Mathematics in Applied Sciences and Engineering
Volume 6, Number 2, June 2025, pp.125-137
https://ojs.lib.uwo.ca/mase
https://doi.org/10.5206/mase/21259

DIVISOR EQUITABLY STRONG NON-SPLIT DIVISOR EQUITABLE DOMINATION IN GRAPHS

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Abstract. Ensuring an equitable distribution of vertices between the two subsets and preventing any one subset from growing overly large is the key aspect of the split equitable domination idea. This approach can be applied practically in areas where fairness and equal coverage are essential, such as urban planning, network design, and resource distribution. We present the idea of non-split divisor equitable domination in graphs as a way to optimize medical networks. Let $\mathscr{Q}$ be a graph with vertex set $\mathscr{R}(\mathscr{Q})$ and edge set $\mathscr{E}(\mathscr{Q})$. Two vertices h, t are known as degree divisor equitable if $gcd(d_{\mathscr{Q}}(h), d_{\mathscr{Q}}(t)) = 1$. $\mathbb{F} \subset \mathscr{R}(\mathscr{Q})$ is known as divisor equitable dominating set of $\mathscr{Q}$ if $\forall\ h \in \mathscr{R} \backslash \mathbb{F}$, there exists a $t \in \mathbb{F}$ such that h and t are adjacent and degree divisor equitable. The divisor equitable domination number of a graph $\gamma_{de}(\mathscr{Q})$ of $\mathscr{Q}$ is the minimum cardinality of a divisor equitable dominating set of $\mathscr{Q}$. In this paper, we introduce the concept of a non-split divisor equitable dominating set, divisor equitably strong non-split divisor equitable dominating set, and divisor equitable independent set and divisor equitable clique number. We also explore the concepts of a divisor equitable vertex dominating set, complement divisor equitable graph, and divisor equitable vertex cut.

1. Introduction

Real-world challenges often spark innovative ideas about domination. Nodes with nearly identical capacities can communicate more effectively within a network. People with comparable social statuses are typically friendlier. In an industry, workers with comparable authority often form associations and collaborate closely. Equality in position, power, wealth, and other attributes is desirable in any group of people and is a key aspect of the constitution of a democratic country. Analyzing this concept requires the construction of a graph model. Building on this idea, Venkatasubramanian swaminathan and Kuppusamy Markandan [18] developed the concept of degree equitable domination and proposed various graph equitabilities, such as degree equitability, inward equitability, outward equitability, equitability in terms of the number of equal degree neighbors, or in terms of the number of strong degree neighbors, etc. On the other hand, certain social statuses, like marriage, must be evaluated on an individual basis. To ensure uniqueness, financial information and password security must also be independent. These issues have inspired the idea of independent domination, which was formalized by Berge [3] and Ore [12], A set is independent if no two vertices in it are adjacent. An independent dominating set $\mathbb{F}$ is a set that is both dominating and independent in $\mathscr{Q}$. The independent domination number of $\mathscr{Q}$, indicated by $i(\mathscr{Q})$ is the minimum size of an independent dominating set were studied by Cockayne and Hedetniemi [6, 7].

Received by the editors 1 October 2024; accepted 4 February 2025; published online 22 February 2025.
2020 *Mathematics Subject Classification.* 05C38, 05C40, 05C69.
Key words and phrases. Equitable domination, equitable non-split domination, divisor equitable domination, divisor equitable non-split domination.

V. R. Kulli and B. Janakiram [10] found strong non - split domination in a graph. A dominating set $\mathbb{F}$ of a graph $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$ is a nonsplit dominating set if the induced subgraph $\langle \mathscr{R} \backslash \mathbb{F} \rangle$ is connected. The nonsplit domination number $\gamma^{ns}(\mathscr{Q})$ of $\mathscr{Q}$ is the minimum cardinality of a nonsplit dominating set.

Divisor equitable domination is a concept in graph theory that extends the idea of domination by incorporating the concept of divisibility. The inclusion of this divisibility constraint ensures that the distribution of influence or resources across the graph is balanced, modular, and structured in a mathematically controlled way. Many authors studied divisor equitable domination, divisor 2- equitable domination and divisor equitable edge domination in fuzzy graphs [9, 14, 15]. Following [9, 14, 15], we define a divisor equitable independent set and a divisor equitable clique number, and introduce the notions of a non-split divisor equitable dominating set and a divisor equitably strong non-split divisor equitable dominating set. We also explore the concepts of a divisor equitable vertex dominating set and relation between divisor equitable independent set and divisor equitable complement of $\mathscr{Q}$.

2. Preliminaries

Definition 2.1. [16] A graph $\mathscr{Q}$ is a finite, non-empty set of objects known as vertices, along with a set of unordered pairs of distinct vertices in $\mathscr{Q}$, referred to as edges. The vertex set and edge set of $\mathscr{Q}$ are denoted by $\mathscr{R}(\mathscr{Q})$ and $\mathscr{E}(\mathscr{Q})$ respectively. A graph $\mathscr{Q}$ with vertex set and edge set is represented as $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$.

Definition 2.2. [4] The degree $d_{\mathscr{Q}}(s)$ of a vertex s in $\mathscr{Q}$ is the no. of edges in $\mathscr{Q}$ incident with s, each loop counting as two edges. The notation $\delta(\mathscr{Q})$ and $\Delta(\mathscr{Q})$ denotes minimum and maximum degrees of vertices respectively.

Definition 2.3. [13] Let $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$ be a graph and $l \in \mathscr{R}$. The open neighborhood $\mathfrak{N}(l)$ and the closed neighborhood $\mathfrak{N}[l]$ of $\mathscr{R}$ are defined by $\mathfrak{N}(l) = \{t \in \mathscr{R} : tl \in \mathscr{E}\}$ and $\mathfrak{N}[l] = \mathfrak{N}(l) \cup \{l\}$. The elements within $\mathfrak{N}(l)$ are referred to as neighbors of l. If $\mathbb{B} \subseteq \mathscr{R}$, then

$$\mathfrak{N}(\mathbb{B}) = \bigcup_{l \in \mathbb{B}} \mathfrak{N}(l) \text{ and } \mathfrak{N}[\mathbb{B}] = \mathfrak{N}(\mathbb{B}) \cup \mathbb{B}.$$

Through out this paper, we use dominating set as D-set, equitable dominating set as ED-set, equitable independent set as EI-set, divisor equitable as dE, divisor equitable dominating set as dED-set, nonsplit divisor equitable dominating set as nsdED-set, strong nonsplit divisor equitable dominating set as snsdED-set, strong nonsplit equitable dominating set as snsED-set, divisor equitable independent set as dEI-set.

Definition 2.4. [11] Let $\mathscr{Q}$ be a graph with vertex set $\mathscr{R}(\mathscr{Q})$ and edge set $\mathscr{E}(\mathscr{Q})$. Two vertices h, t are considered degree equitable if $|d_{\mathscr{Q}}(h) - d_{\mathscr{Q}}(t)| \leq 1$. A subset $\mathbb{F}$ of $\mathscr{R}(\mathscr{Q})$ is known as ED-set of $\mathscr{Q}$ if for every $h \in \mathscr{R} \backslash \mathbb{F}$, $\exists$ a $t \in \mathbb{F} \ni h$ and t are adjacent and degree equitable. The ED number $\gamma_e(\mathscr{Q})$ of a graph $\mathscr{Q}$ is the minimum cardinality of an ED-set of $\mathscr{Q}$.

Definition 2.5. [2] If a vertex $h \in \mathscr{R} \ni |d_{\mathscr{Q}}(h) - d_{\mathscr{Q}}(r)| \geq 2 \; \forall \; r \in \mathfrak{N}(h)$, then h is in every ED-set and the points are referred to as equitable isolates. The set of all equitable isolates indicated as I_e.

Definition 2.6. [18] Let $h \in \mathscr{R}$. The equitable neighbourhood of h is indicated by $\mathfrak{N}_e(h)$ is described as $\mathfrak{N}_e(h) = \{w \in \mathscr{R} : w \in \mathfrak{N}(h), |d_{\mathscr{Q}}(h) - d_{\mathscr{Q}}(w)| \leq 1\}$ and $h \in I_e \iff \mathfrak{N}_e(h) = \emptyset$. The cardinality of $\mathfrak{N}_e(h)$ is indicated by $d^e_{\mathscr{Q}}(h)$.

The minimum and maximum equitable degree of a point in $\mathscr{Q}$ are indicated respectively by $\delta_e(\mathscr{Q})$ and $\Delta_e(\mathscr{Q})$. That is

$$\delta_e(\mathscr{Q}) = \min_{h \in \mathscr{R}(\mathscr{Q})} |\mathfrak{N}_e(h)|, \Delta_e(\mathscr{Q}) = \max_{h \in \mathscr{R}(\mathscr{Q})} |\mathfrak{N}_e(h)|.$$

Definition 2.7. [18] A subset $\mathbb{F}$ of $\mathscr{R}$ is known as equitable independent set(EI-set), if for any $h \in \mathbb{F}, t \notin \mathfrak{N}^e_{\mathscr{Q}}(h)$ for all $t \in \mathbb{F}\backslash\{h\}$. The maximum cardinality of a EI-set is indicated by β_e. The minimum cardinality of a EI-set is indicated by i_e.

Definition 2.8. [10] A D-set $\mathbb{F}$ of a graph $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$ is a nonsplit dominating set(D-set) if the induced subgraph $\langle \mathscr{R}\backslash\mathbb{F}\rangle$ is connected. The nonsplit domination number $\gamma^{ns}(\mathscr{Q})$ of $\mathscr{Q}$ is the minimum cardinality of a nonsplit D-set.

Definition 2.9. [11] Let $\mathbb{F}$ be an ED-set of $\mathscr{Q}$. $\mathbb{F}$ is said to be equitably strong non-split ED-set if $\langle \mathscr{R}\backslash\mathbb{F}\rangle$ is equitably complete. (i.e, any two vertices of $\mathscr{Q}$ are adjacent and all the vertices of $\mathscr{R}\backslash\mathbb{F}$ are degree equitable in $\mathscr{Q}$). The minimum(maximum) cardinality of a minimal strong non-split ED-set of $\mathscr{Q}$ is known as strong non-split equitable domination number of $\mathscr{Q}$ (upper strong non-split ED number of $\mathscr{Q}$) and is referred by $\gamma_e^{sns}(\mathscr{Q})(\Gamma_e^{sns}(\mathscr{Q}))$.

Definition 2.10. [1] An equitable vertex cut (or an equitable separating set) of $\mathscr{Q}$ is a set $\mathbb{F} \subset \mathbb{R}(\mathscr{Q})$ such that $\mathscr{Q}\backslash\mathbb{F}$ is equitable disconnected. The connectivity, $k_e(\mathscr{Q})$, is the smallest number of vertices in any equitable vertex cut of $\mathscr{Q}$. A vertex whose removal increases the number of equitable components of $\mathscr{Q}$ is known as an equitable cut-vertex (or point of equitable articulation). The maximal equitable connected subgraph of $\mathscr{Q}$ that has no equitable cut-vertex is known as an equitable block of $\mathscr{Q}$.

Definition 2.11. [17] Let $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$ be a graph. The equitable complement graph of $\mathscr{Q}$ indicated by $\bar{\mathscr{Q}}^e$ is the graph with the same vertices as $\mathscr{Q}$ and any two vertices m, t are adjacent if m and t are not equitable adjacent in $\mathscr{Q}$.

Definition 2.12. [5] A clique in graph $\mathscr{Q}$ is a complete subgraph of $\mathscr{Q}$ (i.e., a subgraph in which any two vertices are adjacent). A clique of n vertices is denoted by $\mathbb{K}_n$. The clique number of $\mathscr{Q}$, indicated $\omega(\mathscr{Q})$, is the cardinality of the largest clique in $\mathscr{Q}$.

Definition 2.13. [8] A subset $\mathbb{F}$ of $\mathscr{R}$ is said to be equitable vertex covering of $\mathscr{Q}$ if for every equitable edge $e = wt$, either $w \in \mathbb{F}$ or $t \in \mathbb{F}$. The minimum cardinality of an equitable vertex cover of $\mathscr{Q}$ is known as equitable covering number of $\mathscr{Q}$ and is indicated by $\alpha_e(\mathscr{Q})$.

3. Main Results

Definition 3.1. Let $\mathscr{Q}$ be a graph with vertex $\mathscr{R}(\mathscr{Q})$ and edge set $\mathscr{E}(\mathscr{Q})$. Two vertices h, t are known to be degree dE if $gcd(d_{\mathscr{Q}}(h), d_{\mathscr{Q}}(t)) = 1$. $\mathbb{F} \subset \mathscr{R}(\mathscr{Q})$ is described as dED-set of $\mathscr{Q}$ if $\forall\ h \in \mathscr{R}\backslash\mathbb{F}$, $\exists$ a $t \in \mathbb{F} \ni h$ and t are adjacent and degree dE. The dED number of a graph $\gamma_{de}(\mathscr{Q})$ of $\mathscr{Q}$ is the minimum cardinality of a dED-set of $\mathscr{Q}$.

Definition 3.2. A dED-set $\mathbb{F}$ of a graph $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$ is a nonsplit divisor equitable dominating set(nsdED-set) if the induced subgraph $\langle \mathscr{R}\backslash\mathbb{F}\rangle$ is divisor equitably connected. The nonsplit divisor equitable domination number $\gamma_{de}^{ns}(\mathscr{Q})$ of $\mathscr{Q}$ is the minimum cardinality of a nsdED-set.

Example 3.1. *Let $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$ be a graph and defined as follows.*

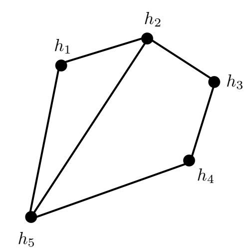

FIGURE 1. Example of nsdED-set.

Here $\mathbb{F} = \{h_4, h_5\}$ is nsdED-set and $\langle \mathscr{R} \backslash \mathbb{F} \rangle$ is divisor equitably connected. $gcd(d_{\mathscr{Q}}(h_1), d_{\mathscr{Q}}(h_2)) = gcd(1, 2) = 1$. *The nsdED number $\gamma_{de}^{ns}(\mathscr{Q})$ of $\mathscr{Q}$ is 2.*

Definition 3.3. Let $\mathbb{F}$ be a dED-set of $\mathscr{Q}$. $\mathbb{F}$ is called a divisor equitably strong non-split divisor equitable dominating set(snsdED-set) if $\langle \mathscr{R} \backslash \mathbb{F} \rangle$ is divisor equitably complete. (i.e., any two vertices of $\mathscr{Q}$ are adjacent and all the vertices of $\mathscr{R} \backslash \mathbb{F}$ are degree divisor equitable in $\mathscr{Q}$). The minimum(maximum) cardinality of a minimal snsdED-set of $\mathscr{Q}$ is known as snsdED number of $\mathscr{Q}$ (upper snsdED number of $\mathscr{Q}$) and is indicated by $\gamma_{de}^{sns}(\mathscr{Q})(\Gamma_{de}^{sns}(\mathscr{Q}))$.

Example 3.2. *Consider $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$ as graph and defined as follows.*

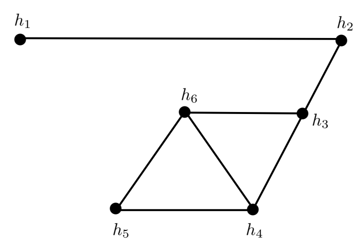

FIGURE 2. Example of snsED-set but not snsdED-set.

Here $\mathbb{F} = \{h_1, h_2, h_5\}$ is equitable and divisor equitable dominating set which is snsED-set but not snsdED-set, where $\langle \mathscr{R} \backslash \mathbb{F} \rangle$ is complete equitable but not complete divisor equitable.

Example 3.3. *Consider $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$ as graph and defined as follows.*

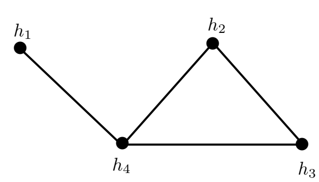

FIGURE 3. Example of both snsED-set and snsdED-set.

Here $\mathbb{F} = \{h_1, h_4\}$ is equitable and divisor equitable dominating set, which is both snsED-set, snsdED-set, where $\langle \mathscr{R} \backslash \mathbb{F} \rangle$ is both complete equitable and complete divisor equitable.

Note 1. $\gamma_{de}(\mathscr{Q}) \leq \gamma_{de}^{ns}(\mathscr{Q}) \leq \gamma_{de}^{sns}(\mathscr{Q})$.

Definition 3.4. If a vertex $f \in \mathscr{R} \ni gcd(d_{\mathscr{Q}}(f), d_{\mathscr{Q}}(w)) \geq 2 \ \forall \ w \in \mathfrak{N}(f)$, then f is in every dED-set and the elements are referred to as dE isolates. The set of all dE isolates indicated as I_{de}.

Theorem 3.4. *A snsdED-set $\mathbb{F}$ is minimal iff for every $h \in \mathbb{F}$ one of the following equality holds.*

(i) *h is a dE isolate $\langle \mathbb{F} \rangle$.*
(ii) *$\mathfrak{N}_{de}(t) \cap \mathbb{F} = \{h\}$ for some t in $\mathscr{R} \backslash \mathbb{F}$.*
(iii) *h is not divisor equitably adjacent with some vertex of $\mathscr{R} \backslash \mathbb{F}$.*

Proof. Obvious. □

Definition 3.5. A subset $\mathbb{F}$ of $\mathscr{R}$ is termed as dEI-set, if for any $h \in \mathbb{F}, w \notin \mathfrak{N}_{de}(h)$ for all $w \in \mathbb{F} \backslash \{h\}$. The maximum cardinality of a dEI-set is indicated by β_{de}. The minimum cardinality of a dEI-set is denoted by i_{de}.

Proposition 3.5. $\beta_{de}(\mathscr{Q}) \leq \gamma_{de}^{sns}(\mathscr{Q})$.

Proof. Let $\mathbb{F}$ be a $\gamma_{de}^{sns}(\mathscr{Q})$ - set of $\mathscr{Q}$ and $\mathscr{W}$ be a dEI-set of $\mathscr{Q}$. Then $|\mathscr{W} \cap (\mathscr{R} \backslash \mathbb{F})| \leq 1$. There is, either $\mathscr{W}$ is a subset of $\mathbb{F}$ or $\mathscr{W}$ contains at most one vertex from $\mathscr{R} \backslash \mathbb{F}$ and the remaining elements are from $\mathbb{F}$. If $\mathscr{W}$ contains a vertex t from $\mathscr{R} \backslash \mathbb{F}$, then $\mathscr{W}$ cannot include vertices in $\mathbb{F}$ that are divisor equitably adjacent to t. As at least one vertex of $\mathbb{F}$ is divisor equitably adjacent with $t, \mathscr{W}$ can contain at most $|\mathbb{F}| - 1$ vertices from $\mathbb{F}$. □

Remark 3.1. When $\mathscr{Q} = \mathbb{K}_2, \beta_{de}(\mathscr{Q}) = \gamma_{de}^{sns}(\mathscr{Q}) = 1$.

Definition 3.6. A subset $\mathbb{Y}$ of $\mathscr{R}(\mathscr{Q})$ is defined as dE clique if $\langle \mathbb{Y} \rangle$ is complete and all vertices of $\mathbb{Y}$ are degree dE in $\mathscr{Q}$. The maximum cardinality of a dE clique of $\mathscr{Q}$ is known as dE clique number of $\mathscr{Q}$ and is represented by $\omega_{de}(\mathscr{Q})$.

Example 3.6. *Let $\mathscr{Q}$ be $\mathbb{K}_2$ by attaching two pendent vertices at exactly one vertex of $\mathbb{K}_2$. Then $\omega(\mathscr{Q}) = 2$ and $\omega_{de}(\mathscr{Q}) = 2$.*

Remark 3.2. $\omega_{de}(\mathscr{Q}) \leq \omega(\mathscr{Q})$.

Theorem 3.7. $n - \omega_{de}(\mathscr{Q}) \leq \gamma_{de}^{sns}(\mathscr{Q}) \leq n - \omega_{de}(\mathscr{Q}) + 1$.

Proof. Consider $\mathbb{F}$ as $\gamma_{de}^{sns}(\mathscr{Q})$- set of $\mathscr{Q}$. Then $\langle \mathscr{R} \backslash \mathbb{F} \rangle$ is a divisor complete subgraph of $\mathscr{Q}$. Thus, $|\langle \mathscr{R} \backslash \mathbb{F} \rangle| \leq \omega_{de}(\mathscr{Q})$. That is, $n - \gamma_{de}^{sns}(\mathscr{Q}) \leq \omega_{de}(\mathscr{Q})$ which implies $n - \omega_{de}(\mathscr{Q}) \leq \gamma_{de}^{sns}(\mathscr{Q})$.

Consider $\mathbb{Y}$ as $\omega_{de}(\mathscr{Q})$ subset of $\mathscr{Q}$. Then for any $h \in \mathbb{Y}, \mathbb{Y} \backslash \{h\}$ is a dE clique of $\mathscr{Q}$. As h is divisor equitably adjacent with every element of $\mathbb{Y} \backslash \{h\}, (\mathscr{R} \backslash \mathbb{Y}) \cup \{h\}$ is a strong D-set whose complement is dE complete. Hence, $\gamma_{de}^{sns}(\mathscr{Q}) \leq |(\mathscr{R} \backslash \mathbb{Y}) \cup \{h\}| = n - \omega_{de}(\mathscr{Q}) + 1$. □

Remark 3.3. Consider $\mathscr{Q}_1$ be the graph from $\mathbb{K}_3$ by connecting one pendent vertex at exactly two of the vertices of $\mathbb{K}_3$. Then $\omega_{de}(\mathscr{Q}_1) = 2$ and $\gamma_{de}^{sns}(\mathscr{Q}_1) = 3$. $|\mathscr{R}(\mathscr{Q}_1)| - \omega_{de}(\mathscr{Q}_1) = 5 - 2 = 3 = \gamma_{de}^{sns}(\mathscr{Q}_1)$. Hence $n - \omega_{de}(\mathscr{Q}_1) = \gamma_{de}^{sns}(\mathscr{Q}_1)$.

Consider $\mathscr{Q}_2$ be the graph from $\mathbb{K}_4$ by connecting a pendent vertex at exactly one vertex of $\mathbb{K}_4$. Then $\omega_{de}(\mathscr{Q}_2) = 2$ and $\gamma_{de}^{sns}(\mathscr{Q}_2) = 3$. $|\mathscr{R}(\mathscr{Q}_2)| - \omega_{de}(\mathscr{Q}_2) = 5 - 2 = 3 = \gamma_{de}^{sns}(\mathscr{Q}_2)$. Hence $n - \omega_{de}(\mathscr{Q}_2) = \gamma_{de}^{sns}(\mathscr{Q}_2)$.

Consider $\mathscr{Q}_3$ be the graph from $\mathbb{K}_3$ by connecting one pendent vertex at exactly one vertex of $\mathbb{K}_3$. Then $\omega_{de}(\mathscr{Q}_3) = 2$ and $\gamma_{de}^{sns}(\mathscr{Q}_3) = 2$. $|\mathscr{R}(\mathscr{Q}_3)| - \omega_{de}(\mathscr{Q}_3) = 4 - 2 = 2 = \gamma_{de}^{sns}(\mathscr{Q}_3)$. Hence $n - \omega_{de}(\mathscr{Q}_3) = \gamma_{de}^{sns}(\mathscr{Q}_3)$.

Definition 3.7. Let $t \in \mathscr{J}$. The divisor equitable neighbourhood of t is indicated by $\mathfrak{N}_{de}(t)$ is described as $\mathfrak{N}_{de}(t) = \{w \in \mathscr{J} / w \in \mathfrak{N}(t), gcd(d_{\mathscr{Q}}(t), d_{\mathscr{Q}}(w)) = 1\}$ and $t \in \mathfrak{I}_{de} \iff \mathfrak{N}_{de}(t) = \emptyset$. The cardinality of $\mathfrak{N}_e(h)$ is termed as equitable degree of h and it is indicated as $d^{de}_{\mathscr{Q}}$. The minimum and maximum equitable degree of a point in $\mathscr{Q}$ are indicated respectively by $\delta_{de}(\mathscr{Q})$ and $\Delta_{de}(\mathscr{Q})$. That is

$$\delta_{de}(\mathscr{Q}) = \min_{t\in \mathscr{J}(\mathscr{Q})} |\mathfrak{N}_{de}(t)| \text{ and } \Delta_{de}(\mathscr{Q}) = \max_{t\in \mathscr{J}(\mathscr{Q})} |\mathfrak{N}_{de}(t)|.$$

Theorem 3.8. *Let $\mathscr{Q}$ be a graph with $\omega_{de}(\mathscr{Q}) \geq \delta_{de}(\mathscr{Q})$. Then, $\gamma^{sns}_{de}(\mathscr{Q}) \leq n - \delta_{de}(\mathscr{Q})$.*

Proof. Assume $\omega_{de}(\mathscr{Q}) \geq \delta_{de}(\mathscr{Q}) + 1$. Then $-\omega_{de}(\mathscr{Q}) \leq -\delta_{de}(\mathscr{Q}) - 1$. From the Theorem 3.7,

$$\begin{aligned}\gamma^{sns}_{de}(\mathscr{Q}) &\leq n - \omega_{de}(\mathscr{Q}) + 1\\ &\leq n - \delta_{de}(\mathscr{Q}).\end{aligned}$$

Assume $\omega_{de}(\mathscr{Q}) = \delta_{de}(\mathscr{Q})$, and $\mathbb{Y}$ be a $\omega_{de}(\mathscr{Q})$- set of $\mathscr{Q}$. Then for $h \in \mathbb{Y}$, we have $d^{de}_{\mathscr{Q}}(h) \geq \delta_{de}(\mathscr{Q})$. Since, $|\mathbb{Y}\backslash\{h\}| < \delta_{de}(\mathscr{Q})$, h is divisor equitably adjacent with at least one vertex of $\mathscr{J}\backslash\mathbb{Y}$. Thus, $\mathscr{J}\backslash\mathbb{Y}$ is divisor equitably snsdED-set of $\mathscr{Q}$. Hence,

$$\gamma^{sns}_{de}(\mathscr{Q}) \leq |\mathscr{J}\backslash\mathbb{Y}| = n - |\mathbb{Y}| = n - \delta_{de}(\mathscr{Q}).$$

□

Corollary 3.9. *The above two theorems are attained iff the below conditions is satisfied: $\omega_{de}(\mathscr{Q}) = \delta_{de}(\mathscr{Q})$.*

Proof. Assume $\omega_{de}(\mathscr{Q}) = \delta_{de}(\mathscr{Q})$. By Theorem 3.7 $n - \omega_{de}(\mathscr{Q}) \leq \gamma^{sns}_{de}(\mathscr{Q})$, we have $n - \delta_{de}(\mathscr{Q}) \leq \gamma^{sns}_{de}(\mathscr{Q})$. From the Theorem 3.8, $\gamma^{sns}_{de}(\mathscr{Q}) \leq n - \delta_{de}(\mathscr{Q})$. Hence, $\gamma^{sns}_{de}(\mathscr{Q}) = n - \delta_{de}(\mathscr{Q})$.

Conversely, assume $\gamma^{sns}_{de}(\mathscr{Q}) = n-\delta_{de}(\mathscr{Q})$. Then $n-\delta_{de}(\mathscr{Q}) \geq n- \omega_{de}(\mathscr{Q})$. Hence, $\omega_{de}(\mathscr{Q}) \geq \delta_{de}(\mathscr{Q})$. While, $\gamma^{sns}_{de}(\mathscr{Q}) \leq n-\omega_{de}(\mathscr{Q})+1$, $n-\delta_{de}(\mathscr{Q}) = \gamma^{sns}_{de}(\mathscr{Q}) \leq n-\omega_{de}(\mathscr{Q})+1$. There is, $\omega_{de}(\mathscr{Q}) \leq \delta_{de}(\mathscr{Q})+1$. Thus, $\omega_{de}(\mathscr{Q}) = \delta_{de}(\mathscr{Q})$. □

Corollary 3.10. *The above two theorems are attained iff the below conditions are satisfied.*

$\omega_{de}(\mathscr{Q}) = \delta_{de}(\mathscr{Q}) + 1$ and every $\omega_{de}(\mathscr{Q})$- set $\mathbb{Y}$ contains a vertex which is not divisor adjacent with any vertex of $\mathscr{J}\backslash\mathbb{Y}$.

Proof. Suppose, $\gamma^{sns}_{de}(\mathscr{Q}) = n-\delta_{de}(\mathscr{Q})$. From the Theorem 3.7, $n-\omega_{de}(\mathscr{Q}) \leq \gamma^{sns}_{de}(\mathscr{Q}) \leq n-\omega_{de}(\mathscr{Q})+1$. Hence, $\omega_{de}(\mathscr{Q}) = \delta_{de}(\mathscr{Q})$ or $\delta_{de}(\mathscr{Q}) + 1$. Assume there is a $\omega_{de}(\mathscr{Q})$ - set $\mathbb{Y}$ with $|\mathbb{Y}| = \delta_{de}(\mathscr{Q}) + 1 \ni$ all vertex in $\mathbb{Y}$ is dE adjacent with some vertex in $\mathscr{J}\backslash\mathbb{Y}$. Then, $\mathscr{J}\backslash\mathbb{Y}$ is a divisor equitably snsdED-set of $\mathscr{Q}$. Thus, $\gamma^{sns}_{de}(\mathscr{Q}) \leq| \mathscr{Q}\backslash\mathbb{Y} |= n - (\delta_{de}(\mathscr{Q}) + 1)$, a contradiction to our assumption $\gamma^{sns}_{de}(\mathscr{Q}) = n - \delta_{de}(\mathscr{Q})$. Hence, every $\omega_{de}(\mathscr{Q})$ - set $\mathbb{Y}$ with $|\mathbb{Y}| = \delta_{de}(\mathscr{Q})+1$ is such that $\mathbb{Y}$ contains a vertex not divisor equitably adjacent to any vertex of $\mathscr{J}\backslash\mathbb{Y}$.

Conversely, consider, $\omega_{de}(\mathscr{Q}) = \delta_{de}(\mathscr{Q})+1$ and every $\omega_{de}(\mathscr{Q})$-set $\mathbb{Y}$ with $|\mathbb{Y}| = \delta_{de}(\mathscr{Q})+1 \ni \mathbb{Y}$ contains a vertex not divisor equitably adjacent to any vertex of $\mathscr{J}\backslash\mathbb{Y}$. So, $\mathscr{J}\backslash\mathbb{Y}$ is not a divisor equitably snsdED-set of $\mathscr{Q}$. Thus, $\gamma^{sns}_{de}(\mathscr{Q}) > |\mathscr{J}\backslash\mathbb{Y}| = n - (\delta_{de}(\mathscr{Q}) + 1)$. That is, $\gamma^{sns}_{de}(\mathscr{Q}) > n - \omega_{de}(\mathscr{Q})$. But, $\gamma^{sns}_{de}(\mathscr{Q}) \leq n - \omega_{de}(\mathscr{Q}) + 1$. Therefore, $\gamma^{sns}_{de}(\mathscr{Q}) = n - \omega_{de}(\mathscr{Q}) + 1$. □

Definition 3.8. A subset $\mathbb{F}$ of $\mathscr{R}(\mathscr{Q})$ is defined as dE vertex D-set of $\mathscr{Q}$, if for any $\mathbb{Y} \subset \mathscr{Q}\backslash\mathbb{F}$, $\exists$ a vertex $u \in \mathbb{F} \ni \mathbb{Y} \cup \{u\}$ is divisor equitably connected. The minimum cardinality of a dE vertex D-set of $\mathscr{Q}$ is called as dE vertex domination number of $\mathscr{Q}$ and is denoted by $\gamma^{vs}_{de}(\mathscr{Q})$.

Example 3.11. *Consider a graph $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$ and defined as follows.*

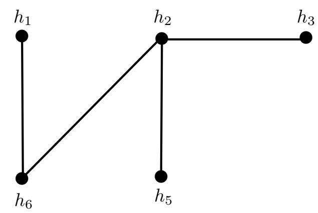

FIGURE 4. Example of dE vertex dominating set.

Here $\mathbb{F} = \{h_1, h_2\}$ is dE vertex dominating set. Then for any subset $\mathbb{Y} = \{h_3, h_5, h_6\}$, where $\mathbb{Y} \cup \{h_2\} = \{h_6, h_2, h_5, h_3\}$ is divisor equitably connected. $\gamma_{de}^{vs}(\mathscr{Q}) = 2$.

Theorem 3.12. *In any graph $\mathscr{Q}$, $\gamma_{de}^{vs}(\mathscr{Q}) \leq \gamma_{de}^{sns}(\mathscr{Q})$.*

Proof. Let $\mathbb{F}$ be a $\gamma_{de}^{sns}(\mathscr{Q})$-set of $\mathscr{Q}$. Then, for any $\mathbb{Y} \subset \mathscr{R} \backslash \mathbb{F}$, $\langle \mathbb{Y} \rangle$ is dE complete and for any $h \in \mathbb{Y}$, $\exists$ a vertex $t \in \mathbb{F}$ which is divisor equitably adjacent with h. Hence, $\mathbb{Y} \cup \{t\}$ is divisor equitably connected. So, $\mathbb{F}$ is a dE vertex dominating set of $\mathscr{Q}$ and thus $\gamma_{de}^{vs}(\mathscr{Q}) \leq |\mathbb{F}| = \gamma_{de}^{sns}(\mathscr{Q})$. □

Example 3.13.
Let $\mathscr{Q}$ be the graph obtained from $\mathbb{K}_3$ by adding a pendant vertex to one of the vertices of $\mathbb{K}_3$ and connecting the pendant vertex to one of the other two vertices of $\mathbb{K}_3$. Then $\gamma_{de}^{sns}(\mathscr{Q})$- sets of $\mathscr{Q}$ and $\gamma_{de}^{sns}(\mathscr{Q}) = 2$. The $\gamma_{de}^{vs}(\mathscr{Q})$- sets of $\mathscr{Q}$ and $\gamma_{de}^{vs}(\mathscr{Q}) = 1$.
Thus, $\gamma_{de}^{vs}(\mathscr{Q}) = 1 < 2 = \gamma_{de}^{sns}(\mathscr{Q})$

Corollary 3.14. *For any equtiable connected tree $\mathscr{T}$, $n - \delta_{de}(\mathscr{T}) \leq \gamma_{de}^{sns}(\mathscr{Q})$.*

Proof. As $\gamma_{de}^{vs}(\mathscr{Q}) \leq \gamma_{de}^{sns}(\mathscr{Q})$ and for any dE connected tree, $\gamma_{de}^{sns}(\mathscr{Q}) = n - \delta_{de}(\mathscr{T})$, the result follows. □

Definition 3.9. The graph diameter of a graph is the length $max_{h,b} d(h, b)$ of the longest shortest path between any two graph vertices (h, b), where $d(h, b)$ is a graph distance.

Theorem 3.15. *If $\mathscr{Q}$ has no dE isolate and $\mathscr{Q}$ has a snsdE, dEI, dED-set, then $diam_{de}(\mathscr{Q}) \leq 3$.*

Proof. Consider $\mathscr{Q}$ as graph without dE isolate and $\mathbb{F}$ be a minimum snsdE, dEI, dED-set.
Case 1: If $t, w \in \mathscr{R} \backslash \mathbb{F}$. Then $d_{de}(t, w) = 1$.
Case 2: If $t \in \mathbb{F}, w \in \mathscr{R} \backslash \mathbb{F}$. Since $\mathscr{Q}$ has no dE isolates and $\mathbb{F}$ is dEI, there is $r \in \mathscr{R} \backslash \mathbb{F} \ni t$ and r are divisor equitably adjacent. Hence, $d_{de}(t, w) \leq d_{de}(t, r) + d_{de}(r, w) = 2$.
Case 3: If $t, w \in \mathbb{F}$. Then there exist h_1, h_2 in $\mathscr{R} \backslash \mathbb{F}$, such that t and h_1 are divisor equitably adjacent and w, h_2 are divisor equitably adjacent. Thus, $d_{de}(t, w) \leq d_{de}(t, h_1) + d_{de}(h_1, h_2) + d_{de}(h_2, w) = 3$. Hence $diam_{de}(\mathscr{Q}) \leq 3$. □

Theorem 3.16. *In $\mathscr{Q}$ has no dE isolate and if $\gamma_{de}^{sns}(\mathscr{Q}) = \gamma_{de}(\mathscr{Q})$, then $diam_{de}(\mathscr{Q}) \leq 3$.*

Proof. Consider $\mathbb{F}$ as a $\gamma_{de}^{sns}(\mathscr{Q})$ - set of $\mathscr{Q}$. As $\mathbb{F}$ is also a dED-set of $\mathscr{Q}$ of cardinality $\gamma_{de}(\mathscr{Q})$ and $\mathscr{Q}$ has no dE isolate, $\mathscr{R} \backslash \mathbb{F}$ is a dED-set of $\mathscr{Q}$ as $\mathscr{R} \backslash \mathbb{F}$ dE complete set of $\mathscr{Q}$. Hence all vertex of $\mathbb{F}$ is dE adjacent with some vertex of $\mathscr{R} \backslash \mathbb{F}$. Proceeding as in Theorem 3.15, we get $diam_{de}(\mathscr{Q}) \leq 3$. □

Definition 3.10. For any graph $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$, the complement divisor equitable graph of $\mathscr{Q}$ indicated by $\overline{\mathscr{Q}_{de}}$, has the same vertices as $\mathscr{Q}$ and two vertices m, t are adjacent if m and t are not dE adjacent in $\mathscr{Q}$.

Theorem 3.17. *Let $\mathbb{F}$ be a divisor equitably independent set in $\mathscr{Q}$ and $|\mathbb{F}| < n - \Delta_{de}(\mathscr{Q})$. Then, $\langle \mathscr{R} \backslash \mathbb{F} \rangle$ is a D-set of $\overline{\mathscr{Q}_{de}}$ is divisor equitable complement of $\mathscr{Q}$.*

Proof. Consider $\mathbb{F}$ as a divisor equitably independent set in $\mathscr{Q}$ and $|\mathbb{F}| < n - \Delta_{de}(\mathscr{Q})$. Then $|\mathscr{R} \backslash \mathbb{F}| > \Delta_{de}(\mathscr{Q})$. So, all vertex of $\mathbb{F}$ is not divisor equitably adjacent with at least one vertex in $\mathscr{R} \backslash \mathbb{F}$. Thus, $\mathscr{R} \backslash \mathbb{F}$ is a D-set of $\overline{\mathscr{Q}_{de}}$. □

Definition 3.11. A graph $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$ is termed as dE connected if there is at least one dE path between every pair of vertices in $\mathscr{Q}$. Otherwise, $\mathscr{Q}$ is dE disconnected.

The dE disconnected graph clearly is made up of two or more dE connected graphs. Each of these dE connected subgraphs is known as dE component. It is evident that a dE connected graph is connected, but in general, the opposite is not true.

Definition 3.12. A dED-set $\mathbb{F}$ of a divisor equitably connected graph $\mathscr{Q}$ is known as strong split divisor equitably dominating set(ssdED-set) of $\mathscr{Q}$ if $\langle \mathscr{R} \backslash \mathbb{F} \rangle$ is totally divisor equitably disconnected. The minimum cardinality of such a set is known as ssdED number of $\mathscr{Q}$ and is denoted by $\gamma_{de}^{ss}(\mathscr{Q})$.

Example 3.18. *Consider $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$ as graph, shown below.*

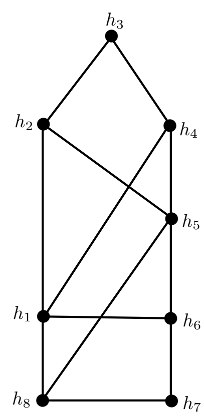

FIGURE 5. Example of ssdED-set but not ssED-set.

Here $\mathbb{F} = \{h_2, h_4, h_7\}$ is equitable and divisor equitable dominating set which is ssdED-set but not ssED-set, $\langle \mathscr{R} \backslash \mathbb{F} \rangle$ is totally divisor equitably disconnected but not equitably disconnected.

Example 3.19. *Consider $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$ as graph, shown below.*

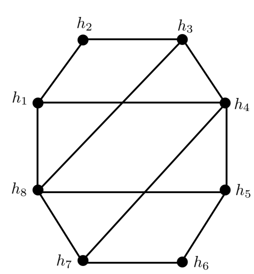

FIGURE 6. Example of ssED-set but not ssdED-set.

Here $\mathbb{F} = \{h_2, h_5, h_6, h_8\}$ is equitable and divisor equitable dominating set which is ssED-set but not ssdED-set, where $\langle \mathscr{R} \backslash \mathbb{F} \rangle$ is totally equitably disconnected but not divisor equitably disconnected.

Example 3.20. *Let $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$ be a graph as shown below.*

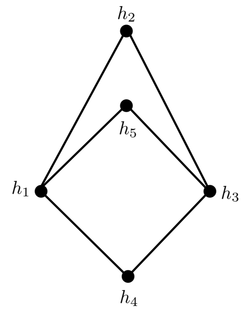

FIGURE 7. Example of both ssED-set and ssdED-set.

Here $\mathbb{F} = \{h_1, h_3\}$ is equitable and divisor equitable dominating set which is ssED-set and ssdED-set, where $\langle \mathscr{R} \backslash \mathbb{F} \rangle$ is totally equitably disconnected and divisor equitably disconnected.

Definition 3.13. Consider a dE connected graph $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$. A dE vertex cut (or a dE separating set) of $\mathscr{Q}$ is a set $\mathbb{F} \subset \mathscr{R}(\mathscr{Q})$ such that $\mathscr{Q} \backslash \mathbb{F}$ is dE disconnected. The connectivity, $k_{de}(\mathscr{Q})$, is defined as the smallest number of vertices in any dE vertex cut of $\mathscr{Q}$. A vertex that, when removed, increases the number of dE components of $\mathscr{Q}$ is called a dE cut-vertex. The maximal dE connected subgraph of $\mathscr{Q}$ that has no dE cut-vertex is referred to as dE block of $\mathscr{Q}$.

Theorem 3.21. *Let $\mathscr{Q}$ be a graph without dE isolate. If $diam_{de}(\mathscr{Q}) \leq 3$, then $\gamma_{de}^{sns}(\mathscr{Q}) \leq n - h$, where h is the number of dE cut vertices of $\mathscr{Q}$.*

Proof. Consider H as a set of all dE cut vertices. Then $|H| = h$. If $h = 0$ or 1, then the result is obviously true.

Let $h \geq 2$ and $w, t \in H$. If w and t are not adjacent dED-set, Then there exists a divisor component, indicated as $\mathscr{R}_1$, containing t in $\mathscr{R}$ excluding $\{w\}$. As w and t are dE non- isolates, there exists a vertex w_1 divisor equitably adjacent with w in a differnt divisor component, indicated as $\mathscr{R}_2$ distinct from $\mathscr{R}_1$. Additionally, $\exists$ a vertex t_1 in $\mathscr{R}_1$ divisor equitable adjacent with t. So, $d_{de}(w_1, t_1) \geq 4$, a contradiction, since $diam_{de}(\mathscr{Q}) \leq 3$. Hence, any two vertices of H are divisor equitably adjacent. Also, all vertex in H is divisor equitably adjacent with atleast one vertex in $\mathscr{R} \backslash H$. Thus, $\mathscr{R} \backslash H$ is a divisor equitably snsdED-set of $\mathscr{Q}$. That is, $\gamma_{de}^{sns}(\mathscr{Q}) \leq |\mathscr{R} \backslash H| = n - h$. □

Theorem 3.22. *Let $\mathscr{Q}$ be a graph without equitable isolate and every vertex of $\mathscr{Q}$ is either a dE cut vertex or a vertex of dE degree one with $\omega_{de}(\mathscr{Q}) = k$. Then $\gamma_{de}^{ns}(\mathscr{Q}) = \gamma_{de}^{sns}(\mathscr{Q}) = n - k$.*

Proof. Consider H as set of all dE cut vertices with $|H| = k$. From Theorem 3.21 we get, H is dE complete. Additionally every vertex in H is dE adjacent with a vertex of dE degree one. Hence, $\mathscr{R} \backslash H$ is a nsdED-set as well as dE snsdED-set of $\mathscr{Q}$ and $|\mathscr{R} \backslash H| = n - k$. If $\gamma_{de}^{sns}(\mathscr{Q}) < n - k$, then the complement of a $\gamma_{de}^{sns}(\mathscr{Q})$- set will contain least $k + 1$ elements which form an induced dE complete subgraph, a contradiction while $\omega_{de}(\mathscr{Q}) = k$. Thus, $\gamma_{de}^{ns}(\mathscr{Q}) = n - k$. Additionally, if $\gamma_{de}^{ns}(\mathscr{Q}) < n - k$, then the complement of a $\gamma_{de}^{sns}(\mathscr{Q})$- set say $\mathbb{F}$ will contain least $k+1$ elements in the complement and so atleast one vertex of dE degree one which can be dominated only by a dE cut vertex. Hence, $\mathbb{F}$ contains a dE cut vertex which implies that $\langle \mathscr{Q} \backslash \mathbb{F} \rangle$ is divisor equitably disconnected, a contradiction while $\mathbb{F}$ is a $\gamma_{de}^{ns}(\mathscr{Q})$- set of $\mathscr{Q}$. Thus $\gamma_{de}^{ns}(\mathscr{Q}) = n - k$. □

Theorem 3.23. *Let $\mathscr{Q}$ be a graph without dE isolates and with $\Delta_{de}(\mathscr{Q}) \leq n-2$ and $\mathbb{F}$ be a snsdED-set of $\mathscr{Q}$ and $|\mathbb{F}| \leq \delta_{de}(\mathscr{Q})$. Then $\mathbb{F}$ is minimal and $\mathscr{R} \backslash \mathbb{F}$ is also a minimal divisor equitably snsdED-set of $\mathscr{Q}$.*

Proof. Assume, that there is a vertex in $\mathbb{F}$ which is divisor equitably adjacent with all vertex of $\mathscr{R}\backslash\mathbb{F}$. There is, $\Delta_{de}(\mathscr{Q}) = n-1$, a contradiction. So, all vertex in $\mathbb{F}$ is not adjacent with some vertex of $\mathscr{R}\backslash\mathbb{F}$. Thus, $\mathbb{F}$ is minimal.

If $|\mathbb{F}| \leq \delta_{de}(\mathscr{Q})$, then each vertex in $\mathbb{F}$ is divisor equitably adjacent with some vertex of $\mathscr{R}\backslash\mathbb{F}$. Hence, $\mathscr{R}\backslash\mathbb{F}$ is a divisor equitably snsdED-set of $\mathscr{Q}$.

Additionally, based on the above argument, not all vertex in $\mathscr{R}\backslash\mathbb{F}$ is devisor equitably adjacent to a vertex in $\mathbb{F}$. Thus, $\mathscr{R}\backslash\mathbb{F}$ is also a minimal divisor equitably snsdED-set of $\mathscr{Q}$. □

Definition 3.14. A subset $\mathbb{F}$ of $\mathscr{R}$ is defined as dE vertex cover of $\mathscr{Q}$ if for every dE edge $e = wt$, either $w \in \mathbb{F}$ or $t \in \mathbb{F}$. The minimum cardinality of a dE vertex cover of $\mathscr{Q}$ is known the dE covering number of $\mathscr{Q}$ and is indicated by $\alpha_{de}(\mathscr{Q})$.

Theorem 3.24. *For a graph $\mathscr{Q} = (\mathscr{R}, \mathscr{E})$, the following are equivalent*

(i) *$B \subseteq \mathscr{R}$ is a dEI-set in $\mathscr{Q}$.*
(ii) *$S = \mathscr{R}\backslash B$ is dE vertex cover for $\mathscr{Q}$.*

Proof. $(i) \implies (ii)$ Suppose S is not a dE vertex cover. Then, there is an edge wt with $wt \notin S$. Hence, $w, t \in B$. This is in contradiction with B being a dEI-set.

$(ii) \implies (i)$ Suppose B is not a dEI-set. Then, there is an edge wt with $w, t \in B$. Hence, $w, t \notin S$. This is in contradiction with S being a dE vertex cover. □

Theorem 3.25. *Let $\mathscr{Q}$ be a graph without dE isolate with $\Delta_{de}(\mathscr{Q}) < \alpha_{de}(\mathscr{Q})$. Then $\gamma_{de}^{sns}(\overline{\mathscr{Q}_{de}}) = n - \omega_{de}(\overline{\mathscr{Q}_{de}})$.*

Proof. Consider B as minimum dE vertex cover of $\mathscr{Q}$ so that $|B| = \alpha_{de}(\mathscr{Q})$. If $\mathscr{Q} = \mathbb{K}_n$, then $\beta_{de}(\mathscr{Q}) = 1$, $\alpha_{de}(\mathscr{Q}) = n-1 = \Delta_{de}(\mathscr{Q})$. By the hypothesis, $\Delta_{de}(\mathscr{Q}) < \alpha_{de}(\mathscr{Q})$. Hence, $\mathscr{Q} \neq \mathbb{K}_n$. $\mathscr{R}\backslash B$ is dEI. Since $|\mathscr{R}\backslash B| = \beta_{de}(\mathscr{Q})$ and $\mathscr{Q} \neq \mathbb{K}_n$, $|\mathscr{R}\backslash B| \geq 2$. If a vertex say $w \in \mathscr{R}\backslash B$ is divisor equitably adjacent with every vertex of B, then $d_{\mathscr{Q}}(w) = |B| = \alpha_{de}(\mathscr{Q}) > \Delta_{de}(\mathscr{Q})$, a contradiction. So, all vertex of $\mathscr{R}\backslash B$ is not adjacent with atleast one vertex of B. Thus, B is a divisor equitably snsdED-set of $(\overline{\mathscr{Q}_{de}})$. Hence, $\gamma_{de}^{sns}(\overline{\mathscr{Q}_{de}}) \leq |B| = \alpha_{de}(\mathscr{Q}) = n - \beta_{de}(\mathscr{Q}) \leq n - \omega_{de}(\overline{\mathscr{Q}_{de}})$. But, $\gamma_{de}^{sns}(\overline{\mathscr{Q}_{de}}) \geq n - \omega_{de}(\overline{\mathscr{Q}_{de}})$. Therefore, $\gamma_{de}^{sns}(\overline{\mathscr{Q}_{de}}) = n - \omega(\overline{\mathscr{Q}_{de}})$. □

4. Applications

The concept of non-split divisor equitable domination has the potential to optimize various systems that rely on balancing loads, resources, or responsibilities. From telecommunications to healthcare, supply chains, and energy distribution, this concept can help ensure that resources are distributed equitably, minimizing system inefficiencies and promoting fairness. However, practical implementation requires sophisticated systems for data management, monitoring, and continual adjustment, which can be challenging in dynamic environments.

4.1. **Application of non - split divisor equitable domination in medical network.** In a telemedicine network, healthcare providers in urban centers serve as hubs for remote or underserved regions. Major hospitals and specialized medical centers in cities have telemedicine infrastructure, forming the dominant set in the network. Small clinics in rural areas and individual patients in remote locations make up the non-dominant set, requiring support and connectivity to urban hubs.

Illustration

Consider a three major urban hospitals (H_1, H_2, H_3) with telemedicine capabilities. There are 8 smaller clinics $(C_1, C_2, C_3, \ldots, C_8)$ in rural areas that require regular consultations and specialist services from urban hospitals.

Divisor equitable dominating set:

Equitable distribution ensures each major urban hospital supports an equal number of smaller clinics.

H_1 supports(dominates) C_1, C_2, C_3;

H_2 supports(dominates) C_4, C_5;

H_3 supports (dominates) C_6, C_7, C_8.

This balanced distribution prevents overloading any single hospital with excessive patients or specialist requests, ensuring fair access to telemedicine services.

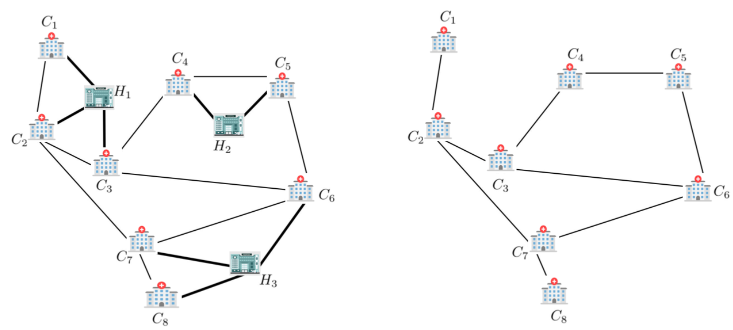

FIGURE 8. Example of non-split divisor equitable dominating set in telemedicine networks.

Here non-split divisor equitable dominating set is $\{H_1, H_2, H_3\}$. The non-split divisor equitable domination ensures that rural clinics $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8$ remain connected, even after removing the non-split divisor equitable dominating set. In Figure 8, if an urban hospitals H_1, H_2, H_3 becomes overwhelmed, the connected rural clinics can collaborate or seek support from nearby clinics linked to other hospitals ensuring no clinic is isolated or unsupported. The Greatest Common Divisor (GCD) is used to determine optimal subnetwork sizes that evenly divide the total network.

This method provides distribution creates a balanced telemedicine network where each urban hospital has a manageable number of rural clinics; rural clinics have dedicated access to specialist services; patient load is distributed to prevent overwhelming any single hospital; specialist resources are allocated efficiently. This structure ensures continuous healthcare delivery even during system stress or partial failure.

4.2. **Application of strong non - split divisor equitable domination in medical network.** Consider a scenario with six hospitals $h_1, h_2, h_3, h_4, h_5, h_6$, where four h_1, h_2, h_3, h_4 are situated in city and two h_5, h_6 are located in the rural areas. Each rural hospital is connected to the city hospitals, creating a network of healthcare facilities. By implementing a divisor equitable condition, this setup ensures a balanced distribution of responsibilities among healthcare providers.

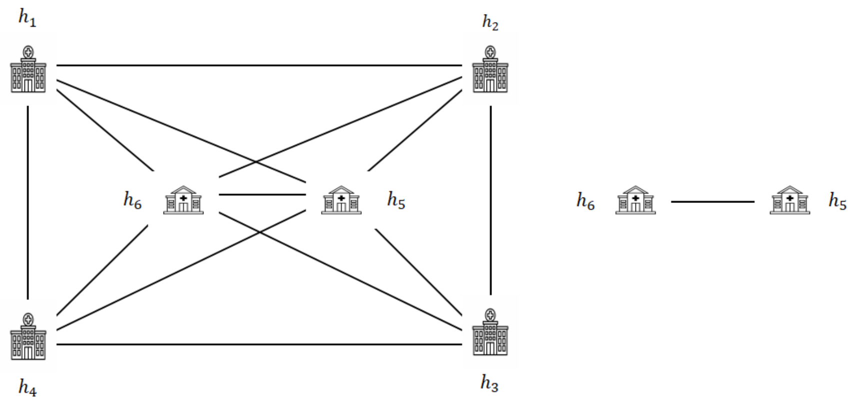

FIGURE 9. Example of strong non-split divisor equitable dominating set in healthcare.

In this scenario, the set of hospitals $\mathbb{F} = \{h_1, h_2, h_3, h_4\}$ constitutes a strong non-split divisor equitable dominating set. When these hospitals are removed from the network, the remaining facilities h_5 and h_6 remain connected and fully operational. The reason for their removal is that these four hospitals are experiencing a shortage of beds and healthcare resources, leading to an overload of patients. Once these hospitals are removed, patients can be redirected to the city hospitals, where they can be treated in a manner that maintains divisor equitable conditions.

5. Conclusion

Divisor equitable domination is a refined and specialized extension of domination theory that introduces an additional layer of fairness and divisibility to the allocation or distribution process in networks or systems. While it's primarily a theoretical tool in graph theory, its application could significantly improve load balancing, resource distribution, and fairness in real-world networks such as telecommunications, supply chains, healthcare, and task optimization systems. The concept ensures that not only is the load distributed equitably but also in a way that respects divisibility, adding an extra layer of control and structure. In this paper, non-split divisor equitable dominating set, a divisor equitably strong non-split divisor equitable dominating set were discussed and defined a divisor equitable independent set and a divisor equitable clique number. Additionally, we developed the concepts of a divisor equitable vertex dominating set, a complement divisor equitable graph, and a divisor equitable vertex cut.

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