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SINGLE SERVER MARKOVIAN QUEUEING SYSTEM WITH WORKING VACATION, IMPATIENT CUSTOMERS AND DISASTERS

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Abstract. This paper examines the single server Markovian queueing system that incorporates working vacations, disasters, and impatient customers. The server initiates a Vacation when no customers are present in the system, during which it continues to serve arriving customers at a reduced rate. Disasters occurs while the main server is either busy or on a working vacation; leading to the removal of all customers from the system and causing the server to break down. When the main server is down, it is sent for repairs, and a substitute server provides service at a reduced rate until the main server is restored. When a customer arrives while the server is busy, undergoing repairs, or on Working Vacation, an impatience timer is activated. Additionally, arriving customers activate an impatience timer if the server is busy, under repair, or on a working vacation. If service is not completed before the timer expires, the customer abandons the system. The paper further presents a cost analysis and numerical illustrations to evaluate system performance.

1. Introduction

Queueing system with disasters and impatient customers are occur in our routine life. The WV period, when the system has no customers then the server takes a vacation in order to attend secondary jobs through this period with various rates.

Zhang and Shan [17] study the $M/M/1$ queue with disasters and impatient customers. Swathi and Kumar [14] describes for single and multiple vacation models during variant states like as busy, breakdown and repair periods is presented. Manoharan and Ashok [10] analyse an infinite capacity of single server queueing model with WV and customer's impatience. Gupta and Kumar [3] analyse an $M/M/1$ retrial model with a waiting server subject to repair and breakdown under WV. Sudhesh and Azhagappan [12] study an $M/M/1$ queueing model with server vacation and impatient customers. Sudesh et al. [13] study an single server vacation queueing system with a system disaster. Karthick and Suvitha [6] investigates the $M/M/3$ queueing system with multiple vacation using Bessel functions. Yohapriyadharsini and Suvitha [16] deals with $M/M/c$ queueing system with two kinds of working vacation and impatient customers using Probability Generating Functions. Ibe and Isijola [4] discussed the differentiated vacation queueing model with dependent service rate.

Parimala [11] study the $M/M(a,b)/(2,1)$ bulk service queueing system of two heterogeneous servers with different service rates. Bounkhel et al. [2] analyse flexible single-server queueing system. Kumar and Sharma [8] analyse the transient solution of a Markovian queuing system with two heterogeneous servers and retention of reneging customers. Jain et al. [5] examine the characteristics to $M/M/1$ queueing model possessing infinite capacity with functioning vacation. Kumar and Shinde [9] deals with bulk arrival and service under vacation and its interruption. Laxmi et al. [15] analyse an $M/M/1$ queue with infinite capacity, WV and customer's impatience during WV period. Baby Saroja and Suvitha

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[1] examines $M/M/1$ queueing model with three service types, where customers arrive according to a Poisson process and the service time of each customer follows a Hyper-Erlang distribution. Kumar and Arivudainambi [7] discussed a $M/M/1$ queue with catastrophes.

A comparison table of existing queueing model and our model are discussed below.

TABLE 1. Comparison with the existing queueing models

S. No	**Model**	**Author**	**Methodology**
1	$M/M/1$ queue with disasters and impatient customers	Zhang and Shan [17]	Probability Generating Function
2	Single and multiple vacation models during busy, breakdown and repair	Swathi and Kumar [14]	Recursive method
3	Infinite capacity of $M/M/1$ queueing model with WV and customer's impatience	Manoharan and Ashok [10]	Probability Generating Function
4	$M/M/1$ retrial model with a waiting server, repair and breakdown under WV	Gupta and Kumar [3]	Probability Generating Function
5	$M/M/1$ queue with server vacation and impatient customers	Sudhesh and Azhagappan [12]	Bessel functions and confluent hyper geometric functions
6	Feedback multiple vacation queueing system with differentiated vacations, and impatient customers.	Sudesh et al. [13]	Laplace transform and Recursive method
7	$M/M(a,b)/(2,1)$ bulk service queue of two servers with different service rates	Parimala [11]	Recursive method
8	Flexible single-server queueing system	Bounkhel et al. [2]	Probability Generating Function
9	Markovian queue with two heterogeneous servers and retention of reneging customers.	Kumar and Sharma [8]	Probability Generating Function and Bessel function
10	$M/M/1$ queueing model possessing infinite capacity with functioning vacation	Jain et al. [5]	Probability Generating Function
11	Bulk arrival and service under vacation and its interruption	Kumar and Shinde [9]	Probability Generating Function
12	$M/M/1$ queue with infinite capacity, WV and customer's impatience	Laxmi et al. [15]	Probability Generating Function
13	$M/Ph/1$ working vacation queue with breakdown and standby server	Baby Saroja and Suvitha. [1]	Matrix Geometric Method

Table 1 *Continued*

14	$M/M/1$ differentiated multiple vacation queue with vacation	Ibe and Isijola. [4]	Recursive Method
15	Time-dependent behaviour of an $M/M/3$ queueing system with multiple vacations and disaster	Karthick and Suvitha. [6]	Bessel function
16	$M/M/c$ queue with two kinds of WV and impatient customers	Yohapriyadharsini and Suvitha. [16]	Probability Generating Function
17	Transient solution of an $M/M/1$ queue with catastrophes	Kumar and Arivudainambi. [7]	Probability Generating Function
18	$M/M/1$ queue with Working Vacation, Impatient Customers and Disasters	Proposed Model	Probability Generating Function

Based on Table 1, the majority of the authors used the PGF approach. In engineering, the PGF approach serves as the formal foundation for uncertainty analysis. Risk assessment, reliability analysis, and cost analysis are frequently used techniques in civil engineering.

Consider a system like a mobile or laptop, which receives notifications(emails and messages) from somewhere consequently. The notifications form a queue and router providing the network connection. Here, the notifications are customers, and the router is a server. For some network issues, the notifications are not received by the system. We assumed that the network issue was a disaster here. If the network is a little slow, the notifications are received slowly, which means the server is in WV. When the router provides the network properly, then the notifications are correctly received, which implies that the server is in a busy state.

The structure of the paper is as follows: Section 2 defines the model and analysis. Section 3 presents the solutions of differential equations, Section 4 describes performance measures, Section 5 gives cost analysis. Section 6 provides numerical analysis and Section 7 presents conclusion.

2. The Model and Analysis

We consider the multiple WV in $M/M/1$ queueing system with customer's impatient due to the server's vacation and disasters. We assume the following:

- The arrival of customers according to a Poisson process with heterogeneous arrival rate $\lambda_i, (i = 1, 2, 3)$ where

$$\lambda_i = \begin{cases} \lambda_1, & \text{arrival rate during busy period;} \\ \lambda_2, & \text{arrival rate during WV;} \\ \lambda_3, & \text{arrival rate during repair.} \end{cases}$$

- The service times are exponentially distributed with rates μ_1, μ_2 and μ_3 which are described as for busy, WV and repair state respectively.
- If no customers are waiting in the system, then the server begins the working vacation and the server serves in a vacation period with slow service rates. After the WV completion, the server will continue the service in busy period. The WV time is exponentially distributed with rate γ.
- When a customer arrives and finds the server is busy or vacation, then the server activates a timer, which is exponentially distributed with parameter ϵ_0 for repair state ϵ_1 for busy state and ϵ_2 for WV.

- The system suffers disasters, occurring when the server is in busy or WV, at a Poisson rate η. When the system fails, all present customers are rejected and lost. Upon failure, a repair process starts immediately. The repair time is exponentially distributed with rate α for busy period.

Let $N(t)$ be the number of customers in the system at the time t and let

$$J(t) = \begin{cases} 1, \text{system is in WV} \\ 2, \text{system is in busy state} \\ 3, \text{system is in repair} \end{cases}$$

The process $\{N(t), J(t); t \geq 0\}$ is defined as a continuous-time Markov process with the state space, $\Omega = \{(n, j), j = 1, 2, 3; n \geq 0\}$.

The balance equations are given below:

$$(\lambda_1 + \eta)P_{0,1} = \alpha P_{0,3} \tag{2.1}$$

$$(\lambda_1 + \mu_1 + n\epsilon_1 + \eta)P_{n,1} = \lambda_1 P_{n-1,1} + \alpha P_{n,3} + (\mu_1 + (n+1)\epsilon_1)P_{n+1,1} + \gamma P_{n,2}, n \geq 1 \tag{2.2}$$

$$(\lambda_2 + \eta)P_{0,2} = (\mu_1 + \epsilon_1)P_{1,1} + (\mu_2 + \epsilon_2)P_{1,2} \tag{2.3}$$

$$(\lambda_2 + \mu_2 + n\epsilon_2 + \eta + \gamma)P_{n,2} = \lambda_2 P_{n-1,2} + (\mu_2 + (n+1)\epsilon_2)P_{n+1,2}, n \geq 1 \tag{2.4}$$

$$(\lambda_3 + \alpha)P_{0,3} = (\mu_3 + \epsilon_3)P_{1,3} + \eta(1 - \sum_{n=0}^{\infty} P_{n,3}) \tag{2.5}$$

$$(\lambda_3 + (\mu_3 + n\epsilon_3) + \alpha)P_{n,3} = \lambda_3 P_{n-1,3} + (\mu_3 + (n+1)\epsilon_3)P_{n+1,3}, n \geq 1 \tag{2.6}$$

The normalizing condition as follows,

$$\sum_{n=0}^{\infty} P_{n,0} + \sum_{n=1}^{\infty} P_{n,1} + \sum_{n=0}^{\infty} P_{n,2} = 1.$$

We define the Probability Generating Function as follows,
$G_1(z) = \sum_{n=0}^{\infty} P_{n,1}z^n, G_2(z) = \sum_{n=0}^{\infty} P_{n,2}z^n$ and $G_3(z) = \sum_{n=0}^{\infty} P_{n,3}z^n$
Multiplying equation (2.3) by z^n, using equation (2.2) and summing all possible values of n, we get

$$\begin{aligned} \epsilon_1 z(z-1)G_1'(z) + \Big[\lambda_1 z(1-z) + \mu_1(z-1) + \eta z\Big]G_1(z) = \mu_1 P_{0,1}(z-1) \\ + \alpha z G_3(z) + \gamma z G_2(z) - \gamma z P_{0,2} - (\mu_1 + \epsilon_1)zP_{1,1} \end{aligned} \tag{2.7}$$

In a similar way, we get from equations (2.4) to (2.6)

$$\begin{aligned} &\epsilon_2 z(z-1)G_2'(z) + \Big[\lambda_2 z(1-z) + \mu_2(z-1) + (\eta + \gamma)z\Big]G_2(z) \\ &= \mu_2 P_{0,2}(z-1) + \gamma z P_{0,2} + (\mu_1 + \epsilon_1)zP_{1,1} \end{aligned} \tag{2.8}$$

$$\epsilon_3 z(z-1)G_3'(z) + [\lambda_3 z(1-z) + \mu_3(z-1) + \alpha z]\, G_3(z) = \mu_3(z-1)P_{0,3} + \eta z(1 - \sum_{n=0}^{\infty} P_{n,3}) \tag{2.9}$$

3. The solutions of differential equations

For $z \neq 0$ and $z \neq 1$, equation (2.7) can be written as follows

$$G_1'(z) + \left[\frac{-\lambda_1}{\epsilon_1} + \frac{\mu_1}{z\epsilon_1} + \frac{\eta}{\epsilon_1(z-1)}\right] G_1(z) = \frac{1}{\epsilon_1(z-1)}[\alpha G_3(z) - \gamma P_{0,2} + \gamma G_2(z) - (\mu_1+\epsilon_1)P_{1,1}] + \frac{\mu_1 P_{0,1}}{\epsilon_1 z} \tag{3.1}$$

To solve the first order linear differential equation (2.7), we get an Integrating Factor (IF) as

$$e^{\frac{-\lambda_1 z}{\epsilon_1}} z^{\frac{\mu_1}{\epsilon_1}} (1-z)^{\frac{\eta}{\epsilon_1}}.$$

Multiplying by IF on both sides of equation (2.7), we get

$$\frac{d}{dz}\left[e^{\frac{-\lambda_1 z}{\epsilon_1}} z^{\frac{\mu_1}{\epsilon_1}} (1-z)^{\frac{\eta}{\epsilon_1}} G_1(z)\right] = \left[\frac{\alpha G_3(z) + \gamma G_2(z) - \gamma P_{0,2} - (\mu_1+\epsilon_1)P_{1,1}}{\epsilon_1(z-1)} + \frac{\mu_1 P_{0,1}}{\epsilon_1 z}\right] e^{\frac{-\lambda_1 z}{\epsilon_1}} z^{\frac{\mu_1}{\epsilon_1}} (1-z)^{\frac{\eta}{\epsilon_1}}$$

Integrating both sides of above from 0 to z, we get

$$G_1(z) = e^{\frac{\lambda_1 z}{\epsilon_1}} z^{\frac{-\mu_1}{\epsilon_1}} (1-z)^{\frac{-\eta}{\epsilon_1}} \left[-\frac{(\alpha G_3(z) + \gamma G_2(z) - \gamma P_{0,2} - (\mu_1+\epsilon_1)P_{1,1})}{\epsilon_1(1-z)} + \frac{\mu_1 P_{0,1}}{\epsilon_1 z}\right] e^{\frac{-\lambda_1 z}{\epsilon_1}} z^{\frac{\mu_1}{\epsilon_1}} (1-z)^{\frac{\eta}{\epsilon_1}} \tag{3.2}$$

From equation (2.8),

$$G_2'(z) + \left[\frac{-\lambda_2}{\epsilon_2} + \frac{\mu_2}{z\epsilon_2} + \frac{(\eta+\gamma)}{\epsilon_2(z-1)}\right] G_2(z) = \left[\frac{\mu_2 P_{0,2}}{\epsilon_2 z} + \frac{\gamma P_{0,2} + (\mu_1+\epsilon_1)P_{1,1}}{\epsilon_2(z-1)}\right] \tag{3.3}$$

To solve the first order linear differential equation (2.8), we obtain an IF as

$$e^{\frac{-\lambda_2 z}{\epsilon_2}} z^{\frac{\mu_2}{\epsilon_2}} (1-z)^{\frac{(\eta+\gamma)}{\epsilon_2}}.$$

Multiplying by IF on both sides of equation (2.8), we get

$$\frac{d}{dz}\left[e^{\frac{-\lambda_2 z}{\epsilon_2}} z^{\frac{\mu_2}{\epsilon_2}} (1-z)^{\frac{(\eta+\gamma)}{\epsilon_2}} G_2(z)\right] = \left[\frac{\mu_2 P_{0,2}}{\epsilon_2 z} - \frac{\gamma P_{0,2} + (\mu_1+\epsilon_1)P_{1,1}}{\epsilon_2(1-z)}\right] \times e^{\frac{-\lambda_2 z}{\epsilon_2}} z^{\frac{\mu_2}{\epsilon_2}} (1-z)^{\frac{(\eta+\gamma)}{\epsilon_2}}$$

Integrating both sides of above from 0 to z, we get

$$G_2(z) = e^{\frac{\lambda_2 z}{\epsilon_2}} z^{\frac{-\mu_2}{\epsilon_2}} (1-z)^{-\frac{(\eta+\gamma)}{\epsilon_2}} \int_0^z \left[\frac{P_{0,2}}{\epsilon_2}\left(\frac{\mu_2}{s} - \frac{\gamma}{(1-s)}\right) - \frac{(\mu_1+\epsilon_1)P_{1,1}}{\epsilon_2(1-s)}\right] \times e^{\frac{-\lambda_2 s}{\epsilon_2}} s^{\frac{\mu_2}{\epsilon_2}} (1-s)^{\frac{(\eta+\gamma)}{\epsilon_2}} ds$$

$$G_2(z) = e^{\frac{\lambda_2 z}{\epsilon_2}} z^{\frac{-\mu_2}{\epsilon_2}} (1-z)^{-\frac{(\eta+\gamma)}{\epsilon_2}} \frac{1}{\epsilon_2} [(\mu_2 K_1(z) - \gamma K_2(z))P_{0,2} - (\mu_1+\epsilon_1)P_{1,1}K_2(z)] \tag{3.4}$$

where

$$K_1(z) = \int_0^z e^{-\frac{\lambda_2 s}{\epsilon_2}} s^{\frac{\mu_2}{\epsilon_2}-1} (1-s)^{\frac{(\eta+\gamma)}{\epsilon_2}} ds \text{ and } K_2(z) = \int_0^z e^{-\frac{\lambda_2 s}{\epsilon_2}} s^{\frac{\mu_2}{\epsilon_2}} (1-s)^{\frac{(\eta+\gamma)}{\epsilon_2}-1} ds.$$

Substitute $z=1$ in equation (3.4), we get

$$P_{1,1}=\frac{[\mu_2K_1(1)-\gamma K_2(1)]P_{0,2}}{(\mu_1+\epsilon_1)K_2(1)}$$

From equation (2.9),

$$G_3'(z)+\left[\frac{-\lambda_3}{\epsilon_3}+\frac{\mu_3}{z\epsilon_3}+\frac{\alpha}{\epsilon_3(z-1)}\right]G_3(z)=\left[\frac{\mu_3P_{0,3}}{\epsilon_3z}+\frac{\eta(1-G_3(1))}{\epsilon_3(z-1)}\right] \tag{3.5}$$

To solve the first order linear differential equation (2.9), we get an IF as

$$e^{\frac{-\lambda_3z}{\epsilon_3}}z^{\frac{\mu_3}{\epsilon_3}}(1-z)^{\frac{\alpha}{\epsilon_3}}.$$

Multiplying by IF on both sides of equation (2.9), we get

$$\frac{d}{dz}\left[e^{\frac{-\lambda_3z}{\epsilon_3}}z^{\frac{\mu_3}{\epsilon_3}}(1-z)^{\frac{\alpha}{\epsilon_3}}G_3(z)\right]=\left[\frac{\mu_3P_{0,3}}{\epsilon_3z}+\frac{\eta(1-G_3(1))}{\epsilon_3(z-1)}\right]e^{\frac{-\lambda_3z}{\epsilon_3}}z^{\frac{\mu_3}{\epsilon_3}}(1-z)^{\frac{\alpha}{\epsilon_3}}$$

Integrating both sides of above from 0 to z, we get

$$G_3(z)=e^{\frac{\lambda_3z}{\epsilon_3}}z^{\frac{-\mu_3}{\epsilon_3}}(1-z)^{-\frac{\alpha}{\epsilon_3}}\int_0^z\left[\frac{\mu_3P_{0,3}}{\epsilon_3s}+\frac{\eta(1-G_3(1))}{\epsilon_3(s-1)}\right]e^{\frac{-\lambda_3s}{\epsilon_3}}s^{\frac{\mu_3}{\epsilon_3}}(1-s)^{\frac{\alpha}{\epsilon_3}}ds$$

Substitute $z=1$ in equation (2.9), we get

$$G_3(1)=\frac{\eta}{(\alpha+\eta)} \tag{3.6}$$

$$G_3(z)=e^{\frac{\lambda_3z}{\epsilon_3}}z^{\frac{-\mu_3}{\epsilon_3}}(1-z)^{\frac{-\alpha}{\epsilon_3}}\frac{1}{\epsilon_3}\left[\mu_3P_{0,3}K_3(z)-\frac{\alpha\eta}{\alpha+\eta}K_4(z)\right] \tag{3.7}$$

where

$$K_3(z)=\int_0^z e^{-\frac{\lambda_3s}{\epsilon_3}}s^{\frac{\mu_3}{\epsilon_3}-1}(1-s)^{\frac{\alpha}{\epsilon_3}}ds \text{ and } K_4(z)=\int_0^z e^{-\frac{\lambda_3s}{\epsilon_3}}s^{\frac{\mu_3}{\epsilon_3}}(1-s)^{\frac{\alpha}{\epsilon_3}-1}ds.$$

Substitute $z=1$ in equation (3.7), we get

$$P_{0,3}=\frac{\alpha\eta K_4(1)}{(\alpha+\eta)\mu_3K_3(1)} \tag{3.8}$$

Substitute the equations (3.4), (3.6), (3.7)and (3.8) in equation (3.2), we get

$$\begin{aligned}G_1(z)=e^{\frac{\lambda_1z}{\epsilon_1}}z^{\frac{-\mu_1}{\epsilon_1}}(1-z)^{\frac{-\eta}{\epsilon_1}}\frac{1}{\epsilon_1}\Bigg[&\frac{\mu_1\alpha P_{0,3}K_5(z)}{(\lambda_1+\eta)}-\frac{\alpha}{\epsilon_3}\Big(\mu_3P_{0,3}K_6(z)-\frac{\alpha\eta}{\alpha+\eta}K_7(z)\Big)\\&-\frac{\gamma}{\epsilon_2}\Big(K_8(z)P_{0,2}-(\mu_1+\epsilon_1)P_{1,1}K_9(z)\Big)+(\gamma P_{0,2}+(\mu_1+\epsilon_1)P_{1,1})K_{10}(z)\Bigg]\end{aligned} \tag{3.9}$$

where

$$K_5(z) = \int_0^z e^{-\frac{\lambda_1 s}{\epsilon_1}} (1-s)^{\frac{\eta}{\epsilon_1}} s^{\frac{\mu_1}{\epsilon_1}-1} ds,$$

$$K_6(z) = \int_0^z K_3(s) e^{\left(\frac{\lambda_3 s}{\epsilon_3} - \frac{\lambda_1 s}{\epsilon_1}\right)} (1-s)^{\frac{\eta}{\epsilon_1} - \frac{\alpha}{\epsilon_3} - 1} s^{\left(\frac{\mu_1}{\epsilon_1} - \frac{\mu_3}{\epsilon_3}\right)} ds,$$

$$K_7(z) = \int_0^z K_4(s) e^{\left(\frac{\lambda_3 s}{\epsilon_3} - \frac{\lambda_1 s}{\epsilon_1}\right)} (1-s)^{\frac{\eta}{\epsilon_1} - \frac{\alpha}{\epsilon_3} - 1} s^{\left(\frac{\mu_1}{\epsilon_1} - \frac{\mu_3}{\epsilon_3}\right)} ds,$$

$$K_8(z) = \int_0^z [\mu_2 K_1(s) - \gamma K_2(s)] e^{\left(\frac{\lambda_2 s}{\epsilon_2} - \frac{\lambda_1 s}{\epsilon_1}\right)} (1-s)^{\frac{\eta}{\epsilon_1} - \frac{(\eta+\gamma)}{\epsilon_2} - 1} s^{\left(\frac{\mu_1}{\epsilon_1} - \frac{\mu_2}{\epsilon_2}\right)} ds,$$

$$K_9(z) = \int_0^z K_2(s) e^{\left(\frac{\lambda_2 s}{\epsilon_2} - \frac{\lambda_1 s}{\epsilon_1}\right)} (1-s)^{\frac{\eta}{\epsilon_1} - \frac{(\eta+\gamma)}{\epsilon_2} - 1} s^{\left(\frac{\mu_1}{\epsilon_1} - \frac{\mu_2}{\epsilon_2}\right)} ds,$$

$$K_{10}(z) = \int_0^z e^{-\frac{\lambda_1 s}{\epsilon_1}} (1-s)^{\frac{\eta}{\epsilon_1}-1} s^{\frac{\mu_1}{\epsilon_1}} ds.$$

Substitute $z = 1$ in equation (2.8), we get

$$G_2(1) = \frac{\gamma P_{0,2} + (\mu_1 + \epsilon_1) P_{1,1}}{(\eta + \gamma)} \tag{3.10}$$

Substitute $z = 1$ in equation (2.7), we get

$$G_1(1) = \frac{\alpha}{\alpha + \eta} - \left(\frac{\gamma P_{0,2} + (\mu_1 + \epsilon_1) P_{1,1}}{\eta + \gamma}\right) \tag{3.11}$$

Substitute $z = 1$ in equation (3.9) and using (3.11) we get,

$$P_{0,2} = \left\{\frac{-\alpha^2 \eta a}{\alpha + \eta}\right\} \left\{(\mu_2 K_1(1) - \gamma K_2(1)) \left(\frac{\gamma}{\epsilon_2} K_9(1) + K_{10}(1)\right) + \gamma \left(K_{10}(1) - \frac{K_8(1)}{\epsilon_2}\right)\right\}^{-1}$$

where

$$a = \left[\frac{K_4(1)}{\mu_3 K_3(1)} \left(\frac{\mu_1 K_5(1)}{\lambda_1 + \eta} - \frac{\mu_3 K_6(1)}{\epsilon_3}\right) + \frac{K_7(1)}{\epsilon_3}\right].$$

Lemma 3.1. *The stability condition of our model is*

$$0 < (\alpha + \eta)(\lambda_1 + \eta) \mu_3 \epsilon_2 \epsilon_3 K_2(1) K_3(1) < 1.$$

Proof. For finding the stability, we use the condition $0 < P_{0,2} < 1$. Using the value of $P_{0,2}$ into the above condition and after some algebraic manipulations, we get $0 < (\alpha+\eta)(\lambda_1+\eta)\mu_3\epsilon_2\epsilon_3 K_2(1) K_3(1) < 1$. □

4. Performance Measures

By calculations, we obtain the following

- Mean size of the system when the server is on busy

$$
\begin{aligned}
E_1 &= \lim_{z\to 1} G_1'(z) \\
&= \frac{1}{-(\epsilon_1+\eta)}\Big\{(-\lambda_1+\mu_1)\Big(\frac{\alpha}{\alpha+\eta} - \frac{(\gamma P_{0,2}+(\mu_1+\epsilon_1)P_{1,1})}{\eta+\gamma}\Big) \\
&\quad \frac{\mu_1\alpha P_{0,3}}{\lambda_1+\eta} + \frac{\gamma}{-(\epsilon_2+(\eta+\gamma))}\Big[\frac{(-\lambda_2+\mu_2)(\gamma P_{0,2}+(\mu_1+\epsilon_1)P_{1,1})}{\eta+\gamma} - \mu_2 P_{0,2}\Big] \\
&\quad + \frac{\alpha}{-(\epsilon_3+\alpha)}\Big[\frac{(-\lambda_3+\mu_3)\eta}{\alpha+\gamma} - \mu_3 P_{0,3}\Big]\Big\}.
\end{aligned}
$$

- Mean size of the system when the server is on WV

$$E_2 = \lim_{z\to 1} G_2'(z) = \frac{1}{-(\epsilon_2+(\eta+\gamma))}\Big[\frac{(-\lambda_2+\mu_2)(\gamma P_{0,2}+(\mu_1+\epsilon_1)P_{1,1})}{\eta+\gamma} - \mu_2 P_{0,2}\Big].$$

- Mean size of the system when the server is on repair state

$$E_3 = \lim_{z\to 1} G_3'(z) = \frac{1}{-(\epsilon_3+\alpha)}\Big[\frac{(-\lambda_3+\mu_3)\eta}{\alpha+\gamma} - \mu_3 P_{0,3}\Big].$$

- $E = E_1 + E_2 + E_3$
- $T = \sum_{j=1}^{3}\sum_{n=0}^{\infty} \mu_j P_{n,j}$
- $MDT = \dfrac{E}{T}$

5. Cost Analysis (numerically)

In this subsection, we develop a model for the costs obtained in this queueing system. Let us consider the below notations.

- C_q - Cost per unit time whenever a customer joins the queue and waits for service;
- C_r - Cost per unit time whenever a customer reneges, either during busy, WV or repair;
- C_{s_1} - Service cost per unit time whenever the server is on busy;
- C_{s_2} - Service cost per unit time whenever the server is on WV;
- C_{s_3} - Service cost per unit time whenever the server is on repair;
- R_r - Average rate of reneging;
- $TC = C_qE + C_rR_r + \mu_1 C_{s_1} G_1(1) + \mu_2 C_{s_2} G_2(1) + \mu_3 C_{s_3} G_3(1)$ where $R_r = \epsilon_1 E_1 + \epsilon_2 E_2 + \epsilon_2 E_3$.

The performance of the proposed model is analyzed numerically using MATLAB. For this analysis, the parameter values are fixed as follows:

$$C_q = 8, C_{s_1} = 5, C_{s_2} = 4, C_{s_3} = 3, C_r = 6.$$

and other parameters as

$$
\begin{aligned}
&\lambda = 1, \lambda_1 = 6\lambda, \lambda_2 = 3\lambda, \lambda_3 = 5.5\lambda, \epsilon = 1, \epsilon_1 = 0.8\epsilon, \epsilon_2 = 0.7\epsilon, \epsilon_3 = 0.5\epsilon, \\
&\mu = 1, \mu_1 = 8\mu, \mu_2 = 5\mu, \mu_3 = 3\mu, \eta = 0.4, \alpha = 0.8, \gamma = 0.9.
\end{aligned}
$$

From Figures 1 and 2, the impact of λ on E with the variation of μ and ϵ. We observe that if λ increases, E increases for lowering the values of μ and ϵ. From Figures 3 and 4, the impact of μ on E with the variation of λ and ϵ. We noticed that if μ increases, E decreases for lowering the values of λ and increasing the value of ϵ.

When the arrival rate increases, the mean system size increase, which means that if more customers arrive, the queue grows as the system needs to manage more demand. When the service rate increases, the mean system size lowers, which means that customers are processed more quickly. This reduces the customers waiting time.

Figure 5 shows TC on various values of μ_1 and ϵ_1. It is observed that, TC decreases when ϵ_1 and μ_1 increases. Figure 6 shows TC on various values of μ_2 and ϵ_2. It is observed that, TC decreases when ϵ_2 and μ_2 increases. Figure 7 shows TC on various values of μ_3 and ϵ_3. It is observed that, TC decreases when ϵ_3 and μ_3 increases.

In Table 2, the parameters varies as λ, μ, ϵ varies, T and MDT are obtained. When the values of λ, μ and ϵ are increase then the T increase. Also, MDT increases for increasing λ value but decrease for μ and ϵ. From Table 3, the effect of λ, η and α on probabilities and TC are presented. We achieve the optimal cost values for varying μ from 0.22 to 0.30 at the different values of λ are given in Table 4.

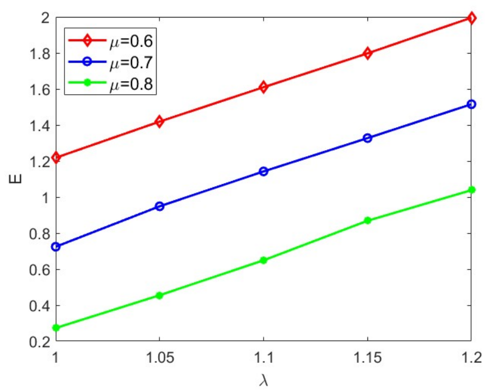

FIGURE 1. λ Vs E by varying μ

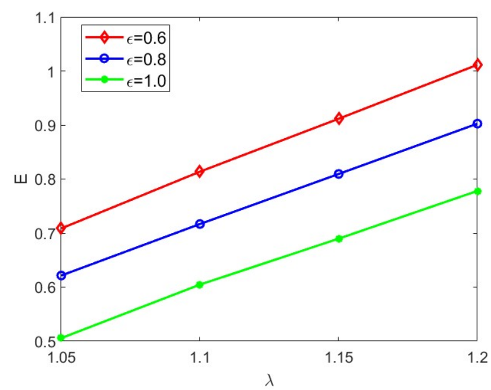

FIGURE 2. λ Vs E by varying ϵ

6. CONCLUSION

In this paper, we analyzed a single-server queueing system incorporating disasters, WV, and impatient customers. Using the probability generating function (PGF) technique, we derived key performance measures and conducted a cost analysis to assess system efficiency. Numerical results obtained through MATLAB illustrate the impact of system parameters on performance. The results highlight the practical applicability of the model in diverse real-world scenarios, such as service centers, industrial systems, telecommunications, and transportation networks. These insights can help optimize queueing systems where service interruptions and customer impatience are significant factors. These insights contribute to the effective design and management of queueing systems, particularly in environments where service interruptions and customer impatience play a crucial role.

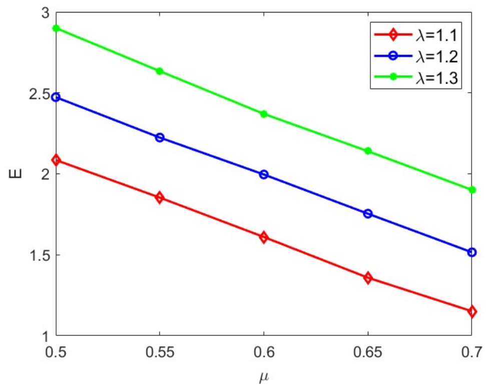

FIGURE 3. μ Vs E by varying λ

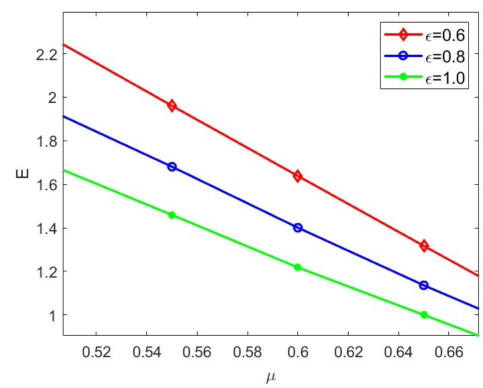

FIGURE 4. μ Vs E by varying ϵ

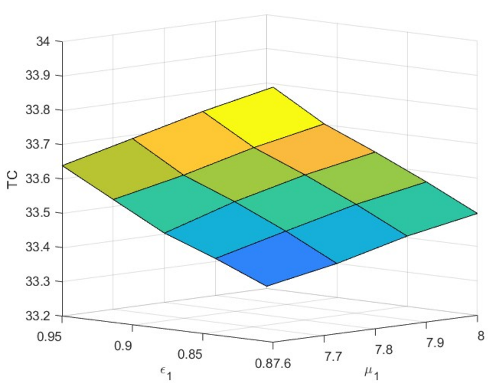

FIGURE 5. Effect of ϵ_1 and μ_1 on TC

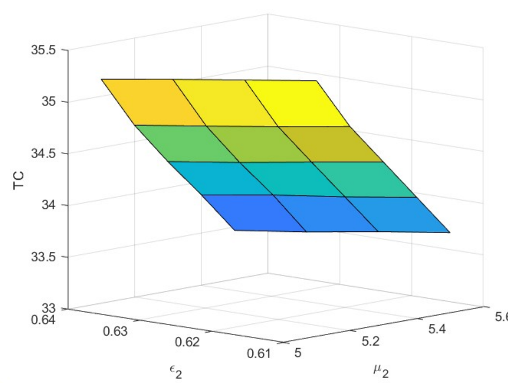

FIGURE 6. Effect of ϵ_2 and μ_2 on TC

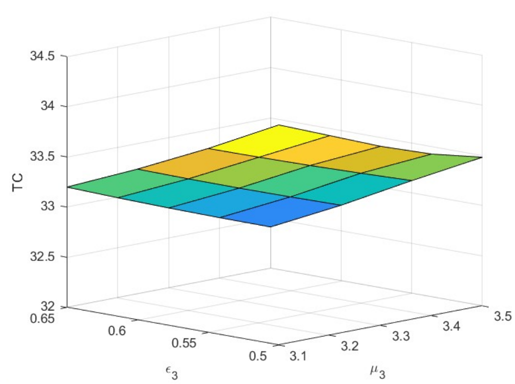

FIGURE 7. Effect of ϵ_3 and μ_3 on TC

λ	1.2	1.3	1.4	1.5	1.6
T	6.3328	6.3330	6.3332	6.3333	6.3334
MDT	0.0183	0.0761	0.1346	0.1937	0.2534
μ	0.1	0.2	0.3	0.4	0.5
T	0.6389	1.2667	1.9000	3.1463	3.8460
MDT	5.8044	2.4986	1.4040	0.5395	0.3167
ϵ	1.3	1.4	1.5	1.6	1.7
T	5.1807	5.1818	5.1840	5.2809	5.4638
MDT	0.0404	0.0397	0.0382	0.0287	0.0130

TABLE 2. Effect of λ, μ, ϵ on T and MDT

		$P_{0,2}$	$P_{0,1}$	$P_{0,3}$	TC
λ	0.5	0.0002	0.0384	0.1241	8.5794
	0.6	0.0008	0.0280	0.1338	14.1392
	0.7	0.0013	0.0209	0.1510	20.0180
η	0.30	0.1745	0.0054	0.05096	34.7054
	0.31	0.3575	0.0056	0.0521	31.3453
	0.32	0.4571	0.0057	0.0533	28.8736
α	0.81	0.0017	0.0066	0.0622	31.8077
	0.82	0.0069	0.0067	0.0622	32.2288
	0.83	0.0280	0.0068	0.0622	32.3411

TABLE 3. Effect of λ, η and α on probabilities and TC

λ \ μ	1.06	1.07	1.08	1.09	1.10
1.10	36.6676	36.2933	**35.9662**	36.5766	37.0508
1.11	37.2977	36.9275	**36.1251**	37.3826	37.8931
1.12	37.9830	37.4126	**37.2226**	37.8314	38.4072
1.13	38.7381	38.2298	**37.7780**	38.5234	39.2104
1.14	39.2404	38.8921	**38.5747**	39.2144	39.7924

TABLE 4. Effect of μ on cost function

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