

## A TWO-STAGE BULK-SERVICE QUEUEING MODEL WITH REWORK, CLOSEDOWN AND MULTIPLE VACATIONS

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**ABSTRACT.** In the manufacturing industry, the quality of products is of great significance in the preservation of the brand image. Quality inspection is defined as a procedure for ensuring the compliance, reliability, uniformity, and standard of a specific product. Through effective inspection, manufacturers are able to detect problems that require correction and prevent errors to ensure quality products for customers. To study such scenarios in queueing systems, we examine a two-stage bulk service single server with batch arrivals, incorporating rework, system closedown, and multiple vacation periods. Once a batch of fabric completes the first stage of regular service, the server proceeds to the second inspection stage for the same batch. If the inspection stage identifies defects, the server returns the batch for rework in the regular service stage with a specific probability. The server initiates regular service for the next batch with a specified probability when inspection yields no defects, and the queue length exceeds " $a$ ". Conversely, if the queue length is below " $a$ " and no defects are present, the server may enter a closedown state with a given probability. After closedown, the server can take multiple vacations of random durations. Using the supplementary variable technique, we develop a model to analyze the probability-generating function of the queue size at any given moment. Additionally, we explore performance metrics through numerical examples, provide a cost analysis, and present the findings with graphical representations.

### 1. INTRODUCTION

In the current and fast growing industrial world, it is important to keep a good check on product quality to avoid the destruction of the business and disappointment of clients. Quality inspection is a process of verifying that a product is uniform, reliable and has met certain requirements. It enables manufacturers to locate poor quality products at an early stage hence reducing the probability of delivering such products into the market thus reducing on the costs of products that have been returned or recalled. Incorporation of the queueing theory in the quality inspection procedures presents a robust framework for measuring and enhancing these activities. More precisely, the model suitable for manufacturing environments when single items are culminated in batches can be the bulk service queue for batch arrival. This concept ensures that most products are handled with ease since they are placed in batches that allow easy and time-saving inspection.

Moreover, addition of rework mechanisms in the queueing model leads to identification of defective items and correction of the same before reaching the next stage in the production process, hence enhancing quality as well as minimizing scrap. Additionally, closedown in manufacturing can affect the manufacturing process regularly, thereby causing a high amount of time wastage, extended wait list of the orders that are pending, and poor utilization of resources. Real-life manufacturing systems too undergo the phenomenon of server downtime which in queueing theory is captured under server vacations. This enabling capacity of the model enables the planning of resource use to control the

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downtimes in the most effective manner. Manufacturers may design comprehensive plans to improve the effectiveness and efficiency of their inspection operations by including these elements: bulk service queues, rework for defects, server vacations and closedown. It enhances the product quality and standard with the help of this extended strategic plan and it will also enhance the brand image and customers' confidence which leads to the manufacturing success in a highly competitive environment.

**1.1. Literature survey.** The following outlines a detailed and systematic review of previous studies to understand various existing research contributions and reveal the current state of research gaps.

Arumuganathan and Jeyakumar [3] explored a bulk queue model that incorporates multiple vacation periods, setup times, an N-policy, and closedown times. Similarly, Jeyakumar and Senthilnathan [9] examined a bulk queueing system with server breakdowns, multiple vacations, and closedown times. Ayyappan and Karpagam [4] studied a queueing system featuring two distinct service types: multiple vacations, closedown times, balking, and the inclusion of a standby server. In another study, Palaniammal and Kumar [21] analyzed a bulk queue system with two-stage heterogeneous services, multiple vacation periods, closedown, server breakdowns, and two types of renovation processes. Nithya and Haridass [18] considered a two-phase bulk service system with batch arrivals, closedown, and interrupted vacations. These models share common characteristics such as bulk arrivals and services, server vacation times, and closedown periods. These studies utilized probability-generating functions to derive the distribution of queue lengths and other performance measures, with applications in manufacturing, chemical processing, and data communication systems.

In a different context, Shapique et al. [23] investigated a differentiated vacation queueing system with closedown periods and impatient customers, focusing on energy-saving strategies for WiMAX and HAP systems. Their analysis included transient and steady-state behavior and a cost-profit analysis using advanced mathematical methods. Laxmi et al. [15] analyzed a bulk input general service queue with a second optional service, along with balking and feedback mechanisms. Their model provided a primary service to all customers, while an optional secondary service was available to those who required it. Begum and Choudhury [6] studied an N-policy queueing system that includes setup times, two-phase heterogeneous services, balking, feedback, and server vacations, deriving the joint probability distribution for the server state and queue length in both elapsed and remaining time scenarios.

Karpagam et al. [11] examined a bulk queueing system with rework processes in the manufacturing sector, considering failures and a single vacation period. Their model accounts for the possibility that specific batches might require rework, with a certain probability of being sent back for this purpose. Niranjana et al. [19] analyzed store-and-forward telemedicine with queueing networks, where feedback acted as structured server vacations that helped maintain balance, reduce waiting times, and stabilize patient flows. Xu et al. [27] introduced a two-phase heterogeneous service retrial queueing model, where regular customers undergo two service phases while priority customers only require one. This model also integrates collisions and delayed vacations, optimizing these elements. Ayyappan and Nithya [5] conducted a steady-state analysis of a priority-based bulk service queueing model with vacations, setup, breakdown and repair, offering sensitivity insights into system reliability.

In the  $M^X/G/1$  framework, Abdollahi and Salehi Rad [1] proposed a batch arrival retrial queue that includes two-phase services, admission, and feedback, where customers may either return to the orbit or exit the system based on specific probabilities. Later, Abdollahi et al. [2] extended this concept to a k-phase service model, where the first phase is mandatory and the others are optional. These models introduce feedback as a probabilistic process, allowing customers to return to the orbit or leave the system after each service phase. This approach is similar to data packet retransmission or rework on a production line. Rao et al. [22] explored a two-phase system in which customers are first served in batches during the initial phase, followed by individual service in the second phase. Keerthiga and

Indhira [14] studied a two-phase service model within an  $M/G/1$  queueing system, focusing on retrial customers in scenarios where servers may experience breakdowns, delayed repairs, or take working vacations.

Mahanta et al. [16] investigated a single-server queueing model with two types of general heterogeneous services. In their model, Bernoulli's feedback determines the choice between treating customers as repeat customers or as new customers for another regular service. This flexibility allows customers to select between two service representatives and decide whether to request the same service again or provide feedback based on specific probabilities. Haridass and Arumuganathan [8] examined a bulk arrival queueing system where the server can accept or reject reservice requests. Their model incorporates a probability of the server accepting these requests, which enhances the system's adaptability. Karpagam and Lokesh [12] presented a novel bulk queueing framework that blends fixed batch services with inspection, rework, and multiple vacations, offering practical insights for improving reliability and productivity in real-world manufacturing systems. Somasundaram and Divya [24] critically examined how vacation periods influenced queueing performance, showing that server downtime balanced efficiency recovery while backup servers ensured continuity and resilience in real-world systems.

Myltalas and Zazanis [17] studied an  $M/G/1$  system with Bernoulli feedback, where customers can rejoin the system after service completion with a certain probability. Their model also incorporates batch arrivals and multiple vacation policies. Varalakshmi et al. [26] explored a single-server queue with immediate feedback, allowing customers to provide a limited number of immediate feedbacks after their service. Karpagam [10] analyzed a batch arrival bulk service queue model with rework processes, server breakdowns, and multiple vacations, mainly focusing on applications in manufacturing industries. Sun et al. [25] contributed to queueing theory by studying customer joining strategies in Markovian queues with batch-limited service and multiple vacation periods. Das and Pradhan [7] analyzed a batch size-dependent queue with server breakdowns, close-down times, and vacations, offering practical insight into real-world service reliability. Karthick and Suvitha [13] examined an  $M/M/3$  heterogeneous server queue under multiple vacations, showing how repeated idle periods affect efficiency and reliability.

**1.2. Research gap.** Section 1.1 provides an overview of the existing research on queueing systems, highlighting the extensive analysis of components such as two-phase services, reserve mechanisms, close-down policies, and multiple vacation strategies. Arumuganathan and Jeyakumar [3] and Jeyakumar and Senthilnathan [9] studied bulk queueing models that include multiple vacation periods and closedown times, while Ayyappan and Karpagam [4] and Palaniammal and Kumar [21] expanded these models by incorporating two heterogeneous bulk services and server breakdown policies.

Nithya and Haridass [18] focused on models with vacation interruptions, two phases of bulk service, and closedown periods. Haridass and Arumuganathan [8] proposed a model where the server can accept or reject reservice requests and batch service queues that feature multiple vacation periods. Karpagam et al. [11] examined rework processes and batch handling in manufacturing settings, incorporating initial failures and a single vacation. Karpagam [10] also investigated bulk queues that include rework processes and two distinct vacation policies.

However, a research gap exists in integrating bulk queues with two-stage service, rework, closedown policies, and multiple vacation strategies into a unified framework. While individual components have been studied, no comprehensive model currently addresses the combined impact of these factors on system performance, reliability, and cost-efficiency. This integration could be particularly valuable in complex industrial systems, such as flexible manufacturing or data communication networks, where service interruptions, resource optimization, and customer reservice requests significantly affect operational efficiency. Exploring this gap would offer new insights into queue dynamics and provide a more holistic approach to managing real-world queueing systems under varying operational constraints.

**1.3. Structure of the paper.** Eight sections organize the remainder of the paper. Section 2 overviews the proposed model, its realistic applications, and the associated notations. Section 3 discusses the steady-state queue size distribution. Section 4 introduces the probability-generating function that describes the queue size distribution in its steady state. Section 5 presents several practical performance metrics. Section 6 develops the cost model using an analytical approach. Section 7 offers numerical results and compares them graphically. Finally, Section 8 concludes the paper.

## 2. DETAILED MODEL DESCRIPTION AND ITS IMPLEMENTATION IN REAL WORLD

Figure 1 shows the pictorial representation of the proposed model, which describes the batch arrival system of two stage general bulk service single server queueing systems with rework, closedown and multiple vacations as explained below.

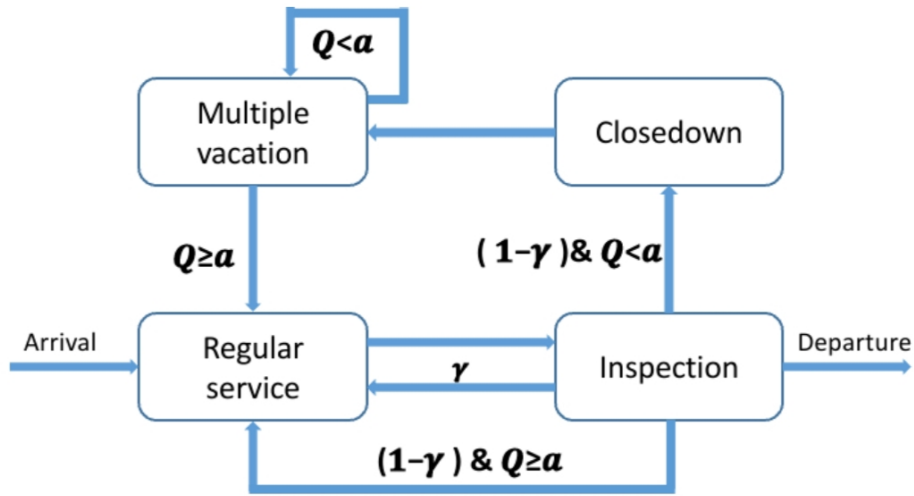


FIGURE 1. Pictorial depiction of the model,  $Q$ -Queue length

Fabrics arrive in groups and proceed to the first stage (regular service) when the queue length reaches or exceeds a specified threshold ( $Q \geq a$ ), which is determined according to the general bulk service rule proposed by Neuts [20]. Under this rule, the server operates with a lower limit  $a$  and an upper limit  $b$  for the batch size. If the queue length  $Q$  lies between  $a$  and  $b$ , the entire group of  $Q$  items is served at once. When  $Q$  reaches or exceeds  $b$ , exactly  $b$  items are processed, while the remaining  $Q - b$  items stay in the queue for the following service round. After regular service, the batch undergoes a second stage (inspection) to evaluate its quality. If defects are detected, the batch is sent back for rework with a probability of  $\gamma$ . If no defects are found, the server evaluates the queue length. If  $Q \geq a$  after inspection, the server continues regular service for the next batch in the queue with a probability of  $(1 - \gamma)$ . However, if  $Q < a$  and no rework is required, the server enters a closedown work with a probability of  $(1 - \gamma)$ , followed by multiple vacations of random lengths until the queue size meets the threshold again. This model ensures efficient resource utilization, minimizes delays, and enhances product quality through iterative rework and inspection processes.

**2.1. Implementation of the model in real world.** In the textile manufacturing process, especially in the areas of dyeing and finishing, these processes are central to the quality of the final product fabric. The process starts with a work-in-progress input of raw fabrics that undergo the dyeing, washing and finishing process so as to achieve the right color, texture and durability. The queueing system is used

during the dyeing process. Fabrics are placed in large dyeing machines, which work in batches or in groups. This process starts as soon as there is a reasonable quantity of fabric ready to be dyed. The dyeing machine will run if the queue contains a minimum load of fabric denoted by ( $a$ ). If the required amount of fabric is not available, the machine does not run, and the operator may experience higher operating costs due to machine inactivity. Once the dyeing machine starts working, it begins the process in which the fabric is treated with a dye bath at a necessary temperature to achieve a uniform coloring. After dyeing, the fabric undergoes many finishing processes like washing, drying and smoothing, exhibiting desired quality characteristics, including ensuring the fabric meets required color specifications, touch and wear ability.

However, if imperfections are found during an inspection stage, such as uniformity in dyeing an improperly implemented color shade or any other imperfection, additional work may be required. This rework can greatly impact the total throughput of the process. For instance, if the fabric does not qualify within the specified color ranges or has defects such as fading uneven dyes, it is returned to the dyeing machine. In reworking the dyeing and finishing, additional time and resources are needed to correct the defects, reducing production efficiency.

After the inspection stage, if the fabric has been processed to the required quality and no further processing is needed, and the fabric quantity is not sufficient to reach the minimum amount, “ $a$ ”, needed to begin the next batch, the operator initiates a closedown work process. This includes setting up the machine for the next run, washing, maintenance, and calibration. After the close-down tasks are done, the operator proceeds to the second type of work, which is known as vacation. In this vacation, operators may perform other supporting duties, for example, general maintenance on other equipment or clerical work, ensuring efficient use of time while waiting for the next batch of fabric to meet the minimum requirement “ $a$ ” and start a new production cycle.

**2.2. Notations.** The following notations describe key parameters and state variables used in the model:

Notation	Description
$\lambda$	Arrival rate
$X$	Batch size random variable
$g_k$	Probability of a batch of size $k$
$X(z)$	Probability generating function (PGF) of batch size
$\aleph_1(\tau)$	Number of items in service station at time $\tau$
$\aleph_2(\tau)$	Number of items in queue at time $\tau$
$\gamma$	Rework probability
$(1 - \gamma)$	Probability of no rework

Below is a summary of the key notations used in the model for Cumulative Distribution Functions (CDF), Probability Density Functions (PDF), Laplace-Stieltjes Transforms (LST), and Remaining Time (RT):

	CDF	PDF	LST	RT
Regular Service	$M_1(\beta)$	$m_1(\beta)$	$\tilde{M}_1(\vartheta)$	$M_1^0(\tau)$
Inspection	$M_2(\beta)$	$m_2(\beta)$	$\tilde{M}_2(\vartheta)$	$M_2^0(\tau)$
Closedown	$C_w(\beta)$	$c_w(\beta)$	$\tilde{C}_w(\vartheta)$	$C_w^0(\tau)$
Vacation	$M_v(\beta)$	$m_v(\beta)$	$\tilde{M}_v(\vartheta)$	$M_v^0(\tau)$

(1)  $\kappa(\tau) = \{1, 2, 3, 4\}$  indicates the server state:

- 1 = Regular service
- 2 = Inspection
- 3 = Closedown
- 4 = Vacation

(2)  $\mathcal{Y}(\tau) = l$  denotes that the server is on its  $l$ -th vacation.

The state probabilities can be expressed as follows:

$$S_{m,p}^{(1)}(\beta, \tau)\Delta\tau = Pr\{\aleph_1(\tau) = m, \aleph_2(\tau) = p, \beta \leq M_1^0(\tau) < \beta + \Delta\tau, \kappa(\tau) = 1\},$$

$$\text{where } a \leq m \leq b, p \geq 0,$$

$$S_{m,p}^{(2)}(\beta, \tau)\Delta\tau = Pr\{\aleph_1(\tau) = m, \aleph_2(\tau) = p, \beta \leq M_2^0(\tau) < \beta + \Delta\tau, \kappa(\tau) = 2\},$$

$$\text{where } a \leq m \leq b, p \geq 0,$$

$$C_u(\beta, \tau)\Delta\tau = Pr\{\aleph_2(\tau) = u, \beta \leq C_w^0(\tau) < \beta + \Delta\tau, \kappa(\tau) = 3\}, \text{ where } u \geq 0,$$

$$M_{l,u}(\beta, \tau)\Delta\tau = Pr\{\aleph_2(\tau) = u, \beta \leq M_v^0(\tau) < \beta + \Delta\tau, \kappa(\tau) = 4, \mathcal{Y}(\tau) = l\},$$

$$\text{where } u \geq 0, l \geq 1.$$

### 3. STEADY STATE QUEUE SIZE DISTRIBUTION

This section obtains the steady-state queue size distribution for the proposed model. We consider the server's behavior in different states: providing regular service, performing inspection, closedown work, or taking vacations. For each state, we define the system's governing equations and apply the Laplace-Stieltjes Transform (LST) and Probability Generating Function (PGF) to obtain the steady-state probabilities.

#### 3.1. Server Busy in Regular Service.

$$-S_{m,0}^{(1)'}(\beta) = -\lambda S_{m,0}^{(1)}(\beta) + \gamma S_{m,0}^{(2)}(0)m_1(\beta) + (1-\gamma) \sum_{r=a}^b S_{r,m}^{(2)}(0)m_1(\beta) + \sum_{l=1}^{\infty} M_{l,m}(0)m_1(\beta),$$

$$a \leq m \leq b, \quad (3.1)$$

$$-S_{m,p}^{(1)'}(\beta) = -\lambda S_{m,p}^{(1)}(\beta) + \gamma S_{m,p}^{(2)}(0)m_1(\beta) + \sum_{k=1}^p S_{m,p-k}^{(1)}(\beta)\lambda g_k, \quad p \geq 1, \quad a \leq m \leq b-1, \quad (3.2)$$

$$-S_{b,p}^{(1)'}(\beta) = -\lambda S_{b,p}^{(1)}(\beta) + \gamma S_{b,p}^{(2)}(0)m_1(\beta) + (1-\gamma) \sum_{r=a}^b S_{r,b+p}^{(2)}(0)m_1(\beta) + \sum_{k=1}^p S_{b,p-k}^{(1)}(\beta)\lambda g_k$$

$$+ \sum_{l=1}^{\infty} M_{l,b+p}(0)m_1(\beta), \quad p \geq 1. \quad (3.3)$$

### 3.2. Server Busy in Inspection.

$$-S_{m,0}^{(2)'}(\beta) = -\lambda S_{m,0}^{(2)}(\beta) + S_{m,0}^{(1)}(0)m_2(\beta), \quad a \leq m \leq b, \quad (3.4)$$

$$-S_{m,p}^{(2)'}(\beta) = -\lambda S_{m,p}^{(2)}(\beta) + S_{m,p}^{(1)}(0)m_2(\beta) + \sum_{k=1}^p S_{m,p-k}^{(2)}(\beta)\lambda g_k, \quad a \leq m \leq b, \quad p \geq 1. \quad (3.5)$$

### 3.3. Server in Closedown.

$$-C'_u(\beta) = -\lambda C_u(\beta) + (1-\gamma) \sum_{r=a}^b S_{r,0}^{(2)}(0)c_w(\beta) + \sum_{k=1}^u C_{u-k}(\beta)\lambda g_k, \quad u \leq a-1, \quad (3.6)$$

$$-C'_u(\beta) = -\lambda C_u(\beta) + \sum_{k=1}^u C_{u-k}(\beta)\lambda g_k, \quad u \geq a. \quad (3.7)$$

### 3.4. Server in Vacation.

$$-M'_{1,0}(\beta) = -\lambda M_{1,0}(\beta) + C_0(0)m_v(\beta), \quad (3.8)$$

$$-M'_{1,u}(\beta) = -\lambda M_{1,u}(\beta) + C_u(0)m_v(\beta) + \sum_{k=1}^u M_{1,u-k}(\beta)\lambda g_k, \quad u \geq 1, \quad (3.9)$$

$$-M'_{l,0}(\beta) = -\lambda M_{l,0}(\beta) + M_{l-1,0}(0)m_v(\beta), \quad l \geq 2, \quad (3.10)$$

$$-M'_{l,u}(\beta) = -\lambda M_{l,u}(\beta) + M_{l-1,u}(0)m_v(\beta) + \sum_{k=1}^u M_{l,u-k}(\beta)\lambda g_k, \quad 1 \leq u \leq a-1, \quad l \geq 2, \quad (3.11)$$

$$-M'_{l,u}(\beta) = -\lambda M_{l,u}(\beta) + \sum_{k=1}^u M_{l,u-k}(\beta)\lambda g_k, \quad u \geq a, \quad l \geq 2. \quad (3.12)$$

The LST is defined as

$$\tilde{S}_{m,p}^{(1)}(\vartheta) = \int_0^\infty e^{-\vartheta\beta} S_{m,p}^{(1)}(\beta) d\beta.$$

Similarly, LST  $\tilde{S}^{(2)}$ ,  $\tilde{C}$  and  $\tilde{M}$  are defined in the same way. Using the LST on both sides of equations (3.1)-(3.12), we obtain

$$\vartheta \tilde{S}_{m,0}^{(1)}(\vartheta) - S_{m,0}^{(1)}(0) = \lambda \tilde{S}_{m,0}^{(1)}(\vartheta) - \gamma S_{m,0}^{(1)}(0) \tilde{M}_1(\vartheta) - (1-\gamma) \sum_{r=a}^b S_{r,m}^{(2)}(0) \tilde{M}_1(\vartheta) - \sum_{l=1}^\infty M_{l,m}(0) \tilde{M}_1(\vartheta), \quad a \leq m \leq b, \quad (3.13)$$

$$\vartheta \tilde{S}_{m,p}^{(1)}(\vartheta) - S_{m,p}^{(1)}(0) = \lambda \tilde{S}_{m,p}^{(1)}(\vartheta) - \gamma S_{m,p}^{(2)}(0) \tilde{M}_1(\vartheta) - \sum_{k=1}^p \tilde{S}_{m,p-k}^{(1)}(\vartheta) \lambda g_k, \quad a \leq m \leq b-1, \quad p \geq 1, \quad (3.14)$$

$$\begin{aligned} \vartheta \tilde{S}_{b,p}^{(1)}(\vartheta) - S_{b,p}^{(1)}(0) &= \lambda \tilde{S}_{b,p}^{(1)}(\vartheta) - \gamma S_{b,p}^{(2)}(0) \tilde{M}_1(\vartheta) - \sum_{k=1}^p \tilde{S}_{b,p-k}^{(1)}(\vartheta) \lambda g_k - \sum_{l=1}^{\infty} M_{l,b+p}(0) \tilde{M}_1(\vartheta) \\ &\quad - (1 - \gamma) \sum_{r=a}^b S_{r,b+p}^{(2)}(0) \tilde{M}_1(\vartheta), \quad p \geq 1, \end{aligned} \quad (3.15)$$

$$\vartheta \tilde{S}_{m,0}^{(2)}(\vartheta) - S_{m,0}^{(2)}(0) = \lambda \tilde{S}_{m,0}^{(2)}(\vartheta) - S_{m,0}^{(1)}(0) \tilde{M}_2(\vartheta), \quad a \leq m \leq b, \quad (3.16)$$

$$\vartheta \tilde{S}_{m,p}^{(2)}(\vartheta) - S_{m,p}^{(2)}(0) = \lambda \tilde{S}_{m,p}^{(2)}(\vartheta) - S_{m,p}^{(1)}(0) \tilde{M}_2(\vartheta) - \sum_{k=1}^p \tilde{S}_{m,p-k}^{(2)}(\vartheta) \lambda g_k, \quad a \leq m \leq b, p \geq 1 \quad (3.17)$$

$$\vartheta \tilde{C}_u(\vartheta) - C_u(0) = \lambda \tilde{C}_u(\vartheta) - (1 - \gamma) \sum_{r=a}^b S_{r,u}^{(2)}(0) \tilde{C}_w(\vartheta) - \sum_{k=1}^u \tilde{C}_{u-k}(\vartheta) \lambda g_k, \quad u \leq a - 1, \quad (3.18)$$

$$\vartheta \tilde{C}_u(\vartheta) - C_u(0) = \lambda \tilde{C}_u(\vartheta) - \sum_{k=1}^u \tilde{C}_{u-k}(\vartheta) \lambda g_k, \quad u \geq a, \quad (3.19)$$

$$\vartheta \tilde{M}_{1,0}(\vartheta) - M_{1,0}(0) = \lambda \tilde{M}_{1,0}(\vartheta) - C_0(0) \tilde{M}_v(\vartheta), \quad (3.20)$$

$$\vartheta \tilde{M}_{1,u}(\vartheta) - M_{1,u}(0) = \lambda \tilde{M}_{1,u}(\vartheta) - C_u(0) \tilde{M}_v(\vartheta) - \sum_{k=1}^u \tilde{M}_{1,u-k}(\vartheta) \lambda g_k, \quad u \geq 1, \quad (3.21)$$

$$\vartheta \tilde{M}_{l,0}(\vartheta) - M_{l,0}(0) = \lambda \tilde{M}_{l,0}(\vartheta) - M_{l-1,0}(0) \tilde{M}_v(\vartheta), \quad l \geq 2, \quad (3.22)$$

$$\vartheta \tilde{M}_{l,u}(\vartheta) - M_{l,u}(0) = \lambda \tilde{M}_{l,u}(\vartheta) - M_{l-1,u}(0) \tilde{M}_v(\vartheta) - \sum_{k=1}^u \tilde{M}_{l,u-k}(\vartheta) \lambda g_k, \quad 1 \leq u \leq a - 1, l \geq 2, \quad (3.23)$$

$$\vartheta \tilde{M}_{l,u}(\vartheta) - M_{l,u}(0) = \lambda \tilde{M}_{l,u}(\vartheta) - \sum_{k=1}^u \tilde{M}_{l,u-k}(\vartheta) \lambda g_k, \quad u \geq a, l \geq 2. \quad (3.24)$$

#### 4. PROBABILITY GENERATING FUNCTION OF THE QUEUE SIZE

The following probability-generating functions (PGFs) are defined in order to derive the PGF for the queue size at an arbitrary time epoch:

$$\left. \begin{aligned} \tilde{S}_h^{(1)}(z, \vartheta) &= \sum_{p=0}^{\infty} \tilde{S}_{h,p}^{(1)}(\vartheta) z^p, \quad S_h^{(1)}(z, 0) = \sum_{p=0}^{\infty} S_{h,p}^{(1)}(0) z^p, \quad a \leq h \leq b, \\ \tilde{S}_h^{(2)}(z, \vartheta) &= \sum_{p=0}^{\infty} \tilde{S}_{h,p}^{(2)}(\vartheta) z^p, \quad S_h^{(2)}(z, 0) = \sum_{p=0}^{\infty} S_{h,p}^{(2)}(0) z^p, \quad a \leq h \leq b, \\ \tilde{C}(z, \vartheta) &= \sum_{u=0}^{\infty} \tilde{C}_u(\vartheta) z^u, \quad C(z, 0) = \sum_{u=0}^{\infty} C_u(0) z^u, \\ \tilde{M}_l(z, \vartheta) &= \sum_{p=0}^{\infty} \tilde{M}_{l,p}(\vartheta) z^p, \quad M_l(z, 0) = \sum_{p=0}^{\infty} M_{l,p}(0) z^p, \quad l \geq 1. \end{aligned} \right\} \quad (4.1)$$

By multiplying equations (3.13) to (3.24) with suitable powers of  $z^p$  and summing over  $p$ , then by using (4.1), we get

$$(\vartheta - u(z))\tilde{S}_m^{(1)}(z, \vartheta) = S_m^{(1)}(z, 0) - \tilde{M}_1(\vartheta) \left[ (1 - \gamma) \sum_{h=a}^b S_{h,m}^{(2)}(0) + \sum_{l=1}^{\infty} M_{l,m}(0) \right] - \gamma S_m^{(2)}(z, 0) \tilde{M}_1(\vartheta),$$

$$a \leq m \leq b - 1, \quad (4.2)$$

$$z^b(\vartheta - u(z))\tilde{S}_b^{(1)}(z, \vartheta) = z^b S_b^{(1)}(z, 0) - \tilde{M}_1(\vartheta) z^b \left[ (1 - \gamma) \sum_{h=a}^b \sum_{p=b}^{\infty} S_{h,p}^{(2)}(0) + \sum_{l=1}^{\infty} \sum_{p=b}^{\infty} M_{l,p}(0) \right]$$

$$- z^b \gamma S_b^{(2)}(z, 0) \tilde{M}_1(\vartheta), \quad (4.3)$$

$$(\vartheta - u(z))\tilde{S}_m^{(2)}(z, \vartheta) = S_m^{(2)}(z, 0) - S_m^{(1)}(z, 0) \tilde{M}_2(\vartheta), \quad a \leq m \leq b, \quad (4.4)$$

$$(\vartheta - u(z))\tilde{C}(z, \vartheta) = C(z, 0) - \tilde{C}_w(\vartheta) (1 - \gamma) \sum_{u=0}^{a-1} \sum_{h=a}^b S_{h,u}^{(2)}(0) z^u, \quad (4.5)$$

$$(\vartheta - u(z))\tilde{M}_1(z, \vartheta) = M_1(z, 0) - C(z, 0) \tilde{M}_v(\vartheta), \quad (4.6)$$

$$(\vartheta - u(z))\tilde{M}_l(z, \vartheta) = M_l(z, 0) - \tilde{M}_v(\vartheta) \sum_{u=0}^{a-1} M_{l-1,u}(0) z^u, \quad l \geq 2. \quad (4.7)$$

Let  $P(z)$  be the PGF of the queue size at an arbitrary time epoch. Then,

$$P(z) = \sum_{h=a}^b \tilde{S}_h^{(1)}(z, 0) \sum_{h=a}^b \tilde{S}_h^{(2)}(z, 0) + \tilde{C}(z, 0) + \sum_{l=1}^{\infty} \tilde{M}_l(z, 0). \quad (4.8)$$

By replacing  $\vartheta = u(z)$  into Equations (4.2)–(4.7), and using the generating function for different states, without loss of generality, Equation (4.8) can be simplified. We denote the following:

$$P(z) = \frac{1}{u(z)y(z)} \left\{ \left[ 1 - \tilde{M}_1(u(z)) \tilde{M}_2(u(z)) \right] \sum_{h=a}^{b-1} d_h [z^b - z^h] + \left[ 1 - \tilde{M}_v(u(z)) \right] A(z) \sum_{h=0}^{a-1} q_h z^h \right.$$

$$\left. + \left[ 1 - \tilde{M}_v(u(z)) \tilde{C}_w(u(z)) \right] A(z) \sum_{h=0}^{a-1} (1 - \gamma) e_h z^h \right\} \quad (4.9)$$

where

$$e_h = \sum_{m=a}^b S_{m,h}^{(2)}(0), \quad q_h = \sum_{l=1}^{\infty} M_{l,h}(0), \quad d_h = (1 - \gamma)e_h + q_h, \quad u(z) = \lambda - \lambda X(z),$$

$$y(z) = z^b \left[ 1 - \gamma \tilde{M}_1(u(z)) \tilde{M}_2(u(z)) \right] - (1 - \gamma) \tilde{M}_1(u(z)) \tilde{M}_2(u(z)), \quad \text{and}$$

$$A(z) = y(z) - \tilde{M}_1(u(z)) \tilde{M}_2(u(z)).$$

**Remark.** The probability-generating function (PGF) must satisfy the condition  $P(1) = 1$ . This requirement is met by setting a term equal to 1 and applying L'Hopital's rule in equation (4.9). Consequently, the steady-state condition for the model's existence is given by

$$\rho = \frac{\lambda X_1 [E(M_1) + E(M_2)]}{b(1 - \gamma)} < 1.$$

**4.1. Computational aspects.** Equation (4.9) contains  $b+a$  unknowns  $e_0, e_1, \dots, e_{a-1}, q_0, q_1, \dots, q_{a-1}$ , and  $d_a, d_{a-1}, \dots, d_{b-1}$ . With the results obtained, we can express  $q_h$  in terms of  $e_h$ , ensuring that only  $b$  constants appear in the numerator. This allows us to derive the probability generating function (PGF) for the number of fabrics, as shown in equation (4.9), with only  $b$  unknowns. By applying Rouché's theorem from complex analysis, we can prove that the polynomial  $y(z)$  has  $b - 1$  zeros inside the unit

circle  $|z| = 1$  and one on the circle. Since  $P(z)$  is analytic both inside and on the unit circle, the numerator must vanish at these points, resulting in  $b$  equations for the  $b$  unknowns. It is worth noting that these equations can be solved using any appropriate numerical method.

4.1.1. *Result.* Let  $q_h$  can be expressed in terms of  $e_h$  as

$$q_h = \sum_{i=0}^n A_i e_{h-i}, \quad h = 0, 1, \dots, a-1.$$

$$A_h = \frac{w_h + \sum_{i=0}^h \beta_i A_{h-i}}{1 - \beta_0}, \quad h = 1, 2, \dots, a-1; \quad A_0 = \frac{\beta_0 \alpha_0}{1 - \beta_0} \quad \text{and} \quad w_h = \sum_{i=0}^h \beta_i \alpha_{h-i}$$

where  $\beta_i$  and  $\alpha_i$  are the probabilities of the 'i' fabrics arrive during vacation and closedown time respectively.

4.2. **PGF of queue size at various epochs.** PGF of queue size at two stage service, closedown and vacation completion epochs

$$M(z) = \frac{1}{u(z)y(z)} \left\{ \text{Big} [1 - \tilde{M}_1(u(z))\tilde{M}_2(u(z))] \left[ \sum_{h=a}^{b-1} d_h [z^b - z^h] - [1 - \tilde{M}_v(u(z))] \sum_{h=0}^{a-1} q_h z^h \right] - [1 - \tilde{M}_v(u(z))\tilde{C}_w(u(z))] \sum_{h=0}^{a-1} (1 - \gamma) e_h z^h \right\} \quad (4.10)$$

$$C_w(z) = \frac{[1 - \tilde{C}_w(u(z))] \sum_{h=0}^{a-1} (1 - \gamma) e_h z^h}{u(z)} \quad (4.11)$$

$$M_v(z) = \frac{[1 - \tilde{M}_v(u(z))] [\tilde{C}_w(u(z)) \sum_{h=0}^{a-1} (1 - \gamma) e_h + \sum_{h=0}^{a-1} q_h] z^h}{u(z)} \quad (4.12)$$

4.3. **Particular case. Case 1:** When there is no inspection and rework then  $P(z)$  becomes

$$P(z) = \frac{1}{(z^b - \tilde{M}_1(u(z)))(-u(z))} \left\{ [\tilde{M}_1(u(z)) - 1] \sum_{h=a}^{b-1} d_h [z^b - z^h] + (z^b - 1) [\tilde{M}_v(u(z)) - 1] \sum_{h=0}^{a-1} q_h z^h + (z^b - 1) [\tilde{M}_v(u(z))\tilde{C}_w(u(z)) - 1] \sum_{h=0}^{a-1} (1 - \gamma) e_h z^h \right\} \quad (4.13)$$

which coincide with the PGF of Palaniammal and Kumar [21] if no two-stage heterogeneous services and server breakdown.

**Case 2:** When closedown time is zero, there is no inspection and rework then  $P(z)$  becomes,

$$P(z) = \frac{(\tilde{M}_1(u(z)) - 1) \sum_{h=a}^{b-1} d_h (z^b - z^h) + (\tilde{M}_v(u(z)) - 1) (z^b - 1) \sum_{h=0}^{a-1} q_h z^h}{(z^b - \tilde{M}_1(u(z)))(-u(z))} \quad (4.14)$$

This result is consistent with Haridass and Arumuganathan [8] for no setup and server's choice of admitting reservice.

## 5. PERFORMANCE MEASURES

In this section, we derive various performance metrics of the concerned queueing model are obtained as follows.

5.1. **Expected queue length.** The expected queue length  $E(Q)$  at an arbitrary time epoch is

$$E(Q) = \left( \frac{dP(z)}{dz} \right)_{z=1} = \frac{Dr^{(II)}Nr^{(III)} - Dr^{(III)}Nr^{(II)}}{3(Dr^{(II)})^2} \quad (5.1)$$

where

$$\begin{aligned} Dr^{(II)} &= 2(-\lambda X_1)K_1, \quad Dr^{(III)} = 3(-\lambda X_1)K_2 + 3(-\lambda X_2)K_1, \quad K_1 = b(1 - \gamma) - M_1M_2, \\ K_2 &= b(b - 1)(1 - \gamma) - \gamma b[2M_1 + 2M_2] - M_{12} - 2M_1M_2 - M_{21}, \\ Nr^{(II)} &= 2(-M_1 - M_2)(b - h) \sum_{h=a}^{b-1} d_h - 2K_3 \sum_{h=0}^{a-1} \left[ (M_{v1} + C_{w1})(1 - \gamma)e_h + M_{v1}q_h \right], \\ Nr^{(III)} &= 3 \left[ (-M_1 - M_2)[b(b - 1) - h(h - 1)] - (M_{12} + M_{21} + 2M_1M_2)(b - h) \right] \sum_{h=a}^{b-1} d_h \\ &\quad + 3 \left[ -(M_{v1} + C_{w1})K_5 - (M_{v2} + C_{w2} + 2M_{v1}C_{w1})K_3 \right] \sum_{h=0}^{a-1} (1 - \gamma)e_h \\ &\quad + 3 \left[ (-M_{v1})K_5 - M_{v2}K_3 \right] \sum_{h=0}^{a-1} q_h, \quad K_3 = K_1 - M_1 - M_2, \\ K_4 &= K_2 - (M_{12} + M_{21} + 2M_1M_2), \quad K_5 = 2h(K_1 - M_1 - M_2) + K_4 \\ M_1 &= \lambda X_1 E(M_1), \quad M_{12} = \lambda X_2 E(M_1) + (-\lambda X_1)^2 E(M_1^2), \quad M_2 = \lambda X_1 E(M_2), \\ M_{21} &= \lambda X_2 E(M_2) + (\lambda X_1)^2 E(M_2^2), \quad C_{w1} = \lambda X_1 E(C_w), \quad M_{v1} = \lambda X_1 E(M_v), \\ C_{w2} &= \lambda X_2 E(C_w) + (\lambda X_1)^2 E(C_w^2), \quad M_{v2} = \lambda X_2 E(M_v) + (\lambda X_1)^2 E(M_v^2). \end{aligned}$$

5.2. **Expected waiting time in the queue.** Using Little's formula, we get expected waiting time at arbitrary epoch is

$$E(W) = \frac{E(Q)}{\lambda X_1} \quad (5.2)$$

5.3. **The length of the server's expected busy time:** The expected length of the busy period  $E(B)$  is given by

$$E(B) = \frac{E(M_1) + E(M_2)}{(1 - \gamma) \sum_{h=0}^{a-1} e_h}, \quad (5.3)$$

where  $E(M_1)$  =mean regular service time and  $E(M_2)$  =mean inspection time,

5.4. **Expected Duration of the Dormant Period.** As a result of multiple vacation processes, let  $I$  be the random variable for the inactive period of time. Then, the anticipated duration of the inactive period is provided by

$$\begin{aligned} E(I) &= E(I_1) + E(C_w), \\ E(I_1) &= \frac{E(M_v)}{1 - \sum_{h=0}^{a-1} \sum_{i=0}^h \sum_{p=0}^{h-i} \beta_p \alpha_{h-i-p} (1 - \gamma) e_i}. \end{aligned} \quad (5.4)$$

where  $E(M_v)$  =mean vacation time and  $E(C_w)$  = mean closedown time.

## 6. COST MODEL

This section describes how the costs are assessed, the effects of various operational decisions, and the measures taken to optimize total expected costs while maintaining quality and productivity of the system. Let us define the following cost parameters:

- $C_s$  : startup cost per unit cycle
- $C_h$  : holding cost per unit customer
- $C_o$  : operating cost per unit time
- $C_r$  : the reward per unit cycle due to vacation
- $C_c$  : closedown cost per unit time

Given that the cycle's length is the sum of its busy and idle periods. The expected cycle duration is calculated as,

$$\mathbf{E}(T_c) = \mathbf{E}(B) + \mathbf{E}(I)$$

Let TC be the total expected cost per unit time of the system. Thus

$$TC = \left[ C_s + C_c E(C_w) - C_r \mathbf{E}(I) \right] \frac{1}{\mathbf{E}(T_c)} + C_h \mathbf{E}(Q) + C_o \rho.$$

## 7. NUMERICAL EXAMPLES

To illustrate the influence of varying system parameters on key performance indicators, a numerical analysis is conducted under carefully chosen example scenarios. This approach sheds light on the interplay between parameters and their impact on performance metrics and the system's cost-efficiency.

**7.1. Parameters and Assumptions.** Below are the specific assumptions and parameter values used in the numerical analysis.

1. Distribution setup: Regular service times and inspection times: 2-Erlang distribution with  $\mu_1$  and  $\mu_2$ . Arrival batch size distribution: Geometric with a mean of 2. Vacation time and closedown time: Exponential with  $\zeta$  and  $\delta$ .

2. Cost parameters:

- $C_s = \text{Re.4}$  (startup cost per unit cycle)
- $C_h = \text{Re.0.50}$  (holding cost per unit item)
- $C_o = \text{Re.5}$  (operating cost per unit time)
- $C_r = \text{Re.1}$  (reward per unit vacation cycle)
- $C_c = \text{Re.0.25}$  (closedown cost per unit time)

3. Iterative Computation: For varying service rates, compute the metrics iteratively to identify the effect on  $\mathbf{E}(Q)$ ,  $\mathbf{E}(W)$ ,  $\mathbf{E}(B)$ ,  $\mathbf{E}(I)$ , and TC. Similarly, analyze the impact of varying arrival rates, and rework probabilities on performance.

4. Graphical Visualization: Plot the results using MATLAB to display the effect of system parameters on the performance measures and costs. The system's stability is ensured by meeting the condition  $\rho$ , which guarantees feasibility for the chosen parameters. Once this prerequisite is satisfied, MATLAB serves as the computational engine to evaluate both the performance and cost structure of the model.

Table 1 shows the analysis of the arrival rate and performance measures. For the numerical results in Table 1, the default parameters are set as follows:  $\mu_1 = 10$ ,  $\mu_2 = 8$ ,  $\zeta = 6$ ,  $\delta = 6$  and  $\gamma = 0.2$ .  $\mathbf{E}(Q)$ ,  $\mathbf{E}(W)$  and  $\mathbf{E}(B)$  increase as we increase the values of  $\lambda$ , but  $\mathbf{E}(I)$  decreases.

TABLE 1. Arrival rate vs Performance measures

$\lambda$	$\rho$	$E(Q)$	$E(W)$	$E(B)$	$E(I)$
4.5	0.6328	28.0110	3.1123	0.5950	0.3848
5.0	0.7031	31.8046	3.1805	0.6974	0.3726
5.5	0.7734	38.6853	3.5168	0.8732	0.3614
6.0	0.8438	52.5868	4.3822	1.2177	0.3515
6.5	0.9141	90.4609	6.9585	2.1401	0.3427

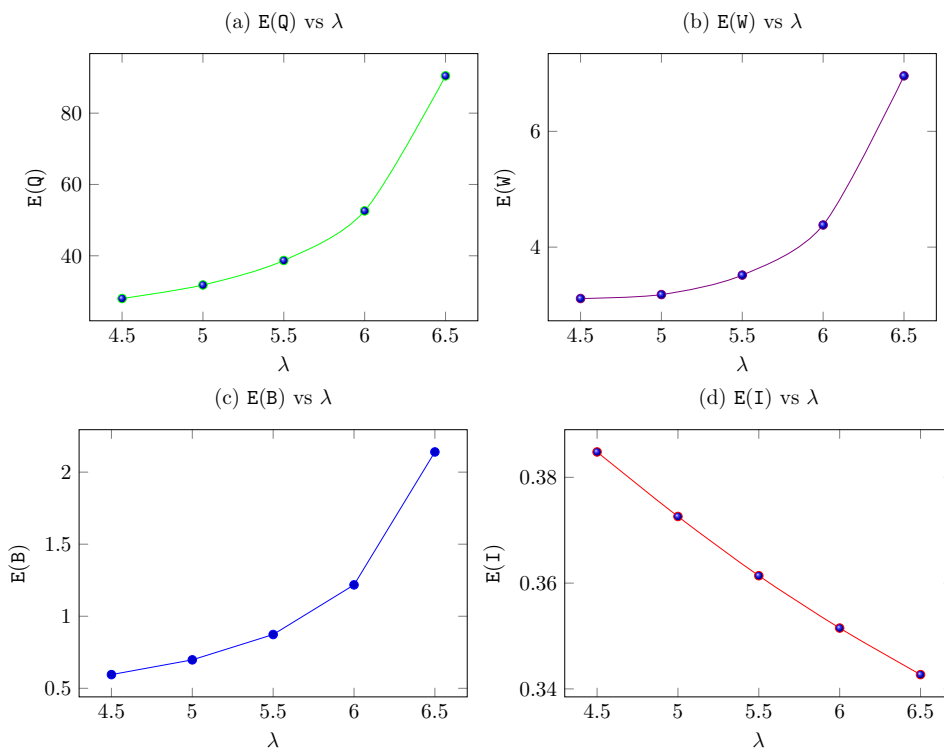
FIGURE 2. Performance metrics as a function of  $\lambda$ 

Figure 2 demonstrates how the arrival rate ( $\lambda$ ) affects key performance indicators in the textile dyeing and finishing process. As  $\lambda$  increases, the expected idle period ( $E(I)$ ) decreases, indicating that the dyeing machines remain idle for shorter durations due to a consistent supply of fabric batches meeting the minimum threshold. This reduction in downtime enhances machine utilization and prevents delays in batch processing. However, with further increases in  $\lambda$ , the system experiences higher workloads, leading to increased expected busy periods ( $E(B)$ ), queue lengths ( $E(Q)$ ), and waiting times ( $E(W)$ ). These increases result from congestion in machine capacity, especially during high-demand periods, causing longer delays for fabric batches and a higher likelihood of rework or defects due to rushed processing.

Table 2 shows the effect of regular service rates on the performance measures. The analysis was conducted with the following parameters:  $\lambda = 5$ ,  $\mu_2 = 8$ ,  $\zeta = 2$ ,  $\delta = 2$  and  $\gamma = 0.2$ .

Figure 3 shows the effect of different service rates ( $\mu_1$ ) on the performance measures of the textile dyeing and finishing process where machines treat batches of fabrics under the constraint of queues. The first

TABLE 2. Regular service rate vs Performance measures

$\mu_1$	$\rho$	$E(Q)$	$E(W)$	$E(B)$	$E(I)$
10.0	0.7031	30.3833	3.0383	1.2277	1.0253
10.5	0.6882	29.2401	2.9240	1.1424	1.0267
11.0	0.6747	28.3091	2.8309	1.0712	1.0280
11.5	0.6624	27.5215	2.7521	1.0115	1.0292
12.0	0.6510	26.8485	2.6849	0.9606	1.0303
12.5	0.6406	26.2735	2.6274	0.9165	1.0313

subplot progresses as the  $E(Q)$  reduces when  $\mu_1$  increases implying better throughput. In the second subplot illustrated here,  $E(W)$  is reduced, indicating that processing is occurring more quickly on a batch basis. The third subplot shows  $E(B)$  reducing, which means that the operation times are becoming more efficient. Last of all,  $E(I)$  also rises slightly, thereby indicating efficient segregation of maintenance and preparation work during machine idle time. These trends underscore the fact that changes in service rates enhance productivity and reduce time wastage in dyeing to produce quality results.

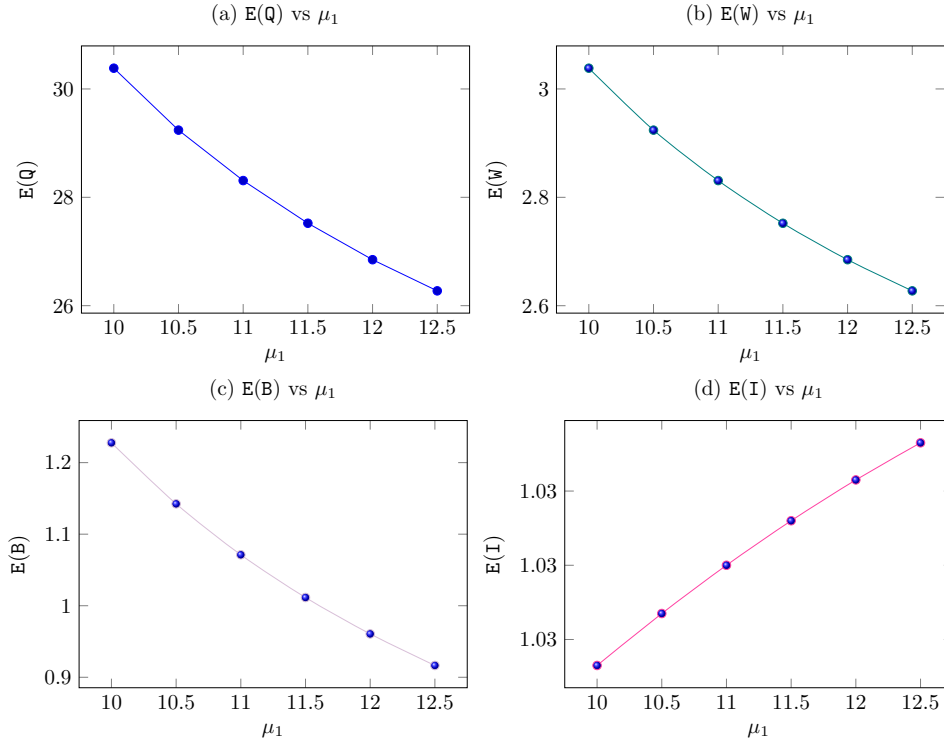
FIGURE 3. Performance metrics as a function of service rate ( $\mu_1$ )

Table 3 shows the effect of the rework probability ( $\gamma$ ) on performance metrics. For this analysis, the default parameters are set to represent typical operating conditions:  $\lambda = 4$ ,  $\mu_1 = 24$ ,  $\mu_2 = 20$ ,  $\zeta = 2$  and  $\delta = 2$ . The following table shows how different probabilities of reworking fabrics impact the major performance indicators in the production system.

TABLE 3. Rework probability vs Performance measures

$\gamma$	$\rho$	$E(Q)$	$E(W)$	$E(B)$	$E(I)$
0.1	0.2037	16.2001	2.0250	0.1936	1.1126
0.2	0.2292	17.5943	2.1993	0.2206	1.0986
0.3	0.2619	19.3503	2.4188	0.2548	1.0847
0.4	0.3056	21.6902	2.7113	0.2987	1.0707
0.5	0.3667	25.1142	3.1393	0.3549	1.0563
0.6	0.4583	31.0266	3.8783	0.4215	1.0411
0.7	0.6111	45.8630	5.7329	0.4645	1.0242

It is depicted in figure 4 that how percentage points of key performance indicators fluctuates with the change in rework probability ( $\gamma$ ) in the textile dyeing and finishing process where  $\gamma$  consider the possibility of fabric batches being reworked due to imperfect dyeing and finishing nature etc. When  $\gamma$  increases, the value of the expected queue length ( $E(Q)$ ) and waiting time ( $E(W)$ ) increases also, indicating the fact that rework requirements intensify congestion. Likewise, the expected busy period ( $E(B)$ ) is affected to ascend implying that the rate of machine utilization has raised because of recurrence of defective batches. On the other hand, the expected idle time ( $E(I)$ ) declines gradually, indicating improved no-operation as rework continues to utilize the equipment. These trends call for improved quality assurance to reduce chance of rework, improve reliability of machinery and avoid the threat of being off-schedule.

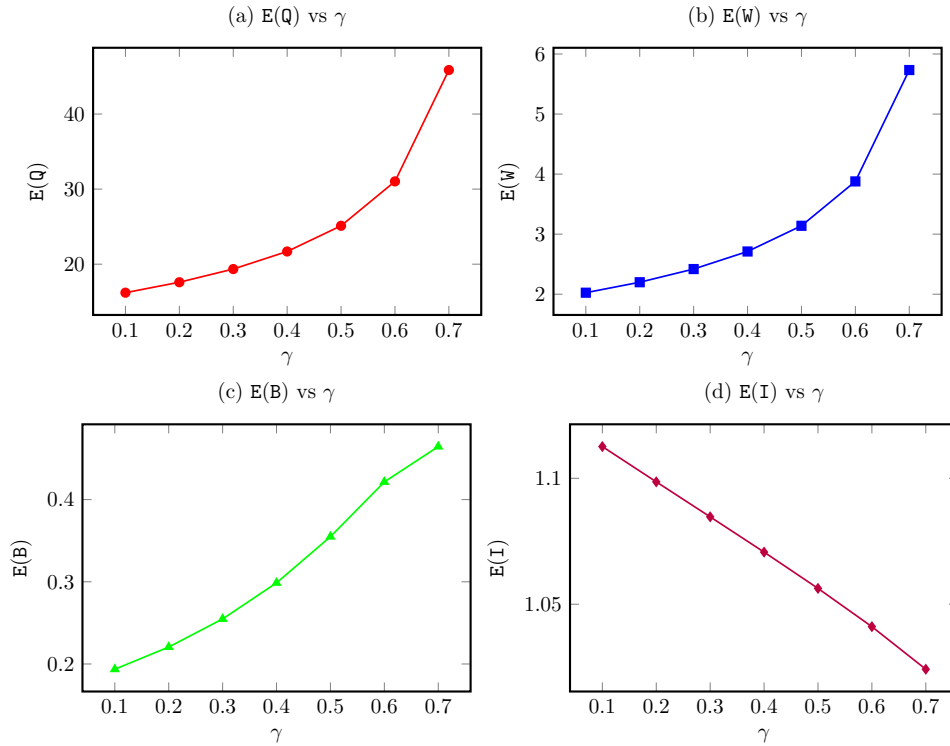


FIGURE 4. Performance metrics as a function of rework probability ( $\gamma$ )

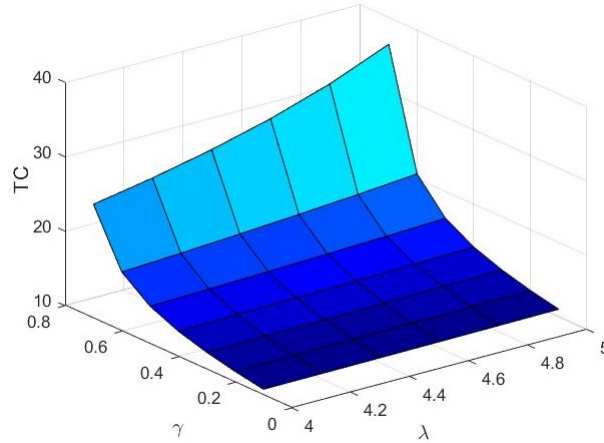


FIGURE 5. Total expected cost with respect to  $\lambda$  and  $\gamma$

Figure 5 is a 3D plot that depicts the rework probability ( $\gamma$ ), arrival rate ( $\lambda$ ), and Total average Cost (TC). When  $\gamma$  increases, the TC increases in turn because a batch involves rework that is attributed to defective products that undergo several processing cycles. Similarly, as  $\lambda$  increases we see that the TC also increases due to the reason that it costs more to manage the arriving batches. The interaction of both variables shows that rework probability and batch arrival control are essential for reducing the total cost in the system. This flowchart illustrates the dilemma of quality assurance and quantity output in the textile production line.

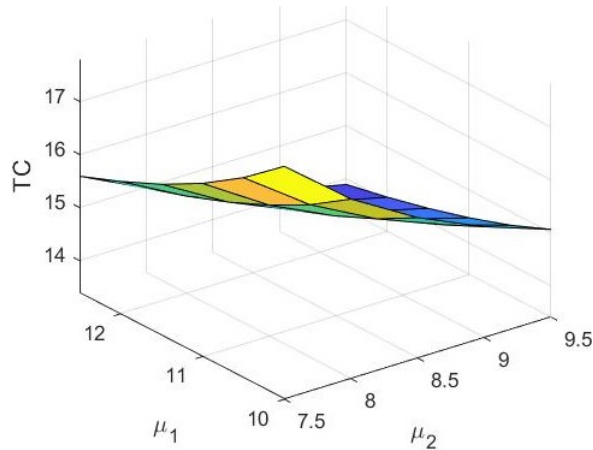


FIGURE 6. Total expected cost with respect to  $\mu_1$  and  $\mu_2$

Figure 6 shows the interaction between the regular service rate ( $\mu_1$ ), the inspection rate ( $\mu_2$ ), and the Total expected Cost (TC) in a textile manufacturing system especially during the dyeing and finishing processes. When the regular service rate ( $\mu_1$ ) is high the TC reduces due to non-stop batch processing leading to short holding periods. Similarly, higher inspection rate ( $\mu_2$ ) guarantees that if there are defects then they will be detected early and hopefully dealt with in a bid to reduce later cycle costs. However, the cost surface shows that there is an interaction between  $\mu_1$  and  $\mu_2$ , which means that the dyeing speed and the inspection efficiency should be optimized to achieve the lowest cost. This figure

underlines the importance of adjusting these rates in the textile production to achieve high quality output at the lowest possible cost.

## 8. CONCLUSION

This paper examined on a batch arrival single server queueing system with two stage general bulk service with rework, closedown and multiple vacations. The proposed model corrects the manufacturing defects by passing the defective batches through rework after inspection so that only the qualified products go for delivery. This is different from the conventional manufacturing order quantity models where defective processes and order fulfillment processes run concurrently, this model adopts inspection and rework. It offers useful information to manufacturing engineers working in environments with high levels of defects or undesirable production environments. The model evaluates average queue length, waiting time, server states, busy periods, and idle periods with the help of numerical examples. Furthermore, the role of the inspection rate in relation to the rate of production is illustrated to assist in the appropriate determination of the lot size. The future research prospects include the incorporation of more realistic control policies such as N-policy and T-policy, the study of standby servers and customers behaviour, and the generalization of the model to multi-server facilities with heterogeneous servers.

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