\frac{\hat{\psi}^n(s)}{a_3}$$

$$+\Big((\mu_2+\mu_3)\hat{G}_{51}+\mu_1\cdot\hat{G}_{41}(s)+(\theta_1+\theta_2)\hat{G}_{21}(s)\Big)\cdot b_2^{n-1}\frac{\hat{\psi}^n}{a_2}(s)\bigg]b_5^{n-1}\frac{\hat{\psi}^n}{a_5}(s)$$

$$+\theta_2\bigg[\Big(\mu_2\hat{G}_{41}(s)+(\theta_2+\theta_3)\hat{G}_{11}(s)\Big)b_1^{n-1}\frac{\hat{\psi}^n(s)}{a_1}$$

$$+\Big((\mu_1+\mu_3)\hat{G}_{61}+\mu_2\hat{G}_{51}(s)+(\theta_2+\theta_3)\hat{G}_{31}(s)\Big)b_3^{n-1}\frac{\hat{\psi}^n}{a_3}(s)\bigg]b_6^{n-1}\frac{\hat{\psi}^n}{a_6}(s)\bigg]$$

$$+\bigg[\sum_{m=1}^{\infty}\bigg[\theta_3\bigg[\Big(\mu_3\hat{G}_{52}(s)+(\theta_1+\theta_3)\hat{G}_{22}(s)\Big)b_2^{n-1}\frac{\hat{\psi}^n(s)}{a_2}$$

$$+\Big((\mu_1+\mu_2)\hat{G}_{42}+\mu_3\hat{G}_{62}(s)+(\theta_1+\theta_3)\hat{G}_{12}(s)\Big)b_1^{n-1}\frac{\hat{\psi}^n}{a_1}(s)\bigg]b_4^{n-1}\frac{\hat{\psi}^n}{a_4}(s)$$

$$+\theta_1\bigg[\Big(\mu_1\hat{G}_{62}(s)+(\theta_1+\theta_2)\hat{G}_{32}(s)\Big)b_3^{n-1}\frac{\hat{\psi}^n(s)}{a_3}+\Big((\mu_2+\mu_3)\hat{G}_{42}+\mu_1\hat{G}_{42}(s)+(\theta_1+\theta_2)\hat{G}_{22}(s)\Big)$$

$$\times b_2^{n-1}\frac{\hat{\psi}^n}{a_2}(s)\bigg]b_5^{n-1}\frac{\hat{\psi}^n}{a_5}(s)+\theta_2\bigg[\Big(\mu_2\hat{G}_{42}(s)+(\theta_2+\theta_3)\hat{G}_{12}(s)\Big)b_1^{n-1}\frac{\hat{\psi}^n(s)}{a_1}$$

$$+\Big((\mu_1+\mu_3)\hat{G}_{62}+\mu_2\hat{G}_{52}(s)+(\theta_2+\theta_3)\hat{G}_{32}(s)\Big)b_3^{n-1}\frac{\hat{\psi}^n}{a_3}(s)\bigg]b_6^{n-1}\frac{\hat{\psi}^n}{a_6}(s)$$

$$+2\bigg(\frac{\beta\lambda^m}{(s+\lambda+\beta)^m}+\frac{2\lambda\beta\theta_2}{s+\lambda+\beta}b_1^{n-1}\frac{\hat{\psi^n(s)}}{a_1}\bigg)\bigg]\bigg]\hat{P}_{0,8}(s), \tag{5.33}$$

Inverting equations (5.5), (5.23)-(5.33) leads to,

$$P_{0,1}(t) = \beta e^{-(\lambda+\alpha)t} P_{0,5}(t),$$

$$P_{0,0}(t) = e^{-(\lambda+\alpha)t}\Big(\delta(t) + \mu_1 G_{11}(t) + \mu_2 G_{21}(t) + \mu_3 G_{31}(t) + \big(1 + \mu_1 G_{12}(t) + \mu_2 G_{22}(t) + \mu_3 G_{32}(t)\big) P_{0,8}(t)\Big),$$

$$P_{n,0}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-(\lambda+\alpha+\theta_1+\theta_2+\theta_3)t} e^{-(\lambda+\alpha)t}\Big(\delta(t) + \mu_1 G_{11}(t) + \mu_2 G_{21}(t) + \mu_3 G_{31}(t) + \big(1 + \mu_1 G_{12}(t) + \mu_2 G_{22}(t) + \mu_3 G_{32}(t)\big) P_{0,8}(t)\Big),$$

$$\hat{P}_{n,1}(t) = \sum_{m=1}^{\infty} \theta_1 l_3 X_1(t) + \Big[2b_1^{n-1}\frac{\psi(t)}{a_1}\Big(\mu_2 \hat{G}_{41}(t) + \mu_3 \hat{G}_{61}(t) + (\theta_2+\theta_3)\hat{G}_{11}(t)\Big)\Big] + \Big[2b_1^{n-1}\frac{\hat{\psi}(t)}{a_1}\Big(\mu_2 \hat{G}_{42}(t) + \mu_3 \hat{G}_{62}(t) + (\theta_2+\theta_3)\hat{G}_{12}(t)\Big) + \sum_{j=0}^{\infty}\frac{\beta\lambda^m t^{m-1}}{(m-1)!^m} e^{-(\lambda+\beta)t} X_1(t) + 2\lambda\beta e^{-(\lambda+\beta)t} b_1^{n-1}\frac{\psi^n(t)}{a_1}\Big]\hat{P}_{0,8}(t),$$

$$\hat{P}_{n,2}(t) = \sum_{m=1}^{\infty} \theta_2 l_3 X_2(t) + \Big[2b_2^{n-1}\frac{\psi(t)}{a_2} * \Big(\mu_1 \hat{G}_{41}(t) + \mu_3 \hat{G}_{51}(t) + (\theta_1+\theta_3)\hat{G}_{21}(t)\Big)\Big] + \Big[2b_2^{n-1}\frac{\psi(t)^n}{a_1}\Big(\mu_1 \hat{G}_{42}(t) + \mu_3 \hat{G}_{52}(t) + (\theta_1+\theta_3)\hat{G}_{22}(t)\Big)\Big]\hat{P}_{0,8}(t),$$

$$\hat{P}_{n,3}(t) = \sum_{m=1}^{\infty} \theta_3 l_3 \hat{X}_3(s) + \Big[2b_3^{n-1}\frac{\psi(t)}{a_2} * \Big(\mu_1 \hat{G}_{51}(t) + \mu_2 \hat{G}_{61}(t) + (\theta_1+\theta_2)\hat{G}_{31}(t)\Big)\Big] + \Big[2b_2^{n-1}\frac{\psi(t)^n}{a_1}\Big(\mu_1 \hat{G}_{52}(t) + \mu_2 \hat{G}_{62}(t) + (\theta_1+\theta_2)\hat{G}_{32}(t)\Big)\Big]\hat{P}_{0,8}(t),$$

$$\hat{P}_{n,4}(t) = \sum_{m=2}^{\infty}\Big[\theta_1\theta_2 l_3\Big(X_1(t) + X_2(t)\Big)\Big]X_4(t) + \sum_{m=1}^{\infty}\Big[\Big(\mu_3 G_{51}(t) + (\theta_1+\theta_3)G_{21}(t)\Big) b_2^{n-1}\frac{\hat{\psi}^n(t)}{a_2} + \Big((\mu_1+\mu_2)G_{41}(t) + \mu_3 G_{61}(t) + (\theta_1+\theta_3)G_{11}(t)\Big) b_1^{n-1}\frac{\psi^n}{a_1}(t)\Big] b_4^{n-1}\frac{\psi^n}{a_4}(s) + \Big[\sum_{m=1}^{\infty}\Big[\Big(\mu_3 G_{52}(t) + (\theta_1+\theta_3)G_{22}(t)\Big) b_2^{n-1}\frac{\psi^n(t)}{a_2} + \Big((\mu_1+\mu_2)G_{42} + \mu_3 G_{62}(t) + (\theta_1+\theta_3)G_{12}(t)\Big) b_1^{n-1}\frac{\psi^n}{a_1}(s)\Big] b_4^{n-1}\frac{\psi^n}{a_4}(t) + \frac{\beta\lambda^m t^{m-1}}{(m-1)!} e^{-(\lambda+\beta)t} + 2\lambda\beta\theta_2 \times e^{-(\lambda+\beta)t} b_1^{n-1}\frac{\hat{\psi^n}(t)}{a_1}\Big]\hat{P}_{0,8}(t),$$

$$\hat{P}_{n,5}(t) = \sum_{m=2}^{\infty}\Big[\theta_2\theta_3 l_3\Big(X_2(t) + X_3(t)\Big)\Big]X_5(t) + \sum_{m=1}^{\infty}\Big[\Big(\mu_1 \hat{G}_{61}(t) + (\theta_1+\theta_2)\hat{G}_{31}(t)\Big) b_3^{n-1}\frac{\hat{\psi}^n(t)}{a_3} + \Big((\mu_2+\mu_3)\hat{G}_{51}(t) + \mu_1 \hat{G}_{41}(t) + (\theta_1+\theta_2)\hat{G}_{21}(t)\Big) b_2^{n-1}\frac{\hat{\psi(t)}^n}{a_2}(t)\Big] b_5^{n-1}\frac{\hat{\psi(t)}^n}{a_5}(t)$$

$$
\begin{aligned}
&+\sum_{m=1}^{\infty}\bigg[\Big(\mu_1\hat{G}_{62}(t)+(\theta_1+\theta_2)\hat{G}_{32}(t)\Big)b_3^{n-1}\frac{\hat{\psi}^n(t)}{a_3}\\
&+\Big((\mu_2+\mu_3)\hat{G}_{42}(t)+\mu_1\hat{G}_{42}(t)+(\theta_1+\theta_2)\hat{G}_{22}(t)\Big)b_2^{n-1}\frac{\hat{\psi}^n}{a_2}(t)\bigg]b_5^{n-1}\frac{\hat{\psi}^n}{a_5}(t)\hat{P}_{0,8}(t),\\
\hat{P}_{n,6}(t)=&\sum_{m=2}^{\infty}\bigg[\theta_1\theta_3 l_3\Big(X_1(t)+X_3(t)\Big)\bigg]X_6(t)+\sum_{m=1}^{\infty}\bigg[\Big(\mu_2\hat{G}_{41}(t)+(\theta_2+\theta_3)\hat{G}_{11}(t)\Big)b_1^{n-1}\frac{\hat{\psi}^n(t)}{a_1}\\
&+\Big((\mu_1+\mu_3)\hat{G}_{61}(t)+\mu_2\hat{G}_{51}(t)+(\theta_2+\theta_3)\hat{G}_{31}(t)\Big)b_3^{n-1}\frac{\hat{\psi}^n}{a_3}(t)\bigg]b_6^{n-1}\frac{\hat{\psi}^n}{a_6}(t)\\
&+\bigg[\sum_{m=1}^{\infty}\bigg[\Big(\mu_2\hat{G}_{42}(t)+(\theta_2+\theta_3)\hat{G}_{12}(t)\Big)b_1^{n-1}\frac{\hat{\psi}^n(t)}{a_1}\\
&+\Big((\mu_1+\mu_3)\hat{G}_{62}(t)+\mu_2\hat{G}_{52}(t)+(\theta_2+\theta_3)\hat{G}_{32}(t)\Big)b_3^{n-1}\frac{\hat{\psi}^n}{a_3}(t)\bigg]b_6^{n-1}\\
&\times\frac{\hat{\psi}^n}{a_6}(t)+\frac{\beta\lambda^m t^{m-1}}{(m-1)!}e^{-(\lambda+\beta)t}+2\lambda\beta\theta_2 e^{-(\lambda+\beta)t}b_1^{n-1}\frac{\hat{\psi^n(t)}}{a_1}\bigg]\hat{P}_{0,8}(s),\\
\hat{P}_{n,7}(t)=&\sum_{m=2}^{\infty}\bigg[\theta_1\theta_2\theta_3 l_3\Big(\big(X_1(t)+X_2(t)\big)X_4(t)+\big(X_2(t)+X_3(t)\big)X_5(t)\big(X_1(t)+X_3(t)\big)X_6(t)\Big)\bigg]\\
&\times\hat{X}_7(t)+\sum_{m=1}^{\infty}\bigg[\theta_3\bigg[\Big(\mu_3 G_{51}(t)+(\theta_1+\theta_3)G_{21}(t)\Big)b_2^{n-1}\frac{\psi^n(t)}{a_2}+\Big((\mu_1+\mu_2)G_{41}(t)+\mu_3 G_{61}(t)\\
&+(\theta_1+\theta_3)G_{11}(t)\Big)b_1^{n-1}\frac{\psi^n}{a_1}(t)\bigg]b_4^{n-1}\frac{\psi^n}{a_4}(t)+\theta_1\bigg[\Big(\mu_1 G_{61}(t)+(\theta_1+\theta_2)G_{31}(t)\Big)b_3^{n-1}\frac{\psi^n(s)}{a_3}\\
&+\Big((\mu_2+\mu_3)G_{51}(t)+\mu_1 G_{41}(t)+(\theta_1+\theta_2)G_{21}(t)\Big)b_2^{n-1}\frac{\psi^n}{a_2}(t)\bigg]b_5^{n-1}\frac{\psi^n}{a_5}(t)+\theta_2\bigg[\Big(\mu_2 G_{41}(t)+(\theta_2\\
&+\theta_3)G_{11}(t)\Big)b_1^{n-1}\frac{\psi^n(t)}{a_1}+\Big((\mu_1+\mu_3)G_{61}(t)\\
&+\mu_2 G_{51}(t)+(\theta_2+\theta_3)G_{31}(t)\Big)b_3^{n-1}\frac{\psi^n}{a_3}(t)\bigg]b_6^{n-1}\frac{\hat{\psi}^n}{a_6}(t)\bigg]+\bigg[\sum_{m=1}^{\infty}\bigg[\theta_3\bigg[\Big(\mu_3 G_{52}(t)+(\theta_1+\theta_3)\\
&\times G_{22}(t)\Big)b_2^{n-1}\frac{\psi^n(t)}{a_2}+\Big((\mu_1+\mu_2)G_{42}(t)+\mu_3 G_{62}(t)+(\theta_1+\theta_3)G_{12}(t)\Big)b_1^{n-1}\frac{\psi^n}{a_1}(t)\bigg]b_4^{n-1}\\
&\times\frac{\psi^n}{a_4}(t)+\theta_1\bigg[\Big(\mu_1 G_{62}(t)+(\theta_1+\theta_2)G_{32}(t)\Big)b_3^{n-1}\frac{\psi^n(s)}{a_3}+\Big((\mu_2+\mu_3)G_{42}(t)+\mu_1 G_{42}(t)\\
&+(\theta_1+\theta_2)G_{22}(t)\Big)b_2^{n-1}\frac{\psi^n}{a_2}(t)\bigg]b_5^{n-1}\frac{\psi^n}{a_5}(t)\\
&+\theta_2\bigg[\Big(\mu_2 G_{42}(t)+(\theta_2+\theta_3)G_{12}(t)\Big)b_1^{n-1}\frac{\psi^n(t)}{a_1}+\Big((\mu_1+\mu_3)G_{62}(t)+\mu_2 G_{52}(t)+(\theta_2+\theta_3)G_{32}(t)\Big)\cdot\\
&\cdot b_3^{n-1}\frac{\psi^n}{a_3}(t)\bigg]b_6^{n-1}\frac{\psi(t)^n}{a_6}+2\bigg(\frac{\beta\lambda^m t^{m-1}}{(m-1)!}e^{-(\lambda+\beta)t}+2\lambda\beta\theta_2 e^{-(\lambda+\beta)t}b_1^{n-1}\frac{\hat{\psi^n(t)}}{a_1}\bigg)\bigg]\bigg]\hat{P}_{0,8}(s),\\
l_3=&\frac{\theta_1\lambda^m t^m}{(m-1)!}e^{-(\lambda+\alpha+\theta_1+\theta_2+\theta_3)t}e^{-(\lambda+\alpha)t}\Big(\delta(t)+\mu_1 G_{11}(t)+\mu_2 G_{21}(t)+\mu_3 G_{31}(t)\\
&+\big(1+\mu_1 G_{12}(t)+\mu_2 G_{22}(t)+\mu_3 G_{32}(t)\big)P_{0,8}(t)\Big).
\end{aligned}
$$

Here all the probabilities are purely expressed in terms of $\hat{P}_{0,8}(t)$. Using (2.18) we can find $\hat{P}_{0,8}(t)$ in the following manner: substituting $\hat{P}_{n,8}(s)$ in (5.6) we get

$$\hat{P}_{0,8}(s) = \frac{\alpha}{s(s+\lambda+\alpha+\beta)} - \frac{\alpha}{s+\lambda+\alpha+\beta}\sum_{n=1}^{\infty}\frac{\lambda^n}{(s+\lambda+\beta)^n}\hat{P}_{0,8}(s).$$

On inversion, we obtain

$$P_{0,8}(t) = \left[\sum_{j=0}^{\infty} - \left(\alpha e^{-(\lambda+\alpha+\beta)t}\sum_{n=1}^{\infty}\frac{\lambda^n t^{n-1}e^{-(\lambda+\beta)t}}{(n-1)!}\right)^j\right]\left[\frac{\alpha}{\lambda+\alpha+\beta}\left(1-e^{-(\lambda+\alpha+\beta)t}\right)\right].$$

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