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STUDY ON MULTIDIMENSIONAL FUZZY GRAPHS THROUGH MODIFIED PARTIAL ORDERING

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Abstract. This paper introduces the concepts of multidimensional fuzzy graphs and edge-powered multidimensional fuzzy graphs, which employ a hybrid structure that combines multidimensional fuzzy sets and graphs. This study redefines the axioms of multidimensional $t-$ norms and $t-$ conorms by providing a more general partial order that can link more components of the range set $\mathcal{J}_\infty\big([0,1]\big)$. More studies on various operations such as direct product, composition, tensor product, join, etc. are conducted with relevant illustrations. A novel complement operator approach is also investigated to link the multidimensional fuzzy graph and the edge-powered multidimensional fuzzy graph. Finally, defining the infimum and supremum of an arbitrary family in $\mathcal{J}_\infty\big([0,1]\big)$ introduces many notions such as vertex degree, $min-$ vertex degree, $max-$ vertex degree, path strength, etc.

1. Introduction

The introduction of fuzzy sets by Zadeh [30, 6] was a significant milestone in dealing with uncertainty and imprecision in problem-solving approaches. Although fuzzy sets are useful in dealing with specific real-world circumstances, their ability to accurately represent complex scenarios with changing levels of ambiguity and vagueness is restricted. As a result, several mathematical models have been developed to encompass a wide range of phenomena, building upon the fundamental idea of fuzziness. The main expansions of Zadeh's fuzzy model include interval-valued fuzzy sets [31], hesitant fuzzy sets [29], intuitionistic fuzzy sets [3], n-dimensional fuzzy sets [27], and m-polar fuzzy sets [1]. Nevertheless, m-polar, intuitionistic, and n-dimensional fuzzy sets, among others, fail to provide sufficient membership values for individual attributes, and models like interval-valued and hesitant fuzzy sets, with their intricate operations, make real-world problem-solving difficult and less meaningful. The above fuzzy models have the major drawback of providing a simple presentation of data with variable ambiguity.

The n-dimensional fuzzy sets and m-polar fuzzy sets were created to provide greater flexibility in portraying the overall vagueness and ambiguity of the material. However, it later encountered a significant limitation in its ability to give personalized attention to each data member while taking into account the uncertainty associated with them. [14]Annaxsuel and Palmeira were driven by this challenge to propose a new model called multidimensional fuzzy sets (MDFS). This model allows for individual attention to each element by allowing the usage of any number of membership values ranging from 0 to 1. The elements in MDFS gain independence and freedom, which enhances the simplicity and accuracy of data display. MDFS is an encompassing model that includes interval-valued fuzzy sets, intuitionistic fuzzy sets, n-dimensional fuzzy sets, m-polar fuzzy sets, and other similar models. Consequently, any findings produced in MDFS may be readily applied to these models and their respective application processes.

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In MDFS each element in the data can choose any number of values from $[0,1]$ based on the vagueness that lies with them. This presentation can be easily modeled as a function from the underlying set to $\mathcal{J}_\infty([0,1])$ that contains all the members from $[0,1]^n$, $n \in \mathbb{N}$ arranged in ascending order. The supremacy of MDFS can be seen by considering instances where elements need individual attention. For instance, let's imagine a situation where an interviewer assesses candidates' proficiency in several disciplines and uses MDFS to analyze their scores. Then the interviewer can represent the overall individual score using membership vectors such as $(0.30, 0.35, 0.40)/(0.29, 0.33, 0.35, 0.39)/(0.40, 0.45)$. However, we need to limit the presentation to a fixed dimension m in an m-polar fuzzy set. If we represent the above data using a 3-polar fuzzy set or a 3-dimensional fuzzy set, all the data members must have membership values such as $(0.30, 0.32, 0.33)/(0.35, 0.36, 0.38)$. This negatively impacts the natural representation of the data according to the interviewer's perception.

The multidimensional fuzzy framework provides a flexible solution to a range of practical problems, such as decision-making, granular computing, and image processing, while maintaining the accuracy of the data. Moreover, substantial progress in MDFS research has produced vital knowledge on aggregation operators, De Morgan's rule, and other fundamental features. A complete grasp of the fundamental processes driving mdfs operations may be obtained by referring to the work of Josen and John [13], where detailed explanations of these important features can be found. Multiattribute decision-making algorithms through MDFS can be seen in [26]. Some novel approaches to a rough approximation of MDFS using distance measures and aggregation operators are studied by Josen, Sunil, and Jobish in [12]. For more hybrid structures in multidimensional fuzzy sets the reader may refer to [11, 10]

Graph theory is a significant field of applied mathematics with wide-ranging applications in various disciplines such as analysis of systems, artificial neural networks, electrical engineering, management theory, mode of transport, building design, and communication [17]. Even though graph theory was able to demonstrate and handle many real-life problems, it failed to deal with problems that contained ambiguity and uncertainty. Hence, in 1975 Rosenfeld [25] defined a new hybrid branch called fuzzy graphs, in which fuzzy sets were used to handle the uncertainty part. Later, Bhattacharya [4] and Bhutani [5] studied connectedness and automorphism-related concepts of fuzzy graphs. In [19] Mordeson introduced various operators in a fuzzy graph and discussed their properties. The hybrid fuzzy graph has been extended to many other structures, such as intuitionistic fuzzy graphs [9], Pythagorean fuzzy graphs [16], Bi-polar fuzzy graphs [24], $m-$polar fuzzy graphs [20, 24], and so on.

This work offers a more generalized form of a fuzzy graph called Multidimensional Fuzzy Graph (MDFG) and Edge-Powered Multidimensional Fuzzy Graph (EPMDFG), which use multidimensional $t-$norms and $t-$conorms respectively, to interconnect the ideas of the graph and MDFS.. Following the preliminary Section 2, we redefine the partial ordering of $\mathcal{J}_\infty([0,1])$ in Section 3 so that many more elements in the latter set can be interrelated and the monotonicity of $t-$norms and $t-$conorms can be obtained without any conditions on the cardinality of membership vectors. We define MDFG and EPMDFG using the redefined aggregation operations, with suitable demonstrations in Section 4. Various operators of MDFG and EPMDFG, including direct product, composition, tensor product, and normal product, are defined, and their properties are explored. Multidimensional fuzzy complements are used to examine various relationships between MDFG and EPMDFG. In Section 5, the supremum and infimum of an arbitrary family of elements in $\mathcal{J}_\infty([0,1])$ are defined. Using them, more properties of MDFG and EPMDFG such as vertex degree, $min-$vertex degree, $max-$vertex degree, path strength, etc. are introduced, and their specialist in connection with the above operators are studied. Finally, in Section 6, the investigations conducted in this study are completed, and ideas for future efforts are presented. vertex degree, path strength, etc. are introduced, and their specialties in connection with the above operators are studied. Finally, in Section 6, the studies made in this paper are concluded, and ideas regarding future works of this paper are provided.

2. Preliminaries

2.1. Multidimensional fuzzy sets.

First of all, we will establish the basic notions that are required to present a MDFS as a function to $\mathcal{J}_\infty\big([0,1]\big)$

Definition 2.1. [14]

Let $\mathcal{J}_n\big([0,1]\big) = \big\{ (y_1,\dots,y_n) \in [0,1]^n \mid y_1 \le y_2 \le \cdots \le y_n \big\}$ where $n \in \mathbb{N}$ and let

$$\mathcal{J}_\infty\big([0,1]\big) = \bigcup_{n=1}^{\infty} \mathcal{J}_n\big([0,1]\big)$$

Then a multidimensional fuzzy set on a set U is defined as a function $\phi : U \to \mathcal{J}_\infty\big([0,1]\big)$

Let $\mathcal{W} \in \mathcal{J}_\infty\big([0,1]\big)$ then $|\mathcal{W}|$ denote the $n \in \mathbb{N}$ such that $\mathcal{W} \in \mathcal{J}_n\big([0,1]\big)$ named as the cardinality of $\mathcal{W}$. From the definition of MDFS, it can be seen that each member of the set U can have a membership value of any cardinality independent of other elements. This a great benefit that MDFS has when we compare it with $m-$polar fuzzy sets or $n-$dimensional fuzzy sets.

Further let $\overline{1} = \{/1/n, n \in \mathbb{N}\}, \overline{0} = \{/0/n, n \in \mathbb{N}\}$,where $/p/ = (p, p..., p) \in \mathcal{J}_n\big([0,1]\big)$, $\dot{1}$ and $\dot{0}$ denote arbitrary element of $\overline{1}$ and $\overline{0}$ respectively.

The extension of partial ordering of $n-$dimensional fuzzy sets to MDFS is defined as follows,

For $\mathcal{V}, \mathcal{W} \in \mathcal{J}_\infty\big([0,1]\big)$, $\mathcal{V} \le_\infty \mathcal{W} \Leftrightarrow |\mathcal{V}| = |\mathcal{W}| = n$ and $\mathcal{V} \le_n^p \mathcal{W}$ where $\le_n^p$ is the partial order on $\mathcal{J}_n([0,1])$, $n \in \mathbb{N}$ given by $\mathcal{V} \le_n^p \mathcal{W} \Leftrightarrow t_1 \le s_1, ...t_n \le s_n$ $(\mathcal{V} = (t_1 \dots t_n)$, $\mathcal{W} = (s_1 \dots s_n))$. Also, if $\mathcal{N}_1$ and $\mathcal{N}_2$ are two MDFS we say that $\mathcal{N}_1 \subseteq \mathcal{N}_2$ if $\mathcal{N}_1(y) \le_\infty \mathcal{N}_2(y)$ for every y.

2.2. Operators on mdfs.

After creating a strong framework for storing and displaying data, the next important step is to incorporate aggregation operators that can link different data structures together for thorough analysis. The influential study, [13], extensively examined t-norms and t-conorms in the context of multidimensional frameworks. This investigation was conducted using two separate functions, referred to as J_1 and J_2, as described below.

$J_1 : \mathcal{J}_\infty([0,1]) \times \mathcal{J}_\infty([0,1]) \to \mathbb{N}$ by

$$J_1(\mathcal{W},\mathcal{V}) = \begin{cases} |\mathcal{W}| \text{ if } \exists p \in \mathbb{N} & : \ t_i = s_i \text{ for all } i = 1,2,\dots,p-1 \text{ and } t_p < s_p \\ |\mathcal{V}| \text{ if } \exists p \in \mathbb{N} & : \ t_i = s_i \text{ for all } i = 1,2,\dots,p-1 \text{ and } t_p > s_p \\ \min\Big\{|\mathcal{W}|, |\mathcal{V}|\Big\} & : \ \text{if there is no such } p \text{ exists.} \end{cases}$$

$J_2 : \mathcal{J}_\infty([0,1]) \times \mathcal{J}_\infty([0,1]) \to \mathbb{N}$ by

$$J_2(\mathcal{W},\mathcal{V}) = \begin{cases} |\mathcal{W}| \text{ if } \exists p \in \mathbb{N} & : \ t_{n+1-i} = s_{m+1-i} \text{ for all } i = 1,2,\dots,p-1 \\ & \quad \text{and } t_{n+1-p} > s_{m+1-p} \\ |\mathcal{V}| \text{ if } \exists p \in \mathbb{N} & : \ t_{n+1-i} = s_{m+1-i} \text{ for all } i = 1,2,\dots,p-1 \\ & \quad \text{and } t_{n+1-p} < s_{m+1-p} \\ \min\Big\{|\mathcal{W}|, |\mathcal{V}|\Big\} & : \ \text{if there is no such } p \text{ exists.} \end{cases}$$

where $\mathcal{W} = (t_1 \dots t_n)$ *and* $\mathcal{V} = (s_1 \dots s_m)$ $n, m \in \mathbb{N}$.

Definition 2.2. [13]
Multidimensional t-norm is a function $\mathbb{R} : \mathcal{J}_\infty\big([0,1]\big) \times \mathcal{J}_\infty\big([0,1]\big) \to \mathcal{J}_\infty\big([0,1]\big)$ satisfying following axioms:

$$
\begin{array}{ll}
(R1) & \big|\mathbb{R}(\mathcal{W},\mathcal{V})\big| = J_1(\mathcal{W},\mathcal{V}) \\
(R2) & \mathbb{R}(\mathcal{W},\mathcal{V}) = \mathbb{R}(\mathcal{V},\mathcal{W}) \\
(R3) & \text{If } \mathcal{V} \leq_\infty \mathcal{Z} \text{ then } \mathbb{R}(\ \mathcal{W},\mathcal{V}) \leq_\infty \mathbb{R}(\mathcal{W},\mathcal{Z}), \text{ given } |\mathcal{W}| = |\mathcal{V}| = |\mathcal{Z}| \\
(R4) & \mathbb{R}(\mathcal{W},\ \dot{1}\ ) = \mathcal{W}, \mathcal{W} \notin \bar{1} \setminus \{\dot{1}\} \\
(R5) & \mathbb{R}\Big(\mathcal{W}, \mathbb{R}(\mathcal{V},\mathcal{Z})\Big) = \mathbb{R}\Big(\mathbb{R}(\mathcal{W},\mathcal{V}),\mathcal{Z}\Big), \text{ whenever } |\mathcal{W}| = |\mathcal{V}| = |\mathcal{Z}|
\end{array}
$$

Let $\mathcal{W} = (t_1, \dots t_n), \mathcal{V} = (s_1, \dots s_m) \in \mathcal{J}_\infty([0,\ 1])$ and let $J_1(\mathcal{W},\mathcal{V}) = l$. then standard multidimensional $t-$norm is given by, $\min(\mathcal{W},\mathcal{V}) = (t_1 \wedge s_1 \dots t_l \wedge s_l)$, where $t_i = s_i = 1 \forall i > l$.

Definition 2.3. [13]
Multidimensional t-conorm is a function $\mathbb{S} : \mathcal{J}_\infty\big([0,1]\big) \times \mathcal{J}_\infty\big([0,1]\big) \to \mathcal{J}_\infty\big([0,1]\big)$ satisfying following axioms:

$$
\begin{array}{ll}
(S1) & |\mathbb{S}(\mathcal{W},\mathcal{V})| = J_2(\mathcal{W},\mathcal{V}) \\
(S2) & \mathbb{S}(\mathcal{W},\mathcal{V}) = \mathbb{S}(\mathcal{V},\mathcal{W}) \\
(S3) & \text{If } \mathcal{V} \leq_\infty \mathcal{Z} \text{ then } \mathbb{S}(\mathcal{W},\mathcal{V}) \leq_\infty \mathbb{S}(\mathcal{W},\mathcal{Z}), \text{ given } |\mathcal{W}| = |\mathcal{V}| = |\mathcal{Z}| \\
(S4) & \mathbb{S}(\mathcal{W},\ \dot{0}) = \mathcal{W}, \mathcal{W} \notin \bar{0} \setminus \{\dot{0}\} \\
(S5) & \mathbb{S}\Big(\mathcal{W}, \mathbb{S}(\mathcal{V},\mathcal{Z})\Big) = \mathbb{S}\Big(\mathbb{S}(\mathcal{W},\mathcal{V}),\mathcal{Z}\Big), \text{ whenever } |\mathcal{W}| = |\mathcal{V}| = |\mathcal{Z}| \\
 & \text{where } \mathcal{W},\mathcal{V},\mathcal{Z} \in \mathcal{J}_\infty\big([0,1]\big)
\end{array}
$$

The standard multidimensional $t-$conorm is given by: $\max(\mathcal{W},\mathcal{V}) = (t_{n-(l-1)} \vee s_{m-(l-1)}, \dots, t_n \vee s_m)$, where $t_{n-i} = s_{m-j} = 0 \forall i \geq n$ and $j \geq m$ where $J_2(\mathcal{W},\mathcal{V}) = l$.

Definition 2.4. A multidimensional complement is a function $O : \mathcal{J}_\infty([0,1]) \to \mathcal{J}_\infty([0,1])$ satisfying the following conditions,

$$
\begin{array}{l}
O1\ :\ |\ O(\mathcal{W})\ | = |\ \mathcal{W}\ | \text{ for all } \mathcal{W} \in \mathcal{J}_\infty([0,1]) \\
O2\ :\ \mathrm{O}(\dot{0}) = \dot{1} \text{ and } \mathrm{O}(\dot{1}) = \dot{0} \text{ for all } \dot{0} \in \bar{0} \ and\ \dot{1} \in \bar{1} \\
O3\ :\ \text{If } \quad \mathcal{W} \leq_\infty \mathcal{V} \quad then \quad O(\mathcal{V})\ \leq_\infty\ C(\mathcal{W}), \text{for all } \mathcal{W},\mathcal{V} \in \mathcal{J}_\infty([0,\ 1])
\end{array}
$$

$O_s(b_1, b_2, \cdots, b_n) = (1 - b_n, 1 - b_{n-1}, \cdots, 1 - b_1)$ is called standard multidimensional complement.

2.3. **Fuzzy Graphs.** Even though we can see different kinds of definitions for fuzzy graphs, we follow the definition given below to extend the work on MDFS.

Definition 2.5. [18] Let X be a non-empty set and Y be a subset of $X \times X$, such that (X, Y) is a simple graph. Then a fuzzy graph on X is a quadruple $G = (X, Y, \psi, \phi)$, where ψ is a fuzzy set on X and ϕ is a fuzzy set on Y satisfying $\phi(x, y) \leq \min(\psi(x), \psi(y))$ for all $x, y \in X$.

There are several types of products and operators, such as union and join, that may be utilized to represent diverse real-life data and their interconnections. The subsequent section provides fundamental definitions of these terms, which may be subsequently referenced in MDFS works.

Definition 2.6. [18] Let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two fuzzy graphs, then their direct product is given by $G = (X, Y, \psi, \phi)$, where $X = Z_1 \times Z_2$, ψ is a fuzzy set defined on X by $\psi(z_1, z_2) = \min(\psi(z_1), \psi(z_2))$ and ϕ is a fuzzy set defined on Y where $Y = \{(v, t_1), (v, t_2), v \in Z_1, t_1t_2 \in W_2\} \bigcup \{(s_1, w), (s_2, w), s_1s_2 \in W_1, w \in Z_2\}$ and $\phi((v, t_1), (v, t_2)) = \min\{\psi_1(v), \phi_2(t_1t_2)\}$ and $\phi((s_1, w), (s_2, w)) = \min\{\psi_2(w), \phi_1(s_1s_2)\}$

Definition 2.7. [18] Let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two fuzzy graphs, then their Composition is given by $G = (X, Y, \psi, \phi)$, where $X = Z_1 \times Z_2$, ψ is a fuzzy set defined on X by $\psi(z_1, z_2) = \min(\psi(z_1), \psi(z_2))$ and ϕ is a fuzzy set defined on Y where

$$\begin{aligned}
&Y = \{(v, t_1)(v, t_2), v \in Z_1, t_1t_2 \in W_2\} \bigcup \{(s_1, w)(s_2, w), s_1s_2 \in W_1, w \in Z_2\} \bigcup \\
&\qquad \bigcup \{(s_1, t_1)(s_2t_2), s_1s_2 \in W_1, W_1 \neq t_2\}, \\
&\phi((v, t_1), (v, t_2)) = \min\{\psi_1(v), \phi_2(t_1t_2)\}, \\
&\phi((s_1, w), (s_2, w)) = \min\{\psi_2(w), \phi_1(s_1s_2)\}, \\
&\phi((s_1, t_1)(s_2, t_2)) = \min\{\psi_2(t_1), \psi_2(t_2), \phi_1(s_1s_2)\}.
\end{aligned}$$

Definition 2.8. [8] Let $G = (X, Y, \psi, \phi)$ be a fuzzy graph. Then the vertex degree of a vertex $v \in X$ is given by

$$\delta(v) = \sum_{vu \in Y} \phi(vu)$$

Definition 2.9. [15] The path strength of a path $P = r_1r_2...r_n$ in $G = (X, Y, \psi, \phi)$ is given by

$$\beta(P) = \min\{\phi(r_ir_{i+1})\}$$

Definition 2.10. [15] Let u and v be two vertices in $G = (X, Y, \psi, \phi)$, then path strength between them is given by

$$\max\{\beta(P_i)\}$$

where P_i is a path joining u and v.

For more such operators, products, and results, refer to [18, 21, 8, 15]

3. A Novel Partial Order for $\mathcal{J}_\infty([0,1])$

One of the main problems that we face when dealing with real-life application problems through MDFS is that, only a subcollection of elements having the same cardinality from $\mathcal{J}_\infty([0,1])$ are connected by the natural partial ordering. In this section, we redefine the natural partial ordering of $\mathcal{J}_\infty([0,1])$ obtained from $n-$ dimensional fuzzy sets in order to relate more elements from the same set. We use the functions J_1 and J_2 defined on [13] for defining the new partial order $\leq_\triangle$ as an extension of $\leq_\infty$.

Let $\mathcal{X}, \mathcal{Y} \in \mathcal{J}_\infty([0,1])$ and $|\mathcal{X}| = |\mathcal{Y}|$ then $\mathcal{X} \leq_\triangle \mathcal{Y}$ if and only if $\mathcal{X} \leq_\infty \mathcal{Y}$.

Now, if $\mathcal{X} = (x_1, x_2, ...x_n)$ and $\mathcal{Y} = (y_1, y_2, ...y_m)$ $(n \neq m)$, with

$$J_1(\mathcal{X}, \mathcal{Y}) \neq J_2(\mathcal{X}, \mathcal{Y}) \tag{3.1}$$

then we say that
$\mathcal{X} \leq_\triangle \mathcal{Y}$ if and only if $x_1 \leq y_1, x_2 \leq y_2 ... x_k \leq y_k$ and $x_n \leq y_m, x_{n-1} \leq y_{n-1}, ... x_{n-k-1} \leq y_{n-k-1}$ where $k = \min\{n, m\}$. It is easy to verify that $\leq_\triangle$ reflexive and (3.1) makes $\leq_\triangle$ asymmetric. Transitivity of $\leq_\triangle$ can also be proved from the definition of $\leq_\triangle$. We denote $\mathcal{X} \geq_\triangle \mathcal{Y}$ if and only if $\mathcal{Y} \leq_\triangle \mathcal{X}$.

3.1. **Redefinition of Multidimensional t-norms and t-conorms.** One of the main drawbacks of multidimensional $t-$norms and $t-$conorms is that, it is not generally true that, for a multidimensional $t-$norm $\mathbb{R}$, $\mathbb{R}(\mathcal{X},\mathcal{Y}) \leq_\infty \mathcal{X}$ or $\mathbb{R}(\mathcal{X},\mathcal{Y}) \leq_\infty \mathcal{Y}$ (true only for the case $|\mathcal{X}| = |\mathcal{Y}|$). This drawback affects MDFS in handling many real-life problems. It can be seen that from the application part of [13] using the min operation. The newly introduced partial order is capable of solving such problems as we have for any $\mathcal{X},\mathcal{Y} \in \mathcal{J}_\infty([0,1])$, $\min(\mathcal{X},\mathcal{Y}) \leq_\triangle \mathcal{X}$ and $\min(\mathcal{X},\mathcal{Y}) \leq_\triangle \mathcal{Y}$. Similarly, for the *max* operator also we have $\mathcal{X} \leq_\triangle \max(\mathcal{X},\mathcal{Y})$ and $\mathcal{Y} \leq_\triangle \max(\mathcal{X},\mathcal{Y})$. Thus, we can redefine the axioms R_3 and S_3 by, for $\mathcal{V},\mathcal{Z},\mathcal{W} \in \mathcal{J}_\infty([0,1])$

R_3 : If $\mathcal{V} \leq_\triangle \mathcal{Z}$ then $\mathbb{R}(\mathcal{W},\mathcal{V}) \leq_\triangle \mathbb{R}(\mathcal{W},\mathcal{Z})$,

S_3 : If $\mathcal{V} \leq_\triangle \mathcal{Z}$ then $\mathbb{S}(\mathcal{W},\mathcal{V}) \leq_\triangle \mathbb{S}(\mathcal{W},\mathcal{Z})$,

Thus, for a $t-norm$ $\mathbb{R}$ and $t-conorm$ $\mathbb{S}$ using the axioms R_3, R_4 and S_3, S_4 respectively, we have

Proposition 3.1. $\mathbb{R}(\mathcal{X},\mathcal{Y}) \leq_\infty \mathcal{X}$ *and* $\mathcal{X} \leq_\infty \mathbb{S}(\mathcal{X},\mathcal{Y})$

Proposition 3.2. *Let R be an idempotent multidimensional $t-$norm and $\mathcal{W} \leq_\triangle \mathcal{X}$ and $\mathcal{W} \leq_\triangle \mathcal{Y}$ then $\mathcal{W} \leq_\triangle R(\mathcal{X},\mathcal{Y})$*

Proof. From the axiom R_3 we have,

$$\mathcal{W} = R(\mathcal{W},\mathcal{W}) \leq_\triangle R(\mathcal{X},\mathcal{W}) \leq_\triangle R(\mathcal{X},\mathcal{Y})$$

□

4. MULTIDIMENSIONAL FUZZY GRAPHS

From the definition of the fuzzy graph $G = (X, Y, \psi, \phi)$, we can see that for any edge xy, $\phi(xy) \leq \min(\psi(x), \psi(y))$. One of the major motivations and applications that fuzzy graphs have is in the field of transportation. Consider a scenario where a number of cities are connected by bridges, and the membership function corresponding to the number of vehicles that each city can accommodate at a time is given by ψ. Similarly, let $\phi(xy)$ denote the membership function corresponding to the number of vehicles that can travel through bridge xy connecting cities x and y, then clearly it is obvious that we should take $\phi(xy) \leq \min(\psi(x), \psi(y))$.

We can find some other occasions in which another form of fuzzy graphs has very practical applications. For example, consider a case where some industries were connected by bridges and $\psi(x)$, $\psi(y)$ denote the membership values corresponding to the weights in tons of the material formed at x and y at a time. Let $\phi(xy)$ denote the membership function corresponding to the weight capacity of bridge xy joining x and y. Then, from the scenario, it is clear that we should have $\phi(xy) \geq \max(\psi(x), \psi(y))$.

Thus, different types of practical problems require different types of fuzzy graph models. Hence, in this paper, we will introduce two types of fuzzy graphs. The first one is called the multidimensional fuzzy graph(MDFG) and the second one is the edge-powered multidimensional fuzzy graph(EPMDFG).

Definition 4.1. Let (X, Y) be a graph and R be a multidimensional $t-$norm, then a multidimensional fuzzy graph is quadruple (X, Y, ψ, ϕ) where ψ and ϕ are MDFS on X and Y respectively satisfying $\phi(xy) \leq_\triangle R(\psi(x), \psi(y))$, where $xy \in Y$ and $x, y \in X$.

Example 1. *In figure (1) and figure (2) we can see illustrations for* MDFG *using standard multidimensional $t-$norm.*

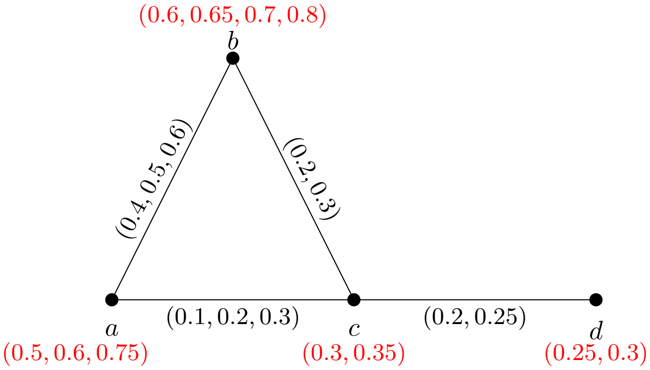

FIGURE 1. MDFG 1

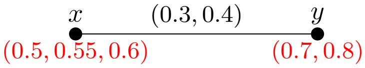

FIGURE 2. MDFG 2

When we compare MDFG with other recent fuzzy graph extensions, such as $m-$polar fuzzy graph we have,

Definition 4.2. [2] Let (X, Y) be a graph, then an $m-$polar fuzzy graph is a pair of m-polar fuzzy sets A, B defined on X and Y respectively, such that for each projection function $P_i, i = 1, 2, ...m$ we have

$$P_i(A(xy)) \leq \min(P_i(B(x)), P_i(B(y)))$$

, $\forall i = 1, 2, ...m$

Thus, if we restrict the dimension of membership values of each element in our MDFG to a fixed number m and if we use the standard multidimensional $t-$norm min operators as R, then the $m-$polar fuzzy graph is nothing but a special case of MDFG.

Next, we define the second kind of multidimensional graph, called edge-powered multidimensional fuzzy graph as follows,

Definition 4.3. Let (X, Y) be a graph and S be a multidimensional $t-$conorm, then an edge-powered multidimensional fuzzy graph is quadruple (X, Y, ψ, ϕ) where ψ and ϕ are MDFS on X and Y respectively satisfying $S(\psi(x), \psi(y)) \leq_{\triangle} \phi(xy)$, where $xy \in Y$ and $x, y \in X$.

Example 2. *In figure (3) and figure (4) we can see illustrations for* EPMDFG *using standard multidimensional $t-$conorm.*

It is very obvious from the definitions of MDFG and EPMDFG that they have a wide scope in many application fields, as we only use aggregation operators and multidimensional fuzzy sets for the definitions of MDFG and EPMDFG.

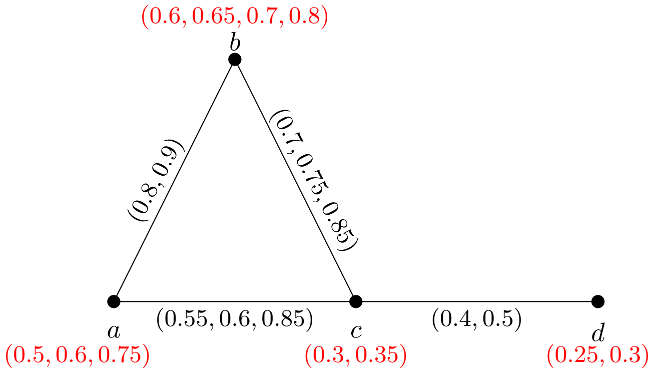

FIGURE 3. EPMDFG 1

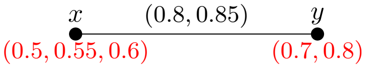

FIGURE 4. EPMDFG 2

4.1. **Operators on Multidimensional Fuzzy Graphs.** In this section, we will discuss and study the properties of various operators on multidimensional fuzzy graphs such as direct product, tensor product, composition, union, etc. These operators have a wide range of applications in areas such as modeling, chemical engineering, etc.

Direct Product A model for concurrency in multiprocessor systems is one of many possible uses for the direct product. Additionally, direct product is utilized for research that combines data from two separate graphs. Here is the formal way to define the direct product of MDFG:

Definition 4.4. Let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two MDFG and R be a multidimensional $t-$norm, then their direct product is given by $G = (X, Y, \psi, \phi)$, where $X = Z_1 \times Z_2$, ψ is a MDFS defined on X by $\psi(z_1, z_2) = R(\psi_1(z_1), \psi_2(z_2))$ and ϕ is a MDFS defined on Y where $Y = \big\{(v, t_1), (v, t_2), v \in Z_1, t_1t_2 \in W_2\big\} \bigcup \big\{(s_1, w), (s_2, w), s_1s_2 \in W_1, w \in Z_2\big\}$ and $\phi((v, t_1), (v, t_2)) = R\{\psi_1(v), \phi_2(t_1t_2)\}$ and $\phi((s_1, w), (s_2, w)) = R\{\psi_2(w), \phi_1(s_1s_2)\}$

Theorem 4.1. *The direct product of two* MDFG *with the same multidimensional $t-$norm R again a* MDFG *with the same multidimensional $t-$norm, provided R is idempotent.*

Proof. Let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two MDFG with respect to a multidimensional $t-$norm R and let $G = (X, Y, \psi, \phi)$ be the direct product of G_1 and G_2.
Now for $(v, t_1), (v, t_2) \in Y$, we have

$$\phi((v, t_1), (v, t_2)) = R\{\psi_1(v), \phi_2(t_1t_2)\} = R\Big\{\psi_1(v), R\Big(\psi_2(t_1), \psi_2(t_2)\Big)\Big\} \leq_\triangle \psi_1(v)$$

by proposition (3.1). By using the same proposition we have,

$$\phi\Big((v, t_1), (v, t_2)\Big) \leq_\triangle \psi_2(t_1), \psi_2(t_2)$$

Thus we have, $\phi\Big((v, t_1), (v, t_2)\Big) \leq_\triangle R\Big(\psi_1(v), \psi_2(t_1)\Big)$ and $\phi\Big((v, t_1), (v, t_2)\Big) \leq_\triangle R\Big(\psi_1(v), \psi_2(t_2)\Big)$

$$\Rightarrow \phi\Big((v, t_1), (v, t_2)\Big) \leq_\triangle R\Big(R(\psi_1(v), \psi_2(t_1)), R\Big(\psi_1(v), \psi_2(t_2)\Big)\Big)$$

$$= R(\psi(v,t_1),\psi(v,t_2))$$

Similarly, we can show that,

$$\phi\Big((s_1,w),(s_2,w)\Big) \leq_{\triangle} R\Big(\psi(s_1,w),\psi(s_2,w)\Big)$$

Thus $G=(X,Y,\psi,\phi)$ is again a MDFG. □

Example 3. *Figure 5 gives the illustration of the direct product of* MDFG *in figure 1 and figure 2.*

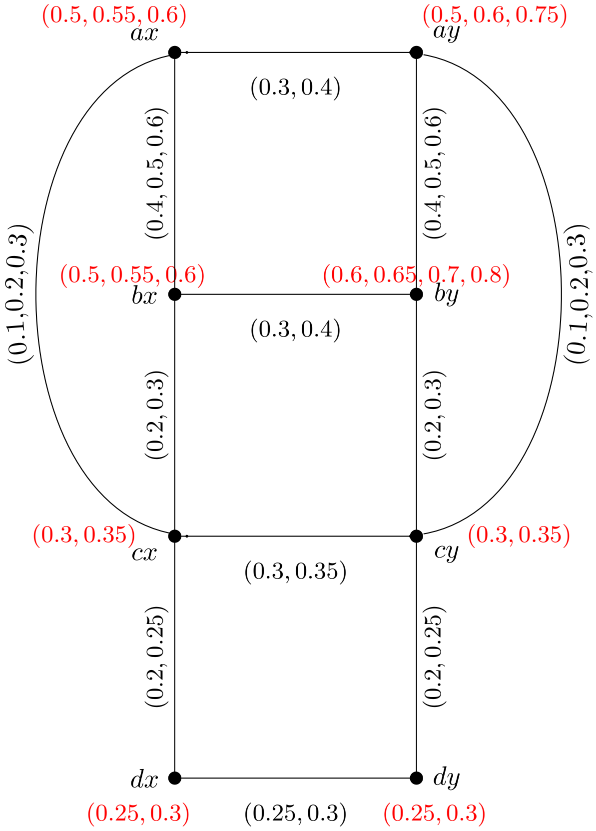

FIGURE 5. Direcrt product

Edge-Powered Direct Product

Next, we define the direct product for the EPMDFG in such a way that the resulting one is also a EPMDFG.

Definition 4.5. Let $G_1=(Z_1,W_1,\psi_1,\phi_1)$ and $G_2=(Z_2,W_2,\psi_2,\phi_2)$ be two EPMDFG and S be a multidimensional $t-$conorm, then their edge-powered direct product is given by $G=(X,Y,\psi,\phi)$, where $X=Z_1\times Z_2$, ψ is a MDFS defined on X by $\psi(z_1,z_2)=S\Big(\psi(z_1),\psi(z_2)\Big)$ and ϕ is a MDFS defined on Y where $Y=\Big\{(v,t_1),(v,t_2), v\in Z_1, t_1t_2\in W_2\Big\}\bigcup\Big\{(s_1,w),(s_2,w), s_1s_2\in W_1, w\in Z_2\Big\}$ and $\phi\Big((v,t_1),(v,t_2)\Big)=S\Big\{\psi_1(v),\phi_2(t_1t_2)\Big\}$ and $\phi\Big((s_1,w),(s_2,w)\Big)=S\Big\{\psi_2(w),\phi_1(s_1s_2)\Big\}$

Theorem 4.2. *The edge-powered direct product of two* EPMDFG *with the same multidimensional $t-$conorm is again a* EPMDFG *with the same multidimensional $t-$conorm(idempotent).*

Proof. Let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two MDFG with respect to a multidimensional $t-$conorm S and let $G = (X, Y, \psi, \phi)$ be the direct product of G_1 and G_2. Now for $(v, t_1), (v, t_2) \in Y$ we have, $\phi((v, t_1), (v, t_2)) = S\{\psi_1(v), \phi_2(t_1 t_2)\} = S\{\psi_1(v), S\Big(\psi_2(t_1), \psi_2(t_2)\Big)\} \geq_\triangle \psi_1(v), \psi_2(t_1), \psi_2(t_2)$ by proposition (3.1). Now by using proposition (3.2), in a similar way that of theorem (4.1) we have,

$$\phi\Big((v, t_1), (v, t_2)\Big) \geq_\triangle S\Big(S(\psi_1(v), \psi_2(t_1)), S(\psi_1(v), \psi_2(t_2))\Big) = S\Big(\psi(v, t_1), \psi(v, t_2)\Big)$$

Similarly, we have,

$$\phi\Big((s_1, w), (s_2, w)\Big) \geq_\triangle S\Big(\psi(s_1, w), \psi(s_2, w)\Big)$$

Hence, the edge$-$powered direct product of two EPMDFG with the same multidimensional $t-$conorm is again a EPMDFG. □

Tensor Product

Tensor products are important tools in classical mechanics, quantum mechanics, graph theory, and image processing[7, 28, 23]. Now we define the tensor product of MDFG and EPMDFG.

Definition 4.6. Let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two MDFG and R be a multidimensional $t-$norm, then their tensor product is given by $G = (X, Y, \psi, \phi)$, where $X = Z_1 \times Z_2$, ψ is a MDFS defined on X by $\psi(z_1, z_2) = R\{\big(\psi(z_1), \psi(z_2)\big)\}$ and ϕ is a MDFS on Y, where $Y = \big\{(s_1, t_1)(s_2, t_2), s_1 s_2 \in W_1, t_1 t_2 \in W_2\big\}$ and $\phi\big((s_1, t_1)(s_2, t_2)\big) = R\big(\phi_1(s_1 s_2), \phi_2(t_1 t_2)\big)$

Theorem 4.3. *The tensor product of two* MDFG *with the same multidimensional $t-$norm R again a* MDFG *with the same multidimensional $t-$norm, provided R is idempotent.*

Proof. Let $G = (X, Y, \psi, \phi)$ be the tensor product of two MDFG $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$. Let $(s_1, t_1)(s_2, t_2) \in Y$, then

$$\begin{aligned} \phi\big((s_1, t_1)(s_2, t_2)\big) &= R\big(\phi_1(s_1 s_2), \phi_2(t_1 t_2)\big) \\ \leq_\triangle \phi_1(s_1 s_2) \leq_\triangle R\big(\psi_1(s_1)\big), \psi_1(s_2)\big)\big) &\leq_\triangle \psi_1(s_1), \psi_1(s_2) \end{aligned} \tag{4.1}$$

Similarly, we have

$$\phi\big((s_1, t_1)(s_2, t_2)\big) \leq_\triangle \psi_2(t_1), \psi_2(t_2) \tag{4.2}$$

From (4.1) and (4.2), we have

$$\begin{aligned} \phi\big((s_1, t_1)(s_2, t_2)\big) &\leq_\triangle R\big(\psi_1(s_1), \psi_2(t_1)\big), R\big(\psi_1(s_2), \psi_2(t_2)\big) \\ \Rightarrow \phi\big((s_1, t_1)(s_2, t_2)\big) &\leq_\triangle R\big(R(\psi_1(s_1), \psi_2(t_1)\big), R\big(\psi_1(s_2), \psi_2(t_2)\big)\big) \\ &= R\big(\psi(s_1, t_1), \phi(s_2, t_2)\big) \end{aligned}$$

Thus, the tensor product of two MDFG with the same multidimensional $t-$norm R again a MDFG. □

Example 4. *Figure 6 is the tensor product of* MDFG *figure 1 and figure 1.*

Edge Powered Tensor Product

Definition 4.7. Let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two EPMDFG and S be a multidimensional $t-$conorm, then their edge powered tensor product is given by $G = (X, Y, \psi, \phi)$, where $X = Z_1 \times Z_2$, ψ is a MDFS defined on X by $\psi(z_1, z_2) = S(\psi(z_1), \psi(z_2))$ and ϕ is a MDFS on Y, where $Y = \big\{(s_1, t_1)(s_2, t_2), s_1 s_2 \in W_1, t_1 t_2 \in W_2\big\}$ and $\phi((s_1, t_1)(s_2, t_2)) = S(\phi_1(s_1 s_2), \phi_2(t_1 t_2))$

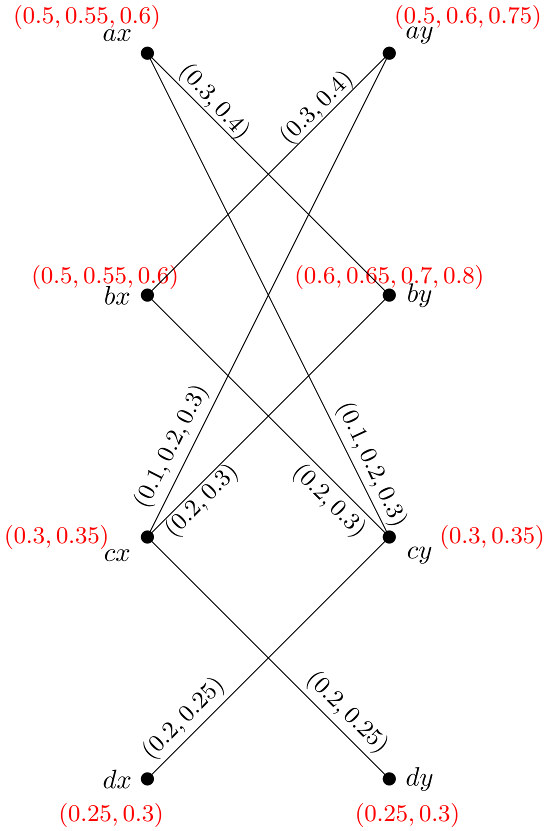

FIGURE 6. Tensor product

Theorem 4.4. *The edge-powered tensor product of two* MDFG *with the same multidimensional $t-$conorm S again a* MDFG *with the same multidimensional $t-$conorm, provided S is idempotent.*

Proof. The proof is similar to that of Theorem (4.3). □

Composition

The composition of graphs is very helpful in many number theoretical aspects, such as the partition of numbers. Next, we define the composition of MDFG and EPMDFG as follows.

Definition 4.8. Let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two MDFG and R be a multidimensional $t-$norm, then their composition $G = G_1 o G_2$ is given by $G = (X, Y, \psi, \phi)$, where $X = Z_1 \times Z_2$, ψ is a MDFS defined on X by $\psi(z_1, z_2) = R\Big(\psi(z_1), \psi(z_2)\Big)$ and ϕ is a MDFS defined on Y where

$$Y = \Big\{(v, t_1), (v, t_2), v \in Z_1, t_1 t_2 \in W_2\Big\} \bigcup \{(s_1, w), (s_2, w), s_1 s_2 \in W_1, w \in Z_2\} \bigcup$$
$$\bigcup \{(s_1, t_1), (s_2, t_2), s_1 s_2 \in W_1, t_1 \neq t_2 \in W_2\},$$

and

$$\phi\Big((v,t_1),(v,t_2)\Big) = R\Big\{\psi_1(v),\phi_2(t_1t_2)\Big\},$$
$$\phi\Big((s_1,w),(s_2,w)\Big) = R\Big\{\psi_2(w),\phi_1(s_1s_2)\Big\},$$
$$\phi\Big((s_1,t_1),(s_2,t_2)\Big) = R\Big(\phi_1(s_1s_2),R\Big(\psi_2(t_1),\psi_2(t_2)\Big)\Big).$$

Theorem 4.5. *The composition of two* MDFG *with the same multidimensional $t-$norm R again a* MDFG *with the same multidimensional $t-$norm, provided R is idempotent.*

Proof. From the proof theorem (4.1), it remains only to prove that for $(s_1,t_1),(s_2,t_2)\in Y$ with $s_1s_2\in W_1, t_1\neq t_2\in W_2$, $\phi((s_1,t_1),(s_2,t_2))\leq_\triangle R(\psi(s_1,t_1),\psi(s_2,t_2))$. Now

$$\phi((s_1,t_1),(s_2,t_2)) = R\Big(\phi_1(s_1s_2),R\Big(\psi_2(t_1),\psi_2(t_2)\Big)\Big)$$

$$\leq_\triangle R\Big(R\Big(\psi_1(s_1),\psi_2(s_2)\Big),R\Big(\psi_2(t_1),\psi_2(t_2)\Big)\Big)$$

Now by applying the proposition (3.1) repeatedly, we have,

$$\phi\Big((s_1,t_1),(s_2,t_2)\Big) \leq_\triangle \psi_1(s_1),\psi_1(s_2),\psi_2(t_1),\psi_2(t_2))$$

Thus, by proposition (3.2)

$$\phi\Big((s_1,t_1),(s_2,t_2)\Big) \leq_\triangle R\Big(\psi_1(s_1),\psi_2(t_1)\Big),R\Big(\psi_1(s_2),\psi_2(t_2)\Big)$$

$$\Rightarrow \phi\Big((s_1,t_1),(s_2,t_2)\Big) \leq_\triangle R\Big(R\Big(\psi_1(s_1),\psi_2(t_1)\Big),R\Big(\psi_1(s_2),\psi_2(t_2))\Big)$$

$$= R\Big(\psi(s_1,t_1),\psi(s_2,t_2)\Big)$$

Hence, the composition of two MDFG with the same multidimensional $t-$norm R again a MDFG. □

Example 5. *Figure 7, gives the composition of* EPMDFG *figure 1 and figure 2.*

Edge-Powered Composition

Definition 4.9. Let $G_1=(Z_1,W_1,\psi_1,\phi_1)$ and $G_2=(Z_2,W_2,\psi_2,\phi_2)$ be two EPMDFG and S be a multidimensional $t-$conorm, then their edge-powered direct product is given by $G=(X,Y,\psi,\phi)$, where $X=Z_1\times Z_2$, ψ is a MDFS defined on X by $\psi(z_1,z_2)=S(\psi(z_1),\psi(z_2))$ and ϕ is a MDFS defined on Y where $Y=\big\{(v,t_1),(v,t_2),v\in Z_1,t_1t_2\in W_2\big\}\bigcup\big\{(s_1,w),(s_2,w),s_1s_2\in W_1,w\in Z_2\big\}\bigcup\big\{(s_1,t_1),(s_2,t_2),s_1s_2\in W_1,t_1\neq t_2\in W_2\big\}$
and $\phi((v,t_1),(v,t_2))=S\big\{\psi_1(v),\phi_2(t_1t_2)\big\}$, $\phi\big((s_1,w),(s_2,w)\big)=S\big\{\psi_2(w),\phi_1(s_1s_2)\big\}$
and $\phi\big((s_1,t_1),(s_2,t_2)\big)=S\big(\phi_1(s_1s_2),S\big(\psi_2(t_1),\psi_2(t_2)\big)\big)$

Theorem 4.6. *The composition of two* EPMDFG *with the same multidimensional $t-$conorm S again a* EPMDFG *with the same multidimensional $t-$conorm, provided S is idempotent.*

Proof. The proof is similar to the proof of the Theorem (4.5). □

Normal Product

The normal product of MDFG is a combined operation of directproduct and tensor product and they are usually used in networking and image processing [22].

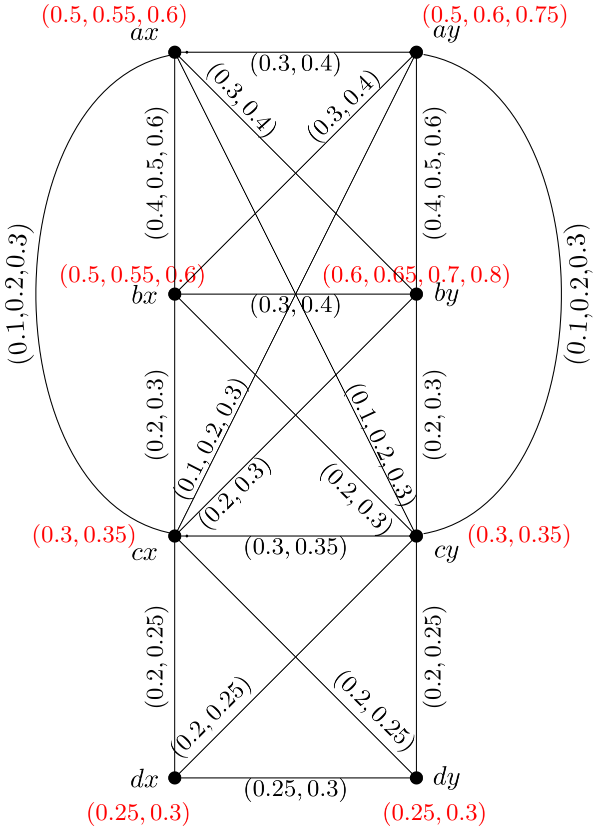

Figure 7. Composition

Definition 4.10. Let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two MDFG and R be a multidimensional $t-$norm, then their normal product is given by $G = (X, Y, \psi, \phi)$, where $X = Z_1 \times Z_2$, ψ is a MDFS defined on X by $\psi(z_1, z_2) = R\big(\psi(z_1), \psi(z_2)\big)$ and ϕ is a MDFS on Y, where $Y = \big\{(v, t_1), (v, t_2), v \in Z_1, t_1t_2 \in W_2\big\} \bigcup \big\{(s_1, w), (s_2, w), s_1s_2 \in W_1, w \in Z_2\big\} \bigcup \big\{(s_1, t_1)(s_2, t_2), s_1s_2 \in W_1, t_1t_2 \in W_2\big\}$ and $\phi\big((v, t_1), (v, t_2)\big) = R\big\{\psi_1(v), \phi_2(t_1t_2)\big\}$ and $\phi\big((s_1, w), (s_2, w)\big)$ $= R\big\{\psi_2(w), \phi_1(s_1s_2)\big\}$, $\phi\big((s_1, t_1)(s_2, t_2)\big) = R\big(\phi_1(s_1s_2), \phi_2(t_1t_2)\big)$

Theorem 4.7. *The normal product of two* MDFG *with the same multidimensional $t-$norm R again a* MDFG *with the same multidimensional $t-$norm, provided R is idempotent.*

Proof. The proof follows from the Theorem 4.1 and Theorem 4.3. □

Edge-powered Normal Product

Definition 4.11. Let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two EPMDFG and s be a multidimensional $t-$conorm, then their edge-powerd normal product is given by $G = (X, Y, \psi, \phi)$, where $X = Z_1 \times Z_2$, ψ is a MDFS defined on X by $\psi(z_1, z_2) = S(\psi(z_1), \psi(z_2))$ and ϕ is a MDFS on Y, where $Y = \big\{(v, t_1), (v, t_2), v \in Z_1, t_1t_2 \in W_2\big\} \bigcup \big\{(s_1, w), (s_2, w), s_1s_2 \in W_1, w \in Z_2\big\} \bigcup \big\{(s_1, t_1)(s_2, t_2), s_1s_2 \in W_1, t_1t_2 \in W_2\big\}$ and $\phi((v, t_1), (v, t_2)) = S\{\psi_1(v), \phi_2(t_1t_2)\}$ and $\phi\big((s_1, w), (s_2, w)\big) = S\big\{\psi_2(w), \phi_1(s_1s_2)\big\}$, $\phi\big((s_1, t_1)(s_2, t_2)\big) = S\big(\phi_1(s_1s_2), \phi_2(t_1t_2)\big)$

Theorem 4.8. *The edge-powered normal product of two* EPMDFG *with the same multidimensional $t-$conorm S again a* EPMDFG *with the same multidimensional $t-$conorm, provided S is idempotent.*

Proof. The proof follows from the Theorem 4.2 and Theorem 4.4. □

Union

A fuzzy graph union is an operation that is used to combine the information from two different fuzzy graphs.

Definition 4.12. Let S be a multidimensional $t-$conorm and let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two MDFG with respect to the multidimensional $t-$norm R, then their union with respect S is given by $G = (X, Y, \psi, \phi)$, where $X = Z_1 \cup Z_2$, $Y = W_1 \cup W_2$ and ψ, ϕ are defined as follows,

$$\psi(z) = \begin{cases} \psi_1(z) \; if \; z \in Z_1 \setminus Z_2 \\ \psi_2(z) \; if \; z \in Z_2 \setminus Z_1 \\ S(\psi_1(z), \psi_2(z)) \; if \; z \in Z_1 \cap Z_2 \end{cases}$$

$$\phi(zw) = \begin{cases} \phi_1(zw) \; if \; zw \in W_1 \setminus W_2 \\ \phi_2(zw) \; if \; zw \in W_2 \setminus W_1 \\ S\big(\phi_1(zw), \phi_2(zw)\big) \; if \; zw \in W_1 \cap W_2 \end{cases}$$

Theorem 4.9. *Union of two* MDFG *is again a* MDFG.

Proof. First, assume that $zw \in W_1 \setminus W_2$,if $z, w \in Z_1$ then nothing to prove. Hence without loss of generality assume that $z \in Z_1$ and $w \in Z_1 \cap Z_2$. Then we have

$$\begin{aligned} \phi(zw) = \phi_1(zw) &\leq_\triangle R(\psi_1(z), \psi_1(w)) = R(\psi(z), \psi_1(w)) \\ &\leq_\triangle R(\psi(z), S(\psi_1(w), \psi_2(w))) = R(\psi(z), \psi(w)) \end{aligned}$$

Now it is enough to prove the case where $zw \in W_1 \cap W_2$. In that case, we have,

$$\begin{aligned} \phi(zw) = S\big(\phi_1(zw), \phi_2(zw)\big) &\leq_\triangle S\big(R(\psi_1(z), \psi_1(w)), R\big(\psi_2(z), \psi_2(w)\big)\big) \\ &\leq_\triangle S\big(\psi_1(z), \psi_2(z)\big) = \psi(z) \end{aligned} \tag{4.3}$$

Similarly, we have

$$\phi(zw) \leq_\triangle S\big(\psi_1(w), \psi_2(w)\big) = \psi(w) \tag{4.4}$$

Thus from (4.3), (4.4) and Proposition 3.2, we have

$$\phi(zw) \leq_\triangle R\big(\psi(z), \psi(w)\big)$$

□

Edge-powered Union

Definition 4.13. Let S_1 be a multidimensional $t-$conorm and let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two EPMDFG with respect to the multidimensional $t-$conorm S_2, then their union with respect S_1 is given by $G = (X, Y, \psi, \phi)$, where $X = Z_1 \cup Z_2$, $Y = W_1 \cup W_2$ and ψ, ϕ are defined as follows,

$$\psi(z) = \begin{cases} \psi_1(z) \; if \; z \in Z_1 \setminus Z_2 \\ \psi_2(z) \; if \; z \in Z_2 \setminus Z_1 \\ S_1\big(\psi_1(z), \psi_2(z)\big) \; if \; z \in Z_1 \cap Z_2 \end{cases}$$

$$\phi_(zw)=\begin{cases}\phi_1(zw)\ if\ zw\in W_1\setminus W_2\\ \phi_2(zw)\ if\ zw\in W_2\setminus W_1\\ S_1\big(\phi_1(zw),\phi_2(zw)\big)\ if\ zw\in W_1\cap W_2\end{cases}$$

Theorem 4.10. *Union of two* EPMDFG *is again a* EPMDFG.

Proof. The proof is similar to the proof of Theorem (4.9). ☐

Join

Join is an operator that is an extension of the union and it can connect vertices that are not joined in both of the graphs. All the information that is in two graphs G_1 and G_2 are also available in their join and some extra information is also available.

Definition 4.14. Let S be a multidimensional $t-$conorm, R_1 be a multidimensional $t-$norm and let $G_1=(Z_1,W_1,\psi_1,\phi_1)$ and $G_2=(Z_2,W_2,\psi_2,\phi_2)$ be two MDFG with respect to the multidimensional $t-$norm R_2, then their join with respect S,R_1 is given by $G=(X,Y,\psi,\phi)$, where $X=Z_1\cup Z_2$, $Y=W_1\cup W_2\cup\big\{(zw),z\in Z_1,w\in Z_2,zw\notin W_1\cup W_2\big\}$ and ψ,ϕ are defined as follows:

$$\psi(z)=\begin{cases}\psi_1(z)\ if\ z\in Z_1\setminus Z_2\\ \psi_2(z)\ if\ z\in Z_2\setminus Z_1\\ S_1\big(\psi_1(z),\psi_2(z)\big)\ if\ z\in Z_1\cap Z_2,\end{cases}$$

$$\phi(zw)=\begin{cases}\phi_1(zw)\ if\ zw\in W_1\setminus W_2\\ \phi_2(zw)\ if\ zw\in W_2\setminus W_1\\ S(\phi_1(zw),\phi_2(zw))\ if\ zw\in W_1\cap W_2\\ R_1\big(\psi_1(z),\psi_2(w)\big)\ if\ zw\notin W_1\cup W_2.\end{cases}$$

Theorem 4.11. *Join of two* MDFG *is again a* MDFG.

Proof. From Theorem (4.9), it is enough to prove the case where $zw\notin W_1\cup W_2(z\in Z_1,w\in Z_2)$. In that case, we have,

$$\phi(zw)=R_1\big(\psi_1(z),\psi_2(w)\big)\leq_\triangle R_1\big(\psi(z),\psi(w)\big)$$

Hence, the join of two MDFG is again a MDFG. ☐

Edge-powered Join

Definition 4.15. Let S_1 be a multidimensional $t-$conorm, R be a multidimensional $t-$norm and let $G_1=(Z_1,W_1,\psi_1,\phi_1)$ and $G_2=(Z_2,W_2,\psi_2,\phi_2)$ be two EPMDFG with respect to the multidimensional $t-$conorm S_2, then their join with respect S_1,R is given by $G=(X,Y,\psi,\phi)$, where $X=Z_1\cup Z_2$, $Y=W_1\cup W_2\cup\big\{(zw),z\in Z_1,w\in Y,zw\notin W_1\cup W_2\big\}$ and ψ,ϕ are defined as follows

$$\psi(z)=\begin{cases}\psi_1(z)\ if\ z\in Z_1\setminus Z_2\\ \psi_2(z)\ if\ z\in Z_2\setminus Z_1\\ S_1\big(\psi_1(z),\psi_2(z)\big)\ if\ z\in Z_1\cap Z_2,\end{cases}$$

$$\phi(zw)=\begin{cases}\phi_1(zw)\ if\ zw\in W_1\setminus W_2\\ \phi_2(zw)\ if\ zw\in W_2\setminus W_1\\ S_1\big(\phi_1(zw),\phi_2(zw)\big)\ if\ zw\in W_1\cap W_2\\ R\big(\psi_1(z),\psi_2(w)\big)\ if\ zw\notin W_1\cup W_2.\end{cases}$$

Theorem 4.12. *Join of two* EPMDFG *is again a* EPMDFG.

Intersection

The intersection of two fuzzy graphs was used to get combined information in which both graphs agree with each other in membership values.

Definition 4.16. Let R_1 be a multidimensional $t-$norm and let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two MDFG with respect to the multidimensional $t-$norm R_2, then their intersection with respect R_1, R_2 is given by $G = (X, Y, \psi, \phi)$, where $X = Z_1 \cap Z_2$, $Y = W_1 \cap W_2$ and ψ, ϕ are defined as $\psi(z) = R_1\big(\psi_1(z), \psi_2(z)\big)$ and $\phi(zw) = R_1\big(\phi_1(zw), \phi_2(zw)\big)$.

Theorem 4.13. *Intersection of two* MDFG *is again a* MDFG.

Proof. Let $G = (X, Y, \psi, \phi)$ be the intersection of two MDFG $G_1 = (Z_1, W_1, \psi_1, \phi_1), G_2 = (Z_2, W_2, \psi_2, \phi_2)$ with respect to the multidimensional $t-$norm R_2. Then

$$\begin{aligned}\phi(zw) = R_1\big(\phi_1(zw), \phi_2(zw)\big) &\leq_\triangle R_1\big(R_2\big(\psi_1(z), \psi_1(w)\big), R_2\big(\psi_2(z), \psi_2(w)\big)\big)\\ &\leq_\triangle \psi_1(z), \psi_1(w), \psi_2(z), \psi_2(w)\big)\\ \implies \phi(zw) &\leq_\triangle R_2\big(\psi_1(z), \psi_2(z)\big), R_2\big(\psi_1(w), \psi_2(w)\big)\\ \implies \phi(zw) \leq_\triangle R_1\big(R_2\big(\psi_1(z), \psi_2(z)), R_2(\psi_1(w), \psi_2(w))\big) &= R_1\big(\psi(z), \psi(w)\big).\end{aligned}$$

Hence the intersection of two MDFG is again a MDFG. □

Edge-powered Intersection

Definition 4.17. Let R_1 be a multidimensional $t-$norm and let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be two EPMDFG with respect to the multidimensional $t-$conorm S, then their intersection with respect R, S is given by $G = (X, Y, \psi, \phi)$, where $X = Z_1 \cap Z_2$, $Y = W_1 \cap W_2$ and ψ, ϕ are defined as $\psi(z) = R\big(\psi_1(z), \psi_2(z)\big)$ and $\phi(zw) = R\big(\phi_1(zw), \phi_2(zw)\big)$

Theorem 4.14. *Intersection of two* EPMDFG *is again a* EPMDFG.

4.2. **Complement of multidimensional fuzzy graphs.** In this section, we define and study a novel complement operator for multidimensional fuzzy graphs that can connect several properties of MDFG and EPMDFG. In the following studies, we can see that most of the operators are related by De Morgan's triple [13]. (That is a triplet (R, S, C) of multidimensional $t-$norm, multidimensional $t-$conorm, and multidimensional complement, respectively, that satisfies De Morgan's law.)

Definition 4.18. Let $G = (X, Y, \psi, \phi)$ be a MDFG, then multidimensional fuzzy complement (MDFC) of G is defined as $G^c = (X^*, Y^*, \psi^*, \phi^*)$ where $X^* = X, Y^* = Y, \psi^* = \psi^C, \phi^* = \phi^C$ where ψ^C and ϕ^C are complements of ψ and ϕ respectively.

Theorem 4.15. *multidimensional fuzzy complement of a* MDFG *is a* EPMDFG *provided the corresponding multidimensional $t-$norm (R), multidimensional $t-$conorm (S) and multidimensional complement (C) form a De Morgan's triple. The converse is also true.*

Proof. Let $G = (X, Y, \psi, \phi)$ be a MDFG then we have $G^c = (X, Y, \psi^C, \phi^C)$. Now, For $xy \in Y$ we have,

$$\begin{aligned}\phi(xy) \leq_\triangle R\big(\psi(x), \psi(y)\big) &\Rightarrow \phi^C(xy) \geq_\triangle R^C\big(\psi(x), \psi(y)\big)\\ &\implies \phi^C(xy) \geq_\triangle S\big(\psi^C(x), \psi^C(y)\big).\end{aligned}$$

Thus, $G^c = (X, Y, \psi^C, \phi^C)$ is a EPMDFG. Similarly, we can prove the converse as well. □

Example 6. *Complement of the* MDFG *figure 1 is given in 8 using the standard multidimensional complement.*

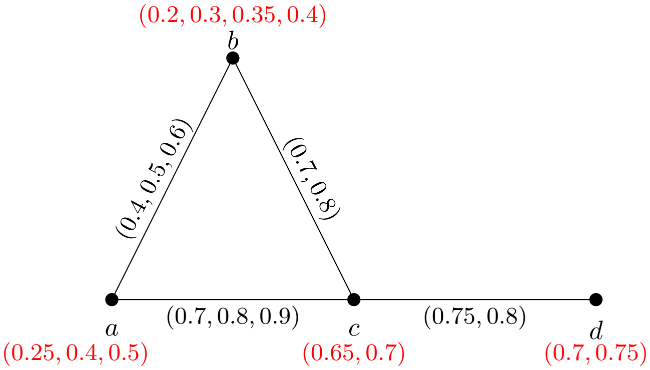

FIGURE 8. Complement of MDFG 1

Next, we will study the relationship between several multidimensional fuzzy operators and edge-powered multidimensional fuzzy operators with respect to their complements.

Theorem 4.16. *Let G be the direct product of two* MDFG *G_1 and G_2, then the* EPMDFG *G^C is the edge-powered direct product of* EPMDFG, *G_1^C and G_2^C.*

Proof. Let $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ and $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ be the two EPMDFG and let $G = (X, Y, \psi, \phi)$, $G^c = (X, Y, \psi^C, \phi^C)$. Then

$$\psi(z_1z_2) = R\big(\psi_1(z_1), \psi_2(z_2)\big) \Rightarrow \psi^C(z_1z_2) = R^C\big(\psi_1(z_1), \psi_2(z_2)\big) = S\big(\psi_1^C(z_1), \psi_2^C(z_2)\big) \tag{4.5}$$

Now, $\phi\big((v, t_1), (v, t_2)\big) = R\big(\psi_1(v), \phi_2(t_1t_2)\big)$

$$\Longrightarrow \phi^C\big((v, t_1), (v, t_2)\big) = R^C\big(\psi_1(v), \phi_2(t_1t_2)\big) = S\big(\psi_1^C(v), \phi_2^C(t_1t_2)\big). \tag{4.6}$$

Similarly, we can show that,

$$\phi^C\big((s_1, w), (s_2, w)\big) = S\big(\psi_2^C(w), \phi_1^C(s_1s_2)\big). \tag{4.7}$$

Thus, from (4.5), (4.6), (4.7) it is clear that G^C is the edge-powered direct product of G_1^C and G_2^C. □

Theorem 4.17. *Let G be the composition of two* MDFG *G_1 and G_2, then the* EPMDFG *G^C is the edge-powered composition of* EPMDFG, *G_1^C and G_2^C.*

Proof. From the Theorem (4.16), it is enough to prove the edges of the form $(s_1, t_1)(s_2, t_2), s_1s_2 \in W_1, t_1 \neq t_2$. We have

$$\phi\Big((s_1, t_1), (s_2, t_2)\Big) = R\Big(\phi_1(s_1s_2), R\Big(\psi_2(t_1), \psi_2(t_2)\Big)\Big)$$

$$\Longrightarrow \phi^C\Big((s_1, t_1), (s_2, t_2)\Big) = S\Big(\phi_1^C(s_1s_2), R^C\Big(\psi_2(t_1), \psi_2(t_2)\Big)\Big) = S\Big(\phi_1^C(s_1s_2), S\Big(\psi_2^C(t_1), \psi_2^C(t_2)\Big)\Big).$$

Thus, G^C is the edge-powered composition of G_1^C and G_2^C. □

Theorem 4.18. *Let G be the tensor-product of two* MDFG *G_1 and G_2, then the* EPMDFG *G^C is the edge-powered tensor-product of* EPMDFG, *G_1^C and G_2^C.*

Proof. We have, $\phi\Big((s_1,t_1)(s_2,t_2)\Big) = R\Big(\phi_1(s_1s_2), \phi_2(t_1t_2)\Big)$

$$\Longrightarrow \phi^C\Big((s_1,t_1)(s_2,t_2)\Big) = R^C\Big(\phi_1(s_1s_2), \phi_2(t_1t_2)\Big) = S\Big(\phi_1^C(s_1s_2), \phi_2^C(t_1t_2)\Big).$$

Thus from the definition of edge-powered tensor-product we have, G^C as the edge-powered tensor-product of EPMDFG, G_1^C, and G_2^C. ☐

Theorem 4.19. *Let G be the normal-product of two* MDFG *G_1 and G_2, then the* EPMDFG *G^C is the edge-powered normal-product of* EPMDFG*, G_1^C and G_2^C.*

Proof. The proof follows from the theorem (4.16) and the theorem (4.18). ☐

Note: In a similar way, we can prove that the union, intersection, and join also follow the same properties with the multidimensional fuzzy complement.

5. INFIMUM AND SUPREMUM ON $\mathcal{J}_\infty([0,1])$

This section will define the infimum and supremum for a finite family of elements from $J_\infty[0,1]$ so that these definitions can be used for studying further properties of multidimensional fuzzy sets. We define the supremum and infimum as a generalization of the standard multidimensional $t-$norm and multidimensional $t-$conorm as follows,

Definition 5.1. Let $\{s_i\}_{i=1}^n \subset J_\infty[0,1]$ where $s_i = (s_i^1, s_i^2, ...s_i^{k(s_i)})$ and $k : \{s_i\} \longrightarrow \mathbb{N}$ be the function that gives the cardinality of s_i as the output, then we define two functions K_1 and K_2 as follows.

- $K_1(\{s_i\}) = |s_j|$, if either of the following is satisfied:
  (1-1): there exist $l \in \mathbb{N}$ such that $s_j^1 = s_i^1, s_j^2 = s_i^2, ...s_j^l < s_i^l$ for every $i \neq j$;
  (1-2): $k(s_j) \leq k(s_i)$ for every $i = 1, 2, ...n$ and $s_j^1 = s_i^1, s_j^2 = s_i^2, ...s_j^{k(s_j)} = s_i^{k(s_i)}$.
- $K_2(\{s_i\}) = |s_j|$, if either of the following is satisfied:
  (2-1): there exist $q \in \mathbb{N}$ such that $s_i^{k(s_i)} = s_j^{k(s_j)}, s_i^{k(s_i)-1} = s_j^{k(s_j)-1}, ...s_i^{k(s_i)-q} < s_j^{k(s_j)-1}$ for every $i \neq j$;
  (2-2): $k(s_j) \leq k(s_i)$ for every $i = 1, 2...n$ and $s_j^1 = s_i^1, s_j^2 = s_i^2, ...s_j^{k(s_j)} = s_i^{k(s_i)}$.

Now using K_1 and K_2, we define the infimum and supremum of the family $\{s_i\}$ respectively as follows,

Definition 5.2. For the family $\{s_i\}$ in Definition (5.1), their infimum is given by,

$$\inf\{s_i\} = \big(\min\{s_i^1\}_{i=1}^n, \min\{s_i^2\}_{i=1}^n, \dots \min\{s_i^{k(s_j)}\}_{i=1}^n\big)$$

where we take $s_i^m = 1$, for all $m < k(s_j)$ for all s_i with $k(s_i) > k(s_j)$.

Definition 5.3. For the family $\{s_i\}$ the supremum is given by,

$$\sup\{s_i\} = \big(\{\max\{s_i^{k(s_i)-k(s_j)+1}\}_{i=1}^n)\}_{i=1}^n, \dots \max\{s_i^{k(s_i)-1}\}_{i=1}^n), \max\{s_i^{k(s_i)}\}_{i=1}^n)$$

where we take $s_i^m = 0$, for all $m > k(s_j)$ for all s_i with $k(s_i) > k(s_j)$.

Basic Properties of Infimum and Supremum

Next, we study some basic properties of the infimum and supremum defined above, so they can be used for further studies of MDFG and EPMDFG. The first two results explain why the infimum and supremum are called the generalization of multidimensional $t-$norms and $t-$conorms.

Theorem 5.1.

1. *For any two* $\mathcal{X}, \mathcal{Y} \in J_\infty[0,1]$ $\inf(\mathcal{X}, \mathcal{Y}) = R(\mathcal{X}, \mathcal{Y})$*, where* R *is the standard multidimensional* $t-$*norm.*
2. *For any two* $\mathcal{X}, \mathcal{Y} \in J_\infty[0,1]$ $\sup(\mathcal{X}, \mathcal{Y}) = S(\mathcal{X}, \mathcal{Y})$*, where* S *is the standard multidimensional* $t-$*norm.*
3. *For any finite family,* $\{s_i\}$ *we have*
$$\inf\{s_i\} \leq_\triangle s_i \leq_\triangle \sup\{s_i\}.$$
4. $\inf^C\{s_i\} = \sup\{s_i^C\}$ *, where* C *is the standard multidimensional complement.*
5. *For any finite family,* $\{s_i\}_{i=1}^n$
$$\inf\{s_i\}_{i=1}^n \leq_\triangle \inf\Big\{\inf\{s_i\}_{i=1}^{k_1}, \inf\{s_i\}_{i=k_1+1}^{k_2}, \ldots \inf\{s_i\}_{i=k_m+1}^{n}\Big\}.$$
where $1 < k_1 < k_2, \ldots k_m < n$
6. *For any finite family,* $\{s_i\}_{i=1}^n$
$$\sup\{s_i\}_{i=1}^n \geq_\triangle \sup\Big\{\sup\{s_i\}_{i=1}^{k_1}, \sup\{s_i\}_{i=k_1+1}^{k_2}, \ldots \sup\{s_i\}_{i=k_m+1}^{n}\Big\}$$
where $1 < k_1 < k_2, \ldots k_m < n$
7. *If* $r_1 \leq_\triangle s_i \leq_\triangle r_2$ *for every* i*, then*
$$r_1 \leq_\triangle \inf(s_i) \leq_\triangle \sup(s_i) \leq_\triangle r_2.$$

Using the infimum and supremum of elements in $J_\infty[0,1]$, we can define the infimum and supremum of a finite collection of MDFG and EPMDFG as follows.

Definition 5.4. Let $G_i = (X_i, Y_i, \psi_i, \phi_i), i = 1,2,3\ldots n$ be a finite family of MDFG. Then $\inf(G_i)$ is defined as the MDFG $G = (X, Y, \psi, \phi)$, where $X = \bigcap_{i=1}^n X_i, Y = \bigcap_{i=1}^n Y_i, \psi(x) = \inf\Big(\psi_i(x)\Big), \phi(x) = \inf\Big(\phi_i(x)\Big)$.

Theorem 5.2. *Let* $G_i = (X_i, Y_i, \psi_i, \phi_i), i = 1,2,3\ldots n$ *be a finite family of* MDFG*, then* $\inf(G_i)$ *is again a* MDFG *(where we take the multidimensional* $t-$*norm as the standard one.)*

Proof. The proof is similar to the proof of the Theorem (4.13) by using the properties in Theorem (5.1). ☐

Now the finite union of MDFG can be defined as follows.

Definition 5.5. Let $G_i = (X_i, Y_i, \psi_i, \phi_i), i = 1,2,3\ldots n$ be a finite family of MDFG with respect to standard multidimensional $t-$norm. Then $\sup(G_i)$ is defined as the MDFG $G = (X, Y, \psi, \phi)$, where $X = \bigcup_{i=1}^n X_i, Y = \bigcup_{i=1}^n Y_i$ and ψ, ϕ defined as follows,

$$\psi(x) = \begin{cases} \psi_j(x) \ if \ x \in X_j \setminus \bigcup_{i \neq j} X_i, \\ \sup(\psi_i(x)) \ if \ x \in X_i; \end{cases}$$

$$\phi(xy) = \begin{cases} \phi_j(xy) \ if \ xy \in Y_j \setminus \bigcup_{i \neq j} Y_i, \\ \sup(\phi_i(xy)) \ if \ xy \in Y_i. \end{cases}$$

Theorem 5.3. *Let* $G_i = (X_i, Y_i, \psi_i, \phi_i), i = 1,2,3\ldots n$ *be a finite family of* MDFG*, then* $\sup(G_i)$ *is again a* MDFG*.*

Remark 5.1. In a similar way to the intersection and union of EPMDFG, we can define the infimum and supremum of EPMDFG also. Then it is easy to see that the resulting infimum and supremum are again EPMDFG and for a finite family $\{G_i\}$ of MDFG we have

$$\overset{C}{\sup}(G_i) = \inf(G_i^C) \ \textit{and} \ \overset{C}{\inf}(G_i) = \sup(G_i^C).$$

where C is the standard multidimensional complement.

5.1. **Vertex Degree and Path Strength.** Defining vertex degree, path length, and other related concepts will be challenging since we need to consider the summation of membership values for both vertices and edges, which can be complicated owing to the varying cardinality of these variables. One of the simplest and most effective ways to solve this problem is to take the sum after computing the averages in each membership vector and it can be done in the following way,

Definition 5.6. $G = (X, Y, \psi, \phi)$ be a MDFG/EPMDFG and $v \in X$, then vertex degree of v is given by

$$\delta(v) = \sum_{uv \in Y} A\Big(\psi(u)\Big). \tag{5.1}$$

where $A(\psi(u))$ is the function that gives average of membership values in $\psi(u)$ as the output.

Example 7. $A(0.2, 0.3, 0.6) = 0.55$ *and* $A(0.4, 0.6, 0.9) = 0.55$

Next, we define the path length in a similar way as follows,

Definition 5.7. Let $P = s_1 s_2 ... s_n$ be a path in the graph (X, Y) of MDFG $G = (X, Y, \psi, \phi)$. then the path length is given by

$$L(P) = \sum_{s_i s_{i+1} \in P} A\Big(\phi(s_i s_{i+1})\Big).$$

Example 8. *In figure 1 the length of the path abcd is* 0.975

Using the notion of path length, we can now define the distance between two vertices u and v as follows,

Definition 5.8. Let u, v be two vertices in a MDFG, $G = (X, Y, \psi, \phi)$ then the distance between u and v is given by,

$$d(uv) = \min_{P_i}\{L(P_i)\}.$$

where P_i is a path joining u and v.

Example 9. *In figure 1 the length between the vertices a and d is* 0.425

Next, we define, the notions like min degree,max degree, path strength, strength of vertices, etc. using the infimum and supremum as follows,

Definition 5.9. $G = (X, Y, \psi, \phi)$ be a MDFG/EPMDFG and $v \in X$, then min degree of v is given by

$$\beta(v) = \inf_{vu \in Y} \Big\{\phi(vu)\Big\}.$$

Now the max degree of v is given by

$$\gamma(v) = \sup_{vu \in Y} \Big\{\phi(vu)\Big\}.$$

Definition 5.10. $P = s_1 s_2 ... s_n$ be a path in the graph (X, Y) of MDFG $G = (X, Y, \psi, \phi)$. Then path strength is given by the pair (V_P, E_P) where

$$V_P = \inf_{i=1}^{n} \Big\{ \psi(s_i) \Big\} \textit{ and } E_P = \inf_{i=1}^{n-1} \Big\{ \phi(s_i s_{i+1}) \Big\}.$$

Example 10. *In figure 1 the strenth of the path abcd is* $\Big((0.25, 0.3), (0.2, 0.25)\Big)$

Definition 5.11. Let u, v be two vertices in a MDFG/EPMDFG, $G = (X, Y, \psi, \phi)$ then the edge strength between u and v is given by,

$$s(uv) = \sup_{P_i} \{E_{P_i}\}$$

where P_i is a path joining u and v.

5.2. **Vertex Degree and Operations.** In this section, we will discuss the min and max degrees of MDFG after doing operations like direct product, tensor product, normal product, etc. For further studies, we only use standard multidimensional $t-$norm and t-conorm.

Theorem 5.4. *Let* $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ *and* $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ *be two* MDFG*, where* $Z_1 = \{r_1, r_2, ...r_n\}$ *and* $Z_2 = \{s_1, s_2, ...s_m\}$*. Let* $G = (X, Y, \psi, \phi)$ *be the multidimensional direct product of* G_1 *and* G_2*. Then for* $r_i s_j \in X$*, there holds*

$$\beta(r_i s_j) = \inf \Big\{ \psi_1(r_i), \psi_2(s_j), \phi_1(r_i r_k)_{r_k \in Z_1}, \phi_2(s_j s_k)_{s_k \in Z_2} \Big\}.$$

Proof. From the definition of direct product, we have,

$$\beta(r_i s_j) = \inf \Big\{ R\Big(\psi_1(r_i), \phi_2(s_j s_k)\Big)_{s_k \in Z_2}, R\Big(\psi_2(s_j), \phi_1(r_i r_k)\Big)_{r_k \in Z_1} \Big\} \tag{5.2}$$

$$\geq_\triangle \inf \Big\{ \psi_1(r_i), \psi_2(s_j), \phi_1(r_i r_k)_{r_k \in Z_1}, \phi_2(s_j s_k)_{s_k \in Z_2} \Big\} \quad \text{(Using Theorem (5.1))}. \tag{5.3}$$

Now, from (5.2) we have,

$$\beta(r_i s_j) \leq_\triangle \psi_1(r_i), \psi_2(s_j), \phi_1(r_i r_k), \phi_2(s_j s_k)$$

$$\beta(r_i s_j) \leq_\triangle \inf \Big\{ \psi_1(r_i), \psi_2(s_j), \phi_1(r_i r_k)_{r_k \in Z_1}, \phi_2(s_j s_k)_{s_k \in Z_2} \Big\} \quad \text{(Using Theorem (5.1))}. \tag{5.4}$$

From (5.3) and (5.4), we have

$$\beta(r_i s_j) = \inf \Big\{ \psi_1(r_i), \psi_2(s_j), \phi_1(r_i r_k)_{r_k \in Z_1}, \phi_2(s_j s_k)_{s_k \in Z_2} \Big\}. \tag{5.5}$$

□

Corollary 5.5. *Let* $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ *and* $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ *be two* EPMDFG*, where* $Z_1 = \{r_1, r_2, ...r_n\}$ *and* $Z_2 = \{s_1, s_2, ...s_m\}$*. Let* $G = (X, Y, \psi, \phi)$ *be the edge-powered multidimensional direct product of* G_1 *and* G_2*. Then for* $r_i s_j \in X$*,*

$$\gamma(r_i s_j) = \sup \Big\{ \psi_1(r_i), \psi_2(s_j), \phi_1(r_i r_k)_{r_k \in Z_1}, \phi_2(s_j s_k)_{s_k \in Z_2} \Big\}.$$

Theorem 5.6. *Let* $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ *and* $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ *be two* MDFG*, where* $Z_1 = \{r_1, r_2, ...r_n\}$ *and* $Z_2 = \{s_1, s_2, ...s_m\}$*. Let* $G = (X, Y, \psi, \phi)$ *be the multidimensional composition of* G_1 *and* G_2*. Then for* $r_i s_j \in X$*,*

$$\beta(r_i s_j) = \inf \Big\{ \psi_1(r_i), \phi_1(r_i r_k)_{r_k \in Z_1}, \phi_2(s_j s_k)_{s_k \in Z_2}, \psi_2(s_k)_{s_k \in Z_2} \Big\}.$$

Proof. From the definition of multidimensional composition of MDFG, we have

$$\begin{aligned}\beta(r_is_j) = \inf\Big\{&R\Big(\psi_1(r_i),\phi_2(s_js_k)\Big)_{s_k\in Z_2}, R\Big(\psi_2(s_j),\phi_1(r_ir_k)\Big)_{r_k\in Z_1},\\ &R\Big(\phi_1(r_ir_k)_{r_k\in Z_1}, R\Big(\psi_2(s_j),\psi_2(s_k)\Big)_{s_k\notin Z_2}\Big)\Big\}\end{aligned}$$

$$\implies \beta(r_is_j) \geq \triangle \inf\Big\{\psi_1(r_i),\phi_1(r_ir_k)_{r_k\in Z_1},\phi_2(s_js_k)_{s_k\in Z_2},\psi_2(s_k)_{s_k\in Z_2}\Big\}. \tag{5.6}$$

Now

$$\beta(r_is_j) \leq_\triangle \psi_1(r_i),\psi_2(s_k),\phi_1(r_ir_k),\phi_2(s_js_k)$$

$$\implies \beta(r_is_j) \leq_\triangle \inf\Big\{\psi_1(r_i),\phi_1(r_ir_k)_{r_k\in Z_1},\phi_2(s_js_k)_{s_k\in Z_2},\psi_2(s_k)_{s_k\in Z_2}\Big\}. \tag{5.7}$$

From (5.6) and (5.7), we have

$$\beta(r_is_j) = \inf\Big\{\psi_1(r_i),\phi_1(r_ir_k)_{r_k\in Z_1},\phi_2(s_js_k)_{s_k\in Z_2},\psi_2(s_k)_{s_k\in Z_2}\Big\}.$$

□

Corollary 5.7. *Let* $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ *and* $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ *be two* EPMDFG, *where* $Z_1 = \{r_1, r_2, ...r_n\}$ *and* $Z_2 = \{s_1, s_2, ...s_m\}$. *Let* $G = (X, Y, \psi, \phi)$ *be the edge-powered multidimensional composition of* G_1 *and* G_2. *Then for* $r_is_j \in X$,

$$\gamma(r_is_j) = \sup\Big\{\psi_1(r_i),\phi_1(r_ir_k)_{r_k\in Z_1},\phi_2(s_js_k)_{s_k\in Z_2},\psi_2(s_k)_{s_k\in Z_2}\Big\}.$$

Theorem 5.8. *Let* $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ *and* $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ *be two* MDFG, *where* $Z_1 = \{r_1, r_2, ...r_n\}$ *and* $Z_2 = \{s_1, s_2, ...s_m\}$. *Let* $G = (X, Y, \psi, \phi)$ *be the multidimensional tensor product of* G_1 *and* G_2. *Then for* $r_is_j \in X$,

$$\beta(r_is_j) = \inf\Big\{\phi_1(r_ir_k)_{r_ir_k\in W_1},\phi_2(s_js_k)_{s_js_k\in W_2}\Big\}.$$

Proof. We have,

$$\begin{aligned}\beta(r_is_j) &= \inf\Big\{R\Big(\phi_1(r_ir_k)_{r_ir_k\in W_1},\phi_2(s_js_k)_{s_js_k\in W_2}\Big)\Big\}\\ &\geq_\triangle \inf\Big\{\phi_1(r_ir_k)_{r_ir_k\in W_1},\phi_2(s_js_k)_{s_js_k\in W_2}\Big\}.\end{aligned} \tag{5.8}$$

But we have,

$$\beta(r_is_j) \leq_\triangle \phi_1(r_ir_k)_{r_ir_k\in W_1},\phi_2(s_js_k)_{s_js_k\in W_2}$$

$$\implies \beta(r_is_j) \leq_\triangle \inf\Big\{R\Big(\phi_1(r_ir_k)_{r_ir_k\in W_1},\phi_2(s_js_k)_{s_js_k\in W_2}\Big)\Big\}. \tag{5.9}$$

Hence, from (5.8) and (5.9) we have

$$\beta(r_is_j) = \inf\Big\{\phi_1(r_ir_k)_{r_ir_k\in W_1},\phi_2(s_js_k)_{s_js_k\in W_2}\Big\}.$$

□

Corollary 5.9. *Let* $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ *and* $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ *be two* EPMDFG, *where* $Z_1 = \{r_1, r_2, ...r_n\}$ *and* $Z_2 = \{s_1, s_2, ...s_m\}$. *Let* $G = (X, Y, \psi, \phi)$ *be the edge-powered multidimensional tensor product of* G_1 *and* G_2. *Then for* $r_is_j \in X$,

$$\gamma(r_is_j) = \sup\Big\{\phi_1(r_ir_k)_{r_ir_k\in W_1},\phi_2(s_js_k)_{s_js_k\in W_2}\Big\}.$$

In a similar manner, we can prove the following properties also,

Theorem 5.10. *Let* $G_1 = (Z_1, W_1, \psi_1, \phi_1)$ *and* $G_2 = (Z_2, W_2, \psi_2, \phi_2)$ *be two* MDFG, *where* $Z_1 = \{r_1, r_2, ...r_n\}$ *and* $Z_2 = \{s_1, s_2, ...s_m\}$ *and let* $G = (X, Y, \psi, \phi)$.

- *If G is the normal product of G_1 and G_2, then*

$$\beta(r_i s_j) = \inf \left\{ \psi_1(r_i), \psi_2(s_j), \phi_1(r_i r_k)_{r_k \in Z_1}, \phi_2(s_j s_k)_{s_k \in Z_2} \right\}.$$

- *If G is the intersection of G_1 and G_2, then*

$$\beta(r_i) = \inf \left\{ \phi_1(r_i r_k)_{r_i r_k \in W_1 \cap W_2}, \phi_2(r_i s_k)_{r_i s_k \in W_1 \cap W_2} \right\}.$$

- *If G is the union of G_1 and G_2, then*

$$\gamma(r_i) = \begin{cases} \sup \left\{ \phi_1(r_i r_k)_{r_k \in Z_1}, \phi_2(r_i s_k)_{s_k \in Z_2} \right\} \; if \; r_i \in Z_1 \cap Z_2, \\ \sup \left\{ \phi_1(r_i r_k)_{r_k \in Z_1} \right\} \; if \; r_i \in Z_1 \setminus Z_2. \end{cases}$$

- *If G is the join of G_1 and G_2 then*

$$\gamma(r_i) = \begin{cases} \sup \left\{ \phi_1(r_i r_k)_{r_k \in Z_1}, \phi_2(r_i s_k)_{s_k \in Z_2} \right\} \; if \; r_i \in Z_1 \cap Z_2, \\ \sup \left\{ \psi_1(r_i), \psi_2(s_j)_{r_i s_j \notin W_1 \cap W_2} \phi_1(r_i r_k)_{r_k \in Z_1} \right\} \; if \; r_i \in Z_1 \setminus Z_2. \end{cases}$$

6. Conclusion

This study aims to develop a hybrid framework that integrates multidimensional fuzzy sets and graphs in two unique techniques to handle a variety of real-world challenges. The natural partial ordering of $J_\infty[0,1]$ is extended to connect more members within the same set. This new partial ordering is then applied to the axioms of multidimensional $t-$ norms and $t-$ conorms. The operators described in this paper are important for tackling real-world difficulties, such as combining data from two distinct MDFG/EPMDFG sources. Given that operator and multidimensional fuzzy graphs are exclusively characterized using multidimensional $t-$ norms and $t-$ conorms, the results can be used in a variety of ways. The idea of multidimensional complements determines the complements of MDFG and EPMDFG, allowing us to define their link with the previously stated operators. The supremum and infimum of elements in $J_\infty[0,1]$ are extensions of normal multidimensional $t-$ norms and $t-$ conorms. These definitions are used to investigate important concepts like vertex degree and path strength and can be used for many other types of research in MDFS later. The work focuses on calculating the lowest and maximum degrees of vertices in multidimensional fuzzy graphs following specific operations. The explicit forms of these degrees are also defined. The lack of a systematic approach to assign multidimensional membership values to edges and vertices can be viewed as a drawback of the proposed method. To further examine multidimensional fuzzy graphs, we intend to broaden our research to include a variety of factors such as connectivity index, path connectedness, domination number, and so on.

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