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GLOBAL DYNAMICS OF A ONE-PREDATOR-TWO-PREY MODEL AND ITS TRAVELING WAVE SOLUTIONS

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Abstract. This work investigates the three species of one-predator-two-prey ecological models in Lotka-Volterra type functional response with or without diffusive terms. Without the diffusive effects and under two essential assumptions, we can generically classify all global dynamics completely. The global asymptotic stabilities of three equilibria are shown analytically in each case. Alternatively, with the diffusive term, we establish the existence of traveling wave solutions by the higher-dimensional shooting method, the Wazewski principle. In particular, there are two critical wave speeds $0 < c_2 < c_1$. We show the existence of traveling wave solutions with the wave speed c if $c > c_1$ and the non-existence of traveling wave solutions if $0 < c < c_2$. Finally, a brief discussion, biological interpretations, and numerical simulations are given.

1. Introduction

In this work, we investigate the dynamics of the following diffusive Lotka-Volterra type one-predator-two-prey model,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = r_1 u_1(1-u_1) - \alpha_1 u_1 v, \\ \dfrac{\partial u_2}{\partial t} = r_2 u_2(1-u_2) - \alpha_2 u_2 v, \\ \dfrac{\partial v}{\partial t} = D\Delta v - \mu v + \beta_1 u_1 v + \beta_2 u_2 v, \end{cases} \tag{1.1}$$

and the corresponding reaction equation,

$$\begin{cases} \dfrac{du_1}{dt} = r_1 u_1(1-u_1) - \alpha_1 u_1 v, \\ \dfrac{du_2}{dt} = r_2 u_2(1-u_2) - \alpha_2 u_2 v, \\ \dfrac{dv}{dt} = -\mu v + \beta_1 u_1 v + \beta_2 u_2 v, \end{cases} \tag{1.2}$$

where u_1 and u_2 are two renewable basal resources in the logistic-type growth with birthrates r_1 and r_2, respectively, and normalized environmental carry capacity. The third species, v, is the predator feeding by species u_1 and u_2, where the coefficients of nonlinear interaction α_1, α_2 are the rates of consumption, and β_1, β_2 measure the contribution of the victim (u_1 and u_2) to the growth of the species v.

For past decades, it has been very successful in investigating a class of conventional predator-prey ecosystems with two species for various functional responses, including extinction results, local and global stability of positive equilibrium, existence, uniqueness, and stability of periodic solutions

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[23, 13, 2] because of the help of Poincaré-Bendixson Theorem. Recently, researchers have paid attention to three species of food web models, including two-predators-one-prey models, food chain models, one-predator-two-prey models, and cyclic models [21] with different functional responses. There are multiple difficulties in investigating three species of ecological models. First, the classical two-dimensional Poincaré-Bendixson Theorem can not be applied. Secondly, the interaction of species in the ecological systems is of a predator-prey type, which is not monotone [30]. So the theory of monotone system [30] cannot be applied. Finally, very complex behavior (chaos) may happen [9, 16].

Early works in Lotka-Volterra type one-predator-two-prey models [3, 6, 26, 27, 14] with competition between prey,

$$\begin{cases} \dfrac{du_1}{dt} = r_1u_1(1-u_1) - \alpha_1 u_1 v - a_{12}u_1u_2, \\ \dfrac{du_2}{dt} = r_2u_2(1-u_2) - \alpha_2 u_2 v - a_{21}u_1u_2, \\ \dfrac{dv}{dt} = -\mu v + \beta_1 u_1 v + \beta_2 u_2 v, \end{cases} \tag{1.3}$$

appear in an ecological sense. The models were first investigated numerically by Gilpin [9], and he found some very complex dynamics for particular parameters. Then, more complete numerical studies are performed in [29, 33]. Klebanoff and Hastings generalize previous numerical results to the model with Holling type II functional response [20]. Moreover, they also show chaos dynamics that happen through bifurcation analysis. Recently, some interesting articles reexamine this model in [5, 20, 29, 31, 35] with different functional responses between the predator and prey, and a complete local analysis of boundary and positive equilibria is analyzed in [8, 25]. Bifurcation numerical results are also simulated in [34].

Comparing systems (1.2) and (1.3), there are the direct competition, a_{12}/a_{21}, and the indirect competition, the apparent competition [11, 12] between two prey for (1.3). We drop out the competitive effect between two prey, $a_{12} = a_{21} = 0$, to see the fundamental issue of the apparent competition. Hence, system (1.2) has two trophic levels, and we focus on the indirect competition between species in the same trophic level. Therefore, we assume there is no direct interspecific competition among the prey in isolation from other complicating factors.

For the diffusive one-predator-two-prey model (1.1), the existence of traveling wave solutions, in short TWS, is an important and exciting subject that has attracted considerable attention [22, 15, 1]. In particular, there have been great successes in the existence, uniqueness, stability, and spreading speed of TWS of monotone systems [22]. Unfortunately, the system (1.1), which has essential nonlinear interactions, predator-prey type, between u_1/u_2 and v, is non-monotone. In the past three decades, by using different methods including the shooting method [4, 18, 19, 15], Conley index [7] and upper-lower solutions method [37, 36, 17, 1], the existence of TWS has been established for various predator-prey systems, and please see the references cited therein. In this work, we will use the so-called higher-dimensional shooting method, Wazewski method [4, 18, 15], to show the existence of positive TWS from one unstable equilibrium to a stable one.

The existence of TWS of the two-prey-one-predator models is shown by the method of upper-lower solutions [17, 1]. However, fundamentally different assumptions exist between the upper-lower solutions method and the Wazewski principle. For the upper-lower solutions method, the TWS connecting trivial solution **0** and the positive solution are often discussed, and the per capita function of each species should be positive at **0** [17]. Alternatively, we show the existence of TWS from the boundary equilibrium $(1, 0, 0)$ to the coexistence by the Wazewski principle without any assumption in the per capita function.

Our main contributions are as follows. First, for the corresponding reaction system (1.2), we clarify entirely the existence, non-existence, and all asymptotic states and their global stabilities, which are investigated theoretically for all parameters. Secondly, we show the existence of TWS for the three-species ecosystem with predator-prey interaction by applying the Wazewski principle. Although the method is similar to [18, 19], our system is three-dimensional. Finally, some engaging numerical simulations and biological interpretations are given. In particular, we show the non-existence of the bistability for (1.2).

The rest of this article is organized in the following manner. In Section 2, we first consider the corresponding reaction equations (1.2) of (1.1), which is a system of three ODEs. The positivity and boundedness of solutions of (1.2) are verified, and some well-known relative two-dimensional results are recalled. Then, we establish three global results by differential inequality and the Lyapunov method, respectively. Section 3 obtains the existence and nonexistence of TWS of (1.1) using the higher-dimensional shooting method. The last section gives remarks and discussions on biological meanings, and some engaging numerical simulations are performed and presented.

2. PRELIMINARY

It is clear that the boundaries of the positive cone of $\mathbb{R}^3$,

$$\begin{aligned} H_{12} &= \{(u_1, u_2, 0) : u_1, u_2 > 0\}, \\ H_{13} &= \{(u_1, 0, v) : u_1 > 0, v > 0\}, \\ H_{23} &= \{(0, u_2, v) : u_2 > 0, v > 0\}, \end{aligned}$$

are all invariant with respect to (1.2); hence, solutions of (1.2) are non-negative or positive with non-negative or positive initial conditions. Moreover, we can see that solutions of (1.2) are bounded.

Lemma 2.1. *Solutions of* (1.2) *with non-negative initial conditions are bounded.*

Proof. It is easy to see that $u_1 \le 1$ and $u_2 \le 1$ eventually by differential inequality. Define $w = \frac{\beta_1}{\alpha_1}u_1 + \frac{\beta_2}{\alpha_2}u_2 + v$, and , for time large enough, we have

$$\begin{aligned} \frac{dw}{dt} &\le \frac{\beta_1}{\alpha_1}(r_1u_1(1-u_1)) + \frac{\beta_2}{\alpha_2}(r_2u_2(1-u_2)) - \mu v \\ &\le \frac{2\beta_1 r_1}{\alpha_1}u_1 + \frac{2\beta_2 r_2}{\alpha_2}u_2 - \frac{\beta_1 r_1}{\alpha_1}u_1 - \frac{\beta_2 r_2}{\alpha_2}u_2 - \mu v \\ &\le M - Dw, \end{aligned}$$

where $M = \frac{2\beta_1 r_1}{\alpha_1} + \frac{2\beta_2 r_2}{\alpha_2}$ and $D = \min\{r_1, r_2, \mu\}$. This differential inequality implies w is bounded, which implies species v is bounded. The proof is complete. □

2.1. **Well-Known Results on the Sub-communities of (1.2).**

On H_{12}, H_{13} and H_{23}, system (1.2) can be reduced to the following two-dimensional subsystems,

$$\begin{cases} \dfrac{du_1}{dt} = r_1u_1(1-u_1), \\ \dfrac{du_2}{dt} = r_2u_2(1-u_2), \end{cases} \tag{2.1}$$

$$\begin{cases} \dfrac{du_1}{dt} = r_1u_1(1-u_1) - \alpha_1u_1v, \\ \dfrac{dv}{dt} = -\mu v + \beta_1u_1v, \end{cases} \tag{2.2}$$

and

$$\begin{cases} \dfrac{du_2}{dt} = r_2u_2(1-u_2) - \alpha_2 u_2 v, \\ \dfrac{dv}{dt} = -\mu v + \beta_2 u_2 v, \end{cases} \tag{2.3}$$

respectively.

For the decoupled system (2.1), it is easy to see that the equilibrium $E_{12} = (1, 1)$ is globally asymptotically stable, in short GAS, in H_{12}. For system (2.2), if $\beta_1 > \mu$ then there is a u_1-v co-existent equilibrium $E_{1v} = (\bar{u}_1, \bar{v}_1)$ where

$$\bar{u}_1 = \frac{\mu}{\beta_1} \quad \text{and} \quad \bar{v}_1 = \frac{r_1(1-\bar{u}_1)}{\alpha_1}. \tag{2.4}$$

It is well known that whenever E_{1v} exists, it is also GAS in H_{13}. Similarly, for system (2.3), if $\beta_2 > \mu$ then there is a u_2-v co-existent equilibrium $E_{2v} = (\bar{u}_2, \bar{v}_2)$ which is GAS in H_{23} where

$$\bar{u}_2 = \frac{\mu}{\beta_2} \quad \text{and} \quad \bar{v}_2 = \frac{r_2(1-\bar{u}_2)}{\alpha_2}. \tag{2.5}$$

Without ambiguity, we use the same notations $E_{12} = (1, 1, 0)$, $E_{1v} = (\bar{u}_1, 0, \bar{v}_1)$ and $E_{2v} = (0, \bar{u}_2, \bar{v}_2)$ to denote the essential boundary equilibria of (1.2) in $\mathbb{R}^3_+$. Well-known two-dimensional results are summarized in the following propositions.

Proposition 2.2. *Let $\phi(t) = \big(u_1(t), u_2(t), v(t)\big)$ be a solution of (1.2) with non-negative initial conditions.*

(i) *On H_{12}, system (1.2) can be reduced to the subsystem (2.1), and $\lim_{t\to\infty} \phi(t) = E_{12} = (1, 1, 0)$.*

(ii) *On H_{13}, system (1.2) can be reduced to the subsystem (2.2). If $\mu \geq \beta_1$, then $\lim_{t\to\infty} v(t) = 0$ and $\lim_{t\to\infty} \phi(t) = E_1 = (1, 0, 0)$. Otherwise, if $\mu < \beta_1$, then equilibrium E_{1v} exists and $\lim_{t\to\infty} \phi(t) = E_{1v} = (\bar{u}_1, 0, \bar{v}_1)$.*

(iii) *On H_{23}, similarly, system (1.2) can be reduced to the subsystem (2.3). If $\mu \geq \beta_2$, then $\lim_{t\to\infty} v(t) = 0$ and $\lim_{t\to\infty} \phi(t) = E_2 = (0, 1, 0)$. Otherwise, if $\mu < \beta_2$, then equilibrium E_{2v} exists and $\lim_{t\to\infty} \phi(t) = E_{2v} = (0, \bar{u}_2, \bar{v}_2)$.*

2.2. Boundary Equilibria, Stability and Dynamics of (1.2) in $\mathbb{R}^3_+$.

In this subsection, we start to investigate the dynamics of (1.2) in the positive cone,

$$\mathbb{R}^3_+ \equiv \{(u_1, u_2, v) : u_1 > 0, u_2 > 0, v > 0\},$$

with helps of corresponding dynamics of (1.2) on the boundaries H_{12}, H_{13} and H_{23} of $\mathbb{R}^3_+$. First, we can obtain a simple global result using the comparison principle.

Lemma 2.3. *Let $\phi(t) = \big(u_1(t), u_2(t), v(t)\big)$ be a solution of (1.2) with initial condition $\big(u_1(0), u_2(0), v(0)\big) \in \mathbb{R}^3_+$. If $\mu > \beta_1 + \beta_2$, then $\lim_{t\to\infty} v(t) = 0$ and the equilibrium $E_{12} = (1, 1, 0)$ is GAS in $\mathbb{R}^3_+$.*

Proof. Let $\eta = \mu - \beta_1 - \beta_2 > 0$. Without loss of generality, we may assume that $u_1(t) \leq 1$ and $u_2(t) \leq 1$ for t large enough. Hence, by considering the third equation of (1.2), we have

$$\frac{\dot{v}}{v} \leq -\mu + \beta_1 + \beta_2 = -\eta < 0,$$

which implies $v(t) \leq v(0)e^{-\eta t}$ by integrating both sides of the last equation from 0 to t. Hence we have $\lim_{t\to\infty} v(t) = 0$.

By Markus's Limiting theorem [24], the system (1.2) approaches asymptotically the system (2.1). We conclude that E_{12} is GAS by Proposition 2.2 (i). This completes the proof. □

In the remainder of this work, we always make the following generic assumptions,

(A1) $\mu < \beta_1 + \beta_2$,
(A2) $0 < \beta_1 < \beta_2$,

The hypothesis **(A1)** is to avoid the triviality of system (1.2), since if $\mu > \beta_1 + \beta_2$ then species v will die out eventually because of the result of Lemma 2.3. From a biological point of view, it says that the predator v can not survive if it cannot overcome the mortality rate by getting benefits from the prey, u_1 and u_2. Furthermore, without loss of generality, we may assume that the hypothesis **(A2)** holds since we can get symmetric results if we reverse the order of **(A2)**.

It is clear that boundary equilibria $E_0 = (0,0,0)$, $E_1 = (1,0,0)$, and $E_2 = (0,1,0)$ are all saddle, and their corresponding dynamics are simple. Upon straightforward computations, we have the Jacobian matrix for (1.2),

$$J(u_1,u_2,v) = \begin{bmatrix} r_1 - 2r_1u_1 - \alpha_1 v & 0 & -\alpha_1 u_1 \\ 0 & r_2 - 2r_2u_2 - \alpha_2 v & -\alpha_2 u_2 \\ \beta_1 v & \beta_2 v & -\mu + \beta_1 u_1 + \beta_2 u_2 \end{bmatrix}.$$

The equilibrium $E_{12} = (1,1,0)$ is also saddle because of assumption **(A1)**.

By evaluating the Jacobian matrix at E_{1v} and E_{2v}, respectively, we obtain that

$$J(E_{1v}) = \begin{bmatrix} -r_1\bar{u}_1 & 0 & -\alpha_1\bar{u}_1 \\ 0 & r_2 - \alpha_2\bar{v}_1 & 0 \\ \beta_1\bar{v}_1 & \beta_2\bar{v}_1 & 0 \end{bmatrix}$$

and

$$J(E_{2v}) = \begin{bmatrix} r_1 - \alpha_1\bar{v}_2 & 0 & 0 \\ 0 & -r_2\bar{u}_2 & -\alpha_2\bar{u}_2 \\ \beta_1\bar{v}_2 & \beta_2\bar{v}_2 & 0 \end{bmatrix},$$

respectively. It is easy to see that E_{1v} is asymptotically stable if $r_2 < \alpha_2\bar{v}_1$, which can be rewritten as the equivalent form

$$\frac{r_2\alpha_1}{r_1\alpha_2} < 1 - \frac{\mu}{\beta_1},$$

and define the key parameter $\delta \equiv r_1\alpha_2/r_2\alpha_1$, which measures the indirect competition between u_1 and u_2. For example, the inequality $\delta > 1$ or equivalently $r_1/\alpha_1 > r_2/\alpha_2$ means that u_1 wins the apparent competition with species u_2 [11, 12] under the r-strategy [28]. Because u_1 has a higher birthrate $(r_1 > r_2)$ if u_1 and u_2 the same negative effects $(\alpha_1 = \alpha_2)$ from v. Conversely, if $\delta < 1$, species u_2 wins the apparent competition with species u_1.

We have the following local stability of boundary equilibria, E_{1v} and E_{2v}, in terms of δ.

Lemma 2.4. *The following statements are valid for* (1.2).

(i) *If*

$$\frac{1}{\delta} < 1 - \frac{\mu}{\beta_1} \tag{2.6}$$

then E_{1v} is asymptotically stable.

(ii) *If*

$$\delta < 1 - \frac{\mu}{\beta_2} \tag{2.7}$$

then E_{2v} is asymptotically stable.

2.3. **The existence of the positive equilibrium.** We find the positive equilibrium $E^* = (u_1^*, u_2^*, v^*)$ by solving the following linear system,

$$\begin{cases} r_1(1-u_1) - \alpha_1 v = 0, \\ r_2(1-u_2) - \alpha_2 v = 0, \\ -\mu + \beta_1 u_1 + \beta_2 u_2 = 0. \end{cases} \tag{2.8}$$

With the first and second equations of (2.8), we have

$$\frac{r_1}{\alpha_1}(1-u_1) = \frac{r_2}{\alpha_2}(1-u_2).$$

Hence (2.8) can be reduced to two straight lines in the u_1-u_2 plane,

$$L_1 : \beta_1 u_1 + \beta_2 u_2 = \mu,$$
$$L_2 : u_2 = \delta u_1 + 1 - \delta,$$

and E^* is exact the intersection of L_1 and L_2 in the interior of the first quadrant. It is obviously that L_1 intersects u_1- and u_2-axis at μ/β_1 and μ/β_2, respectively. Please refer to Figure 1. For the straight line L_2 and the special case, $\delta = 1$, it is clear that

$$u_1^* = u_2^* = \frac{\mu}{\beta_1+\beta_2} \quad \text{and} \quad v^* = \frac{r_1(\beta_1+\beta_2-\mu)}{\alpha_1(\beta_1+\beta_2)}.$$

And it is worth noting that (2.6) and (2.7) do not hold simultaneously. In this case, E_{1v} and E_{2v} are unstable, and we will show that E^* is globally asymptotically stable when it exists in Theorem 2.7. However, generic, $\delta = 1$ cannot happen, so we only consider $\delta < 1$ or $\delta > 1$.

If $\delta < 1$, then the inequality $1 - \delta < \mu/\beta_2$ implies the existence of positive equilibrium. Please refer to Figure 1(a). Similarly, if $\delta > 1$, then the inequality $1 - 1/\delta < \mu/\beta_1$ implies the existence of positive equilibrium by referring to Figure 1(b). For these two generic cases, the positive equilibrium has the explicit forms,

$$u_1^* = \frac{\mu - (1-\delta)\beta_2}{\beta_1 + \beta_2\delta}, \qquad u_2^* = \frac{(1-\delta)\beta_1 + \mu\delta}{\beta_1 + \beta_2\delta},$$

and

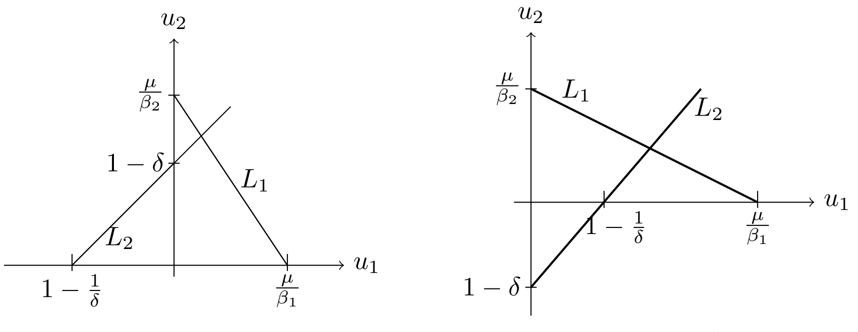
$$v^* = \frac{r_1(\beta_1+\beta_2-\mu)}{\alpha_1(\beta_1+\beta_2\delta)} = \frac{r_2(\beta_1+\beta_2-\mu)}{\alpha_2(\beta_1/\delta+\beta_2)}.$$

Moreover, we also verify that the inequality $u_1^* < 1$, that is, $\mu - (1-\delta)\beta_2 < \beta_1 + \beta_2\delta$, is equivalent to **(A1)**. Similarly, it is easy to see that $u_2^* < 1$ and $v^* > 0$ if and only if assumption **(A1)** holds. Hence, we have the following results.

Lemma 2.5. *Excluding the case $\delta = 1$, the positive equilibrium E^* exists if and only if one of the following conditions holds,*

(i) $\delta < 1$ *and* $1 - \delta < \frac{\mu}{\beta_2}$,
(ii) $\delta > 1$ *and* $1 - \frac{1}{\delta} < \frac{\mu}{\beta_1}$.

2.4. **Parameters Space Classifications.** In this subsection, based on the results of local stabilities of equilibria, we would like to classify the parameter space and find its corresponding global dynamics.



(a) $0<\delta<1$ and $1-\delta<\frac{\mu}{\beta_2}$ (b) $\delta>1$ and $1-\frac{1}{\delta}<\frac{\mu}{\beta_1}$

FIGURE 1. The existence of positive equilibrium for two generic cases of δ.

We consider the generic classification of all parameters in the following ways,

$$\delta < 1-\frac{\mu}{\beta_2}, \tag{2.9}$$

$$\delta > 1-\frac{\mu}{\beta_2}, \tag{2.10}$$

$$\frac{1}{\delta} < 1-\frac{\mu}{\beta_1}, \tag{2.11}$$

$$\frac{1}{\delta} > 1-\frac{\mu}{\beta_1}, \tag{2.12}$$

constrained with the generic relations between β_1, β_2 and μ under hypothesis **(A1)** and **(A2)** which are classified into three cases, $0<\mu<\beta_1$, $\beta_1<\mu<\beta_2$, and $\beta_2<\mu<\beta_1+\beta_2$.

(i) $0<\mu<\beta_1$:
In this case, it is easy to see that (2.9) is equivalent to (2.7), and (2.11) is equivalent to (2.6). The inequalities (2.9) and (2.11) could not hold simultaneously, because the inequality (2.9) implies $\delta<1$ and (2.11) implies $\delta>1$. Moreover, if $\delta\neq 1$, then (2.10) and (2.12) hold if and only if the positive equilibrium E^* exists by **Lemma 2.5**. Hence, we can conclude that if E_{1v} is stable, then the bistability phenomenon could not happen, and E^* does not exist. Similarly, if E_{2v} is stable, the bistability phenomenon could not happen, and E^* does not exist. We summarize these results in the first column of Table 1.

(ii) $\beta_1<\mu<\beta_2$:
In this case, (2.11) cannot be true since $1-\mu/\beta_1<0$. Therefore, if (2.9) and (2.12) hold, then E_{2v} exists and is asymptotically stable by **Lemma 2.4**. However, if (2.10) and (2.12) hold, then E^* exsits and is asymptotically stable by **Lemma 2.5**. These results constitute the second column of Table 1.

(iii) $\beta_2<\mu<\beta_1+\beta_2$:
Finally, in this case, (2.9) and (2.11) do not hold in the same ways. Hence, the only possibility is the existence of E^*, which forms the third column of Table 1.

2.5. **Two Extinction Results and Global Stability of the Positive Equilibrium.** This subsection establishes the following two global extinction results based on Markus's Theorem.

Proposition 2.6. *The the following statements are valid for* (1.2).

(i) *If* $0<\mu<\beta_1$ *and* (2.10) *hold, then* E_{1v} *is GAS.*

TABLE 1. Parameters classification of (1.2) and their corresponding local dynamics

	$0<\mu<\beta_1$	$\beta_1<\mu<\beta_2$	$\beta_2<\mu<\beta_1+\beta_2$
$\delta<1-\frac{\mu}{\beta_2}$ (2.9) $\frac{1}{\delta}<1-\frac{\mu}{\beta_1}$ (2.11)	🟥	🟥	🟥
$\delta>1-\frac{\mu}{\beta_2}$ (2.10) $\frac{1}{\delta}<1-\frac{\mu}{\beta_1}$ (2.11)	E_{1v} $\exists$, stable E_{2v} $\exists$, unstable E^* $\nexists$,	🟥	🟥
$\delta<1-\frac{\mu}{\beta_2}$ (2.9) $\frac{1}{\delta}>1-\frac{\mu}{\beta_1}$ (2.12)	E_{1v} $\exists$, unstable E_{2v} $\exists$, stable E^* $\nexists$,	E_{1v} $\nexists$ E_{2v} $\exists$, stable E^* $\nexists$,	🟥
$\delta>1-\frac{\mu}{\beta_2}$ (2.10) $\frac{1}{\delta}>1-\frac{\mu}{\beta_1}$ (2.12)	E_{1v} $\exists$, unstable E_{2v} $\exists$, unstable E^* $\exists$, stable	E_{1v} $\nexists$, E_{2v} $\exists$, unstable E^* $\exists$, stable	E_{1v} $\nexists$, E_{2v} $\nexists$, E^* $\exists$, stable

(ii) *If* $0<\mu<\beta_2$ *and* (2.12) *hold, then* E_{2v} *is GAS.*

Proof. We only show the case (i). The proof of case (ii) is similar, so we omit it. Let $p=(u_1^0,u_2^0,v^0)\in\mathbb{R}^3_+$ be an arbitrary point, and $\tilde{p}=(u_1^0,0,v^0)$ be the u_1-v plane projection of (u_1^0,u_2^0,v^0). Consider solutions $\phi(t;p)=\big(u_1(t),u_2(t),v(t)\big)$ and $\phi(t;\tilde{p})=\big(\tilde{u}_1(t),\tilde{u}_2(t),\tilde{v}(t)\big)$ of (1.2) with initial conditions p and $\tilde{p}$, respectively. It is easy to see that the $\tilde{u}_2(t)=0$ for all $t\geq 0$ since H_{13} is an invariant subspace with respect to (1.2). By comparing the v-coordinates, $v(t)$ and $\tilde{v}(t)$, of $\phi(t;p)$ and $\phi(t;\tilde{p})$, we have $v(t)\geq\tilde{v}(t)$ for all $t\geq 0$, since $v(0)=v^0=\tilde{v}(0)$ and

$$\dot{v}(t)=-\mu+\beta_1u_1(t)+\beta_2u_2(t)\geq-\mu+\beta_1u_1(t)=\dot{\tilde{v}}(t).$$

Moreover, it is clear that $\lim_{t\to\infty}\phi(t;\tilde{p})=(\bar{u}_1,0,\bar{v}_1)$, by the assumptions and Proposition 2.2 (ii). Therefore, we have $v(t)\geq\bar{v}_1-\varepsilon$ for any positive number ε for a time large enough.

Taking $\varepsilon=(\alpha_2\bar{v}_1-r_2)/2\alpha_2$, there is a positive time $T(\varepsilon)$ such that if $t\geq T(\varepsilon)$ then $v(t)\geq\bar{v}_1-\varepsilon$. Consider

$$\frac{\dot{u}_2}{u_2}=r_2(1-u_2)-\alpha_2v\leq r_2-\alpha_2v\leq r_2-\alpha_2\bar{v}_1+\alpha_2\varepsilon=\frac{r_2-\alpha_2\bar{v}_1}{2}<0,$$

which implies $u_2(t)\to 0$ for $t\to\infty$. By Markus's Theorem, system (1.2) approaches asymptotically the two-dimensional subsystem (2.2). Hence, by Proposition 2.2 (ii) again, we obtain that the equilibrium E_{1v} is globally asymptotically stable. We complete the proof. $\square$

Evaluating Jacobian matrix at E^*, we have

$$J(E^*)=\begin{bmatrix}-r_1u_1^* & 0 & -\alpha_1u_1^*\\ 0 & -r_2u_2^* & -\alpha_2u_2^*\\ \beta_1v^* & \beta_2v^* & 0\end{bmatrix},$$

and the stability of E^* can be established by checking Routh-Hurwitz stability criterion. However, we can show that E^* is GAS by the Lyapunov method whenever it exists.

Proposition 2.7. *If the positive equilibrium* E^* *exists, it is globally asymptotically stable.*

Proof. Consider the Lyapunov function

$$V(u_1,u_2,v)=\frac{\beta_1}{\alpha_1}\int_{u_1(0)}^{u_1(t)}\frac{\eta-u_1^*}{\eta}\,d\eta+\frac{\beta_2}{\alpha_2}\int_{u_2(0)}^{u_2(t)}\frac{\eta-u_2^*}{\eta}\,d\eta+\int_{v(0)}^{v(t)}\frac{\eta-v^*}{\eta}\,d\eta.$$

Then, by using the identities $r_1 = r_1u_1^* + \alpha_1 v^*$, $r_2 = r_2u_2^* + \alpha_2 v^*$ and $\mu = \beta_1 u_1^* + \beta_2 u_2^*$, we have

$$\begin{aligned}\frac{dV}{dt} &= \frac{\beta_1}{\alpha_1}(u_1 - u_1^*)(r_1(1-u_1) - \alpha_1 v) + \frac{\beta_2}{\alpha_2}(u_2 - u_2^*)(r_2(1-u_2) - \alpha_2 v) + \\ &\quad (v - v^*)(-\mu + \beta_1 u_1 + \beta_1 u_2) \\ &= -\frac{\beta_1 r_1}{\alpha_1}(u_1 - u_1^*)^2 - \frac{\beta_2 r_2}{\alpha_2}(u_2 - u_2^*)^2 \le 0.\end{aligned}$$

It is easy to see that the maximal invariant subset of

$$\left\{ \frac{dV}{dt} = 0 \right\} = \{(u_1^*, u_2^*, v) : v > 0\}$$

is the singleton set $\{E^*\}$. Hence, by LaSalle's invariant principle, all solutions converge to E^*. The proof is complete. ☐

3. EXISTENCE OF TRAVELING WAVE SOLUTIONS OF (1.1)

In this section, we always assume that $0 < \mu < \beta_1$, (2.10) and (2.12) hold. The existence of the traveling wave solution of (1.1) from the saddle boundary equilibrium $E_1 = (1, 0, 0)$ to the positive equilibrium E^* by the higher-dimensional shooting method, the Wazewski principle is investigated. Outlines of the methodology are stated as follows.

(i) By using the moving coordinates, the reaction-diffusion system is transformed into an ODE system, and a TWS of (1.1) from E_1 to E^* is equivalent to a heteroclinic orbit of the corresponding ODE system from E_1 to E^*. Therefore, we focus on the ODE system.

(ii) Analyzing the structure of the unstable manifold of the unstable equilibrium E_1, we construct a variant of the Wazewski set Σ with E_1 as its boundary point and containing E^* in the interior of Σ. Clarify the dynamics of the ODE system of all boundaries of Σ, and identify the "exit set" of Σ.

(iii) Next, pick up a curve γ contained in the unstable manifold with two endpoints on the exit set of the boundary of Σ. All solutions with initial conditions on γ will tend to E_1 as $t \to -\infty$. Then, show that a particular point exists on γ, and the solution starting from this point will stay in the interior of Σ for all $t \ge 0$.

(iv) Finally, we define a nonempty subset G of Σ, which contains the point stayed in the interior of Σ for all positive time under the action of the ODE system. Then, we can obtain our main result, a heteroclinic orbit from E_1 to E^*, by constructing a Lyapunov function and using LaSalle's invariance principle on G.

Moreover, we also show the non-existence of the positive TWS of (1.1) by analyzing the eigenvalues of the E_1.

3.1. The ODE forms and the Lienard Transformation.

We consider the solution of (1.2) with moving coordinate $\xi = x + ct$ with the positive wave speed $c > 0$ of the form

$$\begin{cases} u_1(x,t) = U_1(x+ct) = U_1(\xi), \\ u_2(x,t) = U_2(x+ct) = U_2(\xi), \\ v(x,t) = V(x+ct) = V(\xi), \end{cases} \tag{3.1}$$

satisfying the asymptotical boundary conditions from E_1 to E^*, that is,

$$\begin{aligned} \lim_{\xi \to -\infty} (U_1(\xi), U_2(\xi), V(\xi)) &= (1, 0, 0), \\ \lim_{\xi \to \infty} (U_1(\xi), U_2(\xi), V(\xi)) &= (u_1^*, u_2^*, v^*). \end{aligned} \tag{3.2}$$

Direct computations show that $(U_1(\xi), U_2(\xi), V(\xi))$ is a traveling wave solution of (1.1) if and only if $(U_1(\xi), U_2(\xi), V(\xi))$ is a solution of the system,

$$\begin{aligned} cU_1' &= U_1(r_1(1-U_1) - \alpha_1 V), \\ cU_2' &= U_2(r_2(1-U_2) - \alpha_2 V), \\ cV' &= DV'' + V(-\mu + \beta_1 U_1 + \beta_2 U_2), \end{aligned} \tag{3.3}$$

which is equivalent to (1.1) via the transformation (3.1). For a positive wave speed c, we consider the Lienard transformation to make the following changes of variable and scaling:

$$\begin{aligned} X_1(t) &= U_1(ct) \\ X_2(t) &= U_2(ct) \\ Y(t) &= V(ct) \\ Z(t) &= \frac{1}{c}[cV(ct) - DV'(ct)] \end{aligned} \tag{3.4}$$

Then, upon straightforward computations, (3.3) is transformed to a four-dimensional system,

$$\begin{aligned} \dot{X}_1(t) &= X_1(r_1(1-X_1) - \alpha_1 Y), \\ \dot{X}_2(t) &= X_2(r_2(1-X_2) - \alpha_2 Y), \\ \dot{Y}(t) &= \rho(Y - Z), \\ \dot{Z}(t) &= Y(-\mu + \beta_1 X_1 + \beta_2 X_2), \end{aligned} \tag{3.5}$$

where $\rho = c^2/D$, and the corresponding asymptotical boundary conditions from E_1 to E^* are

$$\begin{aligned} \lim_{t\to-\infty} (X_1(t), X_2(t), Y(t), Z(t)) &= (1, 0, 0, 0), \\ \lim_{t\to\infty} (X_1(t), X_2(t), Y(t), Z(t)) &= (u_1^*, u_2^*, v^*, v^*). \end{aligned} \tag{3.6}$$

3.2. **The Wazewski Set and Its Exit Subset.** Let

$$c > c_1 \equiv 2\sqrt{D(\beta_1 + \beta_2 - \mu)}, \tag{3.7}$$

and take σ_1, σ_2 to be positive constants satisfying

$$0 < \frac{\rho - \sqrt{\rho^2 - 4\rho(\beta_1 + \beta_2 - \mu)}}{2\rho} < \sigma_1 < \frac{\rho + \sqrt{\rho^2 - 4\rho(\beta_1 + \beta_2 - \mu)}}{2\rho} < 1$$

and

$$1 < \frac{\rho + \sqrt{\rho^2 + 4\rho\mu}}{2\rho} < \sigma_2,$$

respectively. We define a wedged like region $\Sigma \subset \mathbb{R}^4$ as follows,

$$\Sigma = \{(X_1, X_2, Y, Z) : 0 \le X_1 \le 1, 0 \le X_2 \le 1, Y \ge 0, \sigma_1 Y \le Z \le \sigma_2 Y\}.$$

Then the boundary of Σ consists of surfaces P_1, P_2, $Q_1 \sim Q_5$ represented by

$$\begin{aligned} Q_1 &= \{X_1 = 0, 0 \le X_2 \le 1, Y > 0, \sigma_1 Y < Z < \sigma_2 Y\},\\ Q_2 &= \{X_1 = 1, 0 \le X_2 \le 1, Y > 0, \sigma_1 Y < Z < \sigma_2 Y\},\\ Q_3 &= \{0 \le X_1 \le 1, X_2 = 0, Y > 0, \sigma_1 Y < Z < \sigma_2 Y\},\\ Q_4 &= \{0 \le X_1 \le 1, X_2 = 1, Y > 0, \sigma_1 Y < Z < \sigma_2 Y\},\\ Q_5 &= \{0 \le X_1 \le 1, 0 \le X_2 \le 1, Y = Z = 0\},\\ P_1 &= \{0 \le X_1 \le 1, 0 \le X_2 \le 1, Y > 0, Z = \sigma_1 Y\},\\ P_2 &= \{0 \le X_1 \le 1, 0 \le X_2 \le 1, Y > 0, Z = \sigma_2 Y\}. \end{aligned}$$

The vector field of (3.5) has elementary dynamics in the boundary of Σ, which the following two lemmas can characterize.

Lemma 3.1. *Let $c > c_1$ and $\Phi_t(p)$ be the flow of (3.5), i.e. $\Phi_t(p)$ is a solution of (3.5) satisfying the initial condition $\Phi_0(p) = p \in \mathbb{R}^4$. Then for any $p \in Int(\Sigma)$, $\Phi_t(p)$ cannot leave Σ from a point in the boundary $\cup_{i=1}^5 Q_i$ of Σ at any positive time.*

Proof. It is obvious that the boundary Q_1, Q_3 and Q_5 are invariant sets of (3.5). Hence any solution of (3.5) through a point in the interior of Σ can not leave Σ from a point in the set $Q_1 \cup Q_3 \cup Q_5$.

First suppose $p_0 = (1, X_2(0), Y(0), Z(0)) \in Q_2$, then the first equation of (3.5) yields that at p_0,

$$\dot{X}_1(0) = -\alpha_1 Y(0) < 0.$$

So that the vector field of (3.5) points interior of Σ and hence $\Phi_t(p_0)$ cannot exit Σ at p_0 in the set Q_2.

Next suppose $p_1 = (X_1(0), 1, Y(0), Z(0)) \in Q_4$, then the second equation of (3.5) yields that at p_1,

$$\dot{X}_2(0) = -\alpha_2 Y(0) < 0.$$

So that the vector field of (3.5) points interior of Σ and hence $\Phi_t(p_1)$ cannot exit Σ at p_1 in the set Q_4. □

Next, let us study the vector field at the sets P_1 and P_2.

Lemma 3.2. *Let $c > c_1$, then the vector field of (3.5) at $p \in P_1 \cup P_2$ points outside of Σ.*

Proof. First consider a point $p_1 = (X_1(0), X_2(0), Y(0), Z(0)) \in P_1$. Then $Z(0) = \sigma_1 Y(0) < Y(0)$ and $Y(0) > 0$. By the last two equations of (3.5), we obtain

$$\dot{Y}(0) = \rho[Y(0) - Z(0)] > 0,$$

$$\begin{aligned} \frac{\dot{Z}(0)}{\dot{Y}(0)} &= \frac{Y(0)(-\mu + \beta_1 X_1(0) + \beta_2 X_2(0))}{\rho[Y(0) - Z(0)]}\\ &= \frac{Y(0)(-\mu + \beta_1 X_1(0) + \beta_2 X_2(0))}{\rho(Y(0) - \sigma_1 Y(0))}\\ &= \frac{-\mu + \beta_1 X_1(0) + \beta_2 X_2(0)}{\rho(1 - \sigma_1)}\\ &\le \frac{\beta_1 + \beta_2 - \mu}{\rho(1 - \sigma_1)} < \sigma_1. \end{aligned}$$

It implies that the vector field at the point $p_1 \in P_1$ points outside of Σ. Next, let $p_2 = (X_1(0), X_2(0), Y(0), Z(0)) \in P_2$. Then $Z(0) = \sigma_2 Y(0) > Y(0)$ and $Y(0) > 0$. At p_2, we have

$$\dot{Y}(0) = \rho[Y(0) - Z(0)] < 0,$$

$$\begin{aligned}\frac{\dot{Z}(0)}{\dot{Y}(0)} &= \frac{Y(0)(-\mu+\beta_1X_1(0)+\beta_2X_2(0))}{\rho[Y(0)-Z(0)]}\\ &= \frac{-\mu+\beta_1X_1(0)+\beta_2X_2(0)}{\rho(1-\sigma_2)}\\ &= \frac{\mu-\beta_1X_1(0)-\beta_2X_2(0)}{\rho(\sigma_2-1)}\\ &\leq \frac{\mu}{\rho(\sigma_2-1)} < \sigma_2.\end{aligned}$$

It implies that the vector field at the point $p_2 \in P_2$ points outside of Σ. □

By Lemma 3.1 and Lemma 3.2, we can see that with initial point $p \in \Sigma$, $\Phi_t(p)$ can only leave Σ from a point in $P_1 \cup P_2$ at some positive time.

3.3. **The Unstable Manifold of E_1.** Now we turn to investigate the unstable manifold ε_1^u of the equilibrium E_1. The linearized system of (3.5) at E_1 is

$$\begin{aligned}\dot{X}_1 &= -r_1(X_1-1)-\alpha_1Y,\\ \dot{X}_2 &= r_2X_2,\\ \dot{Y} &= \rho(Y-Z),\\ \dot{Z} &= (\beta_1-\mu)Y,\end{aligned} \tag{3.8}$$

with Jacobian matrix of (3.5) at E_1,

$$\begin{bmatrix} -r_1 & 0 & -\alpha_1 & 0\\ 0 & r_2 & 0 & 0\\ 0 & 0 & \rho & -\rho\\ 0 & 0 & \beta_1-\mu & 0\end{bmatrix}.$$

For $c > c_1$, it is easy to see that the Jacobian matrix has one negative eigenvalue

$$\lambda_0 = -r_1.$$

Moreover, there are three distinct positive eigenvalues

$$\begin{aligned}\lambda_1 &= r_2,\\ \lambda_2 &= \frac{\rho+\sqrt{\rho^2-4\rho(\beta_1-\mu)}}{2},\\ \lambda_3 &= \frac{\rho-\sqrt{\rho^2-4\rho(\beta_1-\mu)}}{2},\end{aligned} \tag{3.9}$$

with the corresponding eigenvectors to λ_1, λ_2 and λ_3,

$$\begin{aligned}v_1 &= [0,1,0,0]^T,\\ v_2 &= \left[\frac{-\alpha_1\lambda_2}{(\lambda_2+r_1)(\beta_1-\mu)},0,\frac{\lambda_2}{\beta_1-\mu},1\right]^T,\\ v_3 &= \left[\frac{-\alpha_1\lambda_3}{(\lambda_3+r_1)(\beta_1-\mu)},0,\frac{\lambda_3}{\beta_1-\mu},1\right]^T.\end{aligned} \tag{3.10}$$

Hence, the unstable manifold ε_1^u of E_1 is tangent to the three-dimensional hypersurface

$$\begin{aligned} P &= \{k_1v_1 + k_2v_2 + k_3v_3 + E_1 : k_1, k_2, k_3 \in \mathbb{R}\} \\ &= \left\{ \begin{bmatrix} x_1(k_1,k_2,k_3) \\ x_2(k_1,k_2,k_3) \\ y(k_1,k_2,k_3) \\ z(k_1,k_2,k_3) \end{bmatrix} \in \mathbb{R}^4 : k_1, k_2, k_3 \in \mathbb{R} \right\}, \end{aligned}$$

where

$$x_1 = 1 - \frac{\alpha_1\lambda_2k_2}{(\lambda_2+r_1)(\beta_1-\mu)} - \frac{\alpha_1\lambda_3k_3}{(\lambda_3+r_1)(\beta_1-\mu)}, \tag{3.11}$$

$$x_2 = k_1, \tag{3.12}$$

$$y = \frac{k_2\lambda_2}{\beta_1-\mu} + \frac{k_3\lambda_3}{\beta_1-\mu}, \tag{3.13}$$

$$z = k_2 + k_3. \tag{3.14}$$

Lemma 3.3. *There are two points $p_1^* \in P_1$ and $p_2^* \in P_2$ and a curve $\gamma \subset \varepsilon_1^u$ which connects p_1^* and p_2^* such that*

$$\gamma \setminus \{p_1^*, p_2^*\} \subset \mathrm{Int}(\Sigma).$$

Proof. Note that the transformations via (3.12)-(3.14), $(k_1,k_2,k_3) \mapsto (x_2,y,z)$, is invertible. Thus, the plane can also be expressed as

$$P = \left\{ \left(1 - \frac{\alpha_1\lambda_2k_2(y,z)}{(\lambda_2+r_1)(\beta_1-\mu)} - \frac{\alpha_1\lambda_3k_3(y,z)}{(\lambda_3+r_1)(\beta_1-\mu)}, x_2, y, z\right) : x_2, y, z \in \mathbb{R} \right\},$$

where

$$\begin{aligned} k_2(y,z) &= \frac{\beta_1-\mu}{\lambda_2-\lambda_3}y - \frac{\lambda_3}{\lambda_2-\lambda_3}z, \\ k_3(y,z) &= -\frac{\beta_1-\mu}{\lambda_2-\lambda_3}y + \frac{\lambda_2}{\lambda_2-\lambda_3}z. \end{aligned}$$

Since the unstable manifold ε_1^u is tangent to P at E_1, there is a small $k > 0$ such that the unstable manifold ε_1^u of E_1 can be expressed as

$$\varepsilon_1^u = \left\{(x_1(x_2,y,z), x_2, y, z) : (x_2,y,z) \in [0,k]^3\right\}$$

where

$$\begin{aligned} x_1(x_2,y,z) = 1 - &\frac{\alpha_1 r_1}{(\lambda_2+r_1)(\lambda_3+r_1)}y - \\ &\frac{\alpha_1\lambda_2\lambda_3}{(\beta_1-\mu)(\lambda_2+r_1)(\lambda_3+r_1)}z + \eta(x_2,y,z), \end{aligned} \tag{3.15}$$

and the function $\eta(x_2,y,z)$ satisfies

$$\eta(x_2,y,z) = O(x_2^2+y^2+z^2) \quad \textit{as} \quad (x_2,y,z) \to (0,0,0). \tag{3.16}$$

Select a sufficiently small $\epsilon > 0$ with $k \geq \sigma_2\epsilon$. It follows from (3.15) and (3.16) that

$$0 < x_1(x_2,y,z) < 1, \text{ for } x_2 \in [0,\epsilon], \ y \in [0,\epsilon], \ z \in [\sigma_1 y, \sigma_2 y]. \tag{3.17}$$

Now, we define $\gamma \subset \varepsilon_1^u$ as

$$\gamma = \{(x_1(\epsilon,\epsilon,z), \epsilon, \epsilon, z) : z \in [\sigma_1\epsilon, \sigma_2\epsilon]\},$$

and let $p_1^* = (x_1(\epsilon,\epsilon,\sigma_1\epsilon), \epsilon, \epsilon, \sigma_1\epsilon)$ and $p_2^* = (x_1(\epsilon,\epsilon,\sigma_2\epsilon), \epsilon, \epsilon, \sigma_2\epsilon)$. Then it is clear that $p_i^* \in P_i$, for $i = 1, 2$ and γ is a curve connecting p_1^* and p_2^*. Moreover, by (3.17), it is obvious that $\gamma \backslash \{p_1^*, p_2^*\} \subset \mathrm{Int}(\Sigma)$ which is denoted the interior set of Σ. □

Lemma 3.4. *For each $c \geq c_1$, there is a point $p_* \in \varepsilon_1^u \cap \mathrm{Int}(\Sigma)$ such that the solution $\Phi_t(p_*)$ of (3.5) through the p_* stays in $\mathrm{Int}(\Sigma)$ for all $t \geq 0$.*

Proof. Let $\gamma \subset \varepsilon_1^u \cap \Sigma$ be defined as in Lemma 3.3. We define two subsets γ_1 and γ_2 of γ as follows:

$$\gamma_i = \{p \in \gamma : \text{ there is a } t \geq 0 \text{ such that } \Phi_t(p) \in P_i \text{ for the first time}\},$$

for $i = 1, 2$. By Lemma 3.3, it is clear that γ_1, γ_2 are nonempty and disjoint. Taking points of the curve γ as initial conditions of (3.5), we will see the evolutionary curve under the vector field of (3.5).

Next, we show that γ_1 and γ_2 are open relative to γ, and the case of γ_1 is verified only. Let p be a point of γ_1 except for the endpoint of γ, and by the Lemma 3.2 the vector field of (3.5) at each point in the plane $P_i (i = 1,\ 2)$ points to the exterior of Σ. Hence there is a positive time $t > 0$ such that $\Phi_t(p) \notin \Sigma$. This induces that there is a numberhood of $\Phi_t(p)$ such that $B_\varepsilon(\Phi_t(p)) \cap \Sigma = \emptyset$ for some positive number ε. By the continuity of solutions on initial values, we can find a positive number δ such that $B_\delta(p) \subset \Sigma$ and $B_\delta(p) \cap \gamma \subset \gamma_1$, which implies that γ_1 is open relative to γ. If p is the endpoint of γ, we can also show it similarly.

It is well known that a connected set cannot be a disjoint union of two open subsets. Hence, $\gamma \backslash (\gamma_1 \cup \gamma_2) \neq \emptyset$, because of connectedness of γ. Let $p_* \in \gamma \backslash (\gamma_1 \cup \gamma_2)$. Then the definitions of γ and γ_i imply that

$$\Phi_t(p_*) \in \mathrm{Int}(\Sigma) \quad \text{for all } t \geq 0.$$

We complete the proof. □

Define the set

$$G := \{p \in \gamma : \phi(t; p) \in \mathrm{int}(\Sigma) \text{ for all } t > 0\},$$

which is non-empty by Lemma 3.4.

Theorem 3.5. *If assumptions $\mu < \beta_1$, (2.10) and (2.12) hold with $c > c_1$, then there exists a traveling wave solution from E_1 to E^*.*

Proof. Consider the Lyapunov function defined on G,

$$\begin{aligned} L(t) =& \frac{\beta_1}{\alpha_1}(X_1 - u_1^* \ln X_1) + \frac{\beta_2}{\alpha_2}(X_2 - u_2^* \ln X_2) + (Y - v^* \ln Y) \\ & - (Y - Z)(1 - \frac{v^*}{Y}) \end{aligned} \tag{3.18}$$

Then

$$\begin{aligned} \frac{dL}{dt} &= \frac{\beta_1}{\alpha_1}(\dot{X}_1 - u_1^* \frac{\dot{X}_1}{X_1}) + \frac{\beta_2}{\alpha_2}(\dot{X}_2 - u_2^* \frac{\dot{X}_2}{X_2}) + (\dot{Y} - v^* \frac{\dot{Y}}{Y}) \\ &= -\frac{\beta_1}{\alpha_1}(X_1 - u_1^*)^2 - \frac{\beta_2}{\alpha_2}(X_2 - u_2^*)^2 \leq 0. \end{aligned}$$

This implies the function L is monotone decreasing for $t \geq 0$, and it is clear that the righthand side of (3.18) approaches to infinity only if

$$X_1(t) \to 0^+,\ X_2(t) \to 0^+,\ Y(t) \to 0^+ \text{ or } Y(t) \to \infty. \tag{3.19}$$

By applying LaSalle's invariant principle, the boundedness of solutions is a requirement.

In addition, for $t \geq 0$,

$$\begin{aligned} L(0) \geq L(t) =& \frac{\beta_1}{\alpha_1}(X_1(t) - u_1^* \ln X_1(t)) + \frac{\beta_2}{\alpha_2}(X_2(t) - u_2^* \ln X_2(t)) \\ &+ (Y(t) - v^* \ln Y(t)) - (Y(t) - Z(t))(1 - \frac{v^*}{Y(t)}) \\ =& \frac{\beta_1}{\alpha_1}(X_1(t) - u_1^* \ln X_1(t)) + \frac{\beta_2}{\alpha_2}(X_2(t) - u_2^* \ln X_2(t)) \\ &- v^* \ln Y(t) + v^* + Z(t)(1 - \frac{v^*}{Y(t)}) \\ \geq& \frac{\beta_1}{\alpha_1}(X_1(t) - u_1^* \ln X_1(t)) + \frac{\beta_2}{\alpha_2}(X_2(t) - u_2^* \ln X_2(t)) \\ &- (\sigma_2 - 1)v^* + \sigma_1 Y(t) - v^* \ln Y(t), \end{aligned}$$

which implies that

$$\begin{aligned} L(0) + (\sigma_2 - 1)v^* \geq& \frac{\beta_1}{\alpha_1}(X_1(t) - u_1^* \ln X_1(t)) + \frac{\beta_2}{\alpha_2}(X_2(t) - u_2^* \ln X_2(t)) \\ &+ \sigma_1 Y(t) - v^* \ln Y(t). \end{aligned}$$

By the above inequality, it is evident that (3.19) could not happen. Therefore, this shows that functions $X_1(t)$, $X_2(t)$, $Y(t)$ are bounded below from zero and bounded above, and it also implies that $Z(t)$ has the same properties. Hence, applying LaSalle's Invariance Principle, the ω-limit set of any solution of (3.5) is contained in the maximal invariant subset of $\{dL/dt = 0\} = \{(u_1^*, u_2^*, Y, Z) : Y, Z > 0\}$, which is the singleton $\{(u_1^*, u_2^*, v^*, v^*)\}$. Hence, by LaSalle's invariant principle, all solutions starting from G converge to E^*

Therefore we take the solution of (3.5), $\Phi_t(p_*)$, with initial point p_* defined on Lemma 3.4. Then $\Phi_t(p_*) \to E_1$ as $t \to -\infty$, since $p_* \in \varepsilon_1^u$, the unstable manifold of E_1. Moreover, $\Phi_t(p_*) \to E^*$ as $t \to \infty$, since $p_* \in G$. The proof is complete. □

Theorem 3.6. *The system (1.1) does not have a positive traveling wave solution connecting* E_1 *and* E_* *for wave speed* $0 < c < c_2 \equiv 2\sqrt{D(\beta_1 - \mu)}$.

Proof. If $0 < c < c_2$, then λ_2 and λ_3 in (3.9) are a pair of complex eigenvalues with positive real parts. By looking at the associated eigenvectors given in (3.10), one can conclude that if a solution is in the unstable manifold of the equilibrium E_1, then its $v(t)$ component can not be nonnegative all the time when the solution converges to E_1 as $t \to -\infty$. This completes the proof. □

4. Numerical Simulations, Biological Implications and Discussions

This work investigates the three-species one-predator-two-prey models (1.1) and (1.2) analytically. For (1.2), with assumptions **(A1)** and **(A2)**, all global dynamics can be classified into three cases, the global stabilities of E_{1v}, E_{2v} and E^*, by the classification of parameters in Table 1. The methodology of the proof for the global stabilities of equilibria, E_{1v} and E_{2v}, is by the comparison principle with the inequalities, (2.6) and (2.7), respectively. Alternatively, we apply LaSalle's invariant principle by constructing the Lyapunov functions to show the global stability of the coexistent equilibrium E^*. In particular, the inequalities (2.9) and (2.11) are essential. They are not only to ensure the stabilities of E_{1v} and E_{2v} (Lemma 2.4, Proposition 2.6), respectively, but inverse of they without equalities are also to guarantee the existence and globally asymptotically stability of the E^* (Lemma 2.5, Proposition 2.7). Moreover, these two inequalities (2.9) and (2.11) cannot hold simultaneously when the equilibria

E_{1v} and E_{2v} exist $(0 < \mu < \beta_1 < \beta_2)$. This fact excludes the possibility of bistability, and we imply that the direct competition between two prey induces bistability.

However, the indirect competition, the apparent competition [11, 12], also happens between two prey. For $0 < \mu < \beta_1 < \beta_2$, the equilibria E_{1v} and E_{2v} exist, and let us rewrite (2.6) and (2.7), which are equivalent to (2.11) and (2.9), with the form,

$$\text{either} \quad \bar{v}_1 > \frac{r_2}{\alpha_2} \quad \text{or} \quad \bar{v}_2 > \frac{r_1}{\alpha_1}. \tag{4.1}$$

We have defined the parameter $\delta = \frac{r_1\alpha_2}{r_2\alpha_1}$ and these two ratios $\frac{r_1}{\alpha_1}$ and $\frac{r_2}{\alpha_2}$ can be taken as parameters of the indirect competition between u_1 and u_2 under the r-strategy [28] of u_1 and u_2. The constant $\bar{v}_1$ is the balance state of v in (2.2) with u_1's birth rate r_1 and consuming rate α_1 in a two-species u_1-v system only. Hence, the left inequality of (4.1) means that the species u_2 loses the apparent competition with u_1. Hence, u_2 dies out eventually. Similarly, the right inequality of (4.1) means that the species u_1 loses apparent competition with u_2. Hence, u_1 dies out eventually. And we have shown that these two inequalities of (4.1) cannot hold simultaneously. Alternatively, if

$$\bar{v}_1 < \frac{r_2}{\alpha_2} \quad \text{and} \quad \bar{v}_2 < \frac{r_1}{\alpha_1}.$$

hold, the species u_1 and u_2 can sustain each other under the apparent competition. Therefore, the positive equilibrium E^* exists and is globally asymptotically stable. For $\beta_1 < \mu < \beta_2$, it is obvious that (2.6)/(2.11) is false. If (2.7) is true, then E_{2v} is GAS and E^* does not exist. Otherwise, E^* exists and is GAS. We can argue similarly for the last case $\beta_2 < \mu < \beta_1 + \beta_2$.

For diffusive system (1.1), by using the high-dimensional shooting method, we have shown the existence of transition between two different spatial homogeneous states, that is the existence of traveling wave solution from E_1 to E^* for $c > c_1 = 2\sqrt{D(\beta_1 + \beta_2 - \mu)}$, and non-existence of TWS for $0 < c < c_2 = 2\sqrt{D(\beta_1 - \mu)}$ which is exact the minimal speed for the classical two-dimensional predator-prey system. It is easy to see that $0 < c_2 < c_1$, and if we treat c_1 and c_2 as functions of β_1 and β_2 then $\lim_{\beta_2 \to 0} c_1(\beta_1, \beta_2) = c_2(\beta_1)$. This means that the wave speed c_2 is the so-called minimal speed when the species u_2 is decoupled from system (1.1). So, it is an interesting issue what the meaning of the gap between c_2 and c_1 is.

Comparing the results of [1], Ai et. al. consider the following diffusive one-predator-n-prey models,

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = d_i \dfrac{\partial^2 u_i}{\partial x^2} + r_i u_i(1 - u_i) - \alpha_i u_i v, & x \in \mathbb{R},\ t > 0, \quad i = 1, \cdots, n, \\ \dfrac{\partial v}{\partial t} = D \dfrac{\partial^2 v}{\partial x^2} - \mu v + v \displaystyle\sum_{i=1}^{n} \beta_i u_i, & x \in \mathbb{R},\ t > 0. \end{cases} \tag{4.2}$$

They first show the existence of a weak traveling wave solution approaching $(1, \cdots, 1, 0)$ as $\xi \to -\infty$ by the method of upper-lower solution and then construct a Lyapunov function to guarantee the convergence to E^*. And they get that system (4.2) has a positive traveling wave solution with speed c connecting E_{12} and E^* if and only if $c \geq c^* \equiv 2\sqrt{D(\Sigma_{i=1}^{n}\beta_i - \mu)}$. So we have the consistent critical wave speeds c_1 and c^* compared to their results, although our TWS are from E_1 and with a different proof strategy.

Similarly, it can also be shown that the existence of a traveling wave solution from E_2 to E^* if $\mu < \beta_2$, (2.10) and (2.12) hold. Motivated by Table 1, it is natural to consider the possibilities of other cases. For example, the traveling wave solution from E_{1v}/E_{2v} to E^* with the same assumptions. We left these open problems to be considered in the following work.

We close this work by performing interesting numerical simulations with `FiPy` [10], an object-oriented PDE solver written in `Python`. In the numerical setting, we consider system (1.1) in one-dimensional bounded space [0, 10] with Neumann boundary condition and parameters, $r_1 = 0.7$, $r_2 = 0.3$, $\alpha_1 = 0.5$, $\alpha_2 = 0.2$, $\beta_1 = 0.5$, $\beta_2 = 0.6$ and $\mu = 0.15$, which are satisfied the assumptions of Theorem 3.5. Moreover, the parameter $\delta = r_1\alpha_1/(r_2\alpha_1) = 0.14/0.15 < 1$, u_2 wins the apparent competition. By straightforward computations, we have the positive equilibrium $E^* = (u_1^*, u_2^*, v^*) \approx (0.10377, 0.16352, 1.2547)$ which is globally asymptotically stable in $\mathbb{R}^3_+$. In the ODE simulation by a standard multistep algorithm (`solve_ivp` from `scipy.integrate`) with the initial condition $(1.0, 0.1, 0.09)$, the time courses of each species are shown in panel (a) of Figure 2. You can see that the simulation starts from a u_1 saturated and u_2, v invading state, and then the system approaches the coexistent state eventually.

In the PDE simulation, we set initial functions: $u_1(x, 0) = 1.0$, $u_2(x, 0) = 0.1$ on [0, 10] and

$$v(x,0) = \begin{cases} 0, & \text{if} \quad 0 \le x \le 9.9, \\ 0.09, & \text{if} \quad 9.9 \le x \le 10, \end{cases}$$

shown in panel (b) of Figure 2. Simulation results are shown in panels (c) and (d). In panel (c), we plot the time course of each species at $x = 5$; that is, we plot $u_1(5, t)$, $u_2(5, t)$, and $v(5, t)$ for time t up to 5000. In panel (d), the transition of all species is shown on $x \in [0, 10]$ for time t up to 5000. You can see almost the same dynamics happening here, although the transition is much slower than ODE because of the diffusive effect.

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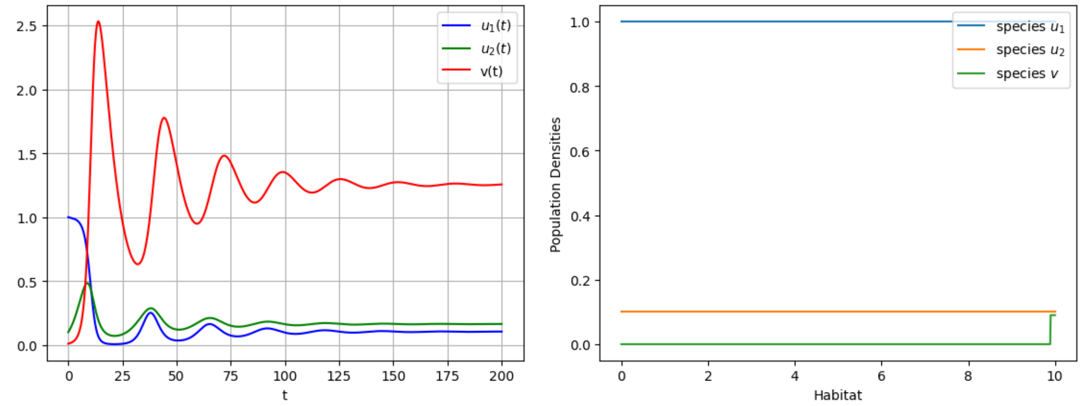
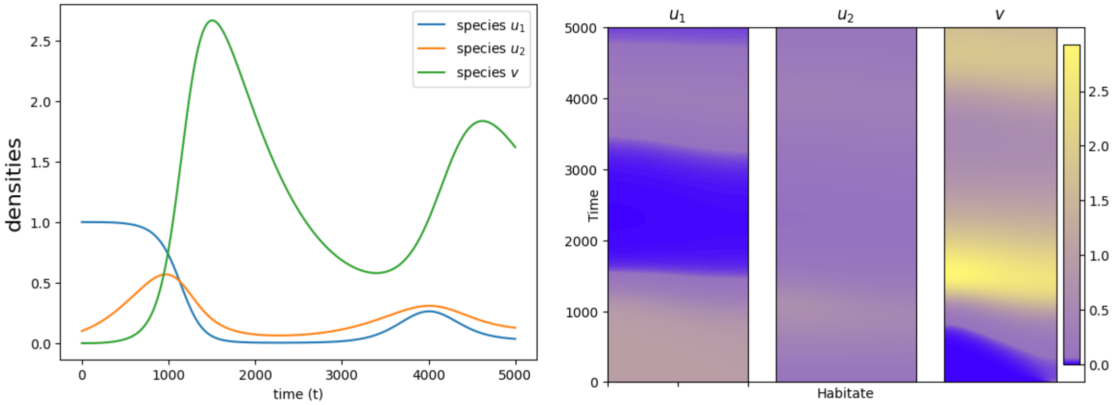


(a) time courses of each species of (1.2). (b) Initial functions of (1.1).



(c) time courses of each species at $x = 5$. (d) simulation results of (1.1)

FIGURE 2. Numerical simulations for parameters $r_1 = 0.7$, $r_2 = 0.3$, $\alpha_1 = 0.5$, $\alpha_2 = 0.2$, $\beta_1 = 0.5$, $\beta_2 = 0.6$ and $\mu = 0.15$. Please see the details in the context.

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