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DUAL PHASE LAG TWO TEMPERATURE FRACTIONAL THERMOELASTICITY IN THE CONTEXT OF GREEN NAGHDI TYPE II

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ABSTRACT. The linear thermoelasticity theory without energy dissipation developed by Green Naghdi has been reconstructed using two-phase lags and two temperature theory in the framework of the fractional time derivative. In half space, the mathematical model for one dimensional wave propagation subject to thermal shock on the bounding surface is discussed. Assume that the bounding surface is traction free. The analytical solutions have been obtained in the Laplace domain. The Gaver-Stehfest technique is simple, efficient, and robust. It has been numerically used to perform an inversion of the Laplace transform, satisfying Kuznetsov's convergence condition in the time domain. The significance of the fractional order parameter on variations of various fields inside the medium is addressed graphically. The utilization of delay time translations in heat flux vector and thermal displacement gradient causes the finite speed of wave propagation and depicts microscopic responses more precisely.

1. INTRODUCTION

Biot [7] invented the classical coupled theory asserting that thermal waves propagate at infinite speed. In an isotropic medium, Lord and Shulman [24] improved the idea of thermal waves propagating at finite speed by taking one relaxation time in generalized thermoelasticity. Green and Lindsay [18] demonstrated the importance of imposing different conditions on the equations of generalized classical thermoelasticity. Chandrasekharaiah [9] gave a review of theories of generalized thermoelasticity. He discussed one dimensional wave problems and also proved uniqueness theorems.

Green and Naghdi [19] developed a new concept of the Green Naghdi-II (GN-II) model. Chen et al. [10, 11, 12] invented a theory of heat conduction dependent on two different temperatures. Warren and Chen [26] examined the wave propagation in thermoelasticity's two temperature theory. El-Karamany and Ezzat [15] proposed two temperature GN thermoelasticity theories. Youssef [27] derived generalized thermoelasticity by considering two temperature theory without energy dissipation. Youssef and Elsibai [28] proposed a one dimensional two temperature thermoelasticity of the GN-II model.

The three different types of Green and Naghdi models are presented by El-Karamany and Ezzat [14]. El-Attar et al. [13] proposed a phase lag of the GN-II model for electro-thermoelasticity. Abouelregal [1] researched the GN-II model with the help of two temperature and two phase delays by taking higher order time derivatives. Abouelregal [2] discussed the dual phase lag GN-II model with the help of two temperature theory for a perfectly conducting spherical cavity. Ezzat et al. [16] proposed a model of phase lag GN thermoelasticity theories that depend on fractional derivatives using a theory of two temperature.

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Abouelregal et al. [4] proposed a two temperature fractional thermoelastic model using the exponential Rabotnov kernel. Zenkour and Abouelregal [29] studied a two temperature generalized thermoelastic model subject to a non-Gaussian laser pulse. Abouelregal et al. [5] proposed a generalized two temperature thermoelastic model using the Caputo–Fabrizio fractional derivative for a non-simple thermoelastic cylinder. Abouelregal and Zenkour [6] studied the effect of fractional thermoelasticity on a two-dimensional mode-I crack problem. Abouelregal [3] proposed a generalized model of thermoelastic heat transfer of a rigid cylinder using phase delays and the two temperature theory. Hendy et al. [21] studied the fractional thermo-viscoelasticity for a two-dimensional problem. Hendy et al. [22] studied fractional Green Naghdi of type III using two temperature magneto-thermo-viscoelasticity theory. Hassaballa et al. [20] modified the Green Naghdi model using fractional order theory for analyzing thermoelectric semispace.

The dual phase lag two temperature fractional thermoelasticity of the GN-II model is a cutting-edge theoretical development that enhances our ability to analyze and predict the behavior of materials. The proposed model represents a novel approach in the field of solid mechanics, thermal engineering, and materials science by considering theories of fractional calculus, dual phase lag, and two temperature within the framework of Green-Naghdi thermoelasticity. By incorporating these theories, the proposed thermoelastic model gives a more accurate and comprehensive description of the interplay between thermal and mechanical fields, especially in materials with complex behaviors like memory effects and non-local interactions.

The fractional order differential operator is a non-local operator and can be interpreted as a measure of memory, and also controls the strength of memory effects. The use of Caputo time fractional derivatives in the proposed model induces a more comprehensive and accurate description of the dynamic behavior of materials. This approach is particularly useful in situations where traditional integer order models fail to capture the observed phenomena due to the presence of memory effects. This theory not only broadens the theoretical understanding of fractional heat conduction but also enhances practical applications in various fields of solid mechanics, thermal engineering, material designing, etc.

In the present article, the linear thermoelasticity theory without energy dissipation developed by Green and Naghdi [19] has been reconstructed by using two phase lags and two temperature theory in the framework of the fractional time derivative of order $\alpha \in (0,1]$ in section 2. In half space, the mathematical model is discussed for one dimensional wave propagation subject to the thermal shock problem at the traction free bounding surface in section 3. In section 4, the analytical solutions have been obtained in the Laplace domain. The Gaver-Stehfest [17, 25] technique has been numerically used for performing the time domain inverse of the Laplace transform, satisfying Kuznetsov's [23] convergence condition. In section 5, numerical results are shown in graphical form. The outcomes of the work are mentioned in section 6.

2. Derivation of FDPL heat conduction equation of GN-II theory for two temperature theory

Following Biot [7], the energy equation is represented as

$$\rho_m c \frac{\partial T}{\partial t} + \gamma T_0 \frac{\partial e}{\partial t} = -\nabla \cdot \mathbf{q} + Q,\ \ c > 0, \tag{2.1}$$

where c represents specific heat, ρ_m represents density, Q is the intensity of the heat source, e represents dilatation, and $\gamma = \alpha_t(3\lambda + 2\mu)$ represents the stress temperature modulus.

Taking the partial derivative of equation (2.1) with respect to the time variable, one obtains

$$\rho_m c \frac{\partial^2 T}{\partial t^2} + \gamma T_0 \frac{\partial^2 e}{\partial t^2} = -\nabla \cdot \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial Q}{\partial t}. \tag{2.2}$$

Following Green and Naghdi [19], the heat conduction law is defined as

$$\mathbf{q}(x,t) = -\kappa^* \nabla \nu(x,t), \tag{2.3}$$

where $\mathbf{q}, t, \kappa^* > 0, \nu, \nabla$ represent heat flux, time, material characteristic, thermal displacement, and gradient operator respectively.

We introduce two phase lags τ_q and τ_ν to vector $\mathbf{q}$ and thermal displacement gradient ν respectively in equation (2.3). Then the generalized constitutive equation is represented as follows

$$\mathbf{q}(x, t+\tau_q) = -\kappa^* \nabla \nu(x, t+\tau_\nu), \ \tau_q, \tau_\nu > 0. \tag{2.4}$$

Using Taylor's series to expand equation (2.4) by keeping terms up to order 2α in phase lag with respect to the Caputo [8] fractional order derivatives, one obtains

$$\left[1 + \frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\mathbf{q} = -\kappa^*\left[1 + \frac{\tau_\nu^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_\nu^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\nabla\nu, \ 0 < \alpha \le 1. \tag{2.5}$$

By applying the partial derivative of equation (2.5) w.r.t. the time variable, one obtains

$$\left[1 + \frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\frac{\partial \mathbf{q}}{\partial t} = -\kappa^*\left[1 + \frac{\tau_\nu^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_\nu^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\nabla T, \ 0 < \alpha \le 1, \tag{2.6}$$

where $\frac{\partial \nu}{\partial t} = T$ satisfies by thermal displacement ν.

Combining equation (2.2) and equation (2.6), one obtains

$$\begin{aligned}&\left[1 + \frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\left[\rho_m c \frac{\partial^2 T}{\partial t^2} + \gamma T_0 \frac{\partial^2 e}{\partial t^2} - \frac{\partial Q}{\partial t}\right] \\ &= \left[1 + \frac{\tau_\nu^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_\nu^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\kappa^* \Delta T, \quad 0 < \alpha \le 1.\end{aligned} \tag{2.7}$$

Operating $(1 - \tilde{a}\nabla^2)$ on both the sides of equation (2.7), one obtains

$$\begin{aligned}&\left[1 + \frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\left[\rho_m c (1 - \tilde{a}\nabla^2)\frac{\partial^2 T}{\partial t^2} + \gamma T_0 (1 - \tilde{a}\nabla^2)\frac{\partial^2 e}{\partial t^2} - (1 - \tilde{a}\nabla^2)\frac{\partial Q}{\partial t}\right] \\ &= \left[1 + \frac{\tau_\nu^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_\nu^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\kappa^* (1 - \tilde{a}\nabla^2)\Delta T, \ 0 < \alpha \le 1.\end{aligned} \tag{2.8}$$

Employing the relation of the two temperature theory developed by Chen et al. [10, 11, 26] to the LHS of equation (2.8), and the relationship between two temperature is

$$T(x,t) - \theta(x,t) = \tilde{a}\Delta T(x,t), \ \tilde{a} > 0, \tag{2.9}$$

where Δ and θ represent the Laplacian operator and thermodynamic temperature respectively.

From equations (2.8) and (2.9), one obtains

$$\begin{aligned}&\left[1 + \frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\left[\rho_m c \frac{\partial^2 \theta}{\partial t^2} + \gamma T_0 (1 - \tilde{a}\nabla^2)\frac{\partial^2 e}{\partial t^2} - (1 - \tilde{a}\nabla^2)\frac{\partial Q}{\partial t}\right] \\ &= \left[1 + \frac{\tau_\nu^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_\nu^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\kappa^* (1 - \tilde{a}\nabla^2)\Delta T, \ 0 < \alpha \le 1,\end{aligned} \tag{2.10}$$

Eliminating the differential coefficient of order higher than ∇^2 in equation (2.10), one obtains

$$\begin{aligned}&\left[1+\frac{\tau_q^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\left[\rho_m c\frac{\partial^2\theta}{\partial t^2}+\gamma T_0\frac{\partial^2 e}{\partial t^2}-\frac{\partial Q}{\partial t}\right]\\&=\left[1+\frac{\tau_\nu^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\frac{\tau_\nu^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\kappa^*\Delta T,\ 0<\alpha\leq 1.\end{aligned} \tag{2.11}$$

Equation (2.11) shows the dual phase lag fractional heat conduction equation of the GN-II theory for two temperature.

2.1. Limiting Cases.

Case 1: If $T=\theta,\ \alpha\to 0, Q=0$, then equation (2.11) represents hyperbolic and leads to the following equation presented by Green and Naghdi [19]

$$\rho_m c\frac{\partial^2 T}{\partial t^2}+\gamma T_0\frac{\partial^2 e}{\partial t^2}=\kappa^*\Delta T. \tag{2.12}$$

Case 2: If $T\neq\theta,\ \alpha\to 0, Q=0$, then equation (2.11) represents hyperbolic and leads to the following equation presented by Youssef [27]

$$\rho_m c\frac{\partial^2 \theta}{\partial t^2}+\gamma T_0\frac{\partial^2 e}{\partial t^2}=\kappa^*\Delta T. \tag{2.13}$$

Case 3: If $T=\theta,\ \alpha\to 1, \tau_q=\tau_0, \tau_q^2\to 0, \tau_\nu=0$, then equation (2.11) leads to the following equation presented by El-Attar et al. [13]

$$\left[1+\tau_0\frac{\partial}{\partial t}\right]\left[\rho_m c\frac{\partial^2 T}{\partial t^2}+\gamma T_0\frac{\partial^2 e}{\partial t^2}-\frac{\partial Q}{\partial t}\right]=\kappa^*\Delta T. \tag{2.14}$$

Case 4: If $T=\theta,\ \alpha\to 1, \tau_\nu^2\to 0$, then equation (2.11) leads to the following equation presented by El-Karamany and Ezzat [14]

$$\left[1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right]\left[\rho_m c\frac{\partial^2 T}{\partial t^2}+\gamma T_0\frac{\partial^2 e}{\partial t^2}-\frac{\partial Q}{\partial t}\right]=\left[1+\tau_\nu\frac{\partial}{\partial t}\right]\kappa^*\Delta T. \tag{2.15}$$

3. Formulation of the Problem

Consider one dimensional half space ($x\geq 0$), which is occupied by homogeneous isotropic thermoelastic material. A bounding surface is to be considered as a unit step function. Suppose that initial conditions are homogeneous and the bounding surface is traction free.

3.1. Mathematical Model.

The following forms are considered by assuming no body forces and heat sources.

Displacement components are given as

$$u_x=u(x,t),\ u_y=0,\ u_z=0. \tag{3.1}$$

The coupled equation of motion is

$$(\lambda+2\mu)\frac{\partial e}{\partial x}-\gamma\frac{\partial}{\partial x}(T-T_0)=\rho_m\frac{\partial^2 u}{\partial t^2}, \tag{3.2}$$

where $e=\frac{\partial u}{\partial x}$.

Stress function $\sigma_{xx}(x,t)$ is represented by

$$\sigma_{xx}=(\lambda+2\mu)e-\gamma(T-T_0), \tag{3.3}$$

where T_0 is a reference temperature.

From equation (2.9), one can express

$$\theta = T - \tilde{a}\frac{\partial^2 T}{\partial x^2},\ \tilde{a} > 0. \tag{3.4}$$

The associated heat conduction equation is represented by

$$\left[1+\frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\left[\rho_m c\frac{\partial^2\theta}{\partial t^2}+\gamma T_0\frac{\partial^2 e}{\partial t^2}\right] = \left[1+\frac{\tau_\nu^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_\nu^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\kappa^*\frac{\partial^2 T}{\partial x^2},\ 0<\alpha\leq 1. \tag{3.5}$$

Equations (3.1) – (3.5) represent governing equations of two temperature dual phase lag fractional heat conduction equations of the GN-II model.

3.2. **Dimensionless Form.** The following dimensionless physical and geometrical parameters are used to mathematically formulate the problem

$$\begin{aligned}&(u^*, x^*, t^*, \tau_q^*, \tau_\nu^*) = c_l\zeta^{'}(u, x, c_l t, c_l\tau_q, c_l\tau_\nu),\ \zeta^{'} = \frac{\rho_m c}{\kappa^*},\ \xi^{'} = \tilde{a}(c_l\zeta^{'})^2,\ \Theta = \frac{(T-T_0)\gamma}{(\lambda+2\mu)},\\ &c_l = \sqrt{\frac{\lambda+2\mu}{\rho_m}},\ \phi = \frac{\theta\gamma}{(\lambda+2\mu)},\ \sigma_{xx}^* = \frac{\sigma_{xx}}{\lambda+2\mu},\ \varepsilon = \frac{\gamma^2 T_0}{\rho_m c(\lambda+2\mu)},\ \beta_0 = \frac{\rho_m c(\lambda+2\mu)}{\rho_m\kappa^*}.\end{aligned} \tag{3.6}$$

By inserting the above variables into the equations (3.2) – (3.5), one obtains (omitting $*$ symbol)

$$\left[1+\frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\left[\beta_0\frac{\partial^2\phi}{\partial t^2}+\beta_0\varepsilon\frac{\partial^2 e}{\partial t^2}\right] = \left[1+\frac{\tau_\nu^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_\nu^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right]\frac{\partial^2\Theta}{\partial x^2},\ 0<\alpha\leq 1, \tag{3.7}$$

$$\frac{\partial e}{\partial x} - \frac{\partial\Theta}{\partial x} = \frac{\partial^2 u}{\partial t^2}, \tag{3.8}$$

$$\sigma_{xx} = \sigma = e - \Theta, \tag{3.9}$$

$$\Theta - \phi = \xi^{'}\frac{\partial^2\Theta}{\partial x^2}. \tag{3.10}$$

Equations (3.7) – (3.10) represent the dimensionless form.

3.3. **Initial and Boundary conditions.** The homogeneous initial conditions are

$$\Theta(x,0) = \frac{\partial\Theta}{\partial t}(x,0) = 0, u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0,\ x \geq 0. \tag{3.11}$$

The regularity conditions are

$$\lim_{x\to\infty}\sigma(x,t) = \lim_{x\to\infty}\Theta(x,t) = 0,\ t>0. \tag{3.12}$$

The thermal and mechanical boundary conditions are given by

$$\sigma(0,t) = 0,\ \Theta(0,t) = \Theta_0 H(t),\ t>0, \tag{3.13}$$

where $H(t)$ denotes a Heaviside unit step function.

The basic goal is to analyze initial boundary problems as determined by equations (3.7) – (3.13).

4. Solution of the Problem

The technique of Laplace transform is implemented to acquire an analytical solution to the initial boundary problem, which is defined in the earlier section. To acquire the time domain solution, the Gaver-Stehfest technique is employed to invert the Laplace transform.

4.1. **Laplace domain Solution.** The governing dimensionless equations (3.7) – (3.10) converted in the Laplace domain as

$$\Upsilon_1\big[\bar{\phi}+\varepsilon\bar{e}\big]=\Upsilon_2\frac{\partial^2\bar{\Theta}}{\partial x^2},\tag{4.1}$$

where

$$\Upsilon_1=\Big[\beta_0p^2+\beta_0p^{2+\alpha}\frac{\tau_q^{\alpha}}{\alpha!}+\beta_0p^{2+2\alpha}\frac{\tau_q^{2\alpha}}{(2\alpha)!}\Big],\Upsilon_2=\Big[1+p^{\alpha}\frac{\tau_\nu^{\alpha}}{\alpha!}+p^{2\alpha}\frac{\tau_\nu^{2\alpha}}{(2\alpha)!}\Big].$$

$$\frac{\partial\bar{e}}{\partial x}-\frac{\partial\bar{\Theta}}{\partial x}=p^2\bar{u},\tag{4.2}$$

$$\bar{\sigma}_{xx}=\bar{e}-\bar{\Theta},\tag{4.3}$$

$$\bar{\Theta}-\bar{\phi}=\xi'\frac{\partial^2\bar{\Theta}}{\partial x^2}.\tag{4.4}$$

By solving equations (4.1) and (4.2), one obtains

$$\frac{d^4\bar{\Theta}}{dx^4}-\left[\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}+p^2\right]\frac{d^2\bar{\Theta}}{dx^2}+\frac{\Upsilon_1p^2}{\Upsilon_2+\Upsilon_1\xi'}\bar{\Theta}=0.\tag{4.5}$$

By considering regularity condition i.e. $\bar{\Theta}\to 0$ when $x\to\infty$. The solution of equation (4.5) is represented as

$$\bar{\Theta}(x,p)=A_1e^{-k_1x}+A_2e^{-k_2x},\tag{4.6}$$

where A_1, A_2 are constants, and k_1, k_2 are positive roots of polynomial equation

$$k^4-\left[\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}+p^2\right]k^2+\frac{\Upsilon_1p^2}{\Upsilon_2+\Upsilon_1\xi'}=0.\tag{4.7}$$

The roots k_1, k_2 are given by

$$k_1^2,k_2^2=\frac{1}{2}\left(\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}+p^2\pm\sqrt{\left[\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}+p^2\right]^2-\frac{4\Upsilon_1p^2}{\Upsilon_2+\Upsilon_1\xi'}}\right).\tag{4.8}$$

The constants A_1, A_2 are obtained by using equation (3.13), one obtains

$$A_1=\frac{\Theta_0}{p\big(k_2^2-k_1^2\big)}\left[k_2^2-\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}\right],A_2=\frac{\Theta_0}{p\big(k_1^2-k_2^2\big)}\left[k_1^2-\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}\right].\tag{4.9}$$

The conductive temperature $\bar{\Theta}(x,p)$ is acquired by taking equation (4.9) in equation (4.6), one obtains

$$\bar{\Theta}(x,p)=\frac{\Theta_0}{p\big(k_2^2-k_1^2\big)}\left\{\left[k_2^2-\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}\right]e^{-k_1x}+\left[-k_1^2+\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}\right]e^{-k_2x}\right\}.\tag{4.10}$$

The thermodynamic temperature $\bar{\phi}(x,p)$ is acquired by taking equation (4.10) in equation (4.4), one obtains

$$\begin{aligned}\bar{\phi}(x,p)=&\frac{\Theta_0}{p\big(k_2^2-k_1^2\big)}\left\{\left[k_2^2-\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}\right]e^{-k_1x}+\left[-k_1^2+\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}\right]e^{-k_2x}\right\}\\&-\frac{\xi'\Theta_0}{p\big(k_2^2-k_1^2\big)}\left\{k_1^2\left[k_2^2-\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}\right]e^{-k_1x}+k_2^2\left[-k_1^2+\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}\right]e^{-k_2x}\right\}.\end{aligned}\tag{4.11}$$

The displacement function $\bar{u}(x,p)$ is acquired from equation (4.2) by using equation (4.10), one obtains

$$\begin{aligned}\bar{u}(x,p)=&\frac{\Theta_0\big(\Upsilon_2+\Upsilon_1\xi'\big)}{p\big(k_2^2-k_1^2\big)\Upsilon_1\varepsilon}\left\{\frac{1}{k_1}\left[\frac{\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}-k_1^2\right]\left[k_2^2-\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}\right]e^{-k_1x}\right.\\&\left.+\frac{1}{k_2}\left[\frac{\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}-k_2^2\right]\left[-k_1^2+\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi'}\right]e^{-k_2x}\right\}.\end{aligned}\tag{4.12}$$

The stress $\bar{\sigma}(x,p)$ is acquired from equation (4.3) by using equations (4.10) and (4.12), one obtains

$$\begin{aligned} \bar{\sigma}(x,p) = \frac{\Theta_0\big(\Upsilon_2+\Upsilon_1\xi^{'}\big)}{p\big(k_2^2-k_1^2\big)\Upsilon_1\varepsilon}\Bigg\{\Bigg[k_1^2-\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi^{'}}\Bigg]\Bigg[k_2^2-\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi^{'}}\Bigg]e^{-k_1x} \\ +\Bigg[k_2^2-\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi^{'}}\Bigg]\Bigg[-k_1^2+\frac{(1+\varepsilon)\Upsilon_1}{\Upsilon_2+\Upsilon_1\xi^{'}}\Bigg]e^{-k_2x}\Bigg\}. \end{aligned} \tag{4.13}$$

Equations (4.10) – (4.13) represent solutions in the Laplace domain.

4.2. **Gaver-Stehfest Algorithm.** Gaver-Stehfest [17, 25] method is used to gain a time domain solution

$$g(t)\approx g_m(t)=\frac{\ln(2)}{t}\sum_{j=1}^{2m}\Bigg\{(-1)^{m+j}\Bigg[\sum_{l=\lfloor\frac{j+1}{2}\rfloor}^{\min(j,m)}\frac{l^{m+1}}{m!}\binom{m}{l}\binom{2l}{l}\binom{l}{j-l}\Bigg]\cdot G\left(\frac{j\ln(2)}{t}\right)\Bigg\}, \tag{4.14}$$

where $\lfloor y\rfloor$ is the flooring function.

4.3. **Convergence Theorem.** Assume that function $g:(0,\infty)\to\mathbb{R}$ is a continuously integrable function, which is exponentially ordered, then the sequence represented in equation (4.14) as

$$g_m(t)\to\ \frac{g(t+0)+g(t-0)}{2},\ \text{ when }\ m\to\infty,$$

where convergence of $g_m(t)$ is based on the value of bounded variation $g(t)$ in the neighborhood of t and Laplace transform $G(p)$ exists for all $p>0$ [23].

5. Numerical Scheme

This section is to explain the effects of the fractional parameter $\alpha = 0, 0.5, 0.75$, and 1 on the field quantities and demonstrate the numerical outcomes of analytical solutions derived in the preceding section. A field quantities, viz. conductive temperature $\Theta(x,t)$, thermodynamic temperature $\phi(x,t)$, displacement $u(x,t)$, and stress $\sigma(x,t)$ distributions are shown graphically at various places of distance x as displayed in figures 1 to 4 by using a fixed value of $t = 0.1, \tau_q = 0.03, \tau_\nu = 0.02$.

5.1. **Material Properties.** The copper material has to be considered with properties such as [14]

$$\begin{aligned} &\mu = 3.86\times10^{10}\ Nm^{-2},\ c = 383.1\ m^2K^{-1},\ \lambda = 7.76\times10^{10}\ Nm^{-2},\ \rho_m = 8954\ kgm^{-3}, \\ &T_0 = 293\ K,\ \kappa^* = 10,\ \xi^{'} = 0.010414,\ \alpha_t = 1.78\times10^{-5}\ K^{-1},\ \varepsilon = 0.0168,\ \beta_0 = 7.99. \end{aligned} \tag{5.1}$$

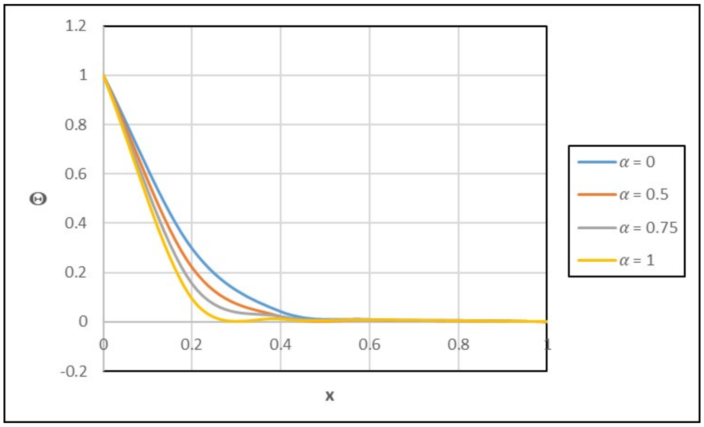

FIGURE 1. Conductive temperature $\Theta(x,t)$ at distinct values of α

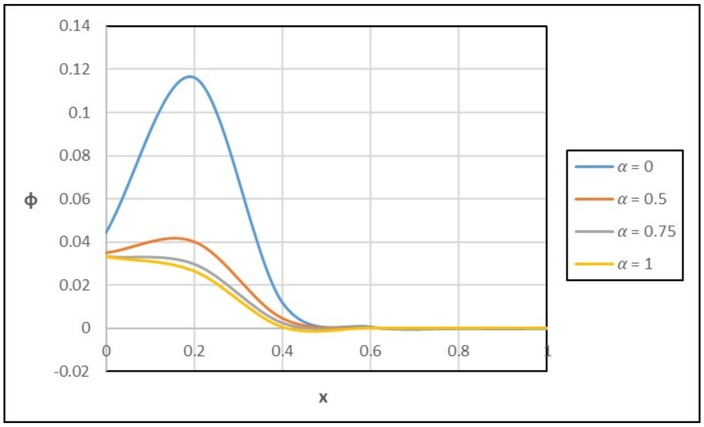

FIGURE 2. Thermodynamic temperature $\phi(x,t)$ at distinct values of α

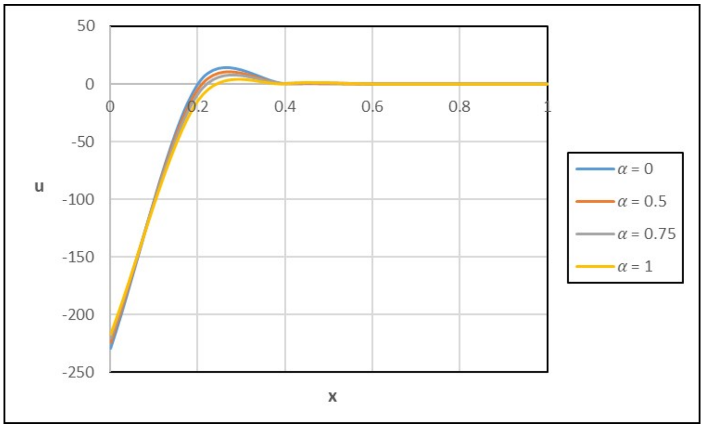

FIGURE 3. Displacement $u(x,t)$ at distinct values of α

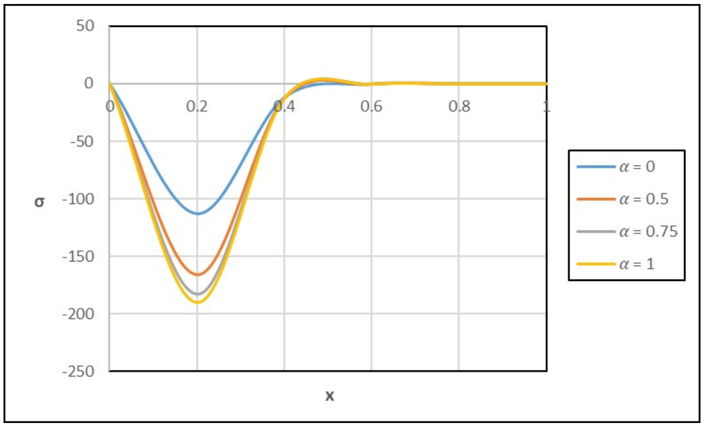

FIGURE 4. Stress $\sigma(x,t)$ at distinct values of α

Figure 1: It exhibits conductive temperature $\Theta(x,t)$ variation with respect to various values of α. The conductive temperature values decrease when parameter α increases, and expansive behavior is observed in $0.2 < x < 0.4$. Temperature values along distance x are maximum for $\alpha = 0.0$ and minimum for $\alpha = 1$.

Figure 2: It exhibits thermodynamic temperature $\phi(x,t)$ variation with respect to various values of α. The thermodynamic temperature values decrease when parameter α increases, and expansive behavior is observed in $0 < x < 0.4$. The thermodynamic values along distance x are maximum for $\alpha = 0$ and minimum for $\alpha = 1$.

Figure 3: It exhibits displacement $u(x,t)$ variation with respect to various values of α. The displacement increases with distance x, reaches a maximum at a certain point, and then decreases to zero. In a range of $x \in (0.2, 0.4)$, displacement values are very compressive in behavior.

Figure 4: It exhibits stress $\sigma(x,t)$ variation with respect to various values of α. It has been found that stress variation satisfies the condition of traction-free ends. The stress function attains minima at $x = 0.2$.

From all figures 1 to 4, one can conclude that the solutions of functions in this proposed model are confined to a bounded area. Beyond the restricted area, the variations of distributions never occur, which indicates that the proposed model's outcomes show the behavior of finite wave propagation.

The fractional order differential operator is a non-local operator and can be interpreted as a measure of memory, and also controls the strength of memory effects. The interpretation of obtained results may help engineers and scientists to design and optimize systems, structures, and materials by understanding how they will behave under realistic operational conditions. This suggests that the behavior of the thermoelastic fields is significantly influenced by fractional order α.

6. CONCLUSIONS

The following are the key outcomes of this work:

(1) The linear thermoelasticity theory without energy dissipation developed by Green Naghdi has been reconstructed using two phase lags and two temperature theory in the framework of the fractional time derivative.

(2) The finite speed of thermal wave is achieved due to the utilization of delay time translation in the heat flux vector. Moreover, delay time translation of thermal displacement gradient helps to study the microscopic response.

(3) The various well-known theories such as thermoelasticity without energy dissipation, two temperature thermoelasticity without energy dissipation, phase lag thermoelasticity without energy dissipation, and dual phase lag thermoelasticity without energy dissipation have been recovered. This shows the proposed theory is the most generalized.

(4) In one dimensional half space, the initial boundary problem of dual phase lag heat conduction equation of two temperature GN-II model without energy dissipation in the framework of the Caputo fractional parameter $0 < \alpha \leq 1$ is addressed.

(5) The analytical solution for the conductive temperature, thermodynamic temperature, displacement, and thermal stresses has been obtained in the Laplace domain and subsequently inverted numerically in the time domain. For specific values of α, the results for various field quantities are computed numerically, satisfying all imposed conditions in this proposed model.

(6) The influence of changing the fractional parameter α on different field quantities, viz. conductive temperature $\Theta(x, t)$, thermodynamic temperature $\phi(x, t)$, displacement $u(x, t)$, and stress $\sigma(x, t)$ has been studied.

(7) According to the results, fractional parameter and phase lags can be used to classify different materials according to how well they transfer heat, and all the field quantities are continuous in nature.

(8) The numerical results show that all the field quantities vanish after a specific distance from the boundary in the given problem. There are no jumps or discontinuities, and sharp points in the curves of the investigated fields. This indicates that the proposed model admits thermal waves have a finite speed of propagation.

(9) The results and proposed model presented in this article can be used in different sectors like geomagnetics, optics, acoustics, new materials design, thermodynamics, geophysics, oil prospecting, etc.

7. NOMENCLATURE

u	Displacement	p	Laplace parameter
x	Cartesian coordinate	T_0	Reference temperature
α	Fractional order	θ	Thermodynamic temperature
ν	Thermal displacement	ϕ	Non-dimensional thermodynamic temperature
$\mathbf{q}$	Heat flux	c	Specific heat capacity
τ_q	Phase-lag of the heat flux	Θ	Non-dimensional conductive temperature
ε	Thermoelastic coupling parameter	$\tilde{a}$	Two temperature parameter
σ	Thermal stress	λ, μ	Lamé constants
κ^*	Material characteristic	∇	Gradient operator
κ	Thermal conductivity of the material	c_l	Speed of iso-thermal elastic wave
t	Time	ξ'	Dimensionless two temperature parameter
ρ_m	Material density	Δ	Laplacian operator
T	Conductive temperature	α_t	Coefficient of linear thermal expansion
e	Cubical dilatation	τ_ν	Phase-lag of the thermal displacement gradient

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