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EXISTENCE OF FORCED WAVES AND GAP FORMATIONS FOR A LOTKA-VOLTERRA COMPETITION SYSTEM WITH NONLOCAL DISPERSAL IN A TIME PERIODIC SHIFT HABITAT

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Abstract. This paper is devoted to the study of a class of time periodic Lotka-Volterra competition system with nonlocal dispersal and shifting habitats. By using some known results of the periodic KPP model and employing the iterative techniques, we prove that there exist two positive numbers $c_0(d_1)$ and $c_0(d_2)$ such that the system admits a forced wave provided that the forcing speed $c \in (-c_0(d_2), c_0(d_1))$. In addition, based on the theoretical results, we show that the gap formations exist for $c > c_0(d_1)$ and $c < -c_0(d_2)$.

1. Introduction

In this paper, we are concerned with the existence of forced traveling wave solutions and gap formations of a Lotka-Volterra competition system with nonlocal dispersal in a time-periodic shifting habitat, which is described by

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\left(J_1 * u - u\right) + u\left[r_1(t, x-ct) - u - a_1 v\right], \\ \dfrac{\partial v}{\partial t} = d_2\left(J_2 * v - v\right) + v\left[r_2(t, x-ct) - a_2 u - v\right], \ t \in \mathbb{R}^+, \ x \in \mathbb{R}. \end{cases} \tag{1.1}$$

Here, $u(t,x)$ and $v(t,x)$ are the population densities of two species under consideration at time t and location x; $d_1 > 0$ and $d_2 > 0$ account for the dispersal coefficients; $c \in \mathbb{R}$ is the shifting speed of the edge of the habitat; $r_1(t, x-ct)$ and $r_2(t, x-ct)$ represent the per capita growth rates, both of which are dependent on time t and the climate shifting variable $x - ct$. Moreover, $r_i(t,z)$, $i = 1, 2$ are T-periodic in the first variable t for some positive number T, that is $r_i(t+T, z) = r_i(t,z)$ for all t $\in \mathbb{R}^+$; $J * u$ is a spatial convolution defined by

$$(J * u)(t,x) = \int_R J(x-y)u(t,y)\mathrm{d}y.$$

Throughout this paper, we always assume that

(J) $J_i \in C\left(\mathbb{R}, \mathbb{R}^+\right)$ is even, satisfying

$$\int_R J_i(y)\mathrm{d}y = 1 \ \text{ and } \ \int_R J_i(y)e^{\lambda y}\mathrm{d}y < +\infty \ \text{ for every } \ \lambda > 0, \ \ i = 1, 2;$$

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(H1) $r_1(t,z)$ and $r_2(t,z)$ are continuous, $r_1(t,z)$ is decreasing and $r_2(t,z)$ is increasing in z. Furthermore,

$$\begin{aligned} &\lim_{z\to-\infty} r_1(t,z)=\theta(t)>0, \quad \lim_{z\to\infty} r_1(t,z)=\beta(t)<0;\\ &\lim_{z\to-\infty} r_2(t,z)=\beta(t)<0, \quad \lim_{z\to\infty} r_2(t,z)=\theta(t)>0 \end{aligned} \tag{1.2}$$

uniformly in t, where $\theta(t)$, $\beta(t)$ are continous and T-periodic functions.

Reaction-diffusion equations can be applied to describe many problems in mathematics, physics, biology, chemistry and other fields. It is commonly used bio-mathematically to describe the spatial structure and quantitative variation of populations [9, 7, 13, 4, 5, 3]. Periodicity is a common phenomenon in mathematical modelings due to seasonal changes typically related to climate changes [16, 1, 18, 6, 10, 17, 8, 14]. Recently, Zhang and Zhao [17] considered the following nonlocal dispersal Fisher-KPP equation in a time-periodic shifting habitat

$$\frac{\partial u}{\partial t}=d(J*u-u)+u[r(t,x-ct)-u],\ t>0,\ x\in\mathbb{R}. \tag{1.3}$$

In [17], the existence, uniqueness, stability and the nonexistence of periodic forced waves as well as the spreading properties for a large class of solutions were addressed. As is well known, populations often have close relationships with each other in the ecosystem. Considering the competitive relationship, Berestycki et al. [2] first proposed that the change of the environment plays an opposite role for the two competing species. In [2], the stationary solution is achieved when c=0, and gap formations are obtained. Wu et al. [12] investigated a Lotka-Volterra competition model with nonlocal dispersal, with the difference being that the monotonicity of the two growth functions is consistent. By analyzing the spatial-temporal dynamics of the system, they identified certain ranges for the worsening speed c. Additionally, Wang and Wu [11] considered the same model as [12], which the growth function conditions are similar to [2]. The results they obtained contained the special case $c=0$. Later, Wang et al. [10] studied the Lotka-Volterra competition system under (H1) in a time-periodic shifting habitat with the following form

$$\begin{cases} \dfrac{\partial u}{\partial t}=d_1\dfrac{\partial^2 u}{\partial x^2}+u\left[r_1(t,x-ct)-u-a_1v\right], \\ \dfrac{\partial v}{\partial t}=d_2\dfrac{\partial^2 v}{\partial x^2}+v\left[r_2(t,x-ct)-a_2u-v\right], \end{cases} \quad t\in\mathbb{R}^+,\ x\in\mathbb{R}. \tag{1.4}$$

They studied the existence, uniqueness and stability of forced pulsating waves as well as gap formation of the reaction-diffusion model. The research of these scholars has greatly enriched the achievements of reaction-diffusion equations in the field of biomathematics.

We incorporate the periodicity into the competition model with nonlocal dispersal. The aim is to consider a more complex and realistic ecosystem dynamic that is closer to nature. In this case, our calculation and proof process will become more difficult.

Obviously, the corresponding limit kinetic system of (1.1) is

$$\begin{cases} u'(t)=u\left[r_1(t,\infty)-u-a_1v\right], \\ v'(t)=v\left[r_2(t,\infty)-a_2u-v\right], \end{cases} \tag{1.5}$$

system (1.5) only has three nonnegative T-periodic solutions (0,0), $(p(t),0)$ and $(0,p(t))$, where $p(t)$ is the nonzero periodic solution of the logistic equations

$$\frac{du}{dt}=u[\theta(t)-u].$$

In fact, it is known through calculation that

$$p(t)=\frac{p_0 e^{\int_0^t \theta(s)\mathrm{d}s}}{1+p_0\int_0^t e^{\int_0^s \theta(\tau)\mathrm{d}\tau}\mathrm{d}s}>0$$

where

$$p_0=\frac{e^{\int_0^T \theta(s)\mathrm{d}s}-1}{\int_0^T e^{\int_0^s \theta(\tau)\mathrm{d}\tau}\mathrm{d}s}.$$

This paper is organized as follows. In Section 2, we introduce some known results of the periodic KPP model. Section 3 is devoted to the existence of the forced waves when $c\in(-c_0(d_2), c_0(d_1))$(see Theorem 3.1). Then, we show the gap formations for $c>c_0(d_1)$ and $c<-c_0(d_2)$ in Section 4 (see Theorem 4.2 and Theorem 4.3). Finally, we present conclusions and expectations in Section 5.

2. Preliminaries

In this paper, we shall use the notation $\bar{f}$ to denote the average value of f on the interval $[0,T]$, namely,

$$\bar{f}=\frac{1}{T}\int_0^T f(x)\mathrm{d}x.$$

For functions $L(t,z)$ and $\gamma(t)$, we also use $\lim\limits_{z\to\infty} L(t,z)=\gamma(t)$ to denote $\lim\limits_{z\to\infty}[L(t,z)-\gamma(t)]=0$ uniformly in t. The forced traveling wave solution of the system (1.1) means a particular solution in the form of

$$(u,v)(t,x)=(U,V)(t,x-ct):=(U,V)(t,z),\ \ z=x-ct, \tag{2.1}$$

satisfying

$$(U,V)(t+T,z)=(U,V)(t,z).$$

By substituting (2.1) into (1.1), we can get the wave profile system as below:

$$\begin{cases} -cU_z=d_1\left(\int_R J_1(y)U(t,z-y)\mathrm{d}y-U\right)+U\left[r_1(t,z)-U-a_1V\right]-U_t,\\ -cV_z=d_2\left(\int_R J_2(y)V(t,z-y)\mathrm{d}y-V\right)+V\left[r_2(t,z)-a_2U-V\right]-V_t,\ \ t\in\mathbb{R}^+,\ \ x\in\mathbb{R}, \end{cases} \tag{2.2}$$

with the boundary conditions at infinity

$$\lim_{z\to-\infty}(U,V)(t,z)=(p(t),0),\quad \lim_{z\to\infty}(U,V)(t,z)=(0,p(t)). \tag{2.3}$$

Then we present two crucial lemmas, centered on exploring the solvability of a spatio-temporal heterogenetic equation which serves as a simplified representation of the original system (2.2).

In view of Zhang and Zhao [17], we can get the following lemma.

Lemma 2.1. *Assume that $r(t,z)$ is a nondecreasing and continuous function in z, and T-periodic in t, satisfying $\lim\limits_{z\to-\infty} r(t,z)<0$, $\lim\limits_{z\to\infty} r(t,z)=\theta(t)>0$ uniformly in t. Then, for any $c>-c_0(d)$ with*

$$c_0(d):=\inf_{\mu>0}\frac{d\left(\int_R J(y)e^{\mu y}\mathrm{d}y-1\right)+\overline{\theta}}{\mu},$$

there exists a unique positive solution $w(t,z)$ which is nondecreasing in z and T-periodic in t for the following boundary problem

$$\begin{cases} d\left(\int_R J(y)w(t,z-y)\mathrm{d}y-w\right)+cw_z+w[r(t,z)-w]-w_t=0,\\ \lim\limits_{z\to-\infty} w(t,z)=0,\ \ \lim\limits_{z\to\infty} w(t,z)=p(t),\ \ t\in\mathbb{R}^+,\ \ z\in\mathbb{R}. \end{cases}$$

Moreover, if one views the solution w as a functional of r, d and J, then $I(r,d,J) := w(r,d,J)(t,z)$ is nondecreasing in the variable r.

By variable substitution, we can obtain the Lemma 2.2 corresponding to Lemma 2.1 as below.

Lemma 2.2. *Assume that $\tilde{r}(t,z)$ is a nonincreasing and continuous function in z, and T-periodic in t, satisfying $\lim_{z\to-\infty}\tilde{r}(t,z)=\theta(t)>0$, $\lim_{z\to\infty}\tilde{r}(t,z)<0$ uniformly in t. Then, for any $c<c_0(d)$, there exists a unique positive solution $\tilde{w}(t,z)$ which is nonincreasing in z and T-periodic in t for the following boundary problem*

$$\begin{cases} d\left(\int_R J(y)\tilde{w}(t,z-y)\mathrm{d}y-\tilde{w}\right)+c\tilde{w}_z+\tilde{w}[\tilde{r}(t,z)-\tilde{w}]-\tilde{w}_t=0, \\ \lim_{z\to-\infty}\tilde{w}(t,z)=p(t),\ \lim_{z\to\infty}\tilde{w}(t,z)=0,\ t\in\mathbb{R}^+,\ z\in\mathbb{R}. \end{cases}$$

Moreover, if one views the solution $\tilde{w}$ as a functional of $\tilde{r}$, d and J, then $\tilde{I}(\tilde{r},d,J) := \tilde{w}(\tilde{r},d,J)(t,z)$ is nondecreasing in the variable $\tilde{r}$.

3. Existence of forced waves

Let

$$c_0\left(d_1\right)=\inf_{\mu>0}\frac{d_1\left(\int_R J_1(y)e^{\mu y}\mathrm{d}y-1\right)+\overline{\theta}}{\mu},\quad c_0\left(d_2\right)=\inf_{\mu>0}\frac{d_2\left(\int_R J_2(y)e^{\mu y}\mathrm{d}y-1\right)+\overline{\theta}}{\mu}.$$

It is worth emphasizing that, in a natural environment free from competitive interference, $c_0\left(d_1\right)$ and $c_0\left(d_2\right)$ serve as the KPP speeds (also known as Fisher-KPP speeds or linear deterministic speeds) for the u-species and v-species, respectively. In addition, if the periodicity is not taken into account, the expression for $c_0\left(d_1\right)$ and $c_0\left(d_2\right)$ will be

$$\inf_{\mu>0}\frac{d_1\left(\int_R J_1(y)e^{\mu y}\mathrm{d}y-1\right)+r_1(-\infty)}{\mu}\text{ and }\inf_{\mu>0}\frac{d_2\left(\int_R J_2(y)e^{\mu y}\mathrm{d}y-1\right)+r_2(+\infty)}{\mu}.$$

We are now fully equipped to present the theorem pertaining to the existence of a forced wave solution for system (1.1).

Theorem 3.1. *Assume that $c\in\left(-c_0\left(d_2\right),c_0\left(d_1\right)\right)$. There exists a T-periodic solution $(U,V)(t,z)$ to the system (2.2)-(2.3). Moreover, $U(t,z)$ is nonincreasing and $V(t,z)$ is nondecreasing in variable z respectively.*

Proof. By Lemma 2.1 and Lemma 2.2, we define two sequences of functions as follows:

$$\begin{aligned} &V_0:=0,\quad U_0:=\tilde{I}\left(r_1,d_1,J_1\right), \\ &V_1:=I\left(r_2-a_2U_0,d_2,J_2\right),\quad U_1:=\tilde{I}\left(r_1-a_1V_1,d_1,J_1\right), \\ &\cdots\cdots \\ &V_{n+1}:=I\left(r_2-a_2U_n,d_2,J_2\right),\quad U_{n+1}:=\tilde{I}\left(r_1-a_1V_{n+1},d_1,J_1\right),\ \forall n\geq1, \end{aligned}$$

which makes sense for $c\in\left(-c_0\left(d_2\right),c_0\left(d_1\right)\right)$. It can be seen from Lemma 2.1 and Lemma 2.2 that $U_n(t,z)$ is nonincreasing and $V_n(t,z)$ is nondecreasing in variable z for each $n\geq1$. Meanwhile, we have $U_n\geq U_{n+1}$, $V_n\leq V_{n+1}$ for all $n\geq0$. By the definition of $\tilde{I}$, one can conclude that $U_0(t,-\infty)=p(t)$ and $U_0(t,\infty)=0$ uniformly in t. Furthermore, for $r_2(t,z)-a_2U_0(t,z)$, we are able to see that $r_2(t,-\infty)-a_2U_0(t,-\infty)=\beta(t)-a_2p(t)<0$ and $r_2(t,\infty)-a_2U_0(t,\infty)=\theta(t)$. Then by the definition of I, it is easy to conclude that $V_1(t,-\infty)=0$ and $V_1(t,\infty)=p(t)$. By using the iteration and induction, we can get that

$$0\leq U_n,\ V_n\leq\max_{t\in[0,T]}p(t)\ \text{ for for all }\ (t,z)\in\mathbb{R}^+\times\mathbb{R},$$

and hence, there exist $U(t,z)$ and $V(t,z)$ such that $(U_n,V_n)(t,z)\to(U,V)(t,z)$ pointwise as $n\to\infty$. Moreover, $U(t,z)$ is nonincreasing but $V(t,z)$ is nondecreasing with respect to z and T-periodic with respect to t. In addition, it follows from the standard regularity analysis on the integral forms of

$$\begin{cases} d_1\left(\int_R J_1(y)U_n\mathrm{d}y-U_n\right)+c\left(U_n\right)_z+U_n\left[r_1(t,z)-U_n-a_1V_n\right]-\left(U_n\right)_t=0,\\ d_2\left(\int_R J_2(y)V_{n+1}\mathrm{d}y-V_{n+1}\right)+c\left(V_{n+1}\right)_z+V_{n+1}\left[r_2(t,z)-a_2U_n-V_{n+1}\right]-\left(V_{n+1}\right)_t=0\end{cases}$$

that $(U,V)(t,z)\in C^1\left(\mathbb{R}^+\times\mathbb{R},\mathbb{R}\right)$ is a solution to the system of (2.2).

Next, we will show that $(U,V)(t,z)$ satisfies the boundary condition (2.3). Since $U(t,z)$ and $V(t,z)$ are bounded and monotone in z, the limits of $(U,V)(t,z)$ at $z=\pm\infty$ exist for each $t\in\mathbb{R}^+$. By letting $z\to\pm\infty$ in (2.2), respectively, we obtain

$$\begin{cases} \dfrac{dU(t,\infty)}{dt}=U(t,\infty)\left[\beta(t)-U(t,\infty)-a_1V(t,\infty)\right],\ t\in\mathbb{R}^+,\\ \dfrac{dV(t,\infty)}{dt}=V(t,\infty)\left[\theta(t)-V(t,\infty)-a_2U(t,\infty)\right],\ t\in\mathbb{R}^+,\\ \dfrac{dU(t,-\infty)}{dt}=U(t,-\infty)\left[\theta(t)-U(t,-\infty)-a_1V(t,-\infty)\right],\ t\in\mathbb{R}^+,\\ \dfrac{dV(t,\infty)}{dt}=V(t,-\infty)\left[\beta(t)-V(t,-\infty)-a_2U(t,-\infty)\right],\ t\in\mathbb{R}^+.\end{cases} \tag{3.1}$$

Noting that U and V are nonnegative and T-periodic, we have from the first and the fourth equation of (3.1) that $U(t,\infty)=V(t,-\infty)=0$. Further, we can infer that $V(t,\infty)>0$ by the monotonicity of the sequence V_n in n. As a result, from the second equation of (3.1), one can conclude that $V(t,\infty)=p(t)$. In the meantime, for each n, we have $V_n(t,z)\le V(t,z)$, it follows that $U_n(t,z)\ge\tilde{I}\left(r_1-a_1V,d_1,J_1\right)>0$. This, together with the third equation of (3.1), shows that $U(t,-\infty)=p(t)$. The proof is complete. □

4. Gap formations

In this section, we focus on the case $c>c_0\left(d_1\right)$ or $c<-c_0\left(d_2\right)$, that is, when there is a rapid change in climate. It was initially proposed in [2]: when the shifting speed c is taken as some values, the densities of species u and v will eliminate with an exponential decaying rate in a region of size $\left[c-c_0\left(d_1\right)\right]t$ (for the case $c>c_0\left(d_1\right)$) or $\left|\left[c+c_0\left(d_2\right)\right]\right|t$ (for the case $c<-c_0\left(d_2\right)$) in the asymptotic sense. We call this region a gap. The reader is also encouraged to refer to [2] for further details.

We first introduce some notations. Let $\mathbb{X}=UC\left(\mathbb{R},\mathbb{R}^2\right)\cap L^\infty\left(\mathbb{R},\mathbb{R}^2\right)$ be the set of all uniformly continuous and bounded vector functions from $\mathbb{R}$ to $\mathbb{R}^2$ equipped with the norm $\|\varphi\|_{\mathbb{X}}:=\|\varphi_1\|+\|\varphi_2\|$, where $\|\varphi_i\|:=\sup_{x\in\mathbb{R}}|\varphi_i(x)|$. Denote

$$\mathbb{X}_+=\left\{\varphi=(\varphi_1,\varphi_2)\in\mathbb{X}:\varphi_1(x)\ge0,\varphi_2(x)\ge0,\ \forall x\in\mathbb{R}\right\}.$$

It follows that $\mathbb{X}_+$ is a closed core of $\mathbb{X}$ and $\mathbb{X}$ is a Banach lattice under the partial ordering induced by $\mathbb{X}_+$. Further, we set

$$\mathbb{X}_\theta=\left\{(\varphi_1,\varphi_2)\in\mathbb{X}:(0,0)\le(\varphi_1,\varphi_2)\le(\theta_{\max},\theta_{\max})\right\},\ \theta_{\max}:=\max_{t\in\mathbb{R}^+}\theta(t)$$

and consider the following Cauchy problem

$$\begin{cases} u_t=d_1\left(J_1*u-u\right)+u\left[r_1(t,x-ct)-u-a_1v\right],\ t\in\mathbb{R}^+,\ x\in\mathbb{R},\\ v_t=d_2\left(J_2*v-v\right)+v\left[r_2(t,x-ct)-v-a_2u\right],\ t\in\mathbb{R}^+,\ x\in\mathbb{R},\\ (u(0,x),v(0,x))=(u_0(x),v_0(x))\in\mathbb{X}_+.\end{cases} \tag{4.1}$$

By slightly revising the proof of Lemma 2.3 in [8], we can get the existence and uniqueness of solutions for problem (4.1).

Theorem 4.1. *Assume that (J) and (H1) hold,* $(u_0, v_0) \in \mathbb{X}_\theta$*, then system (4.1) admits a unique solution* $(u(t,x), v(t,x))$ *with* $(u(0,x), v(0,x)) = (u_0(x), v_0(x))$ *and* $(u,v)(\cdot,x) \in C(\mathbb{R}^+, \mathbb{X}_+)$.

To investigate the gap formation between the species u and v, we further postulate an assumption concerning the nontrivial initial data $u_0(x)$ and $v_0(x)$.

(H2) Let $u_0(x),\ v_0(x) \in L^\infty(\mathbb{R})$ and $(u_0, v_0) \in \mathbb{X}_\theta$. Moreover, there exists a number $A > 0$ large enough so that $u_0(x) = 0$ for all $x \geq A$ and $u_0(x) > 0$ for all $x < A$ and $v_0(x) > 0$ for all $x \in \mathbb{R}$ or $v_0(x)$ is supported on the right of the x-axis.

Theorem 4.2. *Assume that* $c > c_0(d_1)$*, (J), (H1) and (H2) hold. Then the unique solution of (4.1) satisfies*

$$0 \leq u(t,x),\ v(t,x) \leq \theta_{\max}, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Moreover, for any two numbers $c_1,\ c_2$ *with* $c_0(d_1) < c_1 < c_2 < c$ *and any real numbers* b_i*, there exist positive numbers* M_i *and* $\alpha_i (i = 1, 2)$*, such that*

$$\sup_{x \geq c_1 t + b_1} u(t,x) \leq M_1 e^{-\alpha_1 t}, \quad \forall t \in \mathbb{R}^+, \tag{4.2}$$

$$\sup_{x \leq c_2 t + b_2} v(t,x) \leq M_2 e^{-\alpha_2 t}, \quad \forall t \in \mathbb{R}^+. \tag{4.3}$$

Proof. Obviously, Theorem 4.1 ensures the existence and uniqueness of the solution to system (4.1). We will first prove (4.2). Noting that $u(t,x),\ v(t,x) \geq 0$ for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$, by the property of $r_1(t,z)$, we can obtain from the first equation of (4.1) that

$$u_t \leq d_1 (J_1 * u - u) + \theta(t) u.$$

Let

$$\Lambda(\mu) = \frac{d_1 \left(\int_R J_1(y) e^{\mu y} \mathrm{d}y - 1 \right) + \overline{\theta}}{\mu}.$$

In view of (J) and $\overline{\theta} > 0$, it follows that $\Lambda(\mu) > 0$ for all $\mu > 0$. Further, we have

$$\Lambda(\mu) = \sum_{m=1}^{\infty} \frac{d_1 \int_R y^{2m} J_1(y) \mathrm{d}y}{(2m)!} \mu^{2m-1} + \frac{\overline{\theta}}{\mu} \to +\infty$$

as $\mu \to 0$ or $\mu \to \infty$, and

$$\Lambda''(\mu) = \sum_{m=1}^{\infty} \frac{d_1 (2m-1)(2m-2) \int_R y^{2m} J_1(y) \mathrm{d}y}{(2m)!} \mu^{2m-3} + \frac{2\overline{\theta}}{\mu^3} > 0, \quad \forall \mu > 0.$$

Thus $\Lambda(\mu)$ can attain its infimum at the unique critical point $\mu_1^* > 0$, that is $c_0(d_1) = \inf_{\mu > 0} \Lambda(\mu) = \Lambda(\mu_1^*)$. One can verify directly that

$$\begin{aligned} \bar{u}(t,x) &:= \theta_{\max} e^{\int_0^t \left[d_1 \int_R J_1(y) e^{\mu_1^* y} \mathrm{d}y - d_1 + \theta(s) - c_0(d_1) \mu_1^* \right] \mathrm{d}s} e^{-\mu_1^* [x - c_0(d_1) t - A]} \\ &= \theta_{\max} e^{\int_0^t [\theta(s) - \overline{\theta}] \mathrm{d}s} e^{-\mu_1^* [x - c_0(d_1) t - A]} \end{aligned}$$

satisfies

$$\bar{u}_t = d_1 (J_1 * \bar{u} - \bar{u}) + \theta(t) \bar{u}.$$

Since $0 \leq u_0(x) \leq \theta_{\max}$ on $\mathbb{R}$ and $u_0(x) = 0$ for $x \geq A$, we have

$$u_0(x) \leq \bar{u}(0,x) = \theta_{\max} e^{-\mu_1^* (x - A)}.$$

By using the comparison principle, we can conclude that

$$u(t,x) \le \bar{u}(t,x), \quad \forall t \in \mathbb{R}^+,\ x \in \mathbb{R},$$

from which it follows that

$$\sup_{x \ge c_1 t + b_1} u(t,x) \le \theta_{\max} R_1 e^{\mu_1^*(A-b_1)} e^{-\mu_1^*[c_1 - c_0(d_1)]t},$$

where R_1 is the maximum of the function $e^{\int_0^t [\theta(s)-\bar{\theta}]\mathrm{d}s}$ on the interval $[0,T]$. Taking $M_1 := \theta_{\max} R_1 e^{u_1^*(A-b_1)} > 0$ and $\alpha_1 := \mu_1^*[c_1 - c_0(d_1)] > 0$ gives (4.2).

Clearly, for equation $\tilde{u}_t = d_1(J_1 * \tilde{u} - \tilde{u}) + \tilde{u}[r_1(t, x-ct) - \tilde{u}]$, $\theta_{\max}$ is an upper solution and $u(t,x)$ is a lower solution. For equation $\tilde{v}_t = d_2(J_2 * \tilde{v} - \tilde{v}) + \tilde{v}[r_2(t,x-ct) - \tilde{v}]$, $\theta_{\max}$ is an upper solution and $v(t,x)$ is a lower solution. According to the comparison principle again, we have $u(t,x)$, $v(t,x) \le \theta_{\max}$.

Next, we intend to work out the proof of (4.3). Since $r_2(t,z)$ is nondecreasing in $z \in \mathbb{R}$ and $\beta(t) = r_2(t,-\infty) < 0 < r_2(t,+\infty) = \theta(t)$, then for any $\varsigma \in (0, -\beta_{\max})$, there exists $z_0 \in \mathbb{R}$ such that $r_2(t,z) \le \beta(t) + \varsigma$, $\forall z \le z_0$. Further, in view of the second equation of (4.1), there holds

$$v_t \le d_2(J_2 * v - v) + [\beta(t) + \varsigma]v, \quad x \le ct + z_0. \tag{4.4}$$

Then we consider an auxiliary equation

$$w_t = d_2(J_2 * w - w) + [\beta(t) + \varsigma]w, \quad \forall t \in \mathbb{R}^+,\ x \in \mathbb{R}. \tag{4.5}$$

Let

$$\Lambda_1(\mu) = -c\mu - d_2\left(\int_R J_2(y) e^{-\mu y}\mathrm{d}y - 1\right) - [\beta(t) + \varsigma].$$

Noting that $-[\beta(t)+\varsigma] > 0$, we can choose $\mu_1 > 0$ such that $\Lambda_1(\mu_1) > 0$. Select a constant $\gamma \in (0, -(\beta_{\max} + \varsigma))$ and define

$$\tilde{w}(t,x) = \theta_{\max}\left[e^{\mu_1(x-ct-z_0)} + e^{-\gamma t}\right].$$

By a direct calculation, we see that

$$\begin{aligned}
&\tilde{w}_t - d_2(J_2 * \tilde{w} - \tilde{w}) - [\beta(t) + \varsigma]\tilde{w} \\
&= \theta_{\max} e^{\mu_1(x-ct-z_0)} \Lambda_1(\mu_1) + \theta_{\max} e^{-\gamma t}[-\gamma - (\beta(t) + \varsigma)] \ge 0,
\end{aligned}$$

which implies that $\tilde{w}$ is an upper solution of (4.5) satisfying

$$\begin{aligned}
&v(t,x) \le \theta_{\max} \le \tilde{w}(t,x), \quad t \in \mathbb{R}^+,\ x \ge ct + z_0, \\
&v(0,x) \le \theta_{\max} \le \tilde{w}(0,x), \quad x \le z_0.
\end{aligned}$$

Set $\Theta(t,x) = \tilde{w}(t,x) - v(t,x)$, together with (4.4), one can get the following inequalities

$$\begin{cases}
\Theta_t - d_2(J_2 * \Theta - \Theta) - [\beta(t) + \varsigma]\Theta \ge 0,\ t \in \mathbb{R}^+,\ x \le ct + z_0, \\
\Theta(t,x) \ge 0,\ t \in \mathbb{R}^+,\ x \ge ct + z_0, \\
\Theta(0,x) \ge \theta_{\max}\left[e^{\mu_1(x-z_0)} + 1\right] - \theta_{\max} \ge 0,\ x \le z_0.
\end{cases}$$

By the maximum principle for linear nonlocal dispersal problems on the semi-unbounded region (see e.g. Lemma 3.3 in [14] or Lemma 4.7 in [15]), we know that $\Theta(t,x) \ge 0$, namely, $\tilde{w}(t,x) \ge v(t,x)$ for all $x \leq ct + z_0$ and $t \in \mathbb{R}^+$. Choose a suitable time $t_0 \geq 0$ so that $c_2 t + b_2 \leq ct + z_0$ holds for all $t \geq t_0$. As a result, we have

$$\begin{aligned}\sup_{x \leq c_2 t + b_2} v(t,x) &\leq \sup_{x \leq c_2 t + b_2} \theta_{\max} \left[e^{\mu_1 (x - ct - z_0)} + e^{-\gamma t} \right] \\ &\leq \theta_{\max} \left[e^{\mu_1 (b_2 - z_0)} e^{-(c - c_2)\mu_1 t} + e^{-\gamma t} \right] \\ &\leq M_2 e^{-\alpha_2 t},\end{aligned}$$

where

$$M_2 := \max \left\{ \theta_{\max} \left[e^{\mu_1 (b_2 - z_0)} + 1 \right], \theta_{\max} e^{\alpha_2 t_0} \right\}, \ \ \alpha_2 := \min \left\{ (c - c_2) \mu_1, \gamma \right\}.$$

The proof is complete. □

Theorem 4.2 indicates that there exists a region of asymptotic size $[c - c_0(d_1)]\, t$ when $c > c_0(d_1)$. By the transformation $x \to -x$ in (4.1), we can show from Theorem 4.2 that the gap formation also occurs for $c < -c_0(d_2)$.

Theorem 4.3. *Assume that $c < -c_0(d_2)$, (J), (H1) and (H2) hold. Then the unique solution of (4.1) satisfies*

$$0 \leq u(t,x),\ v(t,x) \leq \theta_{\max}, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Moreover, for any two numbers $\hat{c}_1$, $\hat{c}_2$ with $c < \hat{c}_1 < \hat{c}_2 < -c_0(d_2)$ and any real numbers $\hat{b}_i$, there exist positive numbers $\hat{M}_i$ and $\hat{\alpha}_i (i = 1, 2\ )$, such that

$$\sup_{x \geq \hat{c}_1 t + \hat{b}_1} u(t,x) \leq \hat{M}_1 e^{-\hat{\alpha}_1 t}, \quad \forall t \in \mathbb{R}^+, \tag{4.6}$$

and

$$\sup_{x \leq \hat{c}_2 t + \hat{b}_2} v(t,x) \leq \hat{M}_2 e^{-\hat{\alpha}_2 t}, \quad \forall t \in \mathbb{R}^+. \tag{4.7}$$

5. Conclusions and expectations

We have proved the existence of forced waves of (1.1) by using iterative techniques and the gap formations theoretically. This partly enables us to predict the trends of species in habitats. However, in this system, there are still many problems to be studied. For example, in [11], Wang and Wu used some delicate analysis to obtain the asymptotic behaviors at infinity of the forced waves, and presented some numerical simulation results to confirm their theoretical results. In addition, in the Lyapunov sense, the stability of the forced pulsating wave was studied in [10]. Due to the introduction of periodicity, studying these contents has become more difficult for our model. We expect that these issues can be addressed later on.

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References

[1] X. X. Bao and Z. C. Wang, *Existence and stability of time periodic traveling waves for a periodic bistable Lotka-Volterra competition system*, J. Diff. Eqns. **255**(2013), 2402-2435.

[2] H. Berestycki, L. Desvillettes and O. Diekmann, *Can climate change lead to gap formation*? Ecol. Complex. **20**(2014), 264-270.

[3] H. Berestycki and J. Fang, *Forced waves of the Fisher-KPP equation in a shifting environment*, J. Diff. Eqns. **264**(2018), 2157-2183.

[4] H. J. Hu and X. F. Zou, *Existence of an extinction wave in the Fisher equation with a shifting habitat*, Proc. Amer. Math. Soc. **145**(2017), 4763-4771.

[5] H. J. Hu, T. S, Yi and X. F. Zou, *On spatial-temporal dynamics of a Fisher-KPP equation with a shifting environment*, Proc. Amer. Math. Soc. **148**(2020), 213-221.

[6] W. T. Li, J. B. Wang and X. Q. Zhao, *Propagation dynamics in a time periodic nonlocal dispersal model with stage structure*, J. Dyn. Diff. Eqns. **32**(2020), 1027-1064.

[7] G. G. Liu, T. Y. Xu and J. X. Yin, *Forced waves of reaction-diffusion model with density-dependent dispersal in shifting environments*, J. Diff. Eqns. **282**(2021), 127-147.

[8] S. X. Qiao, W. T. Li and J. B. Wang, *Propagation dynamics of nonlocal dispersal competition systems in time-periodic shifting habitats*, J. Diff. Eqns. **378**(2024), 399-459.

[9] H. R. Thieme and X. Q. Zhao, *Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction diffusion models*, J. Diff. Eqns. **195**(2003), 430-470.

[10] H. Y. Wang, C. H. Pan and C. H. Ou, *Propagation dynamics of forced pulsating waves of a time periodic Lotka-Volterra competition system in a shifting habitat*, J. Diff. Eqns. **340**(2022), 359-385.

[11] J. B. Wang and C. F. Wu, *Forced waves and gap formations for a Lotka-Volterra competition model with nonlocal dispersal and shifting habitats*, Nonlinear. Anal. RWA **58**(2021), 103208.

[12] C. F. Wu, Y. W and X. F. Zou, *Spatial-temporal dynamics of a Lotka-Volterra competition model with nonlocal dispersal under shifting environment*, J. Diff. Eqns. **267**(2019), 4890-4921.

[13] J. H. Wu and X. F. Zou, *Traveling wave fronts of reaction-diffusion systems with delay*, J. Dyn. Differ. Eqns. **13**(2001), 651-687.

[14] S.L. Wu and M.D. Huang, *Time periodic traveling waves for a periodic nonlocal dispersal model with delay*, Proc. Amer. Math. Soc. **148**(2020), 4405-4421.

[15] G. B. Zhang and X. Q. Zhao, *Propagation phenomena for a two-species Lotka-Volterra strong competition system with nonlocal dispersal*, Calc. Var. Partial. Diff Eqns. **59**(2020), 1-34.

[16] G. Y. Zhao and S. G. Ruan, *Existence, uniqueness and asymptotic stability of time periodic traveling waves for a periodic Lotka-Volterra competition system with diffusion*, J. Math. Pure. Appl. **95**(2011), 627-671.

[17] G.B. Zhang and X.Q. Zhao, *Propagation dynamics of a nonlocal dispersal Fisher-KPP equation in a time-periodic shifting habitat*, J. Diff. Eqns. **268**(2020), 2852-2885.

[18] Y. Zhang and X.Q. Zhao, *Bistable travelling waves for a reaction and diffusion model with seasonal succession*, Nonlinearity **26**(2013), 691-709.

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