

ANALYSIS OF A MAY-HOLLING-TANNER RATIO-DEPENDENT PREDATOR-PREY MODEL WITH AN ALTERNATIVE FOOD SOURCE FOR THE PREDATOR AND A STRONG ALLEE EFFECT FOR THE PREY

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ABSTRACT. This paper presents a detailed analysis of a bitrophic chain to understand the complex ecological behavior applied to this specific model. The model is described by means of a two-dimensional system of ordinary differential equations. The existence and uniqueness of the solutions of this system are examined, as well as their boundedness and positivity. In addition, through a differentiable equivalence, conditions for local and global stability at biologically relevant critical points are established. Periodic solutions are also explored. Finally, the Python programming language is used to perform a quantitative analysis of these critical points, showing different scenarios of the qualitative analysis previously obtained.

1. INTRODUCTION

Allee effects are named after the American ecologist Warder Clyde Allee, who studied the benefits of cooperative behaviour in small populations [1, 2]. Although Allee did not provide a formal definition of the Allee effect, it was later defined by [48] as a positive relationship between a component of individual adaptation and the number or density of conspecifics. The former can be interpreted as follows: the adaptation of an individual in a small population decreases as the population size also decreases; for more information, see [48, 26]. This phenomenon has been given various names, such as depensation, negative competition effect, inverse or positive density dependence, underpopulation, allelocatalysis, etc. It should be noted that the term depensation is used mainly in fisheries sciences, while the Allee effect is more common in biology and ecology [31].

The Allee effect can be classified into component Allee effect and demographic Allee effect. The former represents the positive relationship between any measurable component of individual fitness and population density. Although the demographic Allee effect is the positive relationship between overall individual fitness and population density. A demographic Allee effect evidences the existence of at least one component Allee effect, while a component Allee effect does not always generate a demographic Allee effect.

Depending on the ecological impact of a demographic Allee effect, two types are distinguished: strong Allee effect and the weak Allee effect. The strong Allee effect is a demographic Allee effect with a critical population size or density. The weak Allee effect is a demographic Allee effect without a critical population size or density. The difference is whether the population in question has a critical population size or density. A population with a weak Allee effect will have a reduced per capita growth rate. However, even with this low population size or density, the population will always show a positive

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per capita growth rate. On the other hand, a population with a strong Allee effect will have a critical population size or density below which the population growth rate will be negative. Therefore, when the population density or size reaches a number below this threshold, the population will be destined to extinction without further help [13].

Some populations may exhibit the Allee effect due to a wide range of biological phenomena, such as reduced vigilance for predators, social thermoregulation, difficulty in mating, and poor feeding at low densities. However, there are other causes that can lead to this phenomenon. For more information, see Table 1 in [8] or Table 2.1 in [13]. The most common mathematical way to describe the Allee effect is the following non-linear differential equation.

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right) (x - m), \quad (1.1)$$

where x and y represent the density of the prey and the predator at time t . In addition, r represents the intrinsic growth rate of the prey population. k denotes the carrying capacity of the prey population. m describes the extinction threshold associated with the strong Allee effect affecting the prey population. Some precedents of the application of the strong Allee effect on ecological mathematical models, not necessarily related to the research style employed throughout this work, are presented below. In [53], a reaction-diffusion model is derived for a population in a bounded one-dimensional habitat, where the population is subject to the strong Allee effect in its natural domain, but obeys a logistic growth in a protection zone. The authors established threshold conditions for population persistence and extinction. In addition, they obtained the influences of the protection zone on long-term population dynamics and proposed strategies to design the optimal location of the protection zone for long-term population persistence. In [14], the complex dynamics of a diffusive predator-prey model, in which the prey is subjected to a strong Allee effect and threshold harvesting, are investigated. The existence and stability of nonnegative steady-state constant steady-state solutions are discussed. The existence and non-existence of non-constant positive steady-state solutions is analysed to identify the ranges of model training parameters. The existence of spatially homogeneous and inhomogeneous Hopf bifurcations and discontinuous Hopf bifurcations is demonstrated.

This paper, [52], is devoted to the study of the dynamical properties and stationary patterns of a diffusive Leslie-Gower prey-predator model with a strong Allee effect on the prey population. First, the authors analyse the nonnegative constant solutions and their stabilities. They also study the dynamical properties of time-dependent solutions. In addition, they investigated stationary patterns induced by diffusions (Turing pattern). In [41], the dynamics of a diffusive predator-prey model with modified Leslie-Gower term and a strong Allee effect on the prey under a homogeneous Neumann boundary condition is considered. First, the authors derive the qualitative properties of the system, including the existence of the global positive solution and the local and global asymptotic stability of the constant equilibria. In addition, they investigate the a priori estimation and non-existence of non-constant positive steady-state solutions. Finally, they establish the existence and local structure of steady-state patterns and time-period patterns for the system.

In the following, some precedents of the application of the strong Allee effect are presented, besides belonging to the same line of research, i.e. models in EDOS. In the paper [54], a Leslie-type predator-prey system with simplified Holling type IV functional response and strong Allee effect on the prey is proposed. The research emphasizes the exploration of bifurcation. In detail, it is shown that several points of positive non-hyperbolic equilibria exist in the system, such as a weak focus of multiplicities one and two, saddle nodes (degenerate) and Bogdanov-Takens singularities (cusp case) of codimensions 2 and 3. In these equilibria, the existence of a bifurcation in the system is demonstrated. Furthermore, it is shown that the system undergoes several types of bifurcations, such as saddle-node bifurcation,

Hopf bifurcation, degenerate Hopf bifurcation, and Bogdanov-Takens bifurcation of codimensions 2 and 3. Also, it is shown that the system has a codimension cusp of at least 4. Therefore, there may be three limit cycles generated from a Hopf bifurcation of codimension 3. In [38], a predator-prey model with Holling type II functional response incorporating the Allee effect and the fear effect in the prey is analyzed. The model includes intraspecific competition between predators. In addition, the authors find out the local dynamics as well as the Hopf bifurcation, considering the level of fear as a bifurcation parameter.

In [15], the authors study a predator-prey system with strong Allee effect in the prey and intraspecific competition in the predator. They further divided the parameter space into several mutually exclusive regions, and for each region gave a fairly complete description of the global dynamics, including equilibrium stability, Hopf bifurcation and its directions, the existence of a heteroclinic orbital loop, a homoclinic orbital loop, and limit cycles. In [39], the authors propose a Leslie-Gower predator-prey system with a strong Allee effect on prey and hunting cooperation among predators. To analyze the impacts of the Allee effect and hunting cooperation, the authors chose the severity of the Allee effect and the coefficient of hunting cooperation as the main control parameters. The stability of all non-negative equilibrium points, including the type of degenerate equilibrium point, is discussed. Based on Sotomayor's theorem, the existence of saddle type bifurcation is deduced. To determine the stability of the limit cycles derived from the Hopf bifurcation, the first Lyapunov number is calculated. Meanwhile, we obtain that the bifurcation limit cycle is stable if the hunting cooperation coefficient and the severity of the Allee effect are sufficiently small. These results show that strong Allee effect and hunting cooperation have significant impacts on the dynamic behaviors of the system.

In [3], the authors analyse a Holling-Tanner predator-prey model with a Holling type II functional response with strong Allee effect associated with the production of the prey species. The extended model exhibits rich dynamics. The authors prove the existence of separatrices in the phase plane that separate basins of attraction related to co-existence and extinction of species. They also demonstrate the existence of a homoclinic curve that degenerates to form a limit cycle and discuss numerous potential bifurcations, such as node-chair, Hopf, and Bogdanov-Takens bifurcations. In the article [20], the study is conducted based on a Leslie-Gower-type predation model, assuming that the prey population is affected by a strong Allee effect and that predators have access to alternative food. The functional response will be assumed to be linear, which is prey dependent and increases monotonically. Among the most important results obtained is that for the same set of parameters, there are different behaviours of the system solutions, since there can be two attractor equilibrium points. Then, the populations can co-exist around fixed population sizes, or the prey population can become extinct. In this article [32], the authors examine the implications of repeated human introductions of invasive species and determine the benefits of policies that reduce introductions. Using the emerald ash borer (EAB) in Ohio as an example and modelling the relationship between this invasive species and the native environment, as well as economic activity in Ohio. The authors demonstrated that when a strong Allee effect is taken into account, the EAB population can be managed to remain below endemic levels; ash tree decline can be slowed. Understanding the interaction between human activity, repeated introductions, and the Allee effect can help policymakers effectively reduce the establishment and spread of invasive species.

Holling proposed, in [23], three types of functional response, which have been extensively studied by different researchers, as in [50, 34, 40, 43], especially Holling's type II functional response, as in [16, 40, 43]. Many biologists have questioned prey-dependent predator-prey models, believing that functional responses should depend on both prey and predator density. Over time, support for this view has become stronger due to biological evidence, as in [4, 21]. Arditi and Ginzburg proposed in [4] a mixed functional response, that is, one that is dependent on both the prey and the predator. This is

called a ratio-dependent functional response, which attempts to modify any prey or predator-dependent functional response $h(x)$ to $h(x/y)$. In [49], JT Tanner proposed the Holling-Tanner [5] or May-Holling-Tanner model [17, 45]. This model is based on the Leslie-Gower-type model [49] or the predator-prey logistic model [6], based on the Leslie-Gower scheme [29] proposed by the ecologist Patrick H. Leslie in 1948 [28]. The importance of the May-Holling-Tanner model was highlighted by Colling in 1997 [12], where he states that this model allows avoiding the paradox of biological control. It is a concept that arises in the field of ecology and pest management.

In summary, the biological control paradox highlights the complexities and unpredictable interactions that can arise when attempting to manage pest populations through the introduction of predatory organisms. Although biological control can be an effective strategy in many cases, before introducing new organisms into an ecosystem, it is crucial to thoroughly investigate and carefully consider the potential consequences. The May-Holling-Tanner model [45, 51, 27], is partially studied in [5] with delay in [46], with proportion-dependent functional response in [30], and with an alternative food source for the predator in [18, 19, 44, 43].

In this work we propose to introduce the strong Allee effect in the model proposed in [43] in order to analyze the implications that this entails. That is, the strong Allee effect is applied to a mathematical predator-prey model, which is conformed by the May-Holling-Tanner model with mixed reason-dependent functional response and alternative food for the predator. This combination of different elements models an ecosystem composed of a prey species, which is the preferred food of the predator species. The population growth of the prey species is limited by the amount of food the ecosystem can supply. In addition, there is a critical minimum threshold density below which the population of the prey species experiences significant difficulties in surviving or reproducing. Whereas if it increases above the threshold density, survival and reproduction rates improve significantly. This threshold density is known as *Allee's threshold density*. Finally, the population growth of the predatory species is limited by the amount of available prey, with the addition of alternative food sources. In other words, in this paper we propose to perturb the proposed model in [43] by incorporating the strong Allee effect on prey. This paper deals with the study of the May-Holling-Tanner model, which corresponds to the Holling-Tanner model with Holling type II functional response, previously studied in [45]. The purpose of this research is to analyze the ecological complex behavior of the proposed case.

The present study consists of a total of eight sections, excluding the introduction, summary and references. Starting with the presentation of the system of ordinary differential equations, which represents the model under study, with the biological meaning of each of the variables and parameters in use, in section *Mathematical model*, followed by section *Positivity, boundedness, existence and uniqueness of the solutions*, these results ensure that the solutions of the system do not leave the region linked to the system, that is, the populations involved do not grow infinitely, but up to a limit. Then, in the section *Topological Equivalence*, a system equivalent to the original, but with greater ease of analysis, is presented. The next section is called *Critical points*, in which all the critical points of the equivalent system are searched and it is ensured that they are positive in order to make biological sense. Subsequently, a local and global stability analysis is applied in the *Qualitative analysis* section. The local stability analysis is achieved by means of the classical linearization and according to what is obtained, the Lyapunov function or the Chetaev criterion is applied for the global stability analysis; the reasons for these decisions are explained at the beginning of this section. Likewise, the existence or non-existence of closed orbits is located in the section *Periodic solutions*, and the reasons for this are explained in the same section. It is followed by the section *Numerical analysis*, where the simulations and the biological interpretation of the simulations, which favor the ecological balance, are presented. Finally, in section

Conclusions, the conclusions of each of the previous sections are presented, in a general way and ending with possible applications, future works and limitations of the model under study.

2. MATHEMATICAL MODEL

The case described is represented in the following system of autonomous two-dimensional ordinary differential equations.

$$X_\sigma(x, y) : \begin{cases} \frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right) (x - m) - \frac{qxy}{x + ay}, \\ \frac{dy}{dt} = y \left(1 - \frac{y}{nx + c}\right) s, \end{cases} \tag{2.1}$$

where x and y represent the density of the prey and the predator at time t . In addition, r represents the intrinsic growth rate of the prey population. This parameter captures the prey’s ability to grow under ideal conditions without external constraints. k denotes the carrying capacity of the prey population, which is the maximum population size the environment can sustainably support, considering the available resources. m describes the extinction threshold associated with the strong Allee effect affecting the prey population; this threshold represents the minimum population density required to avoid extinction due to factors such as difficulties in reproduction or survival at low densities. q represents the intrinsic growth rate of the prey; a is the degree to which the environment provides protection to the prey. In addition, s represents the intrinsic growth rate of the predator; n measures the quality of the prey as food for the predator, while the parameter c describes the size of the alternative food for the predator. The above leads to define the following parameter vector $\sigma = (r, k, m, q, a, s, n, c) \in \mathbb{R}_+^8$. Thus, the system (2.1) is defined in the following region.

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x; 0 < y\}. \tag{2.2}$$

The proof of the existence and uniqueness of the solutions are given in the next section.

3. POSITIVITY, BOUNDEDNESS, EXISTENCE AND UNIQUENESS OF SOLUTIONS

Since we are working on an ecological-biological model, the positivity and boundedness of the solutions to the problem posed in (2.1) will be demonstrated. This ensures that the population does not increase indefinitely.

Definition 3.1. Let $\Omega \subset \mathbb{R}^n$ and $(x(t), y(t))$ be a solution of the following system:

$$\begin{cases} x' = F_1(x, y), \\ y' = F_2(x, y), \end{cases} \tag{3.1}$$

Ω is said to be invariant for the system (3.1) if $(x(0), y(0)) \in \Omega$, then $(x(t), y(t)) \in \Omega$ for all $t \geq 0$.

Proposition 3.1. *The region Ω is invariant by the system (2.1).*

Proof. Let $(x(0), y(0)) \in \Omega$ the initial point of the initial value problem or the initial value problem associated with the system (2.1).

$$\begin{cases} x'(t) = \left(r \left(1 - \frac{x(t)}{k}\right) (x(t) - m) - \frac{qy(t)}{x(t) + ay(t)} \right) x(t), \\ y'(t) = \left(s - \frac{y(t)s}{nx(t) + c} \right) y(t), \\ x(0) > 0, y(0) > 0, \end{cases}$$

solving the previous system, we obtain:

$$\begin{aligned} x(t) &= x(0) \exp \left\{ \int_0^t \left(r \left(1 - \frac{x(\iota)}{k} \right) (x(\iota) - m) - \frac{qy(\iota)}{x(\iota) + ay(\iota)} \right) d\iota \right\} > 0, \\ y(t) &= y(0) \exp \left\{ \int_0^t \left(s - \frac{y(\iota)s}{nx(\iota) + c} \right) d\iota \right\} > 0, \end{aligned}$$

then, we have that $x(t) > 0$ and $y(t) > 0$. Therefore, $(x(t), y(t)) \in \Omega$ for all $t \geq 0$. Thus we see that the region Ω is invariant by the system (2.1). \square

Theorem 3.2. *The solutions $x(t)$ and $y(t)$ of the system (2.1) are bounded for all $t \geq 0$.*

Proof. From the first equation of the system (2.1), we have:

$$\frac{dx}{x(k-x)(x-m)} \leq \frac{r}{k} dt.$$

Integrating we obtain:

$$\begin{aligned} -\frac{1}{km} \int_0^t \frac{dx(\iota)}{x(\iota)} - \frac{1}{k(k-m)} \int_0^t \frac{d(k-x(\iota))}{k-x(\iota)} + \frac{1}{m(k-m)} \int_0^t \frac{d(x(\iota)-m)}{x(\iota)-m} &\leq \int_0^t \frac{r}{k} dt, \\ \frac{1}{km(k-m)} [(m-k) \ln(x(\iota))|_0^t - m \ln(k-x(\iota))|_0^t + k \ln(x(\iota)-m)|_0^t] &\leq \frac{r}{k} t. \end{aligned}$$

Then, from various calculations, we have that

$$\begin{aligned} m \ln \left(\frac{x(t)(k-x(0))}{x(0)(k-x(t))} \right) + k \ln \left(\frac{x(0)(x(t)-m)}{x(t)(x(0)-m)} \right) &\leq rm(k-m)t, \\ \left(1 - \frac{m}{x(t)} \right)^k \left(\frac{k-x(0)}{x(0)} \right)^m &\leq \left(\frac{x(0)-m}{x(0)} \right)^k \left(\frac{k}{x(t)} - 1 \right)^m \exp(rm(k-m)t). \end{aligned} \quad (3.2)$$

This last equation can be divided into the following two equations

$$\left(1 - \frac{m}{x(t)} \right)^k \leq \left(\frac{x(0)-m}{x(0)} \right)^k \exp(rmkt), \quad (3.3)$$

$$\left(\frac{k-x(0)}{x(0)} \right)^m \leq \left(\frac{k}{x(t)} - 1 \right)^m \exp(-rm^2t). \quad (3.4)$$

Note that multiplying the equation (3.3) with (3.4) results in the equation (3.2). Then, ordering appropriately, we obtain

$$x(t) \leq \frac{mx(0)}{x(0) - (x(0) - m) \exp(rmt)}, \quad x(t) \leq \frac{kx(0)}{x(0) + (k - x(0)) \exp(rmt)}.$$

Then, by L'Hôpital's rule, we have that $x(t) \leq m$ or $x(t) \leq k$. Therefore, $x(t) \leq d$, where $d_1 = \max\{m, k\}$ for all $t \geq 0$. Then, the second equation of the system (2.1) will look like this.

$$\frac{dy}{(nd_1 + c - y)y} \leq \frac{s}{nd_1 + c} dt.$$

Ordering and integrating,

$$-\frac{1}{nd_1 + c} \int_0^t \frac{d(nd_1 + c - y(\iota))}{nd_1 + c - y(\iota)} + \frac{1}{nd_1 + c} \int_0^t \frac{dy(s)}{y(\iota)} \leq \frac{1}{nd_1 + c} \int_0^t s dt,$$

$$-\ln(nd_1 + c - y(t))|_0^t + \ln(y(t))|_0^t \leq st.$$

Then, from various calculations, we have that

$$\frac{y(0)(nd_1 + c - y(t))}{y(t)(nd_1 + c - y(0))} \leq \exp(st),$$

$$y(t) \leq \frac{(nd_1 + c)y(0) \exp(st)}{nd_1 + c + y(0)(\exp(st) - 1)}.$$

Then, $y(t) \leq nd_1 + c$ for all $t \geq 0$. □

From the previous Theorem it follows that $(x(t) + y(t)) \leq d_1(n + 1) + c$. Moreover, the region of analysis is bounded, i.e.

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x \leq k; 0 < y \leq nd_1 + c\}.$$

This way of bounding solutions differs from articles such as [42, 40]. According to Corollary 2.2.3, retrieved from [24][page 36]. This indicates that if the system under study is of class C^1 over its associated region, then the existence and uniqueness of the solutions of the I.V.P. associated to the system under study are ensured. Clearly, the system under study (2.1) is of class C^1 . Therefore, according to the above result, it is ensured that the solutions of system (2.1) exist and are unique.

4. TOPOLOGICAL EQUIVALENCE

To reduce the number of parameters involved and facilitate qualitative analysis and numerical simulations, the following change of variables

$$x = uk, \quad y = vnk, \quad t = \frac{1}{rk}\tau$$

and the following rescaling of parameters

$$M = \frac{m}{k}, \quad B = \frac{rk}{qn}, \quad A = \frac{ark}{q}, \quad S = \frac{s}{rk}, \quad C = \frac{c}{nk}$$

are performed. Applying this change of variables in the system (2.1), we obtain the following:

$$U_\varsigma(u, v) : \begin{cases} \frac{du}{d\tau} = u(1 - u)(u - M) - \frac{uv}{Bu + Av} =: F_1(u, v), \\ \frac{dv}{d\tau} = S \left(1 - \frac{v}{u + C} \right) v =: F_2(u, v) \end{cases} \tag{4.1}$$

defined in the region:

$$\bar{\Omega} = \{(u, v) \in \mathbb{R}^2 : 0 < u \leq D; 0 < v \leq D + C\}. \tag{4.2}$$

Where, $D = \max\{1, M\}$; the variable changes made cause a diffeomorphism, see [11, 7, 47] given by $\varphi : \bar{\Omega} \times \mathbb{R} \rightarrow \Omega \times \mathbb{R}$, such that

$$\varphi(u, v, \tau) = \left(uk, vnk, \frac{1}{rk}\tau \right) = (x, y, t).$$

Moreover, $\text{Det}(\varphi'(u, v)) > 0$. Therefore, the diffeomorphism preserves the time orientation and the system (4.1) is differentially equivalent to the system (2.1). It can be seen that the number of parameters is substantially reduced from eight in the system (2.1), to five in (4.1). Therefore, from here on we work with the system of ordinary differential equations (4.1). The mathematical expression φ' represents the Jacobian matrix of φ .

5. CRITICAL POINTS

The u -nullcline of the (4.1) system in Ω is:

$$v = \frac{Bu(1-u)(u-M)}{1-A(1-u)(u-M)},$$

while the v -nullcline of the (4.1) system in Ω is: $v = u + C$. From this, the critical points of the (4.1) system in its Ω domain are: $(1, 0)$, $(M, 0)$, $(0, C)$, and (u^*, v^*) . Where u^* and v^* are determined by the roots of the following cubic polynomial, which is the result of equating u -isocline and v -isocline.

$$\begin{aligned} &u^3(A+B) + u^2(AC - (M+1)(A+B)) + \\ &u(1-AC + M(A+B-AC)) + C(AM+1) = 0. \end{aligned} \quad (5.1)$$

Following the Cardano Method, which aims to solve analytically any cubic equation, see [9]: First, one can convert the polynomial (5.1) to its normal form, dividing by $(A+B)$ and ordering terms, thus obtaining, see [25]:

$$\begin{aligned} &u^3 + u^2 \left(\frac{AC - (M+1)(A+B)}{A+B} \right) + \\ &u \left(\frac{1-AC + M(A+B-AC)}{A+B} \right) + C \left(\frac{AM+1}{A+B} \right) = 0. \end{aligned}$$

Moreover, substituting

$$u = z - \frac{AC - (M+1)(A+B)}{3(A+B)},$$

thus eliminating the quadratic term, we obtain the following form $z^3 + Pz + Q$, where the linear component and its independent term are defined by:

$$\begin{aligned} P &= \frac{1-AC + M(A+B-AC)}{A+B} - \frac{1}{3} \left(\frac{AC - (M+1)(A+B)}{A+B} \right)^2, \\ Q &= \frac{2}{27} \left(\frac{AC - (M+1)(A+B)}{A+B} \right)^3 \\ &\quad - \frac{(AC - (M+1)(A+B))(1-AC + M(A+B-AC))}{3(A+B)^2} \\ &\quad + C \left(\frac{AM+1}{A+B} \right). \end{aligned}$$

Let $\Delta = (Q/2)^2 + (P/3)^3$. Then, the critical points of the system (4.1) have three cases:

(i) One real root ($\Delta > 0$):

$$u_4 = \sqrt[3]{\frac{-Q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-Q}{2} - \sqrt{\Delta}} - \frac{AC - (M+1)(A+B)}{3(A+B)}. \quad (5.2)$$

(ii) Two real roots ($\Delta = 0$):

$$\begin{aligned} u_5 &= 2\sqrt[3]{-\frac{Q}{2}} - \frac{AC - (M+1)(A+B)}{3(A+B)}, \\ u_6 &= -\sqrt[3]{-\frac{Q}{2}} - \frac{AC - (M+1)(A+B)}{3(A+B)}. \end{aligned} \quad (5.3)$$

(iii) Three real roots ($\Delta < 0$):

$$\begin{aligned} u_7 &= 2\sqrt{-\frac{P}{3}} \cos\left(\frac{\theta}{3}\right) - \frac{AC - (M+1)(A+B)}{3(A+B)}, \\ u_8 &= 2\sqrt{-\frac{P}{3}} \cos\left(\frac{\theta + 2\pi}{3}\right) - \frac{AC - (M+1)(A+B)}{3(A+B)}, \\ u_9 &= 2\sqrt{-\frac{P}{3}} \cos\left(\frac{\theta + 4\pi}{3}\right) - \frac{AC - (M+1)(A+B)}{3(A+B)}. \end{aligned} \tag{5.4}$$

Let $\theta = \arccos\left(\frac{3Q}{2P}\sqrt{\frac{-3}{P}}\right)$ ($0 < \theta < \pi$) and

$$\begin{aligned} R_1 &= \sqrt[3]{\frac{-Q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-Q}{2} - \sqrt{\Delta}}, & R_2 &= 2\sqrt[3]{-\frac{Q}{2}}, \\ R_3 &= -\sqrt[3]{-\frac{Q}{2}}, & R_4 &= 2\sqrt{-\frac{P}{3}} \cos\left(\frac{\theta}{3}\right), \\ R_5 &= 2\sqrt{-\frac{P}{3}} \cos\left(\frac{\theta + 2\pi}{3}\right), & R_6 &= 2\sqrt{-\frac{P}{3}} \cos\left(\frac{\theta + 4\pi}{3}\right). \end{aligned}$$

The main parameters are as follows:

$$\begin{aligned} \alpha_1 &= R_1 - \frac{AC - (M+1)(A+B)}{3(A+B)}, & \beta_1 &= \alpha_1 + C, \\ \alpha_2 &= R_2 - \frac{AC - (M+1)(A+B)}{3(A+B)}, & \beta_2 &= \alpha_2 + C, \\ \alpha_3 &= R_3 - \frac{AC - (M+1)(A+B)}{3(A+B)}, & \beta_3 &= \alpha_3 + C, \\ \alpha_4 &= R_4 - \frac{AC - (M+1)(A+B)}{3(A+B)}, & \beta_4 &= \alpha_4 + C, \\ \alpha_5 &= R_5 - \frac{AC - (M+1)(A+B)}{3(A+B)}, & \beta_5 &= \alpha_5 + C, \\ \alpha_6 &= R_6 - \frac{AC - (M+1)(A+B)}{3(A+B)}, & \beta_6 &= \alpha_6 + C. \end{aligned}$$

Therefore, the critical points of the system (4.1) are:

$$(u_1, v_1) = (1, 0), \tag{5.5}$$

$$(u_2, v_2) = (M, 0), \tag{5.6}$$

$$(u_3, v_3) = (0, C), \tag{5.7}$$

$$(u_4, v_4) = (\alpha_1, \beta_1), \tag{5.8}$$

$$(u_5, v_5) = (\alpha_2, \beta_2), \tag{5.9}$$

$$(u_6, v_6) = (\alpha_3, \beta_3), \tag{5.10}$$

$$(u_7, v_7) = (\alpha_4, \beta_4), \tag{5.11}$$

$$(u_8, v_8) = (\alpha_5, \beta_5), \tag{5.12}$$

$$(u_9, v_9) = (\alpha_6, \beta_6). \tag{5.13}$$

The following results ensure that the last six critical points are positive.

Proposition 5.1. *If $Q < 0$, $Q/2 < -\sqrt{\Delta}$ and $AC \leq (M+1)(A+B)$, then the critical point (u_4, v_4) is positive.*

Proof. Let $Q < 0$ and $Q/2 < -\sqrt{\Delta}$. Then $-Q/2 > \sqrt{\Delta} > -\sqrt{\Delta}$. It follows that

$$R_1 = \sqrt[3]{\frac{-Q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-Q}{2} - \sqrt{\Delta}} > 0,$$

From the condition $AC \leq (M+1)(A+B)$, we have

$$-\frac{AC - (M+1)(A+B)}{3(A+B)} > 0,$$

implying

$$u_4 = R_1 - \frac{AC - (M+1)(A+B)}{3(A+B)} > 0.$$

From the above results we have that $v_4 = u_4 + C > 0$. Therefore, the critical point (u_4, v_4) is positive. \square

Similarly, the following results are obtained:

Proposition 5.2. *If $Q < 0$ and $AC \leq (M+1)(A+B)$. Then, the critical point (u_5, v_5) is positive.*

Proof. Let $Q < 0$, then $R_2 > 0$. Moreover, $AC \leq (M+1)(A+B)$ causes

$$\frac{AC - (M+1)(A+B)}{3(A+B)} \leq 0.$$

Then, $u_5 > 0$ and, since $v_5 = \alpha_2 + C$, consequently $v_5 > 0$. Therefore, the critical point (u_5, v_5) is positive. \square

Proposition 5.3. *If $Q > 0$ and $AC \leq (M+1)(A+B)$. Then, the critical point (u_6, v_6) is positive.*

Proof. Let $Q > 0$, then $R_3 > 0$. Moreover, $AC \leq (M+1)(A+B)$ causes

$$\frac{AC - (M+1)(A+B)}{3(A+B)} \leq 0.$$

Then, $u_6 > 0$ and, since $v_6 = \alpha_3 + C$, consequently $v_6 > 0$. Therefore, the critical point (u_6, v_6) is positive. \square

Proposition 5.4. *If $1 \leq AC$, $A+B \leq AC$ and $AC \leq (M+1)(A+B)$. Then, the critical point (u_7, v_7) is positive.*

Proof. Let $1 \leq AC$ and $A+B \leq AC$ then $P < 0$. Note that $0 < \theta < \pi$ implies $\cos(\frac{\theta}{3}) \in (\frac{1}{2}, 1)$. Therefore $R_4 > 0$. Moreover, $AC \leq (M+1)(A+B)$ causes

$$\frac{AC - (M+1)(A+B)}{3(A+B)} \leq 0.$$

Then, $u_7 > 0$ and, since $v_7 = \alpha_4 + C$, consequently $v_7 > 0$. Therefore, the critical point (u_7, v_7) is positive. \square

Proposition 5.5. *If $1 \leq AC$, $A+B \leq AC$, $AC \leq (M+1)(A+B)$ and*

$$R_5 - \frac{AC - (M+1)(A+B)}{3(A+B)} > 0.$$

Then, the critical point (u_8, v_8) is positive.

Proof. Let $1 \leq AC$ and $A + B \leq AC$ then $P < 0$. Note that $0 < \theta < \pi$ implies that $\cos(\frac{\theta+2\pi}{3}) \in \langle -1, -\frac{1}{2} \rangle$. Therefore, $R_5 < 0$. From $AC \leq (M + 1)(A + B)$, it follows that $AC - (M + 1)(A + B) > 0$. Finally

$$R_5 - \frac{AC - (M + 1)(A + B)}{3(A + B)} > 0,$$

ensuring that $\alpha_5 > 0$ and consequently $\beta_5 = \alpha_5 + C > 0$. Therefore, the critical point (u_8, v_8) is positive. \square

Proposition 5.6. *If $1 \leq AC$, $A + B \leq AC$, $AC \leq (M + 1)(A + B)$ y $\frac{\pi}{2} < \theta < \pi$. Then, the critical point (u_9, v_9) is positive.*

Proof. Let $1 < A_1A_2$, $A_1A_4 < A_1 + A_4$ then $P < 0$. From $\frac{\pi}{2} < \theta < \pi$ it follows that $\cos(\frac{\theta+4\pi}{3}) \in \langle 0, -\frac{1}{2} \rangle$. Therefore, $R_6 > 0$, added to the fact that $AC \leq (M + 1)(A + B)$ it is clear that

$$R_6 - \frac{AC - (M + 1)(A + B)}{3(A + B)} > 0.$$

Then it is assured that $\alpha_6 > 0$ and consequently $\beta_6 = \alpha_6 + C > 0$. Therefore, the critical point (u_9, v_9) is positive. \square

6. QUALITATIVE ANALYSIS

In this section we study the local and global stability of the critical points. Since the system (4.1) represents population dynamics, stability options are sought, especially the asymptotically stable case. Mathematics provides a large number of tools for population dynamics, such as linearization or the Hurwitz polynomial, which act on the Jacobian matrix. In addition, other tools such as LaSalle’s Theorem or the generalized Krasovskii Theorem, which act on regions containing the critical point to be studied. The difference between these two types of approaches lies in their consequences. Linearization and the Hurwitz polynomial give conditions for the parameters, but not for the initial value. The opposite is the case for the LaSalle and Krasovskii results. Therefore, these last two results allow us to simulate a scenario closer to the scenario modeled by the system under study by themselves. However, by themselves these results are limited, since they only contemplate the asymptotically stable case. Whereas the linearization and the Hurwitz polynomial take into account the unstable and asymptotically stable cases, thus allowing a larger number of scenarios to be analyzed. The specialized literature in biomathematics dictates that the application of the strong Allee effect on any population inevitably dies out. Therefore, linearization is used in the present paper because of its extensive results. For any critical point $E = (u, v)$, the Jacobian matrix of the system (4.1) evaluated at $E = (u, v)$ takes the form:

$$J = \begin{pmatrix} 2u(1 + M) - M - 3u^2 - \frac{Av^2}{(Bu + Av)^2} & -\frac{Bu^2}{(Bu + Av)^2} \\ \frac{Sv^2}{(u + C)^2} & S \left(1 - \frac{2v}{u + C} \right) \end{pmatrix}. \tag{6.1}$$

In addition, for global stability analysis, the Lyapunov function, retrieved from [22], is applied; for more information, see [7, 47, 11] and the Chetaev criterion [10], from which the Lyapunov instability theorems are derived as particular cases. Moreover, this criterion is less restrictive; for more information, see [33]. These results serve us both to obtain a correct initial value and to ensure that the result obtained holds over time. Of the two options that the linearization has, the critical points found are analyzed

qualitatively, by the criterion of the trace and the determinant. To facilitate the following results, it is considered from the system (4.1), that

$$F_1 = u \left((1-u)(u-M) - \frac{v}{Bu+Av} \right), \quad F_2 = S \left(1 - \frac{v}{u+C} \right) v.$$

6.1. Stability of the critical point (u_1, v_1) . The Jacobian matrix of system (4.1) at the critical point $(u_1, v_1) = (1, 0)$

$$J_1 = \begin{pmatrix} M-1 & -\frac{1}{B} \\ 0 & S \end{pmatrix}. \quad (6.2)$$

Then, $\text{Tr}(J_1) = S + M - 1$ and $\text{Det}(J_1) = S(M - 1)$.

Theorem 6.1. *If $M \neq 1$ and $S + M > 1$. Then the critical point (u_1, v_1) of the system (4.1) is unstable.*

Proof. This demonstration is divided into two parts.

- Let $M < 1$, then, consequently $\text{Det}(J_1) < 0$. Therefore, the critical point (u_1, v_1) of the system (4.1) is an saddle point (unstable).
- Let $M > 1$ and $S + M > 1$, then, consequently $\text{Det}(J_1) > 0$ and $\text{Tr}(J_1) > 0$. Therefore, the critical point (u_1, v_1) of the system (4.1) is unstable. □

Theorem 6.2. *If $M > 1$ and $S + M < 1$. Then the critical point (u_1, v_1) of the system (4.1) is asymptotically stable.*

Proof. Let $M > 1$ and $S + M < 1$, then, consequently $\text{Det}(J_1) > 0$ and $\text{Tr}(J_1) < 0$. Therefore, the critical point (u_1, v_1) of the system (4.1) is asymptotically stable. □

For this same critical point, we consider the scalar function $V_1 = (u - 1)^2/2 + v^2/2S$. Which is of class C^2 . Clearly $V_1 > 0$ in $\bar{\Omega} - \{(1, 0)\}$ and $V_2(1, 0) = 0$. Moreover,

$$\dot{V}_1 = \frac{\partial V_1}{\partial u} \cdot F_1 + \frac{\partial V_1}{\partial v} \cdot F_2 = u(u-1) \left((1-u)(u-M) - \frac{v}{Bu+Av} \right) + \left(1 - \frac{v}{u+C} \right) v^2.$$

Theorem 6.3. *If $M > u > 1$, $v > 0$, $M + C < v$ and $(1-u)(u-M)(Bu+Av) < v$. Then $\dot{V}_1(u, v) < 0$ for all $(u, v) \in \bar{\Omega} - \{(1, 0)\}$.*

Proof. Let $M > u > 1$, $v > 0$ and $M + C < v$ then $u + C - v < M + C - v$ and $(1-u)(u-M) > 0$. Then, from $(1-u)(u-M)(Bu+Av) < v$, it follows that $\dot{V}_1(u, v) < 0$ for all $(u, v) \in \bar{\Omega} - \{(1, 0)\}$. □

Therefore, according to Theorem 1.5. retrieved from [11, page 24], the scalar function V_1 is a strict Liapunov function. Therefore the critical point (u_1, v_1) of the system (4.1) is asymptotically stable. It should be remembered that the local qualitative analysis of this critical point takes into consideration the cases of stability and instability. Therefore, the following result is presented for the global instability of this critical point. We define the following ball $\mathcal{B}_1 = \{(u, v) \in \mathbb{R}^2 : |u - 1| \leq 1; |v| \leq 1\}$ and set $\mathcal{U}_1 = \{(u, v) \in \mathcal{B}_1 : V_1(u, v) > 0\}$.

Theorem 6.4. *If $C \leq 1$, $0 < u < 1$, $0 < v \leq 1$. Then $\dot{V}_1 > 0$ for all $(u, v) \in \mathcal{U}_1$.*

Proof. From $0 < u < 1$ and $0 < v \leq 1$, we have the following results:

$$-1 < u - 1 < 0, \quad -\frac{v}{Bu+Av} < (1-u)(u-M) - \frac{v}{Bu+Av}, \quad 1 - \frac{1}{C} < 1 - \frac{v}{u+C}.$$

From $1 < C$, it follows that $0 < \frac{\partial V_1}{\partial v} \cdot F_2$. Then $\dot{V}_1 > 0$ for all $(u, v) \in \mathcal{U}_1$. □

Therefore, according to Theorem 2.5, retrieved from [33, page 43], the critical point (u_1, v_1) of the system (4.1) is unstable.

6.2. **Stability of the critical point** (u_2, v_2) . The Jacobian matrix of system (4.1) at the critical point $(u_2, v_2) = (M, 0)$

$$J_2 = \begin{pmatrix} M(1-M) & -\frac{1}{B} \\ 0 & S \end{pmatrix}. \quad (6.3)$$

Then, $\text{Tr}(J_2) = M(1-M) + S$ and $\text{Det}(J_2) = M(1-M)S$.

Theorem 6.5. *If $M \neq 1$ and $S + M > M^2$. Then the critical point (u_2, v_2) of the system (4.1) is unstable.*

Proof. This demonstration is divided in two parts.

- Let $1 < M$, then, consequently $\text{Det}(J_2) < 0$. Therefore, the critical point (u_2, v_2) of the system (4.1) is an saddle point (unstable).
- Let $1 > M$ and $S + M > M^2$, then, consequently $\text{Det}(J_2) > 0$ and $\text{Tr}(J_2) > 0$. Therefore, the critical point (u_2, v_2) of the system (4.1) is unstable. □

Theorem 6.6. *If $1 > M$ and $S + M < M^2$. Then the critical point (u_2, v_2) of the system (4.1) is asymptotically stable.*

Proof. Let $1 > M$ and $S + M < M^2$, then, consequently $\text{Det}(J_2) > 0$ and $\text{Tr}(J_2) < 0$. Therefore, the critical point (u_2, v_2) of the system (4.1) is asymptotically stable. □

For this same critical point we consider the following scalar function: $V_2 = (u - M)^2/2 + v^2/2S$. Which is of class C^2 . Clearly $V_2 > 0$ in $\bar{\Omega} - \{(M, 0)\}$ and $V_2(M, 0) = 0$. Moreover,

$$\dot{V}_2 = \frac{\partial V_2}{\partial u} \cdot F_1 + \frac{\partial V_2}{\partial v} \cdot F_2 = (u - M)((1 - u)(u - M) - \frac{v}{Bu + Av})u + (1 - \frac{v}{u + C})v^2.$$

Theorem 6.7. *If $M > u > 1$, $v > 0$, $M + C < v$ and $(1 - u)(u - M)(Bu + Av) > v$. Then $\dot{V}_2(u, v) < 0$ for all $(u, v) \in \bar{\Omega} - \{(M, 0)\}$.*

Proof. Let $M > u > 1$, $v > 0$ and $M + C < v$ then $u + C - v < M + C - v$ and $(1 - u)(u - M) > 0$. Then, from $(1 - u)(u - M)(Bu + Av) > v$, it follows that $\dot{V}_2(u, v) < 0$ for all $(u, v) \in \bar{\Omega} - \{(M, 0)\}$. □

Therefore, according to Theorem 1.5. retrieved from [11, page 24], the scalar function V_2 is a strict Liapunov function. Therefore the critical point (u_2, v_2) of the system (4.1) is asymptotically stable. It should be remembered that the local qualitative analysis of this critical point takes into consideration the cases of stability and instability. Therefore, the following result is presented for the global instability of this critical point. We define the following ball $\mathcal{B}_2 = \{(u, v) \in \mathbb{R}^2 : |u - M| \leq 1; |v| \leq 1\}$ and set $\mathcal{U}_2 = \{(u, v) \in \mathcal{B}_2 : V_2(u, v) > 0\}$.

Theorem 6.8. *If $0 < u < M$, $0 < v \leq 1$, $1 < C$ and $M < 1$. Then $\dot{V}_2 > 0$ for all $(u, v) \in \mathcal{U}_2$.*

Proof. From $0 < u < M$ and $0 < v \leq 1$, we have the following results:

$$-M < u - M < 0, \quad M(M - 1) - \frac{v}{Bu + Av} < (1 - u)(u - M) - \frac{v}{Bu + Av}, \quad 1 - \frac{1}{C} < 1 - \frac{v}{u + C}.$$

From $1 < C$ it follows that $0 < \frac{\partial V_2}{\partial v} \cdot F_2$ and $M < 1$ ensures that $0 < \frac{\partial V_2}{\partial u} \cdot F_1$. Then $\dot{V}_2 > 0$ para todo $(u, v) \in \mathcal{U}_2$. □

Therefore, according to Theorem 2.5, retrieved from [33, page 43], the critical point (u_2, v_2) of the system (4.1) is globally unstable.

6.3. Stability of the critical point (u_3, v_3) . The Jacobian matrix of system (4.1) at the critical point $(u_3, v_3) = (0, C)$

$$J_3 = \begin{pmatrix} -M - \frac{1}{A} & 0 \\ S & -S \end{pmatrix}. \quad (6.4)$$

Then, $\text{Tr}(J_3) = -S - M - \frac{1}{A} < 0$ and $\text{Det}(J_3) = S(M + \frac{1}{A}) > 0$. Then, the critical point (u_3, v_3) of the system (4.1) is asymptotically stable. For this same critical point, we consider the following scalar function: $V_3 = u^2/2 + (v - C)^2/2S$. Which is of class C^2 , clearly $V_3 > 0$ in $\bar{\Omega} - \{(0, C)\}$ and $V_3(0, C) = 0$. Moreover,

$$\dot{V}_3 = \frac{\partial V_3}{\partial u} \cdot F_1 + \frac{\partial V_3}{\partial v} \cdot F_2 = ((1 - u)(u - M) - \frac{v}{Bu + Av})u^2 + v(v - C)(1 - \frac{v}{u + C}).$$

Theorem 6.9. *If $(1 - u)(u - M)(Bu + Av) \neq v$, $0 \neq u$, $v = u + C$ and $C \neq v$. Then $\dot{V}_3(u, v) < 0$ for all $(u, v) \in \bar{\Omega} - \{(0, C)\}$.*

Proof. From $v = u + C$, we have that

$$\dot{V}_3 = ((1 - u)(u - M) - \frac{v}{Bu + Av})u^2.$$

As long as $(1 - u)(u - M)(Bu + Av) - v \neq 0$, $0 \neq u$ and $0 \neq v - C$. It follows that $\dot{V}_3(u, v) < 0$ for all $(u, v) \in \bar{\Omega} - \{(0, C)\}$. \square

Therefore, according to Theorem 1.5. retrieved from [11, page 24], the scalar function V_3 is a strict Liapunov function. Then the critical point (u_3, v_3) of the system (4.1) is asymptotically stable.

6.4. Stability of the critical points (u^*, v^*) . The remaining critical points (5.5), (5.6), (5.7), (5.8), (5.9), (5.10) lend themselves to be analyzed together, ie, instead of analyzing each of the remaining six critical points, they can be analyzed by $(u^*, v^*) = (\alpha_i, \beta_i)$, where $i = \{1, 2, 3, 4, 5, 6\}$. Moreover, due to the $v - nullcline$ isocline in Omega, $\beta_i = \alpha_i + C$. Therefore, the Jacobian matrix of system (4.1) at the critical point $(u^*, v^*) = (\alpha_i, \beta_i)$, where $i = \{1, 2, 3, 4, 5, 6\}$

$$J_4 = \begin{pmatrix} 2\alpha_i(1 + M) - M - 3(\alpha_i)^2 - \frac{A(\alpha_i + C)^2}{(\alpha_i(A + B) + AC)^2} & -\frac{B(\alpha_i)^2}{(\alpha_i(A + B) + AC)^2} \\ S & -S \end{pmatrix}. \quad (6.5)$$

Then, $\text{Tr}(J_4) = 2\alpha_i(1 + M) - \rho_i$ and $\text{Det}(J_4) = S(\mu_i - 2\alpha_i(1 + M))$. Where,

$$\rho_i = M + S + 3(\alpha_i)^2 + \frac{A(\alpha_i + C)^2}{(\alpha_i(A + B) + AC)^2}.$$

$$\mu_i = M + 3(\alpha_i)^2 + \frac{A(\alpha_i + C)^2 + B(\alpha_i)^2}{(\alpha_i(A + B) + AC)^2}.$$

The following results are obtained:

Theorem 6.10. *If $\mu_i \neq 2\alpha_i(1 + M)$ and $2\alpha_i(1 + M) > \rho_i$. Then, the critical point (u_4, v_4) of the system (4.1) is unstable.*

Proof. This demonstration is divided in two parts.

- (1) Let $\mu_i < 2\alpha_i(1 + M)$, then, consequently $\text{Det}(J_4) < 0$. Therefore, the critical point (u_4, v_4) of the system (4.1) is an saddle point (unstable).
- (2) Let $\mu_i > 2\alpha_i(1 + M)$ and $2\alpha_i(1 + M) > \rho_i$, then, consequently $\text{Det}(J_4) > 0$ and $\text{Tr}(J_4) > 0$. Therefore, the critical point (u_4, v_4) of the system (4.1) is unstable.

\square

Theorem 6.11. *If $\mu_i > 2\alpha_i(1 + M)$ and $2\alpha_i(1 + M) < \rho_i$. Then, the critical point (u_4, v_4) of the system (4.1) is asymptotically stable.*

Proof. Let $\mu_i > 2\alpha_i(1 + M)$ and $2\alpha_i(1 + M) < \rho_i$, i.e., $\text{Det}(J_4) > 0$ and $\text{Tr}(J_4) < 0$. Then, the critical point (u_4, v_4) of the system (4.1) is asymptotically stable. \square

For these same critical points, we consider the following scalar function: $V_4 = (u - \alpha_i)^2/2 + (v - \beta_i)^2/2S$. Which is of class C^2 , clearly $V_4 > 0$ in $\bar{\Omega} - \{(\alpha_i, \beta_i)\}$ and $V_4(\alpha_i, \beta_i) = 0$. In addition,

$$\dot{V}_4 = \frac{\partial V_4}{\partial u} \cdot F_1 + \frac{\partial V_4}{\partial v} \cdot F_2 = (u - \alpha_i) \left((1 - u)(u - M) - \frac{v}{Bu + Av} \right) u + v(v - \beta_i) \left(1 - \frac{v}{u + C} \right).$$

Theorem 6.12. *If $(1 - u)(u - M)(Bu + Av) \neq v$, $u \neq \alpha_i$, $v = u + C$ and $\beta_i \neq v$. Then $\dot{V}_4(u, v) < 0$ for all $(u, v) \in \bar{\Omega} - \{(\alpha_i, \beta_i)\}$.*

Proof. Let $v = u + C$ then

$$\dot{V}_4 = (u - \alpha_i) \left((1 - u)(u - M) - \frac{v}{Bu + Av} \right) u.$$

Then, we have two cases:

- Let $(1 - u)(u - M) > \frac{v}{Bu + Av}$ and $u < \alpha_i$. Then it follows that $\dot{V}_4(u, v) < 0$ for all $(u, v) \in \bar{\Omega} - \{(\alpha_i, \beta_i)\}$.
- Let $(1 - u)(u - M) < \frac{v}{Bu + Av}$ and $u > \alpha_i$. Then it follows that $\dot{V}_4(u, v) < 0$ for all $(u, v) \in \bar{\Omega} - \{(\alpha_i, \beta_i)\}$.

completing the proof. \square

Therefore, according to Theorem 1.5. retrieved from [11, page 24], the scalar function V_4 is a strict Liapunov function. Therefore the critical points (u^*, v^*) of the system (4.1) are asymptotically stable. It should be remembered that the local qualitative analysis of these critical points takes into consideration the cases of stability and instability. Therefore, the following result is presented for the global instability of these critical points. Then, the following is defined $\mathcal{B}_4 = \{(u, v) \in \mathbb{R}^2 : |u - \alpha_i| \leq \epsilon; |v - \beta_i| \leq \epsilon\}$ and $\mathcal{U}_4 = \{(u, v) \in \mathcal{B}_4 : V_4(u, v) > 0\}$, where ϵ tends to zero.

Theorem 6.13. *Assume*

$$\frac{\beta_i + \epsilon}{B(\alpha_i - \epsilon) + A(\beta_i - \epsilon)} > (\alpha_i - M - \epsilon)(1 - \epsilon - \alpha_i)$$

where $\beta_i = \alpha_i + C$. Then $\dot{V}_4(u, v) > 0$ for all $(u, v) \in \mathcal{U}_4$.

Proof. From the set \mathcal{B}_4 , it follows that $\alpha_i - \epsilon \leq u \leq \alpha_i$ and $\beta_i - \epsilon \leq v \leq \beta_i$ for each $i \in \{1, 2, 3, 4, 5, 6\}$. According to these results, it is clear that

$$\epsilon(\alpha_i - \epsilon) \left(\frac{\beta_i + \epsilon}{B(\alpha_i - \epsilon) + A(\beta_i - \epsilon)} - (\alpha_i - \epsilon - M)(1 - \alpha_i - \epsilon) \right) \leq \frac{\partial V_4}{\partial u} \cdot F_1. \tag{6.6}$$

It is clear that from the first hypothesis $\frac{\partial V_4}{\partial u} \cdot F_1 \geq 0$. While from the second hypothesis, $\frac{\partial V_4}{\partial v} \cdot F_2 = 2\epsilon^2 > 0$. Therefore, under these conditions $\dot{V}_4(u, v) > 0$, for all elements of \mathcal{B}_4 . In other words \mathcal{U}_4 is different from the vacuum. \square

Therefore, according to Theorem 2.5, retrieved from [33, page 43], the critical points (u^*, v^*) of the system (4.1) are globally unstable.

7. PERIODIC SOLUTIONS

Periodic solutions, also called periodic orbits or closed orbits in population dynamics, are fundamental to understanding how populations of species interact and develop in cyclically changing environments. One of their applications is bifurcations. These are phenomena where a dynamical system exhibits periodic solutions near an equilibrium point. This is relevant in population dynamics, as it can indicate changes in the stability of populations under different environmental conditions. To search for periodic solutions of the system (4.1), the Dulac criterion, retrieved from [11, page 102], is used. This criterion makes use of a Φ function called the Dulac function, which is difficult to find, since having no clear constraints, this can be any smooth function. Therefore, a technique for its construction is used, presented in [35, page 44-45], [37, page 3] and [36, page 3]. Taking the 2.4 section of [42] as an example. The following candidate function is defined as a Dulac function: $\Phi = (uv)^{-1}$.

Proposition 7.1. *Assume*

$$p := \frac{Buv}{(Bu + Av)^2} + u(1 - u) - u(u - M) - \frac{Sv}{u + C} \neq 0 \text{ for all } (u, v) \in \bar{\Omega}.$$

Then the function $\Phi = 1/uv$ is a Dulac function of the system (4.1) in the region (4.2).

Proof. Direct calculation gives

$$F_1 \frac{\partial \Phi}{\partial u} + F_2 \frac{\partial \Phi}{\partial v} = -\frac{1}{uv} \left((1 - u)(u - M) - \frac{v}{Bu + Av} \right) - \frac{S}{uv} \left(1 - \frac{v}{u + C} \right).$$

Dividing this by Φ , we have

$$\frac{1}{\Phi} (F_1 \frac{\partial \Phi}{\partial u} + F_2 \frac{\partial \Phi}{\partial v}) = \frac{v}{Bu + Av} - (1 - u)(u - M) - S \left(1 - \frac{v}{u + C} \right).$$

Note that

$$\operatorname{div}(F_1, F_2) = \frac{Buv}{(Bu + Av)^2} - \frac{Sv}{u + C} + u(1 - u) - u(u - M) - \frac{v}{Bu + Av} + (1 - u)(u - M) + S \left(1 - \frac{v}{u + C} \right).$$

Thus,

$$\frac{1}{\Phi} (F_1 \frac{\partial \Phi}{\partial u} + F_2 \frac{\partial \Phi}{\partial v}) + \operatorname{div}(F_1, F_2) = \frac{Buv}{(Bu + Av)^2} + u(1 - u) - u(u - M) - \frac{Sv}{u + C} = p.$$

Therefore,

$$F_1 \frac{\partial \Phi}{\partial u} + F_2 \frac{\partial \Phi}{\partial v} + \Phi \operatorname{div}(F_1, F_2) = p\Phi \neq 0 \text{ in the region (4.2).}$$

We conclude then that the function Φ is a Dulac function of the system (4.1) in the region (4.2). \square

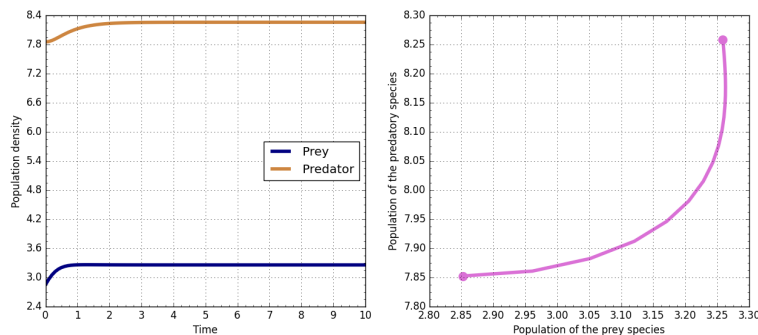
Theorem 7.2. *If the function Φ is a Dulac function. Then $\operatorname{div}(F_1\Phi, F_2\Phi) \neq 0$ for all $(u, v) \in \bar{\Omega}$.*

Proof. You have that

$$\operatorname{div}(F_1\Phi, F_2\Phi) = \frac{B}{(Bu + Av)^2} + \frac{(1 - u)}{v} - \frac{(u - M)}{v} - \frac{S}{u(u + C)} = p\Phi.$$

Then $\operatorname{div}(F_1\Phi, F_2\Phi) \neq 0$ for all $(u, v) \in \bar{\Omega}$. \square

According to Proposition (7.1) and Theorem (7.2), there is no periodic solution for the system (4.1) in the region (4.2).

FIGURE 1. First simulation of critical point (u_7, v_7) .

8. QUANTITATIVE ANALYSIS

In this section we perform the numerical analysis of the critical point that meets the constraints proposed above. This analysis consists of the interpretation of some simulations performed for each critical point, with the help of several Python programming language packages, such as SciPy, Matplotlib and Sympy. It should be noted that the simulations are developed in Google Colab, also known as Colaboratory. Which is a free cloud-based tool that allows users to write, run and share Python code. The simulations are performed on the differentially equivalent system, i.e., on (4.1). The predefined function *odeint* of the sub-package *integrate*, which belongs to the *Scipy* package of *Python*, has been used. This allows to better sketch the temporal evolution of the populations as well as to guarantee better simulations than with a numerical method. In addition, using the *Sympy* package, more complex numerical calculations were solved, such as local and global stability analysis, as well as finding the correct values for each parameter and initial value.

8.1. Quantitative analysis of critical point (u_7, v_7) . The first simulation (1), is conditioned by Proposition (5.4) and Theorems (6.11) and (6.12). The initial point is taken to be $(2.85255, 7.85255)$, $A = 0.4$, $B = 0.5$, $C = 5.0$, $M = 4.0$, $S = 1.6$. Replacing, we have $P = -8.572016460905349$, $Q = 7.30122440684855$, $\Delta = -10.001445127492977$, $R_4 = 1.9266775564022338$, $\alpha_4 = 2.85260348232816$, $\beta_4 = 7.85260348232816$, $\epsilon = 10^{-4}$, $\theta = 2.8932388446055812$, this last result is in radians. In the first plot of the first simulation (1) of the critical point (u_7, v_7) with respect to time, the population dynamics of the prey species is divided into two phases. The first phase occurs in a very short period of time compared to the second phase. In the first stage, the prey species starts with the smallest possible number of members, and then experiences an exponential population increase, while stabilizing as it approaches its second phase. In which the prey population is already stabilized, and where the number of members of its species undergoes small changes. It practically remains constant in this second phase, in addition to remaining at its maximum possible population size. At the same time, the predator species also experiences a population dynamics divided into two phases; in fact, the population dynamics of the predator species behaves similarly to the population dynamics of the prey species. However, this phase lasts longer and the growth of the predator species is not exponential, but practically linear. In other words, the predator species, like the prey species, starts with its smallest possible population size, and then grows at a constant rate, over a longer period of time than the first phase of its prey. It grows in this way until it stabilizes as it reaches its second phase, in which its population dynamics undergoes so few changes that it apparently remains constant for the rest of its second phase. In addition, the predator species in this phase remains at its maximum possible population size. In the

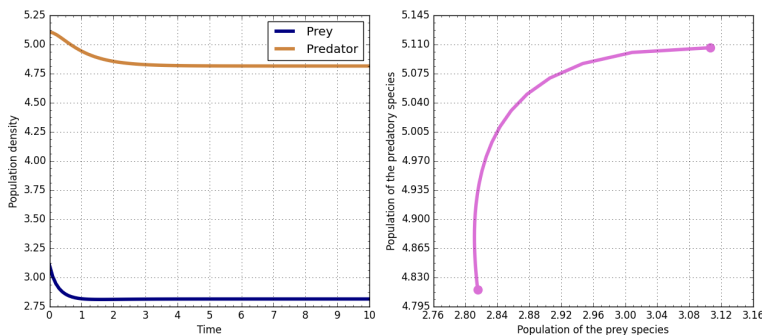
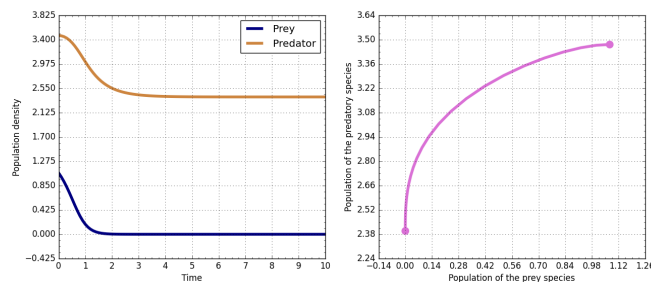


FIGURE 2. Second simulation of the critical point (u_7, v_7) .

second graph of the first simulation (1) of the critical point (u_7, v_7) , the population dynamics of the predator species grows logarithmically with respect to the population dynamics of the prey species, i.e., the members of both species increase, in two phases. The first one starts with the initial value of this simulation $(2.85255, 7.85255)$, where the prey species grows more with respect to the predator species. Then, in the second phase, the roles are reversed, i.e., the predator species grows faster than the prey species, until the maximum possible population size for both species is reached. The second simulation (2), is conditioned by Proposition (5.4) and Theorems (6.11) and (6.12). The initial point is taken to be $(3.10615, 5.10615)$, $A = 0.6$, $B = 0.35$, $C = 2.0$, $M = 3.5$, $S = 1.1$. Replacing, we have $P = -4.623961218836563$, $Q = 2.7933372211692697$, $\Delta = -1.7109832759465338$, $R_4 = 2.027258110491392$, $\alpha_4 = 3.106205478912444$, $\beta_4 = 5.106205478912444$, $\epsilon = 10^{-4}$, $\theta = 1.8466519157852461$, this last result is in radians. In the first plot of the second simulation (2) of the critical point (u_7, v_7) with respect to time, the population dynamics of the prey species is divided into two phases. The first phase occurs in a period of time much shorter than the second. In which it starts with the largest possible number of members; then it undergoes an abrupt population decrease, practically exponential, while it stabilizes as it approaches its second phase. In which the population of the prey is already stabilized, and where the number of members of its species undergoes small changes. It remains practically constant in this second phase, in addition to remaining at its population minimum without becoming extinct. At the same time, the predator species experiences a population dynamics also divided into two phases; in fact, the population dynamics of the predator species behaves similarly to the population dynamics of the prey species, with the exception of the first phase, since the first stage of the population dynamics of the predator species lasts longer and does not occur exponentially, but practically linearly. That is, in the first phase, the predator species suffers losses of its species in a time interval greater than that of its prey. Practically in a linear fashion, it then stabilizes, at the same time that it reaches its second phase, in which its population dynamics undergoes so few changes that it apparently remains constant in the second phase. In addition, its population size remains at its minimum possible over time, without reaching extinction. In the second graph of the second simulation (2) of the critical point (u_7, v_7) , the population dynamics of the predator species decreases exponentially with respect to the population dynamics of the prey species, i.e., the members of both species decrease, in three phases. The first one starts with the initial value of this simulation $(3.10615, 5.10615)$, where the prey species suffers more losses with respect to its predator. Then, in the second phase, both species suffer practically the same amount of losses; this is visualized in the most linear part of this graph. Finally, in the third and last phase, the predator species declines faster than the prey species. Without reaching extinction in both species.

FIGURE 3. First simulation of critical point (u_9, v_9) .

8.2. Quantitative analysis of critical point (u_9, v_9) . The first simulation (3), is conditioned by Proposition (5.4) and Theorems (6.10) and (6.13). The initial point is taken to be $(1.07272, 3.47272)$, $A = 1.0$, $B = 1.1$, $C = 2.4$, $M = 3.0$, $S = 1.4$. Replacing, we have $P = -3.8163265306122445$, $Q = 1.8006694741388616$, $\Delta = -1.2480013757413602$, $R_6 = 0.12039530688724256$, $\alpha_6 = 1.072776259268195$, $\beta_6 = 3.472776259268195$, $\epsilon = 10^{-4}$, $\theta = 1.730989899419405$, this last result is in radians. In the first graph of the first simulation (3) of the critical point (u_9, v_9) with respect to time. The population dynamics of the prey species starts with as many members as possible, and then rapidly and linearly loses them until it reaches zero members, i.e., the prey species becomes extinct. Meanwhile, the predator species experiences a population dynamic divided into two phases. In the first phase, the predator species starts with the largest possible number of members, and then decreases steadily like a Gaussian bell, until it stabilizes, thus initiating its second stage, in which the population dynamics of the predator undergoes so few changes that it apparently remains constant throughout the remainder of the simulation. That is, the number of members of the predator species remains at its minimum possible over time. In the second graph of the first simulation (3) of the critical point (u_9, v_9) , the population dynamics of the predator species decreases exponentially with respect to the population dynamics of the prey species, i.e., the members of both species decrease, in three phases. The first one starts with the initial value of this simulation $(1.07272, 3.47272)$, where the prey species suffers more losses with respect to its predator. Then, in the second phase, both species suffer practically the same amount of losses; this is visualized in the most linear part of this graph. Finally, in the third and last phase, the predator species declines faster than the prey species. However, the prey species becomes extinct, the opposite is the case with its predator.

The second simulation (4), is conditioned by Proposition (5.4) and Theorems (6.10) and (6.13). The initial point is taken to be $(1.24163, 4.04163)$, $A = 1.2$, $B = 0.7$, $C = 2.8$, $M = 3.7$, $S = 2.4$. Replacing, we have $P = -6.949981532779319$, $Q = 2.158496487448229$, $\Delta = -11.268545412430628$, $R_6 = 0.26448791301661856$, $\alpha_6 = 1.2416808954727592$, $\beta_6 = 4.0416808954727586$, $\epsilon = 10^{-4}$, $\theta = 1.8317802914559036$, this last result is in radians. In the first graph of the second simulation, the critical point (u_9, v_9) is plotted with respect to time. The population dynamics of the prey species starts with as many members as possible, and then rapidly loses them until it reaches zero members, i.e., the prey species becomes extinct. It should be noted that this population dynamic is represented as a Gaussian bell, as this simulation appears. Meanwhile, the predator species undergoes a population dynamic divided into two phases. In the first phase, the predator species starts with the largest possible number of members, and then decreases steadily like a Gaussian bell, until it stabilizes, thus initiating its second stage, in which the population dynamics of the predator undergoes so few changes that it

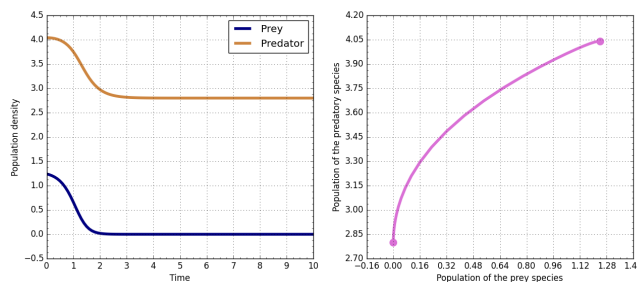


FIGURE 4. Second simulation of critical point (u_9, v_9) .

apparently remains constant for the remainder of the simulation. That is, the number of members of the predator species remains at its minimum possible over time.

In the second graph of the second simulation (3) of the critical point (u_9, v_9) , the population dynamics of the predator species decreases exponentially with respect to the population dynamics of the prey species, i.e., the members of both species decrease, in three phases. The first one starts with the initial value of this simulation $(1.24163, 4.04163)$, where the prey species suffers more losses with respect to its predator. Then, in the second phase, both species suffer practically the same amount of losses; this is visualized in the most linear part of this graph. Finally, in the third and last phase, the predator species declines faster than the prey species. However, the prey species becomes extinct, the opposite is the case with its predator.

9. CONCLUSIONS

The purpose of this research is to analyze the complex ecological behavior of an ecosystem composed of a prey species, which is the preferred food of a predator species. The population growth of the prey species is limited by the amount of food that can be supplied by the ecosystem. In addition, there is a critical minimum threshold density below which the population of the prey species experiences significant difficulties in surviving or reproducing. Whereas if it increases above the threshold density, survival and reproduction rates improve significantly. Finally, the population growth of the predator species is limited by the amount of available prey, with the addition of alternative food sources. This case is represented by a two-dimensional system of ordinary differential equations. In the first instance, it is concluded that the solutions of the system under study are positive, bounded, exist and are unique. Then, it is concluded that the system has nine critical points, three are located on the axes of the plane and the remaining six are the result of the isoclines of the topologically equivalent system, which facilitates the study, thanks to the fact that it has three parameters less than the original system. Then, with the use of mathematical tools such as the trace and determinant criterion, the Liapunov functions, the Chetaev criterion and the Dulac-Bendixon criterion, conditions were found to ensure the asymptotic stability and local and global instability of all the critical points found. These conditions provide different relationships between parameters and initial values. However, only the etiologically attractive cases were simulated, i.e., the critical points resulting from matching the isoclines.

- From the first plot of the first simulation (1) of the critical point (u_7, v_7) , it is concluded that both species grow to their corresponding maximum possible population size. While from the second plot of the first simulation (1) of the critical point (u_7, v_7) , it is concluded that the predatory prey grows exponentially with respect to its prey. Therefore, from the first

simulation (1) of the critical point (u_7, v_7) (5.11), it is generally concluded that both species achieve ecological equilibrium, i.e., neither becomes extinct over time.

- From the first plot of the second simulation (2) of the critical point (u_7, v_7) , it is concluded that both species decrease until they stabilize, reaching their minimum possible population size, without reaching zero members. While from the second plot of the second simulation (2) of the critical point (u_7, v_7) , it is concluded that the predator species decreases exponentially with respect to its prey, without reaching zero members for both species. Therefore, from the second simulation (2) of the critical point (u_7, v_7) (5.11), it is generally concluded that both species achieve ecological equilibrium, i.e., neither achieves extinction over time.
- From the first graph of the first simulation (3) of the critical point (u_9, v_9) , it is concluded that both species decrease until they stabilize, reaching their minimum possible population size. The opposite is the case for the prey, which reaches zero members. Meanwhile, from the second graph of the first simulation (3) of the critical point (u_9, v_9) , it is concluded that the predator species decreases exponentially with respect to its prey, without reaching zero members for the case of the predator, opposite case for its prey. Therefore, from the first simulation (3) of the critical point (u_9, v_9) (5.13), it is generally concluded that the predator species achieves ecological equilibrium, i.e., it does not go extinct over time. However, the prey species does not achieve ecological equilibrium, i.e., it goes extinct. It should be noted that the decrease of both species is sketched as a Gaussian bell.
- From the first graph of the second simulation (4) of the critical point (u_9, v_9) , it is concluded that both species decrease until they stabilize, reaching their minimum possible population size. The opposite is the case for the prey, which reaches zero members. Meanwhile, from the second graph of the second simulation (4) of the critical point (u_9, v_9) , it is concluded that the predator species decreases exponentially with respect to its prey, without reaching zero members for the case of the predator, opposite case for its prey. Therefore, from the second simulation (3) of the critical point (u_9, v_9) (5.13), it is generally concluded that the predator species achieves ecological equilibrium, i.e., it does not go extinct over time. However, the prey species does not achieve ecological equilibrium, i.e., it goes extinct. It is worth noting that the decrease of both species is depicted as a Gaussian bell.
- It is also concluded that of the four presented cases of population versus time, two achieve ecological equilibrium (1), (2) and of the other two cases (2), (3), (4) and (4), (3), (4), only the predator achieves ecological equilibrium, but its prey does not, i.e., only the predator manages to survive over time. In reality, this occurs because in the first two cases the prey species initiates above the critical minimum threshold density, thus ensuring optimal survival and reproduction rates. The opposite is true in the last two cases, where the prey species starts below the minimum critical threshold density, which means very poor survival and reproduction rates. If we add to this the presence of the predator, the reason for the extinction of the prey in the last two cases is clear, but the predator does not follow the same fate, because it has an alternative food.
- It is also concluded that of the four cases presented of population versus population, one is of exponential growth (1) and three are of exponential decrease, one without reaching extinction (2) and two if they do become extinct (3), (4). From this it can be concluded that in one case out of four, both species grow, one out of three decline without going extinct and that two out of three cases decline until the prey goes extinct, but the predator manages to survive.

Furthermore, it is concluded that there are no periodic solutions for this system. According to the four simulated cases, it is concluded that this model ensures ecological equilibrium for the predator, i.e., it manages to survive over time. But not for the prey.

Despite the additions applied. Models like this one are limited. From experimental work such as [16], it is known that the Holling type II functional response can be employed in all kinds of scenarios. However, where it really shines is when the populations it models are invertebrates or small vertebrates. It is recommended that despite being perturbed by the mixed reason-dependent functional response, this same line of study be maintained. In addition, for future work, we can change the strong Allee effect for the weak one, a double Allee effect, take into account cannibalism in the predator, the strategies of the prey to prevent hunting, such as group defense, seeking refuge, or the effect of indirect fear suffered by the prey, and modify this ecological model into an ecoepidemiological model. Instead of an ODE, delay equations or a PDE can be applied. These hints are both for the system under study and the system on which this study was based [43]. While with respect to the periodic solutions, in future works the scenario where they do exist can be proposed and, from this, study the bifurcations of dimension one and/or two both locally and globally.

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