

OPTIMIZATION OF Z-FUZZY SOFT β -COVERING BASED FUZZY ROUGH SETS AND THEIR APPLICATION TO MULTIPLE ATTRIBUTE GROUP DECISION MAKING

S. PAVITHRA AND A. MANIMARAN

ABSTRACT. Fuzzy sets play a crucial role in representing real-world complexities where precise binary logic falls short. Rough sets are closely related to fuzzy sets, and their combined use provides a powerful framework for handling uncertain and incomplete information. The combination of rough sets and fuzzy sets is an intriguing area of research that bridges the gap between crisp and uncertain information. In this paper, three different kinds of fuzzy serial relations are introduced. These relations form new fuzzy soft β -covering based fuzzy rough set models. The main objective of this research is to maximize the lower approximation and minimize the upper approximation of existing models. Eventually, we use the suggested rough set model to resolve MAGDM problems. The proposed MAGDM algorithm demonstrated superior performance when compared to other algorithms. Its effectiveness lies in its ability to find optimal solutions.

1. INTRODUCTION

In order to deal with uncertainty, various theories have been created for data mining. Problems related to uncertainty are addressed using a variety of approaches, including probability, classical set theory, soft sets, etc. All these concepts are susceptible to different limitations and constraints. Introduced by Lotfi Zadeh in 1965, fuzzy sets (FS) have been widely used to handle uncertainty. Zadeh represented FS using singletons, where each element's degree of membership to the set is defined within the range $[0, 1]$.

Pawlak [19] proposed the concept of rough sets. Rough set is applied to data in order to reveal hidden patterns and streamline the analysis process. Several researchers are working on this idea applied to many domains such as [4, 7, 10, 23]. Extension of rough sets are covering rough sets (CRS), which have recently become a significant area of research. Based on CRS, researchers can examine uncertainty and roughness in a broader context. This method is helpful because it permits academics to examine uncertainty and roughness in a more systematic manner. CRS has attracted many researchers due to its wide range of applications and some of its extensions are presented in the following literature [42, 43, 13, 30].

As rough sets and fuzzy sets have proven to be useful scientific tools for dealing with uncertainty, the question of whether the two kinds of sets can be quickly combined arises as a logical consequence. All currently available generalizations of fuzzy approximation sets are founded on various special fuzzy relations. Dubois et al. [6] developed the idea of fuzzy rough sets and rough fuzzy sets. Radzikowska and Kerre [20] proposed a general strategy for fuzzifying rough sets. They construct a general class of

Received by the editors 4 July 2024; accepted 26 February 2025; published online 15 March 2025.

2020 *Mathematics Subject Classification.* 03E72, 90B50.

Key words and phrases. Fuzzy serial relation, fuzzy soft β -neighborhood, fuzzy soft complementary β -neighborhood, fuzzy soft β -adhesion, multiple attribute group decision making.

fuzzy rough sets, which refer to as $(\mathcal{J}, \mathcal{F})$ -fuzzy rough sets. Shaheen et al. [24] initiated an effort to characterise a rough approximation for a set of hesitant fuzzy sets over some universe.

Ma [14] presented two different kinds of fuzzy covering rough sets which could be considered as a bridge between covering rough sets and fuzzy rough sets. Deng [5] presents a lattice theory-based method for fuzzy rough sets. The new model is built on the notions of fuzzy covering. Yeung et al. [31] developed a unique approach that is identical to the Mamdani algorithm and established a topological and lattice structure for fuzzy rough sets. Sun et al. [26] investigate multigranulation structures for rough approximation over fuzzy concepts across two distinct universes. Yang and Hu [28] proposed the concept of fuzzy complementary β -neighborhood. A new fuzzy covering model was presented by Yang [29], which is based on Zadeh's extension concept over two distinct universe. Many academics have been motivated by the concept of a fuzzy rough set, which has led to the proposal of a wide range of models, some of which are presented in [34, 37].

To characterise a useful mathematical technique for handling multi-attribute ambiguity, Molodtsov [17] came up with the idea of soft sets. Maji et al. [15, 16] presented several operators on soft sets. Ali [1] studied the relationships between fuzzy sets, rough sets and soft sets. Roy and Maji [21] proposed a decision-making model by creating a comparison table using fuzzy soft sets. Feng et al. initiated soft P-rough set in [8]. Shabir et al. [22] developed a Modified Soft Rough set (MSR) using Feng's soft set theory. MSR performs better in terms of accuracy than other existing models. Yüksel et al. [32] derived soft covering-based rough sets. Zhan et al. [35] presented five different kinds of soft covering based rough sets.

Zhan and Sun [36] proposed a novel models soft rough fuzzy covering (SRFC) and soft fuzzy rough covering (SFRC). Atef and Nada [2] proposed three different kinds of SFRC. Teams of experts collaborate to solve multiple decision-making challenges to reach the best solution. They have to evaluate potential candidates based on various factors. The idea of fuzzy soft covering based on multigranulation defined in [3] is quite helpful in these circumstances. Zhang and Zhan [39] contributed fuzzy soft β -coverings, fuzzy soft β -neighborhoods and fuzzy soft complement β -neighborhoods and prefaced four different fuzzy soft β -covering based fuzzy rough sets (FS β CFRS). Nawar et al. [18] developed a fuzzy soft β -adhesion and using this concept a new kind of FS β CFRS is defined.

Multiple Attribute Group Decision Making (MAGDM) techniques encompass the process of selecting preferred options from a set of alternatives distinguished by multiple attributes. The utility of these approaches spans diverse domains, including but not limited to professional, managerial, and political decision-making. MAGDM aims to opt for the most optimal alternative or prioritize options by taking into account a multitude of attributes, leading to its extensive utilization across a wide array of disciplines [25, 27]. MAGDM techniques are used to solve decision-making problems involving multiple criteria and multiple decision-makers. These techniques evaluate and rank alternatives based on various attributes, considering the preferences and opinions of a group of decision-makers. The process typically involves structuring the problem, identifying criteria, evaluating alternatives, aggregating preferences, building consensus, and selecting or ranking the best option. MAGDM is widely applied in fields like energy, logistics, and safety management to ensure systematic, transparent, and inclusive decision-making.

MAGDM techniques are extensively applied across various fields to address complex decision-making problems involving multiple criteria and stakeholders. In energy management, MAGDM helps in evaluating and selecting the best energy sources by considering factors like cost, environmental impact, and efficiency. In logistics and supply chain management, it aids in selecting suppliers, optimizing transportation routes, and managing inventory by balancing criteria such as cost, delivery time, reliability, and sustainability. Healthcare applications of MAGDM include selecting treatment plans, evaluating

medical technologies, and making resource allocation decisions, taking into account factors like cost, effectiveness, patient preferences, and risk. In safety management, MAGDM is used to assess and mitigate risks in industries like construction, manufacturing, and aviation by selecting appropriate safety measures and evaluating safety performance.

The primary objective of this research is to define three kinds of Z-fuzzy soft β -covering based ψ -fuzzy rough sets (ψ -Z-FS β CFRS) and three kinds of Z-fuzzy soft β -covering based \mathcal{D} -fuzzy rough sets (\mathcal{D} -Z-FS β CFRS) to generalize the concepts of ψ -soft fuzzy rough covering (ψ -SFRC) and \mathcal{D} -soft fuzzy rough covering (\mathcal{D} -SFRC) models defined in [2, 36]. To maximize the lower approximation and to minimize the upper approximation of ψ -SFRC and \mathcal{D} -SFRC models, we have defined these new kinds of ψ -Z-FS β CFRS and \mathcal{D} -Z-FS β CFRS. Increased lower approximation and reduced upper approximation provide accurate results in the decision process, which is the main reason for defining these models.

The structure of this paper is outlined below: Definitions necessary for understanding the subsequent sections are provided in Section 2. In Section 3, three kinds of ψ -Z-FS β CFRS and three kinds of \mathcal{D} -Z-FS β CFRS are defined. Section 4 provides the relationship between the three kinds of ψ -Z-FS β CFRS and three kinds of \mathcal{D} -Z-FS β CFRS. Section 5 deals with the comparative analysis of these models. Section 6 presents an application to find optimal alternatives using the proposed model. In Section 7, the conclusion is discussed.

2. PRELIMINARIES

In this section, fundamental definitions are presented to facilitate a comprehensive understanding of the subsequent study. Throughout the paper, we assume that Ω represents a finite universal set.

Definition 2.1. [33] A fuzzy set M on Ω is defined by a mapping $M(\bullet) : \Omega \rightarrow [0, 1]$ where $M(v)$ indicates the membership grade of $v \in \Omega$ in M . $F(\Omega)$ indicates the family of fuzzy subsets of Ω .

Definition 2.2. [40] A relation $R = \{(v, u), \mu_R(v, u) : (v, u) \in \Omega \times \Omega\}$ is known as a fuzzy relation on $\Omega \times \Omega$. Then, R is said to be

- (1) reflexive if $\mu_R(v, v) = 1, \forall v \in \Omega$.
- (2) symmetric if $\mu_R(v, u) = \mu_R(u, v), \forall v, u \in \Omega$.
- (3) transitive in the sense of max-min iff
 $\mu_R(v, w) \geq \max_{u \in \Omega} (\min(\mu_R(v, u), \mu_R(u, w))), \forall u, v, w \in \Omega$.

Definition 2.3. [11] Let $M, O \in F(\Omega)$ and $M(v), O(v) \in [0, 1]$, for all $v \in \Omega$. Then, it holds the following properties.

- (1) $M \subseteq O \Leftrightarrow M(v) \leq O(v)$;
- (2) $M = O \Leftrightarrow M \subseteq O$ and $O \subseteq M$;
- (3) $(M \cap O)(v) = M(v) \wedge O(v)$ and $(M \cup O)(v) = M(v) \vee O(v)$;
- (4) $M^c(v) = 1 - M(v)$.

Definition 2.4. [9, 12] Let $\tilde{\mathcal{C}} = \{\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_l\}$ with $\tilde{C}_i \in F(\Omega) (i = 1, 2, \dots, l)$, a fuzzy covering of Ω , if $\left(\bigcup_{i=1}^l \tilde{C}_i\right)(v) = 1$, for every $v \in \Omega$.

Ma [14] defined a fuzzy β -covering as a generalization of fuzzy covering by changing 1 with $\beta (0 < \beta \leq 1)$, we call $\tilde{\mathcal{C}} = \{\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_l\}$ with $\tilde{C}_i \in F(\Omega) (i = 1, 2, \dots, l)$, a fuzzy β -covering of Ω , if $\left(\bigcup_{i=1}^l \tilde{C}_i\right)(v) \geq \beta$, for every $v \in \Omega$. Then $(\Omega, \tilde{\mathcal{C}})_\beta$ is known as a fuzzy β -covering approximation space (F β CAS).

Definition 2.5. [17] Let E be a set of all attributes. The pair $K = (\tilde{F}, B)$ is a soft set over Ω , where $\tilde{F} : B \rightarrow P(\Omega)$ is a set-valued mapping and $B \subseteq E$.

Definition 2.6. [21] If the mapping \tilde{F} is defined by $\tilde{F} : B \rightarrow F(\Omega)$, then (\tilde{F}, B) is known as a fuzzy soft set over Ω .

Definition 2.7. [38] If $(\bigcup_{b \in B} \tilde{F}(b))(v) = 1$, for all $v \in \Omega$, then (\tilde{F}, B) is known as a fuzzy soft covering over Ω . The triad $\mathcal{J} = (\Omega, \tilde{F}, B)$ is known as a fuzzy soft covering approximation space (FSCAS).

Definition 2.8. [39] If $(\bigcup_{b \in B} \tilde{F}(b))(v) \geq \beta$, for all $v \in \Omega$, then $(\tilde{F}, B)_\beta$ is known as a fuzzy soft β -covering over Ω . Here, the triad $\mathcal{J} = (\Omega, \tilde{F}, B)_\beta$ is known as a fuzzy soft β -covering approximation space (FS β CAS).

Definition 2.9. [39] Let $\mathcal{J} = (\Omega, \tilde{F}, B)_\beta$ be a FS β CAS for some $\beta \in (0, 1]$. For all $v \in \Omega$, the fuzzy soft β -neighborhood of v is defined as $\widetilde{SN}_v^\beta = \cap \{ \tilde{F}(b) \in \tilde{F}(B) : \tilde{F}(b)(v) \geq \beta \}$.

Definition 2.10. [39] Let $\mathcal{J} = (\Omega, \tilde{F}, B)_\beta$ be a FS β CAS for some $\beta \in (0, 1]$. Then, $\widetilde{SM}_v^\beta = \widetilde{SN}_u^\beta(v)$ is the fuzzy soft complementary β -neighborhood of v , for all $u, v \in \Omega$.

Definition 2.11. [18] Let $\mathcal{J} = (\Omega, \tilde{F}, B)_\beta$ be a FS β CAS for some $\beta \in (0, 1]$. Then, $\widetilde{SA}_v^\beta = \widetilde{SN}_v^\beta \cap \widetilde{SM}_v^\beta$ is the fuzzy soft β -adhesion of v , for all $u, v \in \Omega$.

3. NEW KINDS OF Z-FS β CFRS

This segment defines three kinds of ψ -Z-FS β CFRS and three kinds of \mathcal{D} -Z-FS β CFRS and their properties are explored.

3.1. ψ -Z₁-FS β CFRS and \mathcal{D} -Z₁-FS β CFRS.

Definition 3.1. Let $(\tilde{F}, B)_\beta$ be a fuzzy soft β -covering over Ω . Let $R_1 = \{ ((v, u), \mathcal{D}_{R_1}^\beta(v, u)) : (v, u) \in \Omega \times \Omega \}$ such that $v = v_i, 1 \leq i \leq p$ and $u = v_j, 1 \leq j \leq p$ is the fuzzy relation where the mapping $\mathcal{D}_{R_1}^\beta : \Omega \times \Omega \rightarrow [0, 1]$ is defined by

$$\mathcal{D}_{R_1}^\beta(v, u) = \frac{\sum_{k=1}^p (\widetilde{SN}_{v_k}^\beta(v) \wedge \widetilde{SN}_{v_k}^\beta(u))}{\sum_{k=1}^p (\widetilde{SN}_{v_k}^\beta(v) \vee \widetilde{SN}_{v_k}^\beta(u))}$$

is the first kind of fuzzy soft β -measure degree (1-FS β MD) of v with respect to u . It is obvious that, $\mathcal{D}_{R_1}^\beta(v, v) = 1$ and $\mathcal{D}_{R_1}^\beta(v, u) = \mathcal{D}_{R_1}^\beta(u, v)$. In addition, $0 \leq \mathcal{D}_{R_1}^\beta(v, u) \leq 1$.

The matrix depiction of 1-FS β MD is given below: if $m_{ij} = \mathcal{D}_{R_1}^\beta(v, u) = \mathcal{D}_{R_1}^\beta(v_i, v_j)$, we present a matrix

$$[m_{ij}]_{p \times p} = \begin{pmatrix} m_{11} & \dots & m_{1p} \\ \vdots & \ddots & \vdots \\ m_{p1} & \dots & m_{pp} \end{pmatrix},$$

as the matrix depiction of 1-FS β MD of order $p \times p$.

By definition 1-FS β MD, we can say that the fuzzy relation R_1 is reflexive and symmetric but not transitive. Therefore, R_1 is a weak fuzzy similarity relation.

Remark 3.1. (i) The fuzzy soft measure degree (FSMD) model introduced by Zhan et al. [36] is a special case of 1-FS β MD model when $\beta = 1$. (ii) 1-FS β MD is a generalization of FSMD.

Example 3.1. Let $\mathcal{J} = (\Omega, \tilde{F}, B)_\beta$ be a FS β CAS and $(\tilde{F}, B)_\beta$ is presented in TABLE 1. For $\beta = 0.6$, we calculate the fuzzy soft β -neighborhood, fuzzy soft complementary β -neighborhood and fuzzy soft β -adhesion as follows.

TABLE 1. Tabular depiction of $(\tilde{F}, B)_\beta$

Ω	b_1	b_2	b_3	b_4	b_5
v_1	0.6	0.2	0.4	0.5	0.7
v_2	0.1	0.8	0.7	0.3	0.2
v_3	0.8	0.4	0.3	0.1	0.6
v_4	0.5	0.6	0.1	0.7	0.2
v_5	0.3	0.4	0.5	0.2	0.8
v_6	0.4	0.7	0.6	0.1	0.2

The matrix form of $\mathcal{D}_{R_1}^\beta$ is given below,

$$[m_{ij}]_{6 \times 6} = \begin{pmatrix} 1 & 0.27 & 0.85 & 0.34 & 0.63 & 0.33 \\ 0.27 & 1 & 0.31 & 0.34 & 0.45 & 0.73 \\ 0.85 & 0.31 & 1 & 0.3 & 0.63 & 0.41 \\ 0.34 & 0.34 & 0.3 & 1 & 0.35 & 0.37 \\ 0.63 & 0.45 & 0.63 & 0.35 & 1 & 0.53 \\ 0.33 & 0.73 & 0.41 & 0.37 & 0.53 & 1 \end{pmatrix}.$$

TABLE 2. Table for $\widetilde{SN}_{v_j}^\beta$

Ω	v_1	v_2	v_3	v_4	v_5	v_6
$\widetilde{SN}_{v_1}^\beta$	0.6	0.1	0.6	0.2	0.3	0.2
$\widetilde{SN}_{v_2}^\beta$	0.2	0.7	0.3	0.1	0.4	0.6
$\widetilde{SN}_{v_3}^\beta$	0.6	0.1	0.6	0.2	0.3	0.2
$\widetilde{SN}_{v_4}^\beta$	0.2	0.3	0.1	0.6	0.2	0.1
$\widetilde{SN}_{v_5}^\beta$	0.7	0.2	0.6	0.2	0.8	0.2
$\widetilde{SN}_{v_6}^\beta$	0.2	0.7	0.3	0.1	0.4	0.6

TABLE 3. Table for $\widetilde{SM}_{v_j}^\beta$

	v_1	v_2	v_3	v_4	v_5	v_6
$\widetilde{SM}_{v_1}^\beta$	0.6	0.2	0.6	0.2	0.7	0.2
$\widetilde{SM}_{v_2}^\beta$	0.1	0.7	0.1	0.3	0.2	0.7
$\widetilde{SM}_{v_3}^\beta$	0.6	0.3	0.6	0.1	0.6	0.3
$\widetilde{SM}_{v_4}^\beta$	0.2	0.1	0.2	0.6	0.2	0.1
$\widetilde{SM}_{v_5}^\beta$	0.3	0.4	0.3	0.2	0.8	0.4
$\widetilde{SM}_{v_6}^\beta$	0.2	0.6	0.2	0.1	0.2	0.6

TABLE 4. Table for $\widetilde{SA}_{v_j}^\beta$

	v_1	v_2	v_3	v_4	v_5	v_6
$\widetilde{SA}_{v_1}^\beta$	0.6	0.1	0.6	0.2	0.3	0.2
$\widetilde{SA}_{v_2}^\beta$	0.1	0.7	0.1	0.1	0.2	0.6
$\widetilde{SA}_{v_3}^\beta$	0.6	0.1	0.6	0.1	0.3	0.2
$\widetilde{SA}_{v_4}^\beta$	0.2	0.1	0.1	0.6	0.2	0.1
$\widetilde{SA}_{v_5}^\beta$	0.3	0.2	0.3	0.2	0.8	0.2
$\widetilde{SA}_{v_6}^\beta$	0.2	0.6	0.2	0.1	0.2	0.6

From the above matrix, we can say that the relation R_1 is reflexive and symmetric but not transitive. For example, let $\Omega = \{v_1, v_2, v_3\}$, then

$$R_1 = \begin{pmatrix} 1 & 0.27 & 0.85 \\ 0.27 & 1 & 0.31 \\ 0.85 & 0.31 & 1 \end{pmatrix}.$$

Since $\mathcal{D}_{R_1}^\beta(v_i, v_j)$ is not always less than or equal to $\mathcal{D}_{R_1}^\beta$. Hence R_1 is not transitive.

By using the concept of 1-FS β MD, we define the first kind of Z-fuzzy soft β -covering based ψ -fuzzy rough sets (ψ -Z₁-FS β CFRS).

Definition 3.2. Let $\mathcal{J} = (\Omega, \widetilde{F}, B)_\beta$ be a FS β CAS, for some $\beta \in (0, 1]$. Let $\mathcal{D}_{R_1}^\beta$ be 1-FS β MD of Ω and $\psi \in [0, 1)$. For each $M \in F(\Omega)$, the sets $L_\psi^1(M)$ and $U_\psi^1(M)$ are known as the first kind of Z-fuzzy soft β -covering ψ -lower approximation and upper approximation (briefly, ψ -Z₁-FS β CLA and ψ -Z₁-FS β CUA), where

$$L_\psi^1(M)(v) = \bigwedge_{u \in \Omega} \left\{ M(u) : \mathcal{D}_{R_1}^\beta(v, u) > \psi \right\} \quad \text{and} \quad U_\psi^1(M)(v) = \bigvee_{u \in \Omega} \left\{ M(u) : \mathcal{D}_{R_1}^\beta(v, u) > \psi \right\} \quad \forall v \in \Omega.$$

If $L_\psi^1(M)(v) \neq U_\psi^1(M)(v)$, then M is known as ψ -Z₁-FS β CFRS.

Example 3.2. (Proceeding from Example 3.1). Take $\psi = 0.5$ and let

$$M = \frac{0.5}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.6}{v_5} + \frac{0.8}{v_6}.$$

Then, we have

$$L_\psi^1(M) = \frac{0.2}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.2}{v_5} + \frac{0.3}{v_6}$$

and

$$U_\psi^1(M) = \frac{0.6}{v_1} + \frac{0.8}{v_2} + \frac{0.6}{v_3} + \frac{0.4}{v_4} + \frac{0.8}{v_5} + \frac{0.8}{v_6}.$$

Theorem 3.1. Let $\mathcal{J} = (\Omega, \widetilde{F}, B)_\beta$ be a FS β CAS. For all $M, O \in F(\Omega)$. Then

- (i) $L_\psi^1(M^c) = \left(U_\psi^1(M) \right)^c$.
- (ii) $U_\psi^1(M^c) = \left(L_\psi^1(M) \right)^c$.
- (iii) If $M \subseteq O$, then $L_\psi^1(M) \subseteq L_\psi^1(O)$.
- (iv) If $M \subseteq O$, then $U_\psi^1(M) \subseteq U_\psi^1(O)$.
- (v) $L_\psi^1(M \cap O) = L_\psi^1(M) \cap L_\psi^1(O)$.

- (vi) $U_{\psi}^1(M \cap O) \subseteq U_{\psi}^1(M) \cap U_{\psi}^1(O)$.
- (vii) $L_{\psi}^1(M \cup O) \supseteq L_{\psi}^1(M) \cup L_{\psi}^1(O)$.
- (viii) $U_{\psi}^1(M \cup O) = U_{\psi}^1(M) \cup U_{\psi}^1(O)$.
- (ix) $L_{\psi}^1(M)(v) \leq M(v) \leq U_{\psi}^1(M)(v)$.

Proof. We shall prove only (i), (iii), (v) and (ix), because the proof of (ii) is similar to (i), (iv) is similar to (iii) and (vi), (vii), (viii) are all similar to (v).

(i) Let

$$\begin{aligned} L_{\psi}^1(M^c) &= \bigwedge_{u \in \Omega} \left\{ M^c(u) : \mathcal{D}_{R_1}^{\beta}(v, u) > \psi \right\} \\ &= \bigwedge_{u \in \Omega} \left\{ 1 - M(u) : \mathcal{D}_{R_1}^{\beta}(v, u) > \psi \right\}. \\ &= 1 - \bigvee_{u \in \Omega} \left\{ M(u) : \mathcal{D}_{R_1}^{\beta}(v, u) > \psi \right\}. \\ &= (U_{\psi}^1(M))^c. \end{aligned}$$

(iii) Let $M, O \in F(\Omega)$ such that $M \subseteq O$ and then we have,

$$\begin{aligned} L_{\psi}^1(M) &= \bigwedge_{u \in \Omega} \left\{ M(u) : \mathcal{D}_{R_1}^{\beta}(v, u) > \psi \right\} \leq \bigwedge_{u \in \Omega} \left\{ O(u) : \mathcal{D}_{R_1}^{\beta}(v, u) > \psi \right\} \\ &= L_{\psi}^1(O). \end{aligned}$$

(v) If $v \in \Omega$, then we have

$$\begin{aligned} L_{\psi}^1(M \cap O)(v) &= \bigwedge_{u \in \Omega} \left\{ (M \cap O)(u) : \mathcal{D}_{R_1}^{\beta}(v, u) > \psi \right\}. \\ &= \bigwedge_{u \in \Omega} \left\{ M(u) : \mathcal{D}_{R_1}^{\beta}(v, u) > \psi \right\} \cap \bigwedge_{u \in \Omega} \left\{ O(u) : \mathcal{D}_{R_1}^{\beta}(v, u) > \psi \right\} \\ &= L_{\psi}^1(M) \cap L_{\psi}^1(O). \end{aligned}$$

(ix) By Definition 3.2. □

By using the concept of 1-FS β MD, we define the first kind of Z-fuzzy soft β -covering based \mathcal{D} -fuzzy rough sets (\mathcal{D} -Z₁-FS β CFRS).

Definition 3.3. Let $\mathcal{J} = (\Omega, \tilde{F}, B)_{\beta}$ be a FS β CAS, for some $\beta \in (0, 1]$. Let $\mathcal{D}_{R_1}^{\beta}$ be 1-FS β MD of Ω . For each $M \in F(\Omega)$, the sets $L_{\mathcal{D}}^1(M)$ and $U_{\mathcal{D}}^1(M)$ are known as the first kind of Z-fuzzy soft β -covering \mathcal{D} -lower approximation and upper approximation (briefly, \mathcal{D} -Z₁-FS β CLA and \mathcal{D} -Z₁-FS β CUA), where

$$L_{\mathcal{D}}^1(M)(v) = \bigwedge_{u \in \Omega} \left\{ (1 - \mathcal{D}_{R_1}^{\beta})(v, u) \vee M(v) \right\} \quad \text{and} \quad U_{\mathcal{D}}^1(M)(v) = \bigvee_{u \in \Omega} \left\{ \mathcal{D}_{R_1}^{\beta}(v, u) \wedge M(v) \right\} \quad \forall v \in \Omega.$$

If $L_{\mathcal{D}}^1(M)(v) \neq U_{\mathcal{D}}^1(M)(v)$, then M is known as \mathcal{D} -Z₁-FS β CFRS.

Example 3.3. (Proceeding from Example 3.1). Let

$$M = \frac{0.5}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.6}{v_5} + \frac{0.8}{v_6}.$$

Then, we have

$$L_{\mathcal{D}}^1(M) = \frac{0.2}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.37}{v_5} + \frac{0.3}{v_6}$$

and

$$U_{\mathcal{D}}^1(M) = \frac{0.6}{v_1} + \frac{0.73}{v_2} + \frac{0.6}{v_3} + \frac{0.4}{v_4} + \frac{0.6}{v_5} + \frac{0.8}{v_6}.$$

Next, we obtain the following results for \mathcal{D} -Z₁-FS β CFRS and the proof of these results is equivalent to Theorem 3.1.

Theorem 3.2. *Let $\mathcal{J} = (\Omega, \tilde{F}, B)_\beta$ be a FS β CAS. For all $M, O \in F(\Omega)$, the following hold.*

- (1) $L_{\mathcal{D}}^1(M^c) = (U_{\mathcal{D}}^1(M))^c$.
- (2) $U_{\mathcal{D}}^1(M^c) = (L_{\mathcal{D}}^1(M))^c$.
- (3) If $M \subseteq O$, then $L_{\mathcal{D}}^1(M) \subseteq L_{\mathcal{D}}^1(O)$.
- (4) If $M \subseteq O$, then $U_{\mathcal{D}}^1(M) \subseteq U_{\mathcal{D}}^1(O)$.
- (5) $L_{\mathcal{D}}^1(M \cap O) = L_{\mathcal{D}}^1(M) \cap L_{\mathcal{D}}^1(O)$.
- (6) $U_{\mathcal{D}}^1(M \cap O) \subseteq U_{\mathcal{D}}^1(M) \cap U_{\mathcal{D}}^1(O)$.
- (7) $L_{\mathcal{D}}^1(M \cup O) \supseteq L_{\mathcal{D}}^1(M) \cup L_{\mathcal{D}}^1(O)$.
- (8) $U_{\mathcal{D}}^1(M \cup O) = U_{\mathcal{D}}^1(M) \cup U_{\mathcal{D}}^1(O)$.
- (9) $L_{\mathcal{D}}^1(M)(v) \leq M(v) \leq U_{\mathcal{D}}^1(M)(v)$.

3.2. ψ -Z₂-FS β CFRS and \mathcal{D} -Z₂-FS β CFRS.

Definition 3.4. Let $(\tilde{F}, B)_\beta$ be a fuzzy soft β -covering over Ω . Let

$$R_2 = \left\{ \left((v, u), \mathcal{D}_{R_2}^\beta(v, u) \right) : (v, u) \in \Omega \times \Omega \right\}$$

such that $v = v_i, 1 \leq i \leq p$ and $u = v_j, 1 \leq j \leq p$, is the fuzzy relation, where the mapping $\mathcal{D}_{R_2}^\beta : \Omega \times \Omega \rightarrow [0, 1]$ is defined by

$$\mathcal{D}_{R_2}^\beta(v, u) = \frac{\sum_{k=1}^p \left(\widetilde{SM}_{v_k}^\beta(v) \wedge \widetilde{SM}_{v_k}^\beta(u) \right)}{\sum_{k=1}^p \left(\widetilde{SM}_{v_k}^\beta(v) \vee \widetilde{SM}_{v_k}^\beta(u) \right)}$$

is the second kind of fuzzy soft β -measure degree (2-FS β MD) of v with respect to u . It is obvious that, $\mathcal{D}_{R_2}^\beta(v, v) = 1$ and $\mathcal{D}_{R_2}^\beta(v, u) = \mathcal{D}_{R_2}^\beta(u, v)$. In addition, $0 \leq \mathcal{D}_{R_2}^\beta(v, u) \leq 1$. Then, we can represent the 2-FS β MD in a matrix form. If $n_{ij} = \mathcal{D}_{R_2}^\beta(v, u) = \mathcal{D}_{R_2}^\beta(v_i, v_j)$, we define a matrix

$$[n_{ij}]_{p \times p} = \begin{pmatrix} n_{11} & \cdots & n_{1p} \\ \vdots & \ddots & \vdots \\ n_{p1} & \cdots & n_{pp} \end{pmatrix},$$

which is the matrix depiction of 2-FS β MD of order $p \times p$.

By the definition of 2-FS β MD, we can say that the fuzzy relation R_2 is reflexive and symmetric but not transitive. Therefore, R_2 is a weak fuzzy similarity relation.

Remark 3.2. (i) The first kind of fuzzy soft measure degree (1-FSMD) model introduced by Atef et al. [2] is a special case of 2-FS β MD model when $\beta = 1$. (ii) 2-FS β MD is a generalization of 1-FSMD.

Example 3.4. Let $\mathcal{J} = (\Omega, \tilde{F}, B)_\beta$ be a FS β CAS and $(\tilde{F}, B)_\beta$ is presented in TABLE 1. For $\beta = 0.6$, the matrix form of $\mathcal{D}_{R_2}^\beta$ is given below

$$[n_{ij}]_{6 \times 6} = \begin{pmatrix} 1 & 0.38 & 1 & 0.34 & 0.74 & 0.38 \\ 0.38 & 1 & 0.38 & 0.35 & 0.38 & 1 \\ 1 & 0.38 & 1 & 0.34 & 0.74 & 0.38 \\ 0.34 & 0.35 & 0.34 & 1 & 0.31 & 0.35 \\ 0.74 & 0.38 & 0.74 & 0.31 & 1 & 0.38 \\ 0.38 & 1 & 0.38 & 0.35 & 0.38 & 1 \end{pmatrix}.$$

From the above matrix, we say that the relation R_2 is reflexive and symmetric but not transitive. For example, let $\Omega = \{v_4, v_5, v_6\}$, then

$$R_2 = \begin{pmatrix} 1 & 0.31 & 0.35 \\ 0.31 & 1 & 0.38 \\ 0.35 & 0.38 & 1 \end{pmatrix}.$$

Since $\mathcal{D}_{R_2}^\beta(v_i, v_j)$ is not always less than or equal to $\mathcal{D}_{R_2}^\beta$. Hence R_2 is not transitive.

By using the concept of 2-FS β MD, we define the second kind of Z-fuzzy soft β -covering based ψ -fuzzy rough sets (ψ -Z₂-FS β CFRS).

Definition 3.5. Let $\mathcal{J} = (\Omega, \tilde{F}, B)_\beta$ be a FS β CAS, for some $\beta \in (0, 1]$. Let $\mathcal{D}_{R_2}^\beta$ be 2-FS β MD of Ω and $\psi \in [0, 1)$. For each $M \in F(\Omega)$, the sets $L_\psi^2(M)$ and $U_\psi^2(M)$ are known as the second kind of Z-fuzzy soft β -covering ψ -lower approximation and upper approximation (briefly, ψ -Z₂-FS β CLA and ψ -Z₂-FS β CUA), where

$$L_\psi^2(M)(v) = \bigwedge_{u \in \Omega} \{M(u) : \mathcal{D}_{R_2}^\beta(v, u) > \psi\} \quad \text{and} \quad U_\psi^2(M)(v) = \bigvee_{u \in \Omega} \{M(u) : \mathcal{D}_{R_2}^\beta(v, u) > \psi\} \quad \forall v \in \Omega.$$

If $L_\psi^2(M)(v) \neq U_\psi^2(M)(v)$, then M is known as ψ -Z₂-FS β CFRS.

Example 3.5. (Proceeding from Example 3.1). Let

$$M = \frac{0.5}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.6}{v_5} + \frac{0.8}{v_6}$$

and take $\psi = 0.5$. Then, we have

$$L_\psi^2(M) = \frac{0.2}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.2}{v_5} + \frac{0.3}{v_6}$$

and

$$U_\psi^2(M) = \frac{0.6}{v_1} + \frac{0.8}{v_2} + \frac{0.6}{v_3} + \frac{0.4}{v_4} + \frac{0.6}{v_5} + \frac{0.8}{v_6}.$$

Here, we obtain the following results for ψ -Z₂-FS β CFRS and the proof of these results is equivalent to Theorem 3.1.

Theorem 3.3. Let $\mathcal{J} = (\Omega, \tilde{F}, B)_\beta$ be a FS β CAS. For all $M, O \in F(\Omega)$. Then,

- (1) $L_\psi^2(M^c) = \left(U_\psi^2(M)\right)^c$.
- (2) $U_\psi^2(M^c) = \left(L_\psi^2(M)\right)^c$.
- (3) If $M \subseteq O$, then $L_\psi^2(M) \subseteq L_\psi^2(O)$.
- (4) If $M \subseteq O$, then $U_\psi^2(M) \subseteq U_\psi^2(O)$.
- (5) $L_\psi^2(M \cap O) = L_\psi^2(M) \cap L_\psi^2(O)$.

- (6) $U_{\psi}^2(M \cap O) \subseteq U_{\psi}^2(M) \cap U_{\psi}^2(O)$.
- (7) $L_{\psi}^2(M \cup O) \supseteq L_{\psi}^2(M) \cup L_{\psi}^2(O)$.
- (8) $U_{\psi}^2(M \cup O) = U_{\psi}^2(M) \cup U_{\psi}^2(O)$.
- (9) $L_{\psi}^2(M)(v) \leq M(v) \leq U_{\psi}^2(M)(v)$.

By using the concept of 2-FS β MD, we define the second kind of Z-fuzzy soft β -covering based \mathcal{D} -fuzzy rough sets (\mathcal{D} -Z₂-FS β CFRS).

Definition 3.6. Let $\mathcal{J} = (\Omega, \tilde{F}, B)_{\beta}$ be a FS β CAS, for some $\beta \in (0, 1]$. Let $\mathcal{D}_{R_2}^{\beta}$ be 2-FS β MD of Ω . For each $M \in F(\Omega)$, the sets $L_{\mathcal{D}}^2(M)$ and $U_{\mathcal{D}}^2(M)$ are known as the second kind of Z-fuzzy soft β -covering \mathcal{D} -lower approximation and upper approximation (briefly, \mathcal{D} -Z₂-FS β CLA and \mathcal{D} -Z₂-FS β CUA), where

$$L_{\mathcal{D}}^2(M)(v) = \bigwedge_{u \in \Omega} \left\{ (1 - \mathcal{D}_{R_2}^{\beta})(v, u) \vee M(v) \right\} \quad \text{and} \quad U_{\mathcal{D}}^2(M)(v) = \bigvee_{u \in \Omega} \left\{ \mathcal{D}_{R_2}^{\beta}(v, u) \wedge M(v) \right\} \quad \forall v \in \Omega.$$

If $L_{\mathcal{D}}^2(M)(v) \neq U_{\mathcal{D}}^2(M)(v)$, then M is known as \mathcal{D} -Z₂-FS β CFRS.

Example 3.6. (Proceeding from Example 3.1). Let

$$M = \frac{0.5}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.6}{v_5} + \frac{0.8}{v_6}.$$

Then, we have

$$L_{\mathcal{D}}^2(M) = \frac{0.2}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.26}{v_5} + \frac{0.3}{v_6}$$

and

$$U_{\mathcal{D}}^2(M) = \frac{0.6}{v_1} + \frac{0.8}{v_2} + \frac{0.6}{v_3} + \frac{0.4}{v_4} + \frac{0.6}{v_5} + \frac{0.8}{v_6}.$$

Here, we obtain the following results for \mathcal{D} -Z₂-FS β CFRS and the proof of these results is equivalent to Theorem 3.1.

Theorem 3.4. Let $\mathcal{J} = (\Omega, \tilde{F}, B)_{\beta}$ be a FS β CAS. For all $M, O \in F(\Omega)$. Then

- (1) $L_{\mathcal{D}}^2(M^c) = (U_{\mathcal{D}}^2(M))^c$.
- (2) $U_{\mathcal{D}}^2(M^c) = (L_{\mathcal{D}}^2(M))^c$.
- (3) If $M \subseteq O$, then $L_{\mathcal{D}}^2(M) \subseteq L_{\mathcal{D}}^2(O)$.
- (4) If $M \subseteq O$, then $U_{\mathcal{D}}^2(M) \subseteq U_{\mathcal{D}}^2(O)$.
- (5) $L_{\mathcal{D}}^2(M \cap O) = L_{\mathcal{D}}^2(M) \cap L_{\mathcal{D}}^2(O)$.
- (6) $U_{\mathcal{D}}^2(M \cap O) \subseteq U_{\mathcal{D}}^2(M) \cap U_{\mathcal{D}}^2(O)$.
- (7) $L_{\mathcal{D}}^2(M \cup O) \supseteq L_{\mathcal{D}}^2(M) \cup L_{\mathcal{D}}^2(O)$.
- (8) $U_{\mathcal{D}}^2(M \cup O) = U_{\mathcal{D}}^2(M) \cup U_{\mathcal{D}}^2(O)$.
- (9) $L_{\mathcal{D}}^2(M)(v) \leq M(v) \leq U_{\mathcal{D}}^2(M)(v)$.

3.3. ψ -Z₃-FS β CFRS and \mathcal{D} -Z₃-FS β CFRS.

Definition 3.7. Let $(\tilde{F}, B)_{\beta}$ be a fuzzy soft β -covering over Ω . Let

$$R_3 = \left\{ \left((v, u), \mathcal{D}_{R_3}^{\beta}(v, u) \right) : (v, u) \in \Omega \times \Omega \right\}$$

such that $v = v_i$, $1 \leq i \leq p$ and $u = v_j$, $1 \leq j \leq p$, is the fuzzy relation, where the mapping $\mathcal{D}_{R_3}^\beta : \Omega \times \Omega \rightarrow [0, 1]$ is defined by

$$\mathcal{D}_{R_3}^\beta(v, u) = \frac{\sum_{k=1}^p \left(\widetilde{S}A_{v_k}^\beta(v) \wedge \widetilde{S}A_{v_k}^\beta(u) \right)}{\sum_{k=1}^p \left(\widetilde{S}A_{v_k}^\beta(v) \vee \widetilde{S}A_{v_k}^\beta(u) \right)}$$

is the third kind of fuzzy soft β -measure degree (3-FS β MD) of v with respect to u . It is obvious that, $\mathcal{D}_{R_3}^\beta(v, v) = 1$ and $\mathcal{D}_{R_3}^\beta(v, u) = \mathcal{D}_{R_3}^\beta(u, v)$. In addition, $0 \leq \mathcal{D}_{R_3}^\beta(v, u) \leq 1$. Then, we can represent the 3-FS β MD in a matrix form. If $K_{ij} = \mathcal{D}_{R_3}^\beta(v_i, v_j)$, we present a matrix

$$[K_{ij}]_{p \times p} = \begin{pmatrix} K_{11} & \dots & K_{1p} \\ \vdots & \ddots & \vdots \\ K_{p1} & \dots & K_{pp} \end{pmatrix},$$

which is the matrix depiction of 3-FS β MD of order $p \times p$.

By definition 3-FS β MD, we can say that the fuzzy relation R_3 is reflexive and symmetric but not transitive. Therefore, R_3 is a weak fuzzy similarity relation.

Remark 3.3. (i) The second kind of fuzzy soft measure degree (2-FSMD) model introduced by Atef et al. [2] is a special case of 3-FS β MD model when $\beta = 1$. (ii) 3-FS β MD is a generalization of 2-FSMD.

Example 3.7. Let $\mathcal{J} = (\Omega, \widetilde{F}, B)_\beta$ be a FS β CAS and $(\widetilde{F}, B)_\beta$ is presented in TABLE 1. For $\beta = 0.6$, the matrix form of $\mathcal{D}_{R_3}^\beta$ is given below

$$[K_{ij}]_{6 \times 6} = \begin{pmatrix} 1 & 0.26 & 0.95 & 0.37 & 0.53 & 0.34 \\ 0.26 & 1 & 0.27 & 0.29 & 0.31 & 0.85 \\ 0.95 & 0.27 & 1 & 0.33 & 0.5 & 0.35 \\ 0.37 & 0.29 & 0.33 & 1 & 0.37 & 0.33 \\ 0.53 & 0.31 & 0.5 & 0.37 & 1 & 0.39 \\ 0.34 & 0.85 & 0.35 & 0.33 & 0.39 & 1 \end{pmatrix}.$$

From the above matrix, we say that the relation R_3 is reflexive and symmetric but not transitive. For example, let $\Omega = \{v_1, v_2, v_3\}$, then

$$R_3 = \begin{pmatrix} 1 & 0.26 & 0.95 \\ 0.26 & 1 & 0.27 \\ 0.95 & 0.27 & 1 \end{pmatrix}.$$

Since $\mathcal{D}_{R_3}^\beta(v_i, v_j)$ is not always less than or equal to $\mathcal{D}_{R_3}^\beta$. Hence R_3 is not transitive.

By using the concept of 3-FS β MD, we define the third kind of Z-fuzzy soft β -covering based ψ -fuzzy rough sets (ψ -Z $_3$ -FS β CFRS).

Definition 3.8. Let $\mathcal{J} = (\Omega, \widetilde{F}, B)_\beta$ be a FS β CAS, for some $\beta \in (0, 1]$. Let $\mathcal{D}_{R_3}^\beta$ be 3-FS β MD of Ω and $\psi \in [0, 1)$. For each $M \in F(\Omega)$, the sets $L_\psi^3(M)$ and $U_\psi^3(M)$ are known as the third kind of Z-fuzzy soft β -covering ψ -lower approximation and upper approximation (briefly, ψ -Z $_3$ -FS β CLA and ψ -Z $_3$ -FS β CUA), where

$$L_\psi^3(M)(v) = \bigwedge_{u \in \Omega} \left\{ M(u) : \mathcal{D}_{R_3}^\beta(v, u) > \psi \right\} \quad \text{and} \quad U_\psi^3(M)(v) = \bigvee_{u \in \Omega} \left\{ M(u) : \mathcal{D}_{R_3}^\beta(v, u) > \psi \right\} \quad \forall v \in \Omega.$$

If $L_\psi^3(M)(v) \neq U_\psi^3(M)(v)$, then M is known as ψ -Z $_3$ -FS β CFRS.

Example 3.8. (Proceeding from Example 3.1). Take $\psi = 0.5$ and let

$$M = \frac{0.5}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.6}{v_5} + \frac{0.8}{v_6}$$

Then, we have

$$L_\psi^3(M) = \frac{0.2}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.2}{v_5} + \frac{0.3}{v_6}$$

and

$$U_\psi^3(M) = \frac{0.6}{v_1} + \frac{0.8}{v_2} + \frac{0.6}{v_3} + \frac{0.4}{v_4} + \frac{0.6}{v_5} + \frac{0.6}{v_6}.$$

Here, we obtain the following results for ψ -Z₃-FS β CFRS and the proof of these results is equivalent to Theorem 3.1.

Theorem 3.5. Let $\mathcal{J} = (\Omega, \tilde{F}, B)_\beta$ be a FS β CAS. For all $M, O \in F(\Omega)$. Then

- (1) $L_\psi^3(M^c) = \left(U_\psi^3(M)\right)^c$.
- (2) $U_\psi^3(M^c) = \left(L_\psi^3(M)\right)^c$.
- (3) If $M \subseteq O$, then $L_\psi^3(M) \subseteq L_\psi^3(O)$.
- (4) If $M \subseteq O$, then $U_\psi^3(M) \subseteq U_\psi^3(O)$.
- (5) $L_\psi^3(M \cap O) = L_\psi^3(M) \cap L_\psi^3(O)$.
- (6) $U_\psi^3(M \cap O) \subseteq U_\psi^3(M) \cap U_\psi^3(O)$.
- (7) $L_\psi^3(M \cup O) \supseteq L_\psi^3(M) \cup L_\psi^3(O)$.
- (8) $U_\psi^3(M \cup O) = U_\psi^3(M) \cup U_\psi^3(O)$.
- (9) $L_\psi^3(M)(v) \leq M(v) \leq U_\psi^3(M)(v)$.

By using the concept of 3-FS β MD, we define the third kind of Z-fuzzy soft β -covering based \mathcal{D} -fuzzy rough sets (\mathcal{D} -Z₃-FS β CFRS).

Definition 3.9. Let $\mathcal{J} = (\Omega, \tilde{F}, B)_\beta$ be a FS β CAS, for some $\beta \in (0, 1]$. Let $\mathcal{D}_{R_3}^\beta$ be 3-FS β MD of Ω . For each $M \in F(\Omega)$, the sets $L_\mathcal{D}^3(M)$ and $U_\mathcal{D}^3(M)$ are known as the third kind of Z-fuzzy soft β -covering \mathcal{D} -lower approximation and upper approximation (briefly, \mathcal{D} -Z₃-FS β CLA and \mathcal{D} -Z₃-FS β CUA), where

$$L_\mathcal{D}^3(M)(v) = \bigwedge_{u \in \Omega} \left\{ (1 - \mathcal{D}_{R_3}^\beta)(v, u) \vee M(v) \right\} \text{ and } U_\mathcal{D}^3(M)(v) = \bigvee_{u \in \Omega} \left\{ \mathcal{D}_{R_3}^\beta(v, u) \wedge M(v) \right\} \quad \forall v \in \Omega.$$

If $L_\mathcal{D}^3(M)(v) \neq U_\mathcal{D}^3(M)(v)$, then M is known as \mathcal{D} -Z₃-FS β CFRS.

Example 3.9. (Proceeding from Example 3.1). Let

$$M = \frac{0.5}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.6}{v_5} + \frac{0.8}{v_6}.$$

Then, we have

$$L_\mathcal{D}^3(M) = \frac{0.2}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.5}{v_5} + \frac{0.3}{v_6}$$

and

$$U_\mathcal{D}^3(M) = \frac{0.53}{v_1} + \frac{0.8}{v_2} + \frac{0.5}{v_3} + \frac{0.4}{v_4} + \frac{0.6}{v_5} + \frac{0.8}{v_6}.$$

Here, we obtain the following results for \mathcal{D} -Z₃-FS β CFRS and the proof of these results is equivalent to Theorem 3.1.

Theorem 3.6. Let $\mathcal{J} = (\Omega, \tilde{F}, B)_\beta$ be a FS β CAS, for all $M, O \in F(\Omega)$. Then

- (1) $L_{\mathcal{D}}^3(M^c) = (U_{\mathcal{D}}^3(M))^c$.
- (2) $U_{\mathcal{D}}^3(M^c) = (L_{\mathcal{D}}^3(M))^c$.
- (3) If $M \subseteq O$, then $L_{\mathcal{D}}^3(M) \subseteq L_{\mathcal{D}}^3(O)$.
- (4) If $M \subseteq O$, then $U_{\mathcal{D}}^3(M) \subseteq U_{\mathcal{D}}^3(O)$.
- (5) $L_{\mathcal{D}}^3(M \cap O) = L_{\mathcal{D}}^3(M) \cap L_{\mathcal{D}}^3(O)$.
- (6) $U_{\mathcal{D}}^3(M \cap O) \subseteq U_{\mathcal{D}}^3(M) \cap U_{\mathcal{D}}^3(O)$.
- (7) $L_{\mathcal{D}}^3(M \cup O) \supseteq L_{\mathcal{D}}^3(M) \cup L_{\mathcal{D}}^3(O)$.
- (8) $U_{\mathcal{D}}^3(M \cup O) = U_{\mathcal{D}}^3(M) \cup U_{\mathcal{D}}^3(O)$.
- (9) $L_{\mathcal{D}}^3(M)(v) \leq M(v) \leq U_{\mathcal{D}}^3(M)(v)$.

4. THE RELATIONSHIP BETWEEN THREE KINDS OF ψ -Z-FS β CFRS AND THREE KINDS OF \mathcal{D} -Z-FS β CFRS

In this section, we present the relationship between the lower and upper approximations of three kinds of ψ -Z-FS β CFRS and three kinds of \mathcal{D} -Z-FS β CFRS.

Proposition 4.1. *Let $\mathcal{J} = (\Omega, \tilde{F}, B)_{\beta}$ be a FS β CAS. For all $M \in F(\Omega)$. Then the following relation holds:*

- (1) $L_{\psi}^1(M) \subseteq L_{\psi}^2(M) \subseteq L_{\psi}^3(M)$.
- (2) $U_{\psi}^3(M) \subseteq U_{\psi}^2(M) \subseteq U_{\psi}^1(M)$.
- (3) $L_{\mathcal{D}}^2(M) \subseteq L_{\mathcal{D}}^1(M) \subseteq L_{\mathcal{D}}^3(M)$.
- (4) $U_{\mathcal{D}}^3(M) \subseteq U_{\mathcal{D}}^1(M) \subseteq U_{\mathcal{D}}^2(M)$.

Proof. The proof is obvious from Definitions 3.2, 3.3, 3.5, 3.6, 3.8 and 3.9. □

5. COMPARISON AND DISCUSSION

In 2019, the idea of a fuzzy soft neighborhood was initiated by Zhan et al. [36]. By using this concept, fuzzy soft measure degree, ψ -soft rough fuzzy covering and \mathcal{D} -soft rough fuzzy covering were formed. A fuzzy soft complementary neighborhood was proposed in [2] to create a new kind of fuzzy soft measure degree, ψ -SFRC and \mathcal{D} -SFRC. Zhan et al. [39] defined fuzzy soft β -neighborhood and fuzzy soft complementary β -neighborhood in order to develop a novel model of FS β CFRS. In 2021, Nawar et al. [18] introduced fuzzy soft β -adhesion and a new kind of FS β CFRS.

As a generalization of the concepts [2, 36], we have defined three kinds of ψ -Z-FS β CFRS and \mathcal{D} -Z-FS β CFRS in Section 3. Now, we compare the lower (ψ -Z-FSCLA, \mathcal{D} -Z-FSCLA) and upper approximations (ψ -Z-FSCUA, \mathcal{D} -Z-FSCUA) of three kinds of ψ -Z-FS β CFRS and \mathcal{D} -Z-FS β CFRS respectively.

Using Example 3.1, the lower and upper approximations of the three kinds of ψ -Z-FS β CFRS and \mathcal{D} -Z-FS β CFRS are displayed in FIGURE 1 and FIGURE 2 respectively. In FIGURE 1, the lower approximation is same for all three kinds of ψ -Z-FS β CFRS, whereas ψ -Z₃-FSCUA has the least upper approximation. In FIGURE 2, we can see that the third kind of \mathcal{D} -Z-FS β CFRS has the least upper approximation and the greatest lower approximation. The graph shows that the upper approximation decreases and the lower approximation increases for the third kind of ψ -Z-FS β CFRS and \mathcal{D} -Z-FS β CFRS. As a result of the above discussion, we can conclude that the third kind of ψ -Z-FS β CFRS and the third kind of \mathcal{D} -Z-FS β CFRS are effective.

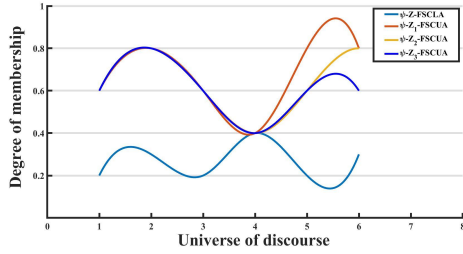


FIGURE 1. The lower and upper approximations of the three kinds of ψ -Z-FS β CFRS

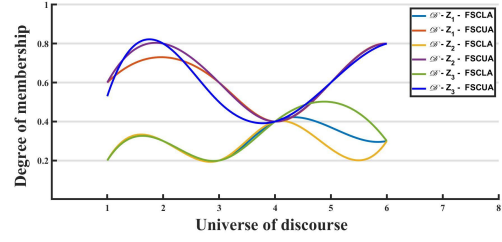


FIGURE 2. The lower and upper approximations of the three kinds of \mathcal{D} -Z-FS β CFRS

6. APPLICATION OF \mathcal{D} -Z₃-FS β CFRS IN MAGDM

This section proposes a MAGDM model using \mathcal{D} -Z₃-FS β CFRS, since the precision of the third kind of \mathcal{D} -Z-FS β CFRS is better than other kinds of \mathcal{D} -Z-FS β CFRS.

6.1. Description and process. For an institute, we need to recruit some skilled people. Let $\Omega = \{v_1, v_2, \dots, v_p\}$ be p candidates and $B = \{b_1, b_2, \dots, b_m\}$ be the attribute set, where $B \subseteq E$. Let β be the critical value. Each applicant must have at least one skill for which the applicant's value is greater than or equal to β . Then $(\tilde{F}, B)_\beta$ is a fuzzy soft β -covering of Ω . Let the fuzzy set M represent the expert's assessment values of each applicant. Calculate 3-FS β MD for each pair of candidates by,

$$\mathcal{D}_{R_3}^\beta(v, u) = \frac{\sum_{i=1}^p \left(\widetilde{S}A_{v_i}(v) \wedge \widetilde{S}A_{v_i}(u) \right)}{\sum_{i=1}^p \left(\widetilde{S}A_{v_i}(v) \vee \widetilde{S}A_{v_i}(u) \right)}.$$

Calculate the lower and upper approximations for M using Definition 3.9. Compute the maximal value for each candidates by, $max(v_i) = M(v_i) + L_{\mathcal{D}}^3(X)(v_i) + U_{\mathcal{D}}^3(X)(v_i)$. Rank the candidates using their maximal values.

6.2. Algorithm. A standard algorithm for the decision-making process is given in this subsection.

Algorithm 1

- 1: Input M , and $(\tilde{F}, B)_\beta$, which denotes the assessment values of the expert group.
 - 2: Using Definition 3.7, calculate 3-FS β MD.
 - 3: By using Definition 3.9, calculate \mathcal{D} -Z₃-FSCLA and \mathcal{D} -Z₃-FSCUA.
 - 4: Compute $max(v_i) = M(v_i) + L_{\mathcal{D}}^3(X)(v_i) + U_{\mathcal{D}}^3(X)(v_i)$.
 - 5: Rank the candidates and obtain the optimal solution.
-

Remark 6.1. The above algorithm is a linear time algorithm whose time complexity is $O(n)$, where n denotes the number of persons in the expert group.

6.3. Illustrative example. The following numerical example illustrates the steps specified in the preceding algorithm.

An interview is going on for the post of assistant professor in a university. Panel consisting of applicants for interview form a set $\Omega = \{v_1, v_2, \dots, v_6\}$ and the attribute set consists of their notable

TABLE 5. Table for $(\tilde{F}, B)_\beta$

Ω	b_1	b_2	b_3	b_4	b_5	b_6	b_7
v_1	0.2	0.7	0.3	0.8	0.4	0.6	0.3
v_2	0.3	0.4	0.1	0.6	0.7	0.5	0.4
v_3	0.6	0.4	0.7	0.3	0.8	0.2	0.5
v_4	0.4	0.6	0.3	0.7	0.2	0.5	0.7
v_5	0.4	0.3	0.2	0.4	0.9	0.6	0.5
v_6	0.7	0.8	0.3	0.6	0.1	0.5	0.4

TABLE 6. Table for $\tilde{S}A_{v_j}^\beta$

Ω	v_1	v_2	v_3	v_4	v_5	v_6
$\tilde{S}A_{v_1}^\beta$	0.6	0.4	0.2	0.3	0.3	0.2
$\tilde{S}A_{v_2}^\beta$	0.4	0.6	0.1	0.2	0.4	0.1
$\tilde{S}A_{v_3}^\beta$	0.2	0.1	0.6	0.2	0.2	0.1
$\tilde{S}A_{v_4}^\beta$	0.3	0.2	0.2	0.6	0.2	0.4
$\tilde{S}A_{v_5}^\beta$	0.3	0.4	0.2	0.2	0.6	0.1
$\tilde{S}A_{v_6}^\beta$	0.2	0.1	0.1	0.4	0.1	0.6

characteristics such as communication skill (b_1), teaching skill (b_2), presentation skill (b_3), experience (b_4), academic records (b_5), time management skills (b_6), and patience (b_7). The organization assigns a critical value of 0.6 to determine whether the individual meets the skill requirements. Using this algorithm, we have to choose the best candidate for the job. Our aim is to select the most deserving candidate. The numerical example of the specified algorithm is executed here. Experts analyze each applicant based on the attributes and provide their observations with actual values. The summary of these actual values are given in TABLE 5.

The overall assessment values of each applicant are given by,

$$M = \frac{0.5}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.6}{v_5} + \frac{0.8}{v_6}.$$

The matrix form of $\mathcal{D}_{R_3}^\beta$ is given below,

$$[K_{ij}]_{6 \times 6} = \begin{pmatrix} 1 & 0.65 & 0.42 & 0.56 & 0.65 & 0.4 \\ 0.65 & 1 & 0.39 & 0.42 & 0.71 & 0.32 \\ 0.42 & 0.39 & 1 & 0.43 & 0.45 & 0.38 \\ 0.56 & 0.42 & 0.43 & 1 & 0.48 & 0.61 \\ 0.65 & 0.71 & 0.45 & 0.48 & 1 & 0.32 \\ 0.4 & 0.32 & 0.38 & 0.61 & 0.32 & 1 \end{pmatrix}$$

TABLE 7. Table for ranking outcomes.

Our proposed models	Obtain a decision
ψ -Z ₁ -FS β CFRS	$v_6 \approx v_5 \approx v_4 \approx v_2 \geq v_1 \geq v_3$
\mathcal{D} -Z ₁ -FS β CFRS	$v_6 \geq v_5 \geq v_4 \approx v_1 \geq v_2 \geq v_3$
ψ -Z ₂ -FS β CFRS	$v_6 \geq v_5 \geq v_1 \approx v_4 \geq v_2 \approx v_3$
\mathcal{D} -Z ₂ -FS β CFRS	$v_6 \approx v_5 \geq v_4 \geq v_1 \geq v_2 \geq v_3$
ψ -Z ₃ -FS β CFRS	$v_6 \geq v_5 \approx v_1 \geq v_4 \approx v_2 \geq v_3$
\mathcal{D} -Z ₃ -FS β CFRS	$v_6 \geq v_5 \geq v_1 \geq v_4 \geq v_2 \geq v_3$

The lower and upper approximations of M are as follows,

$$L_{\mathcal{D}}^3(M) = \frac{0.35}{v_1} + \frac{0.3}{v_2} + \frac{0.2}{v_3} + \frac{0.4}{v_4} + \frac{0.3}{v_5} + \frac{0.4}{v_6},$$

$$U_{\mathcal{D}}^3(M) = \frac{0.6}{v_1} + \frac{0.6}{v_2} + \frac{0.45}{v_3} + \frac{0.61}{v_4} + \frac{0.6}{v_5} + \frac{0.8}{v_6}.$$

Based on the above information, the maximal values for each candidate are $\max(v_1) = 1.45$, $\max(v_2) = 1.2$, $\max(v_3) = 0.85$, $\max(v_4) = 1.41$, $\max(v_5) = 1.5$, $\max(v_6) = 2$. The candidates are arranged according to their maximal values, $v_6 \geq v_5 \geq v_1 \geq v_4 \geq v_2 \geq v_3$.

From the above ranking, we can conclude that the candidate v_6 is more suitable for the job.

6.4. Advantages of the proposed model. Real-life decision-making issues frequently arise in complicated environments with imprecise and unclear data, which makes them challenging to solve. When the data is complicated, hazy, or ambiguous, the offered approaches are quite appropriate, especially when the underlying data is based on fuzzy soft covering. The following are a few benefits of the recommended techniques:

- Prescriptive decision-making methods can be applied to any kind of unsolved decision problem and are easily understood.
- The proposed model will provide accurate results when compared with other existing models, which helps in finding the optimal results in the decision-making process.
- In contrast to the ψ -SFRC and \mathcal{D} -SFRC models defined in [2, 36], the technique proposed in this study has persistent distinguishability, which can aid us in controlling the particular ideal explanation for diverse decision makers safety preferences. Contrarily, the ψ -SFRC and \mathcal{D} -SFRC methodology may result in near-full evaluation values for each scheme, diminishing the method's capacity to discriminate between the schemes and making it more difficult to do so.

6.5. Comparison between the proposed models. In this subsection, we replace Definition 3.9 in Algorithm 1 with Definitions 3.2, 3.3, 3.5, 3.6 and 3.8 to obtain the decision results for each newly proposed model. TABLE 7 shows that the third kind of \mathcal{D} -Z-FS β CFRS provides more accurate results than other models, which is the main reason for selecting this model for application.

6.6. Comparison with other existing techniques. The literature contains a wide range of methods that can be used to solve various decision-making problems. Each of these decision-making strategies has its own advantages and disadvantages. The effectiveness of each technique is determined by the chosen problem. Here, we compare the proposed decision-making strategies with some existing decision-making techniques. In this subsection, we compare our model with Nawar's model [18], Zhan's model [36] and Zhang's model [39] to illustrate the significance of our model in the decision-making process. As shown in TABLE 8, we get $v_1 \approx v_4$ using existing models. Therefore, we cannot determine which

TABLE 8. Table for ranking results

Various models	Obtained result
Nawar's model	$v_6 \geq v_5 \geq v_1 \approx v_4 \approx v_2 \geq v_3$
Zhan's model	$v_6 \geq v_5 \geq v_1 \approx v_4 \geq v_2 \geq v_3$
Zhang's model	$v_6 \geq v_5 \geq v_1 \approx v_4 \geq v_2 \approx v_3$
Our model	$v_6 \geq v_5 \geq v_1 \geq v_4 \geq v_2 \geq v_3$

is better between v_1 and v_4 . However, using our method, we obtained $v_1 \geq v_4$, indicating that v_1 is more effective than v_4 . This is particularly useful for identifying which alternative is better than the other. Our method provides a clear ranking of alternatives from best to worst. Therefore, the newly proposed model gives more precise and accurate results compared to other existing models, which shows the effectiveness and importance of our model.

7. CONCLUSION

In this paper, three kinds of ψ -Z-FS β CFRS and three kinds of \mathcal{D} -Z-FS β CFRS are defined to increase the lower approximation and to decrease the upper approximation of ψ -SFRC and \mathcal{D} -SFRC models. By analyzing these models, we conclude that the third kind of ψ -Z-FS β CFRS and third kind of \mathcal{D} -Z-FS β CFRS are the most effective models. Moreover, the effective model \mathcal{D} -Z₃-FS β CFRS is used to propose a novel MAGDM method to recruit a suitable candidate for the assistant professor job. As a result, we found that the applicant v_6 is the best candidate. A limitation of the study is that the three proposed models provide similar results with little difference in values. To select one of the three models to use in the decision-making process, the problem must first be analyzed. In the future, we plan to learn more about fuzzy soft β -minimal descriptions, fuzzy soft β -maximal descriptions and fuzzy soft β -covering, as well as explore and develop some new kinds of fuzzy soft β -covering based fuzzy rough set. We will develop a novel multi-attribute group decision-making using these new models that will provides clear results for any kind of problem.

REFERENCES

- [1] M. I. Ali, *A note on soft sets, rough soft sets and fuzzy soft sets*, Appl. Soft Comput. **11** (2011), 3329-3332.
- [2] M. Atef and S.I. Nada, *On three types of soft fuzzy coverings based rough sets*, Math. Comput. Simul. **185** (2021), 452-467.
- [3] M. Atef, M. I. Ali and T. M. Al-shami, *Fuzzy soft covering-based multi-granulation fuzzy rough sets and their application*, Comput. Appl. Math. **40** (2021), 1-26.
- [4] E. Atik, A. El Fattah, A. Nawar and M. Atef, *Rough approximation models via graphs based on neighborhood systems*, Granul. Comput. **6** (2021), 1025-1035.
- [5] T. Deng, Y. Chen, W. Xu and Q. Dai, *A novel approach to fuzzy rough sets based on a fuzzy covering*, Inf. Sci. **177** (2007), 2308-2326.
- [6] D. Dubois and H. Prade, *Rough fuzzy sets and fuzzy rough sets*, Int. J. General Syst. **17** (1990), 191-209.
- [7] S. M. El-Ghamrawy and A. I. Eldesouky, *An agent decision support module based on granular rough model*, Int. J. Inf. Technol. Decis. Mak. **11** (2012), 793-820.
- [8] F. Feng, X. Liu, V. Leoreanu-Fotea and Y. B. Jun, *Soft sets and soft rough sets*, Inf. Sci. **181** (2011), 1125-1137.
- [9] T. Feng, S. P. Zhang and J. S. Mi, *The reduction and fusion of fuzzy covering systems based on the evidence theory*, Int. J. Approx. Reason. **53** (2012), 87-103.
- [10] S. Greco, B. Matarazzo and R. Slowinski, *Rough sets theory for multicriteria decision analysis*, Eur. J. Oper. Res. **129** (2001), 1-47.
- [11] G. Klir and B. Yuan, *Fuzzy sets and Fuzzy logic*, Prentice hall, New Jersey, **4** (1995), 1-12.
- [12] G. Liu and W. Zhu, *The algebraic structures of generalized rough set theory*, Inf. Sci. **178** (2008), 4105-4113.

- [13] L. Ma, *On some types of neighborhood-related covering rough sets*, Int. J. Approx. Reason. **53** (2012), 901-911.
- [14] L. Ma, *Two fuzzy covering rough set models and their generalizations over fuzzy lattices*, Fuzzy Sets Syst. **294** (2016), 1-17.
- [15] P. K. Maji, A. R. Roy and R. Biswas, *An application of soft sets in a decision making problem*, Comput. Math. with Appl. **44** (2002), 1077-1083.
- [16] P. K. Maji, R. Biswas and A. R. Roy, *Soft set theory*, Comput. Math. with Appl. **45** (2003), 555-562.
- [17] D. Molodtsov, *Soft set theory-First results*, Comput. Math. with Appl. **37** (1999), 19-31.
- [18] A. S. Nawar, M. Atef and A. M. Khalil, *Certain types of fuzzy soft β -covering based fuzzy rough sets with application to decision-making*, J. Intell. Fuzzy Syst. **40** (2021), 10825-10836.
- [19] Z. Pawlak, *Rough sets*, Int. J. Comput. Inf. **11** (1982), 341-356.
- [20] A. M. Radzikowska and E. E. Kerre, *A comparative study of fuzzy rough sets*, Fuzzy Sets Syst. **126** (2002), 137-155.
- [21] A. R. Roy and P. K. Maji, *A fuzzy soft set theoretic approach to decision making problems*, J. Comput. Appl. Math. **203** (2007), 412-418.
- [22] M. Shabir, M. I. Ali and T. Shaheen, *Another approach to soft rough sets*, Knowl. Based Syst. **40**(2013), 72-80.
- [23] M. Shabir, R. S. Kanwal and M. I. Ali, *Reduction of an information system*, Soft Comput. **24** (2020), 10801-10813.
- [24] T. Shaheen, M. I. Ali and M. Shabir, *Generalized hesitant fuzzy rough sets (GHFRS) and their application in risk analysis*, Soft Comput. **24** (2020), 14005-14017.
- [25] P. Sivaprakasam, and M. Angamuthu, *Generalized Z-fuzzy soft β -covering based rough matrices and its application to MAGDM problem based on AHP method*, Decision Making: Applications in Management and Engineering, **6** (2023), 134-152.
- [26] B. Sun, W. Ma and Y. Qian, *Multi-granulation fuzzy rough set over two universes and its application to decision making*, Knowl. Based Syst. **123** (2017), 61-74.
- [27] G. Xin and L. Ying, (2024). *Multi-attribute decision-making based on comprehensive hesitant fuzzy entropy*, Expert Syst. Appl., **237** (2024), 121459.
- [28] B. Yang and B. Q. Hu, *On some types of fuzzy covering-based rough sets*, Fuzzy Sets Syst. **312** (2017), 36-65.
- [29] B. Yang, *Fuzzy covering-based rough set on two different universes and its application*, Artif. Intell. Rev. **5** (2022), 4717-4753.
- [30] Y. Yang, D. Chen, X. Zhang and Z. Ji, *Covering rough set-based incremental feature selection for mixed decision system*, Soft Comput. **26** (2022), 2651-2669.
- [31] D. S. Yeung, D. Chen, E. C. Tsang, J. W. Lee and W. Xizhao, *On the generalization of fuzzy rough sets*, IEEE Trans. Fuzzy Syst. **13** (2005), 343-361.
- [32] Ş. Yüksel, N. Tozlu and T. H. Dizman, *An application of multicriteria group decision making by soft covering based rough sets*, Filomat **29** (2015), 209-219.
- [33] L. A. Zadeh, *Fuzzy sets*, Inf. Control. **8** (1965), 338-353.
- [34] J. Zhan, B. Sun and J. C. R. Alcantud, *Covering based multi-granulation $(\mathcal{F}, \mathcal{T})$ -fuzzy rough set models and applications in multi-attribute group decision-making*, Inf. Sci. **476** (2019), 290-318.
- [35] J. Zhan and Q. Wang, *Certain types of soft coverings based rough sets with applications*, Int. J. Mach. Learn. Cybern. **10** (2019), 1065-1076.
- [36] J. Zhan and B. Sun, *Covering-based soft fuzzy rough theory and its application to multiple criteria decision making*, Comput. Appl. Math. **38** (2019), 1-27.
- [37] J. Zhan, X. Zhang, Y. Yao, *Covering based multi-granulation fuzzy rough sets and corresponding applications*, Artif. Intell. Rev. **53** (2020), 1093-1126.
- [38] H. Zhang, H. Liang and D. Liu, *Two new operators in rough set theory with applications to fuzzy sets*, Inf. Sci. **166** (2004), 147-165.
- [39] L. Zhang and J. Zhan, *Fuzzy soft β -covering based fuzzy rough sets and corresponding decision-making applications*, Int. J. Mach. Learn. Cybern. **10** (2019), 1487-1502.
- [40] H. J. Zimmermann, *Fuzzy set theory—and its applications*, Springer Science & Business Media, (2011).
- [41] W. Zhu, *Topological approaches to covering rough sets*, Inf. Sci. **177** (2007), 1499-1508.
- [42] W. Zhu and F. Y. Wang, *On three types of covering-based rough sets*, IEEE Trans Knowl Data Eng. **19** (2007), 1131-1144.
- [43] W. Zhu and F. Y. Wang, *The fourth type of covering-based rough sets*, Inf. Sci. **201** (2012), 80-92.

S. PAVITHRA, RESEARCH SCHOLAR, DEPARTMENT OF MATHEMATICS, SCHOOL OF ADVANCED SCIENCES, VELLORE INSTITUTE OF TECHNOLOGY, VELLORE 632 014, TAMIL NADU, INDIA.

Email address: pavithrasivaa1997@gmail.com

A. MANIMARAN, CORRESPONDING AUTHOR, DEPARTMENT OF MATHEMATICS, SCHOOL OF ADVANCED SCIENCES, VELLORE INSTITUTE OF TECHNOLOGY, VELLORE 632 014, TAMIL NADU, INDIA.

Email address: marans2011@gmail.com