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DIVERSITY AND ECOTOURISM ON MULTIPURPOSE MARINE PROTECTED AREAS

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ABSTRACT. Beyond its ecological benefits, there is a growing call for the establishment of marine protected areas (MPAs) for varied economic activities. Due to the decline in the wild catch, there is an increase in demand for farmed fish and a call for the development of aquaculture production in marine protected areas. In addition, the diversity in marine protected areas is suitable for the introduction of a tourism industry. Therefore, we need to design a clear scientific management system that balances the main goal of marine protected areas with the economic benefits, of aquaculture and tourism activities. This paper aims to determine the optimal size of the marine protected areas, the aquaculture size in the protected areas, and the level of tourism activities that maximize the overall benefit of the stakeholders. This is vital for the co-existence and sustainability of different sectors such as aquaculture production, tourism activities, and open-access fisheries.

1. INTRODUCTION

Global interest in creating marine protected areas (MPAs), also known as ocean-protected areas, has increased as a result of human activity-induced degradation of ocean ecosystems [27]. These regions are crucial for preserving the ocean's biodiversity and productivity, particularly for fish populations [10]. Beyond just protecting the environment, MPAs can be extremely helpful in reducing poverty and ensuring food security through the establishment of commercial aquaculture and the use of MPAs as ecotourism destinations. The development of multipurpose MPAs shall consider the ecosystem, the prey-predator makeup, and human activities in and outside the MPAs. In this study, we consider an ecosystem subdivided into two parts: MPAs and open-access fishery. In our case, we consider the International Union for Conservation of Nature (IUCN) category VI MPAs, these VI MPAs allow different activities, and specifically in this research we try to show the possibility of installing aquaculture and tourism activities simultaneously.

In our model, we consider the development of aquaculture farms, tourism activities in the MPAs, and the prey-predator dynamics with shared harvesting efforts outside the MPAs. Studying the dynamic relationship between prey and predator and harvesting predators and their prey has long been a main focus in ecology and mathematical ecology [6]. Researchers considered harvesting prey, predator, or both in fishery systems accounting for the interaction between prey and predator species. [19] discussed the dynamics of harvesting of prey populations whereas [7] described the phenomenon of selective harvesting on a prey-predator system. In this paper, we consider harvesting both prey and predator species with a predetermined effort allocation towards targeting each species based on the condition of the ecological environment under consideration[14, 15, 23, 2, 1, 9].

Aquaculture as an option to meet the increasing demand for fish continues to grow faster than other agricultural sectors [11]. The growth would not be sustainable without appropriate space allocation

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and cooperative management. IUCN category VI protected areas allow aquaculture activities while preserving biodiversity and enhancing sustainable economic development [11]. The development of the tourism industry in MPAs has a positive effect on the well-being of society by creating jobs and alternative sources of income [24]. Careful and optimized planning are all-important for the effectiveness and coexistence of aquaculture and tourism activities in MPAs [4]. Unless we design well-thought-out strategies for site selection, implementation, and expansion plans, the tourism and fishery sectors' coexistence might be fragile and have unintended negative consequences. If managed properly, benefits from tourism can in many cases be far greater than those from fishing. For instance, in Medes Islands Marine Reserve (Spain), annual revenue from tourism is about 20 times greater than fishing revenue whereas, in the Great Barrier Reef annual revenue from tourism can be 36 times greater than fishing revenue[27]. However, the impacts on the surrounding ecosystem and communities can be detrimental when tourism is not well-regulated and sustainable development plans are not in place [5, 26].

The remainder of the paper is organized as follows. in section 2, we set up the dynamic equations of the prey and predator stock in and outside the MPAs with the presence of aquaculture and tourism activities inside MPAs. Then we determine the equilibrium solutions and study their stability. We determine the maximum sustainable yield as a function of effort and find the corresponding sustainable effort level. In section 3, we analyze the control version of the model and numerically illustrate the steady-state optimal solutions for both the control and state variables. Section 4 presents the conclusion of the paper.

2. The Model

In this section, we develop a model, an aggregated and deterministic model of the prey-predator in continuous time. We look at setting up multipurpose MPAs to introduce aquaculture and tourism activities. We also consider prey-predator interactions in and outside the MPAs.

2.1. Prey stock dynamics with aquaculture in MPAs.

We consider an ecosystem (marine environment) consisting of an open-access fishery and marine protected areas (MPAs) that are suitable for aquaculture development and tourism activity (i.e, we assume the entire marine environment is divided into MPAs and open-access fishery areas). In MPAs, we allow aquaculture to be established and introduce tourism activities with no harvest. Outside the MPAs, we assume open-access fishery where we harvest prey and predators with shared effort.

We start with the conventional ecological transition model for prey populations in and outside the MPAs. Let X be the prey population outside the MPAs and the harvest function be $h = \sigma_1\beta_1 EX$, where σ_1 is the catch ability coefficient of the prey, E the shared effort towards both prey and predator with $\beta_1 E$ for harvesting the prey and $(1-\beta_1)E$ for predator harvesting. Then the transition dynamics for X can be described as

$$\frac{dX}{dt} = r_1 X\left(1-\frac{X}{K}\right) - \sigma_1\beta_1 EX, \tag{2.1}$$

where r_1 is the growth rate parameter outside the reserve, $r_1 \leq r_0$, where r_0 is the natural growth rate of the prey, and K is the carrying capacity of the entire environment,

We assume the resource is uniformly distributed over a homogeneous area. A fraction, say, m, $0 < m < 1$, of the total area dedicated for the reserve, leaving the portion $(1-m)$ of the total area outside the reserve. Therefore, the carrying capacity within the reserve is mK and outside the reserve is $(1-m)K$.

After the establishment of the MPAs, we describe the dynamics of the biomass of prey outside the reserve with harvest by

$$\frac{dX}{dt} = r_1 X \left(1 - \frac{X}{(1-m)K}\right) - \sigma_1 \beta_1 E X,\ X(0) = X_0 \tag{2.2}$$

and for the prey in the reserve, Y, without harvest the transition equation is

$$\frac{dY}{dt} = r_2 Y (1 - \frac{Y}{(m-a)K}),\ Y(0) = Y_0 \tag{2.3}$$

where a is the portion of the reserved area dedicated to aquaculture at time t with $a \leq m$, and r_2 is the growth rate parameter inside the reserve, $r_2 \leq r_0$, where r_0 is the natural growth rate of the prey.

Dividing both sides of Equations (2.2) and (2.3) by the carrying capacity, K, we get

$$\frac{dx}{dt} = r_0 x \left(1 - \frac{x}{1-m}\right) - \sigma_1 \beta_1 E x,\ x(0) = x_0 \tag{2.4}$$

$$\frac{dy}{dt} = r_0 y (1 - \frac{y}{m-a}),\ y(0) = y_0, \tag{2.5}$$

where $x = \frac{X}{K}$ and $y = \frac{Y}{K}$ are the density of the prey stock outside and inside the reserve, respectively.

To describe the dynamics of the aquaculture size we assume that the rate of expansion of the aquaculture activity depends on the current ratio of the aquaculture to the size of the reserve, $\frac{a}{m}$, the maximum portion of the reserve that can be assigned to aquaculture, v, and the measure of the relative density of the prey population outside the reserve, $1-x$. Like [5], we assume that the expansion of the aquaculture activity after its initial implementation is governed by:

$$\frac{da}{dt} = (v - \frac{a}{m})(1-x),\ a(0) = a_0, \tag{2.6}$$

where v is an exogenous variable that controls the maximum portion of the reserved area that can be allocated to aquaculture. Like [21], the aquaculture area serves as a prey refuge that may need protection from predators [18, 25]. For simplicity, the dynamics of the prey in the aquaculture area are not included in our analysis.

2.2. **Prey-predators stock dynamics with spillover effect and tourism activities in MPAs.**

In this section, we represent possible interactions between different sectors involved in the system and their impacts in and outside the reserve (i.e. the predator effect, spillover effect, and tourism activities). We consider a density-dependent one-side flow of the prey from MPAs to outside the reserve. The spillover rate depends on the density of the prey in the reserve and the relative size of a and m and it is described as $d_2 \frac{y}{m}(1 - \frac{a}{m})$, where d_2 is the dispersion rate. Therefore, the dynamics of the prey outside the MPAs becomes

$$\frac{dx}{dt} = r_1 \left(1 - \frac{x}{1-m}\right) x - \sigma_1 \beta_1 E x + d_2 \frac{y}{m}\left(1 - \frac{a}{m}\right), \quad x(0) = x_0, \tag{2.7}$$

and the biomass dynamics inside the MPAs is

$$\frac{dy}{dt} = r_2 (1 - \frac{y}{m-a}) y - d_2 \frac{y}{m}(1 - \frac{a}{m}), \quad y(0) = y_0. \tag{2.8}$$

Moreover, we make the following assumptions regarding the prey-predator interactions in and outside, and the effect of tourism activities in the reserve.

A1. We assume that the same effort, E, operates targeting both prey and predator, and harvesting activity is allowed only in the unreserved area. Predators can move in and outside the MPAs without specifying a density-dependent flux [18, 16, 3].

A2. We consider different prey and predator interactions in and outside the MPAs.
Outside of the reserve, we assume Type II interaction

$$\psi_1(x, z) = \left(\frac{b_1 x}{\alpha + x}\right) z,$$

where z is the population of predators, b_1 maximum killing rate parameter and α is the half-saturation (half maximum killing rate) parameter.
In the reserved area, we assume an interaction functional of Type I where consuming food does not interfere with searching for food. The number of prey eaten per predator per unit of time also called the intake rate depends linearly on the prey density possibly with a fixed maximum [22]. In this case, the chance of an encounter between the predator and prey is determined by

$$\psi_2(y, z) = b_2 yz,$$

where b_2 is a number that measures the searching (and capturing) efficiency of the predators in the reserve.

A3. In MPAs, outside the aquaculture site, due to predator abundance and diversity, there is a non-consumptive tourist activity such as scuba diving, mammal-watching tours, sailing trips, recreational fishing, etc. These activities affect the predator population and the impact depends on the number of tourists and their interaction with the predator [8, 20]. Following [20], we assume that the number of tourists, T, depends directly on the reserve size and inversely on the effort level around the reserve:

$$T(m, E) = \lambda(\kappa m)^\eta E^\mu, \tag{2.9}$$

where λ, κ are positive parameters, $\mu < 0$, and $\eta > 0$.

By incorporating all the assumptions made, we set up the predator stock dynamics as

$$\frac{dz}{dt} = \left(\frac{b_3 x}{\alpha + x}\right) z + b_4 yz - \sigma_2\beta_2 Ez - dz - b_5\lambda(\kappa m)^\eta E^\mu z,$$

where d is the natural mortality rate of the predator, as mentioned earlier $(1 - \beta_1)E = \beta_2 E$ is the effort towards harvesting the predator, b_3 and b_4 are conversion factors, b_5 is the predator mortality rate due to tourism activities (i.e loss of predators may be due to over-fishing, boat strikes, bows of larger ships, or changing food habits by offering food to provoke to the proximity of tourists) [8, 12, 17]. In summary, the dynamics of the system can be expressed as:

$$\begin{aligned}
\frac{dx}{dt} &= r_1\left(1 - \frac{x}{1-m}\right)x - \sigma_1\beta_1 Ex + d_2\left(1 - \frac{a}{m}\right)\frac{y}{m} - \left(\frac{b_1 x}{\alpha + x}\right)z\,, \quad x(0) = x_0\\
\frac{dy}{dt} &= r_2\left(1 - \frac{y}{m-a}\right)y - d_2\left(1 - \frac{a}{m}\right)\frac{y}{m} - b_2 yz\,, \quad y(0) = y_0\\
\frac{dz}{dt} &= \left(\frac{b_3 x}{\alpha + x}\right)z + b_4 yz - \sigma_2\beta_2 Ez - dz - b_5\lambda(\kappa m)^\eta E^\mu z, \quad z(0) = z_0\\
\frac{da}{dt} &= \left(v - \frac{a}{m}\right)(1 - x)\,, \quad a(0) = a_0,
\end{aligned} \tag{2.10}$$

where the constants b_1, b_2, b_3 and b_4 satisfy the constraints $0 < b_3 < b_1$ and $0 < b_4 < b_2$.

2.3. **Equilibrium Solutions, Stability, and Maximum Sustainable Yield.**

Equilibrium points of the above system, Equation (2.10), as a function of the control variables, E, m, and v, and the parameters can be found by solving the following system of algebraic equations:

$$r_1\left(1-\frac{x}{1-m}\right)x-\sigma_1\beta_1 Ex+d_2\left(1-\frac{a}{m}\right)\frac{y}{m}-\left(\frac{b_1x}{\alpha+x}\right)z=0 \tag{2.11}$$

$$r_2\left(1-\frac{y}{m-a}\right)y-d_2\left(1-\frac{a}{m}\right)\frac{y}{m}-b_2yz=0 \tag{2.12}$$

$$\left(\frac{b_3x}{\alpha+x}\right)z+b_4yz-\sigma_2\beta_2 Ez-dz-b_5\lambda(\kappa m)^\eta E^\mu z=0 \tag{2.13}$$

$$\left(v-\frac{a}{m}\right)(1-x)=0 \tag{2.14}$$

From (2.14), at the steady-state, we get

$$a_s=vm, \tag{2.15}$$

provided $x_s\neq 1$. This shows that in the long run, $\frac{a}{m}$ asymptotically approaches the management's regulatory limitation, v. For the remaining part of this section, we set $\eta=\kappa=\lambda=1$, and $\nu=-1$. Substituting (2.15) into (2.12) and solving (2.12) for z, we get

$$z_s(y,m)=\frac{r_2(m(1-v)-y)-d_2(1-v)^2}{b_2m(1-v)}=\frac{1}{b_2}\left(r_2(1-\frac{y}{m(1-v)})-d_2(\frac{1-v}{m})\right) \tag{2.16}$$

is positive if $r_2m-d_2(1-v)>\frac{y}{1-v}$.

By substituting (2.16) in (2.13) and solving for x

$$x_s(y,E,m)=\alpha_1\left(\frac{b_3}{b_3+b_4y-d-\sigma_2\beta_2E-\frac{b_5m}{E}}-1\right) \tag{2.17}$$

is positive provided $b_4y-d-\sigma_2\beta_2E-\frac{b_5m}{E}<0$.

From Equation (2.16) we can see that the state of the predator stock mainly depends on the per unit growth rate of the fish inside the reserve and the rate of flow of the prey from the remaining area of the reserve to the open access fishing ground.

We can also express y and z in terms of x by substituting (2.15) into (2.12) and solving (2.12) and (2.13) simultaneously for y and z. Then substituting $y(x)$ and $z(x)$ in (2.11) and simplifying, we get

$$a_1x^4+a_2x^3+a_3x^2+a_4x+a_5=0, \tag{2.18}$$

where

$$\begin{aligned}
a_1 &= -b_2b_4Emr_1(1-v),\\
a_2 &= -b_2b_4Em(1-v)((1-m)\sigma_1\beta_1E-r_1(1-m-2\alpha)),\\
a_3 &= -a_1\alpha^2+b_1(m-1)(b_4E(v-1)(d_2(v-1)+mr_2)+r_2(b_5m+E(d+\sigma_2\beta_2E))),\\
&\quad +2\alpha b_2b_4E^2(m-1)m\sigma_1\beta_1(v-1)-2\alpha b_2b_4Em^2r_1(v-1)+2\alpha_1b_2b_4Emr_1(v-1),\\
&\quad +b_2b_5d_2(m-1)m(v-1)^2+b_2dd_2E(m-1)(v-1)^2+b_2d_2E^2(m-1)\sigma_2\beta_2(v-1)^2,\\
&\quad -b_3E(m-1)\left(b_1r_2+b_2d_2(v-1)^2\right),\\
a_4 &= -\alpha(m-1)\left(b_1(b_4E(v-1)(d_2(v-1)+mr_2)+r_2(b_5m+E(d+\sigma_2\beta_2E)))\right),\\
&\quad +\alpha(m-1)\left(b_2(v-1)\left(\alpha_1b_4Em(\sigma_1\beta_1E-r_1)\qquad +2b_5d_2m(v-1)+2d_2E(v-1)(d+\sigma_2\beta_2E)\right),\\
&\quad -\alpha b_2b_3d_2E(m-1)(v-1)^2,\\
a_5 &= -\alpha^2b_2d_2(m-1)(v-1)^2(b_5m+E(d+\sigma_2\beta_2E))
\end{aligned}$$

Note that since $m < 1$ and $v < 1$, we have $a_1 < 0$ and $a_5 > 0$. Thus, at least one of the solutions is positive by Descartes's Rule of Signs.

Equation (2.18) can be solved for x in terms of the control variables. The expression is very complicated. Instead, we sketched the graph of the prey population density as a function of effort, E, for fixed values of $m = 0.1$ and $v = 0.15$.

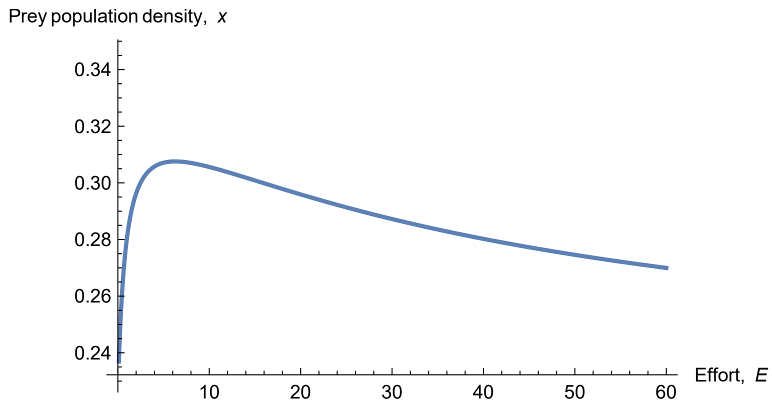

FIGURE 1. Prey population as function of effort for when m = 0.1, v=0.15, $\beta_1 = 0.8$, and $\beta_2 = 0.2$

Numerical Illustrations: Equilibrium Solutions and Stability.

It may be noted that it is quite difficult to have numerical values of the system's parameters based on real-world observations. On the other hand, it is necessary to know the parameters in connection to the observed real system. Therefore, the simulation results should be considered from a qualitative, rather than a quantitative point of view. We, therefore, take some hypothetical data set in Table 1 to solve the system numerically and obtain equilibrium solutions. Then we found the steady-state solutions and analyzed the stability of the solutions. We also graphically represent the transition trajectories of the state variables to the equilibrium states.

TABLE 1. Parameters and their values used for numerical Illustrations

Parameter	Description	Value
r_1	Growth rate parameter of prey outside MPA	1.8
r_2	Growth rate parameter of prey inside MPA	2
σ_1	Catchability coefficient per unit effort	0.015
σ_2	Catchability coefficient per unit effort	0.001
c	Cost per unit effort per unit carrying capacity for wild-catch	0.05
p	Per unit price in US dollars	15
d_2	Dispersion parameter	0.05
b_1	Searching and capturing efficiency of the predators	0.8
b_2	Searching and capturing efficiency of the predators in MPA	0.1
b_3	The efficiency by which the food (prey) is turned into a new predator	0.1
b_4	The efficiency by which the food (prey) is turned into a new predator	0.05
d	mortality rate	0.01
α	half saturation rate	0.1

Using the parameter values included in Table 1, and control variable values: $m = 0.1$, $v = 0.25$, and $E = 83$, we found the equilibrium solution of the system

$$x_s = 0.2267,\ y_s = 0.0538,\ z_s = 0.2016,\ a_s = 0.0350. \tag{2.19}$$

To analyze the stability of the equilibrium solutions, we linearize the above system of equations about an equilibrium point. The Jacobian matrix corresponding to the system at an equilibrium point (x_s, y_s, z_s, a_s) is

$$\begin{pmatrix} A_{11} & \frac{d_2(m-a_s)}{m^2} & -\frac{b_1 x_s}{\alpha + x_s} & -\frac{d_2 y_s}{m^2} \\ 0 & -\frac{r_2 y_s}{m(1-v)} & -b_2 y_s & y_s\left(\frac{d_2}{m^2} - \frac{r_2 y_s}{(a_s-m)^2}\right) \\ \frac{\alpha b_3 z_s}{(\alpha+x_s)^2} & b_4 z_s & A_{33} & 0 \\ -\left(v - \frac{a_s}{m}\right) & 0 & 0 & \frac{x_s - 1}{m} \end{pmatrix} \tag{2.20}$$

where

$$A_{11} = r_1 - \frac{\alpha b_1 z_s}{(\alpha + x_s)^2} - \sigma_1 \beta_1 E - \frac{2 r_1 x_s}{1-m},$$
$$A_{33} = \left(\frac{b_3 x_s}{\alpha + x_s}\right) + b_4 y_s - \sigma_2 \beta_2 E - d - b_5 (\frac{m}{E})$$

The Jacobian matrix at the equilibrium solution becomes

$$\begin{pmatrix} -0.415538 & 0.325 & -0.158417 & -0.267852 \\ 0 & -1.64832 & -0.00535704 & -1.09063 \\ 0.171596 & 0.0133399 & -3.469446951953614\text{*\^{}-18} & 0 \\ 0 & 0 & 0 & -9.75308 \end{pmatrix}$$

and the corresponding eigenvalues are

$$\{-9.75308, -1.64842, -0.333278, -0.0821577\}. \tag{2.21}$$

The real part of all the eigenvalues is negative, implying that the equilibrium point (2.19) is stable.

We also numerically solve the system (2.10). In Figures 2 and 3, we presented trajectories of the prey population outside the reserve and the total predator population for different values of m. From Figures 2 and 3, when m increases the steady-state prey population decreases and the steady-state population of predators increases. This is due to predators having plenty of alternative food in the MPAs and we spend less time catching the predator than the prey.

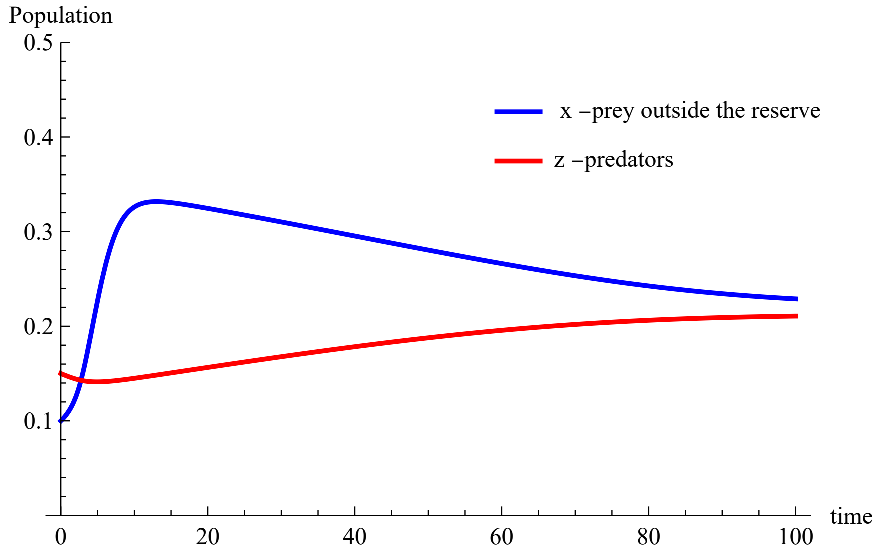

FIGURE 2. Solution curves of the prey outside the MPAs and total predator when m = 0.1 and v=0.25

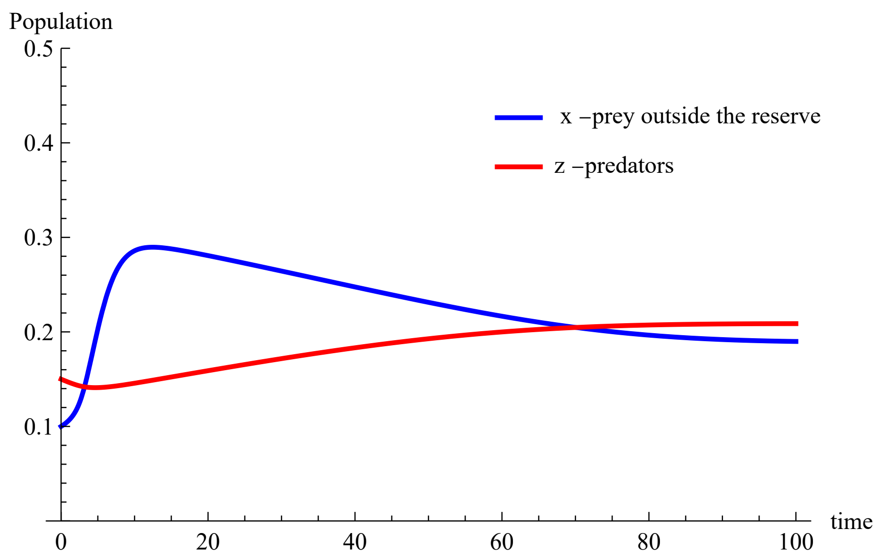

FIGURE 3. Solution curves of the prey outside the MPAs and total predator when m=0.2 and v=0.25

Maximum Sustainable Yield.

In this section, we numerically represent the static maximum sustainable effort that can be theoretically exerted on the ecosystem to attain the maximum sustainable rent from the natural resource. Then, in the next section, we attempt to find the optimized sustainable effort using dynamic optimization techniques. In our case, the sustainable total yield is analyzed when both prey and predator species are subject to shared harvesting effort [13], and the allocation of the effort determined by the management based on the state of the ecosystem and cooperation agreements between the stakeholders. By substituting the non-negative stable steady state stocks, $x_s(E)$ and $z_s(E)$, into the harvest function, the corresponding sustainable harvest (yield) is described as

$$h_s(E) = \sigma_1 E_1 x_s(E) + \sigma_2 E_2 z_s(E),$$

where $E_1 = \beta_1 E$ is the proportion of the effort dedicated to catching prey and $E_2 = \beta_2 E$ is the predator with $\beta_1 + \beta_2 = 1$. We can determine the maximum sustainable effort (E_{MSY}) corresponding to the maximum sustainable yield (MSY) by differentiating $h_s(E)$ in terms of E, setting it to zero, and solving for E. The analytic solution of E_{MSY} and MSY as a function of the parameters involved in the model is not easy to represent algebraically. We graphically illustrate the results for a set of chosen values of the parameters included in Table 1 and a value for the control variable, $m = 0.1$. We also chose different shares of the shared effort towards catching the prey and predator: $\beta_1 = 0.8$ and $\beta_2 = 0.2$, and $\beta_1 = 0.7$ and $\beta_2 = 0.3$.

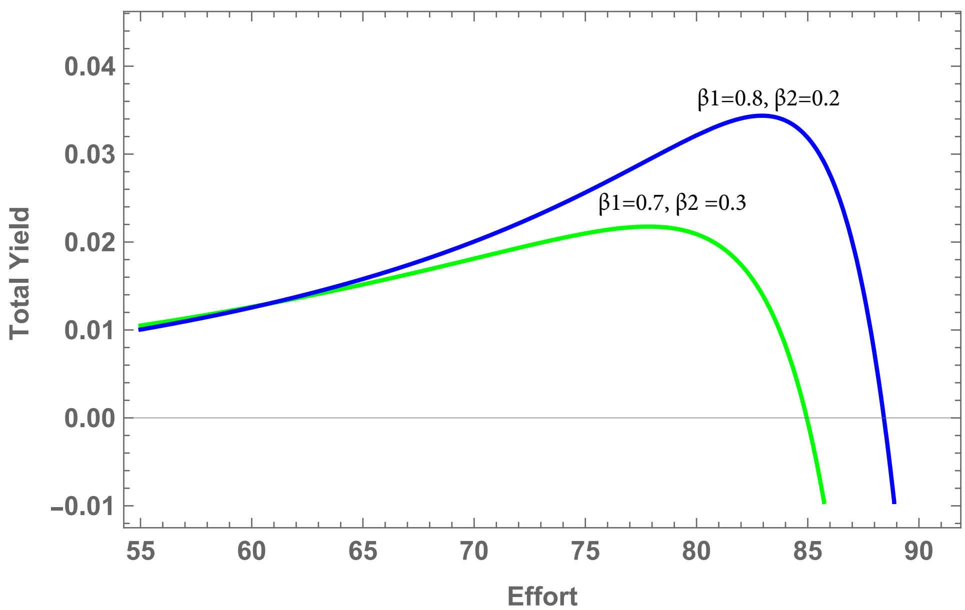

FIGURE 4. Total yield for different allocations of the shared effort towards harvesting prey and predator.

From Figure 4, the E_{MSY} depends on the percentage of the effort dedicated to capture each species. If predator harvest and tourism are more beneficial than the prey catch, we reduce the effort in catching the prey. That resulted in a lower optimal E_{MSY}.

3. Optimized Operations Management

In this section, we consider the optimized version of our problem. We set up the problem as a dynamically efficient utilization of the resources and space by considering E and m as control variables.

We aim to maximize the long-run profit from wild capture fish, including benefits from tourism and aquaculture.

We consider three profit functions related to incomes generated from harvesting both the prey and predators outside the MPAs, aquaculture production in MPAs, and tourism activities. The profit from harvesting both prey and predator using shared effort, E, is

$$\Pi_W = p_1\sigma_1\beta_1 Ex + p_2\sigma_2\beta_2 Ez - cE, \tag{3.1}$$

where p_1 is the unit market price of the prey and p_2 is the unit price of the predator, β_1 and β_2 are portions of the effort towards catching prey and predator, respectively, with $\beta_1+\beta_2 = 1$ and c is the per unit cost. Then, we consider the presence of aquaculture production in MPAs, suppose the production function for aquaculture is $Z(A) = P_aA$ and the cost of production is quadratic $\Phi(A) = c_a(P_a)^2A$, where P_a is per unit area production of farmed fish and c_a is a cost parameter. We can rewrite the production and cost functions in terms of a as $z(a) = P_aa$ and $\phi(a) = c_a(P_a)^2a$, where $z(a) = \frac{Z(A)}{K}$ and $\phi(a) = \dfrac{\Phi(A)}{K}$, K is the carrying capacity. We assume that buyers cannot distinguish between a species that is farmed or caught in the wild. Let the unit price of farmed fish be constant, p. Thus the net profit from aquaculture at time t is

$$\Pi_A = pz(a) - \phi(a). \tag{3.2}$$

We also assume the introduction of tourism activities in MPAs that alters the dynamics of the fishery and its economic benefit plays a crucial role. The revenue from tourism in the MPAs includes a finite number of non-extractive activities (i.e. scuba diving, mammal-watching tours, sailing trips, etc.). Let the number of tourists attracted to the marine reserve, T, be based on a Cobb–Douglas production function, depending directly on the size of the protected area (i.e. m), and inversely to the effort level, E,

$$T(m,E) = \lambda(\kappa m)^\eta E^\mu \tag{3.3}$$

where λ, κ are positive parameters, $\mu < 0$, and $0 < \eta < 1$ [20]. Hence, the net profit from tourism activities, Π_T, using the estimated average net profit per tourist, n_1, is given by:

$$\Pi_T(m,E) = n_1T(m,E) \tag{3.4}$$

Therefore, the total profit from all these activities becomes:

$$MP(m,E,a,z) = \Pi_W(E) + \Pi_A(a) + \Pi_T(m,z) \tag{3.5}$$

where $\Pi_W(E)$, $\Pi_A(a)$ and $\Pi_T(m,z)$ are given by (3.1),(3.2) and (3.4), respectively.

If all future costs and benefits are discounted at a positive social discount rate of δ, the manager's objective is to maximize the overall present value of the stream of surpluses (3.5)

$$\max \mathrm{MP}(m,E,a,z) = \max_{E,m}\int_0^\infty (MP(m,E,a,z))\, e^{-\delta t}dt, \tag{3.6}$$

subject to the dynamic equations (2.10) with control variables $m_{min} \leq m \leq m_{max}$, $0 \leq E \leq E_{max}$. The current value Hamiltonian corresponding to this problem is

$$\begin{aligned}
&H(x, y, z, a, \lambda_1, \lambda_2, \lambda_3, \lambda_4, E, m) \\
&= MP(m, E, a, z) + \lambda_1 \left(r_1 \left(1 - \frac{x}{1-m}\right) x - \sigma_1 \beta_1 E x + d_2 \left(1 - \frac{a}{m}\right) \frac{y}{m} - \left(\frac{b_1 x}{\alpha + x}\right) z \right) \\
&\quad + \lambda_2 \left(r_2 \left(1 - \frac{y}{m-a}\right) y - d_2 \left(1 - \frac{a}{m}\right) \frac{y}{m} - b_2 y z \right) \\
&\quad + \lambda_3 \left(\left(\frac{b_3 x}{\alpha + x}\right) z + b_4 y z - \sigma_2 \beta_2 E z - dz - b_5 \lambda (\kappa m)^{\eta} E^{\mu} z \right) + \lambda_4 \left(\left(v - \frac{a}{m}\right)(1 - x) \right)
\end{aligned}$$

where λ_1 the shadow value of the prey in the open access area, λ_2 is the shadow value of the MPAs, λ_3 the shadow value of the aquaculture, and λ_4 the shadow value of the stock of the predator.

Using Pontryagin's Maximum Principle for an infinite horizon autonomous system, the necessary conditions for optimality are

$$\begin{aligned}
H_E(x, y, z, a, \lambda_1, \lambda_2, \lambda_3, \lambda_4, E, m) &= 0 \\
H_m(x, y, z, a, \lambda_1, \lambda_2, \lambda_3, \lambda_4, E, m) &= 0,
\end{aligned}$$

where H_E and H_m are partial derivatives of the Hamiltonian function with respect to the control variables E and m, and the co-state equations

$$\begin{aligned}
-H_x(x, y, z, a, \lambda_1, \lambda_2, \lambda_3, \lambda_4, E, m) &= \frac{d\lambda_1}{dt} - \delta\lambda_1, \\
-H_y(x, y, z, a, \lambda_1, \lambda_2, \lambda_3, \lambda_4, E, m) &= \frac{d\lambda_2}{dt} - \delta\lambda_2, \\
-H_z(x, y, z, a, \lambda_1, \lambda_2, \lambda_3, \lambda_4, E, m) &= \frac{d\lambda_3}{dt} - \delta\lambda_3 \quad \text{and} \\
-H_a(x, y, z, a, \lambda_1, \lambda_2, \lambda_3, \lambda_4, E, m) &= \frac{d\lambda_4}{dt} - \delta\lambda_4,
\end{aligned} \tag{3.7}$$

where H_x, H_y, H_z and H_a are partial derivatives of the Hamiltonian function with respect to the state state variables, x, y, z, and a respectively.

At steady state

$$\frac{d\lambda_1}{dt} = \frac{d\lambda_2}{dt} = \frac{d\lambda_3}{dt} = \frac{d\lambda_4}{dt} = 0,$$

implying

$$H_x(.) = \delta\lambda_1, H_y(.) = \delta\lambda_2, H_z(.) = \delta\lambda_3, H_a(.) = \delta\lambda_4. \tag{3.8}$$

Therefore, at the steady state, we have

$$\begin{aligned}
&\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = \frac{da}{dt} = 0 \text{ and} \\
&H_E(.) = 0,\ H_m(.) = 0,\ H_x(.) = \delta\lambda_1,\ H_y(.) = \delta\lambda_2,\ H_z(.) = \delta\lambda_3,\ H_a(.) = \delta\lambda_4.
\end{aligned} \tag{3.9}$$

The above system, (3.9), can be solved numerically for the optimal steady-state solutions of the state and control variables: $x^*,\ y^*,\ z^*,\ a^*,\ E^*,$ and m^*.

Numerical results: Optimal Steady-State Solutions

By assigning values for the parameters involved in the model given in Tables 1 and 2, we solve for the optimal steady-state solutions using MATHEMATICA 13.0 ™. We also perform sensitivity analysis of the optimal solutions by changing the values of the exogenous control variable, v, and the net profit from tourism parameter, n_1. The results are summarized in Table 3.

Table 2. Parameters and their values used for numerical simulations

Parameter	Description	Value
P_a	aquaculture production per unit square meter in kg	44.6
c_a	cost parameter for aquaculture	0.00319
c	cost per unit carrying capacity per unit effort	0.005
δ	positive social discount rate	0.05
n_1	a measure of net profit from tourism per tourist	$17, 21$

Table 3. Optimal steady-state solution for different scenarios

n_1	v	x^*	y^*	z^*	a^*	m^*	E^*
17	0.15	0.2739	0.0351	0.1803	0.0095	0.0631	64.9581
17	**0.25**	0.2638	0.0321	0.1847	0.0155	0.0622	64.0682
21	0.15	0.2737	0.0355	0.1804	0.0095	0.0636	64.9656

From Table 3, as we allow more space in the reserve for aquaculture, the stock of prey in and outside the reserve decreases. The total shared effort was also reduced which resulted in an increment in the predator population since we target the prey more than predators (we use $\beta_1 = 0.8$ and $\beta_2 = 0.2$ for our simulation). When the measure of the net profit from tourism, n_1, increases, the reserve size increases, the aquaculture activity remains at the same level and the shared effort increases. Moreover, the stock of prey decreases and the predator increases as long as we keep the same distribution of effort in catching the prey and predators. This in turn boosts the ecotourism sector.

4. Conclusion

Marine protected areas (MPAs) have been integral to marine resource management. Recent developments show that besides their ecological benefits MPAs can be used for economic growth, poverty reduction, and ecotourism improvement. In this paper, we propose the establishment of allocated zones for aquaculture inside MPAs and allow tourism activities to operate in MPAs while protecting the environment and enhancing ecological biodiversity. The mathematical model presented in this paper attempts to include the complex relationship between the biological and physical habitat of the marine ecosystem. Understanding the species diversity of the ecosystem is detrimental to modelling the prey-predator interaction in and outside the MPAs.

In this research, we determine the optimal size of the aquaculture and the optimal level of effort for fishing. Moreover, the optimal effort and reserve size determine the tourism activity. We find the optimal size of aquaculture relative to the size of the reserve and policy restrictions for mutual economic benefits of aquaculture and tourism activities in MPAs. We also show that the predator population benefits more from the reserve. For instance, when the reserve size increases the predator population would be in a better state compared to the prey.

Implementation of aquaculture in MPAs has a positive effect on boosting the supply of fish even though it harms the environment. Tourism activities if properly managed create job opportunities and significantly increase income. In addition, aquaculture and tourism activities reduce the pressure on the open-access fishing ground. Cooperative management tools are essential for the effectiveness of the simultaneous implementation of aquaculture and tourism activities in multipurpose MPAs.

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