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DECISION-MAKING ALGORITHM BASED ON THE ENERGY OF INTERVAL-VALUED FUZZY SOFT SETS

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Abstract. In our work, we continue to explore the properties of interval-valued fuzzy soft sets, which are obtained by combining interval-valued fuzzy sets and soft sets. We introduce the concept of energy of an interval-valued fuzzy soft set, as well as pessimistic and optimistic energy, enabling us to construct an effective decision-making algorithm. Through examples, the paper demonstrates how the introduced algorithm is successfully applied to problems involving uncertainty. Additionally, we compare the proposed method with other methods dealing with similar or related issues, primarily highlighting its advantage in ensuring the existence of an optimal solution.

1. Introduction

The majority of problems in real-life situations, such as economics, engineering, environmental protection, social sciences and medical sciences, often lack clear data. Therefore, due to various forms of uncertainty characterizing these problems, traditional methods are often not sufficiently successful. In 1965, Zadeh [72] introduced the concept of fuzzy sets defining them as mappings from a set to the unit interval on the real line. Fuzzy sets were created as a mathematical approach to describe situations involving ambiguous classes, i.e. "collections" of objects with no clear membership criteria. Since Zadeh formulated fuzzy sets, numerous researchers have explored fuzzy sets and their applications in areas such as automata, languages, decision-making, logic, etc. Alongside these applications, theoretical aspects of fuzzy set theory have been studied. In 1966, Goguen [27] generalizes the concept of fuzzy sets, defining them as mappings from a non-empty set to a suitable partially ordered set, with the most notable results emerging when the poset is a lattice. Brown [10] further demonstrated that Zadeh's fundamental findings still hold when these lattices are Boolean in 1971. Since then, many new theories dealing with imprecision and uncertainty have been developed. Some of these approaches involve expanding the theory of fuzzy sets, while others attempt to address imprecision and uncertainty in different ways. It is worth mentioning the theory of intuitionistic fuzzy sets introduced by Atanassov in 1986 (see [7], [8]), vague set theory [26] proposed by Gau and Buehrer in 1993, as well as rough set theory introduced by Pawlak in the early 1980s (see [54], [55]). All these theories have their limitations when dealing with problems related to uncertainty and imprecision [46]. We assume that the difficulties arise from the inadequacy of tools for parameterizing these theories.

Molodtsov [46] introduced soft sets in 1999 as a mathematical tool for solving problems of uncertainty and imprecision without the aforementioned difficulties. Expanding on Molodtsov's work, Maji,

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Biswas, and Roy [43] defined essential operations for soft sets, such as union, intersection, and complement, adapting classical set operations to manage uncertainty effectively. In their follow-up study [41], they introduced parameter reduction techniques for decision-making, demonstrating how soft sets can streamline complex decision criteria. Cagman and Enginoglu [12] later introduced soft matrix theory, providing an efficient computational structure for soft sets in matrix form to support decision-making applications. Subsequent contributions by Ali [3] and Sezgin [61] extended the algebraic structure of soft sets by introducing operations such as restricted intersections, extended unions, and soft binary relations, thereby increasing the range of applications in areas such as data science and computational mathematics (see [44], [49], [56], [65]).

Fuzzy soft sets, introduced by Maji, Biswas, and Roy [42], provide a mathematical framework that combines fuzzy set theory with soft set theory to effectively manage uncertainties in data. Using the definition of fuzzy soft sets, many researchers have expanded the scope of soft set theory with interesting applications. Roy and Maji [59] explored some of these applications of fuzzy soft sets, while Som [62] introduced soft relations and fuzzy soft relations within the context of soft set theory. Additionally, Yang [66] defined operations on fuzzy soft sets using three key fuzzy logic operators: negation, triangular norm, and triangular conorm. In decision-making, fuzzy soft sets have found valuable applications, particularly in fields like medical diagnosis, engineering, and risk analysis. One decision-making method is presented in [12]. New properties and operations have been developed for fuzzy soft sets, leading to the development of new decision-making algorithms [13], [18], [48]. For efficient application and to address the requirements of specific problems, it is often necessary to generalize fuzzy soft sets. These generalizations extend their theoretical framework, enabling the construction of more versatile models suitable for a wider range of applications. Mohammed [45] proposed a novel decision-making algorithm in the fuzzy soft set environment by hybridizing existing techniques, introducing the concept of absolute scores and a priority table for group decision-making problems. Das [17] proposed a modified group decision-making method for fuzzy soft set-based problems, replacing absolute scores with weighted average ratings, offering greater stability and feasibility compared to the approach by Mohammed. Fuzzy soft multiset extends the concepts of soft sets and multisets, combining their flexibility to address uncertainties effectively. Das [19] introduced the weighted fuzzy soft multiset as a further generalization of fuzzy soft multiset, explored its fundamental properties, and proposed a new adjustable decision-making approach tailored to uncertain environments. Mukherjee [50] defined Einstein product and Einstein sum for fuzzy soft multisets and used these operations to develop an adjustable decision-making approach for handling uncertainty in decision-making based on fuzzy soft multisets. In [20] a method for solving group decision-making problems using intuitionistic fuzzy parameterized intuitionistic multi-fuzzy N-soft sets of dimension q is proposed, introducing the induced intuitionistic fuzzy parameterized hesitant N-soft set as an extension of the multi-fuzzy N-soft set in decision-making methods.

Interval-valued fuzzy sets were independently introduced by Zadeh [73], Grattan-Guiness [29], Jahn [35] and Sambuc [60] in the 1970s, where membership values are intervals. During that period, interval-valued fuzzy sets emerged in the literature under various forms, but it was not until the 1980s, through the work of Gorzalczany [23], [28] and Türksen [64], that their significance and formal terminology were firmly established. Significant contributions to interval-valued fuzzy sets also include Biswas [9] and Li [37] in the study of interval-valued fuzzy subgroups, Mondal [47] in interval-valued fuzzy topology, and researchers like Bustince [11], Chen [14], Yuan [69] and Arnould [6] in approximate reasoning. Additionally, Bustince [11] and Deschrijver [22] focused on interval-valued fuzzy relations and implications.

Fuzzy soft set models, as shown in [42], assume that membership degrees are crisp real values between 0 and 1. In many fuzzy applications, the use of interval-valued data to express membership degrees is more practical and effective, as it accommodates individual variability in membership functions. If we enable membership degrees in fuzzy soft sets to be represented by subintervals of $[0, 1]$, it would provide greater flexibility and precision in modeling uncertainty. As a result, interval-valued fuzzy soft sets (IVFSS) were constructed in [63] and [67] by combining interval-valued fuzzy sets and soft sets, making it a robust tool for handling datasets characterized by fuzziness and ambiguity. Alkhazaleh introduced the concept of generalised interval-valued fuzzy soft set in [4] and fuzzy parameterized interval-valued fuzzy soft set in [5] and gave their application to decision-making and medical diagnosis. Woldie [71] extended the concept of interval-valued fuzzy soft sets by combining them with fuzzy codes.

Interval-valued fuzzy soft sets find the greatest application in decision-making, as well as in data filling, information measure, etc. The decision-making methods utilizing interval-valued fuzzy soft sets were initially introduced by Son [63] and Yang [67]. Yang proposed a decision-making method based on choice value for IVFSS. Ma [39] proposed four distinct types of parameter reduction for interval-valued fuzzy soft sets, analyzing these approaches in terms of computational complexity, practical applicability, and the quality of reduction results. Additionally, in [40] a novel decision-making algorithm based on two table types - the average table and the antitheses table - for interval-valued fuzzy soft sets is introduced, while in [38] an efficient decision-making approach involving additional objects was provided. Qin [53] proposed a new approach to decision-making using IVFSS based on the means of the contrast table. Yang [68] proposed a revised TOPSIS and choice value method for interval fuzzy soft sets with unknown weight information. Peng [57] introduced new information measures (similarity, distance, and entropy) along with their interrelationships, and propose three interval-valued fuzzy soft decision-making approaches. In this paper, the algorithms in [38], [53], [63], [67] and [68] will be analyzed and their potential weaknesses identified.

Graph theory, as a mathematical discipline, represents a fundamental theory and a powerful tool in various fields, including computer science and chemistry. The concept of graph energy, as a numerical parameter, currently attracts the attention of many researchers. Gutman [31] introduced the concept of graph energy in 1978 as the sum of the absolute values of the eigenvalues of the adjacency matrix of the graph. The study of matrices and their properties, especially eigenvalues and singular values, forms the basis for analyzing graph energy. Graph energy enables the study of stability and structural properties of graphs, with wide applications in fields such as coding theory, social network analysis and many others. The introduction of graph energy has brought about many new scientific results (see [21], [32], [33], [36], [51], [52], [74]). Nikoforov's observation [51] that the energy of a graph is essentially the nuclear (or trace) norm of its adjacency matrix naturally leads to extending the concept of energy to matrices. This broadens the understanding of what energy represents. It is important to note that this norm is extensively studied in matrix theory, functional analysis and optimization theory (see for example [34], [58]), which makes graph energy particularly intriguing, as the nuclear norm is already a fundamental parameter in matrix theory. However, in general, these two extensions do not coincide, making it interesting to explore how they are related. The concept of nuclear norm, defined as the sum of the singular values of a matrix, is indispensable in minimization problems. By minimizing certain values, we can indeed determine which alternative contributes the most or the least to a system with multiple alternatives. Inspired by this idea, Mudrić Staniškovski [48], as well as Alcuntad [2], developed efficient decision-making algorithms in fuzzy soft and hesitant fuzzy soft environments. To apply a similar idea within the theory of IVFSS, it is necessary to overcome the specific challenges posed by this theory. Due to its characteristic structure, it is essential to introduce the notions of pessimistic and optimistic energy in order to ultimately ensure the existence of a single numerical feature. Although IVFSS allows membership to be represented as a subinterval of $[0, 1]$, which provides greater flexibility in modeling uncertainty and imprecision, the introduction of the concept of IVFSS energy enables the consolidation of this information into a single numerical value. In this way, IVFSS energy can serve as a criterion for comparing and ranking alternatives, thereby improving the accuracy and stability of the decision-making process.

In multi-criteria decision-making, optimism and pessimism highlight individual differences among decision-makers. Due to their consistency and persistence, they continuously influence decision-making processes, making their recognition a valuable contribution to the multi-criteria analysis and improvement of decision quality. Chen [15] linked optimism and pessimism to multi-criteria decision analysis in the context of intuitionistic fuzzy sets. Chen [16] also developed a method to reduce cognitive dissonance and connect optimism and pessimism in multi-criteria analysis using optimistic and pessimistic point operators in an interval-valued fuzzy decision environment. Feng [25] introduced pessimistic, optimistic, and neutral reduct fuzzy soft sets of interval-valued fuzzy soft sets, proposing flexible decision-making models based on (weighted) interval-valued fuzzy soft sets. Wen [70] introduced optimistic and pessimistic estimates into interval-valued intuitionistic fuzzy soft sets (IVIFSS) for multi-criteria group decision-making by defining the IVIFSS value vector, including IVIFSS weighted averaging, optimistic IVIFSS value, and pessimistic IVIFSS value. All these studies motivated us to introduce the concept of pessimistic and optimistic energy in IVFSS, through which we will define the energy for IVFSS.

In Section 2 of our paper, we present the fundamental concepts of soft set theory, fuzzy soft set theory, as well as interval-valued fuzzy soft set theory. In Section 3, we introduce new concepts, such as pessimistic and optimistic energy of interval-valued fuzzy soft sets, as well as their energy. These energies play a significant role in creating decision-making algorithms. We also provide basic properties and boundaries for the introduced energies. Section 4 is dedicated to the decision-making algorithm based on these energies and its application to a practical problem. The first three subsections of Section 5 discuss the limitations of the decision-making algorithms presented in [63], [53], [67], [38] and [68] along with the advantages of our algorithm compared to them. Subsection 5.4 provides a comparative analysis of the results obtained from all the observed methods on the same example, emphasizing both the strengths and weaknesses of the mentioned algorithms. In Section 6, we summarize the obtained results from which certain conclusions can be drawn. Additionally, guidelines for further investigation of the introduced energies and their potential applications are provided.

2. Preliminaries

In this section, we define the fundamental concepts essential for understanding our research. We will begin by introducing the fundamental concepts of soft sets. Then, we will explore the concept of fuzzy sets and interval-valued fuzzy sets. Finally, we will focus on the fuzzy soft sets and the central concept of our research - interval-valued fuzzy soft sets.

2.1. **Soft sets.** Let U be a non-empty finite universal set, E be the set of parameters and $A \subseteq E$. Using $P(U)$, we denote the power set of the universe U.

Definition 2.1. [46] A soft set F_A over the universe U is a set determined by the mapping $f_A : E \to P(U)$, where $f_A(x) = \emptyset$ whenever $x \notin A$.

The mapping f_A is commonly referred to as the approximating function of the soft set F_A for each $x \in E$. The soft set F_A over U can also be represented using ordered pairs in the following way:

$$F_A = \left\{ (x, f_A(x)) \,\middle|\, x \in E, f_A(x) \in P(U) \right\}.$$

We will denote the collection of all soft sets over the universe U as $S(U)$.

More detailed properties of these sets, along with numerous examples, can be found in [1], [3], [12], [41], [44], [56] and [61].

2.2. **Interval-valued fuzzy sets.** Fuzzy sets allow assigning a degree of membership to an element within the range of 0 to 1, enabling the representation of uncertainty. Interval-valued fuzzy sets extend this approach by allowing membership values to be intervals, providing a more precise way to model uncertainty in complex systems. These sets lead to the structures that will be the primary focus of our research.

Definition 2.2. [72] A fuzzy set X over the universe U is a set determined by the mapping $\mu_X : U \to [0,1]$.

The mapping μ_X is called the membership function of X, while the value $\mu_X(u)$ is called the degree of membership of element $u \in U$ in the fuzzy set X.

A fuzzy set X over U can be represented in the following way:

$$X = \left\{ (\mu_X(u)/u) \,\middle|\, u \in U, \mu_X(u) \in [0,1] \right\}.$$

We will denote the collection of all fuzzy sets over the universe U as $F(U)$.

Definition 2.3. [73] An interval-valued fuzzy set $\hat{X}$ over the universe U is a set defined by mapping $i_{\hat{X}} : U \to Int\,([0,1])$, where $Int\,([0,1])$ is the set of all closed subintervals of $[0,1]$.

The mapping $i_{\hat{X}}$ is called the interval-membership function of $\hat{X}$. If for every $u \in U$ we denote, respectively, with $i^-_{\hat{X}}(u)$ and $i^+_{\hat{X}}(u)$ the lower and upper degrees of membership u to $\hat{X}$, where $0 \leqslant i^-_{\hat{X}}(u) \leqslant i^+_{\hat{X}}(u) \leqslant 1$, then we can denote $i_{\hat{X}}(u)$ as

$$i_{\hat{X}}(u) = \left[i^-_{\hat{X}}(u), i^+_{\hat{X}}(u) \right]$$

and the value $i_{\hat{X}}(u)$ is called the degree of membership an element u to $\hat{X}$.

We can also represent the interval-valued fuzzy set $\hat{X}$ in the following way:

$$\hat{X} = \left\{ \left(i_{\hat{X}}(u)/u \right) \,\middle|\, u \in U, i_{\hat{X}}(u) \in Int\,([0,1]) \right\}.$$

We will denote the collection of all interval-valued fuzzy sets over the universe U as $IVFS(U)$.

For more detailed information on fuzzy sets and interval-valued fuzzy sets, we recommend [24], [30], [72] and [73].

2.3. **Interval-valued fuzzy soft sets.** Before defining the structure that forms the basis of our research, we will briefly recall the concept of fuzzy soft sets, which arises from the hybridization of soft sets and fuzzy sets.

Definition 2.4. [42] A fuzzy soft set Γ_A over the universe U is a set determined by the mapping $\gamma_A : E \to F(U)$, where $\gamma_A(x) = \emptyset$ whenever $x \notin A$.

Then γ_A is called the fuzzy approximating function of the fuzzy soft set Γ_A, while the value $\gamma_A(x)$ is the set called the x-element of the fuzzy soft set for all $x \in E$. Hence, the fuzzy soft set Γ_A over the universe U can be, similarly as soft sets, represented by a set of ordered pairs in the following way:

$$\Gamma_A = \left\{ (x, \gamma_A(x)) \,\middle|\, x \in E, \gamma_A(x) \in F(U) \right\}.$$

We will denote the collection of all fuzzy soft sets over the universe U as $FS(U)$.

By combining interval-valued fuzzy sets and soft sets, we obtain interval-valued fuzzy soft sets.

Definition 2.5. [67] An interval-valued fuzzy soft set (IVFSS) $\mathcal{F}_A$ over the universe U is a set defined by the mapping $\eta_A : E \to IVFS(U)$, where $\eta_A(x) = \emptyset$ whenever $x \notin A$.

Similar to soft sets, the mapping η_A is called the interval-valued fuzzy approximating function of the interval-valued fuzzy soft set $\mathcal{F}_A$, and the value $\eta_A(x)$ is a set called an interval-valued fuzzy value set of the parameter $x \in E$. Then $\mathcal{F}_A$ can be represented in the following way:

$$\mathcal{F}_A = \left\{(x, \eta_A(x)) \,\middle|\, x \in E, \eta_A(x) \in IVFS(U)\right\}.$$

We will denote the collection of all interval-valued fuzzy soft sets over the universe U as $IVFSS(U)$. In the following, we will use the notations $\mathcal{F}_A, \mathcal{F}_B, \mathcal{F}_C, \ldots$ for interval-valued fuzzy soft sets and $\eta_A, \eta_B, \eta_C, \ldots$ for their interval-valued fuzzy approximating functions, respectively.

We will provide an example of an interval-valued fuzzy soft set from the work [67], which we will use to illustrate the main concepts in our work.

Example 2.1. [67] *Suppose that $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ is the set of the houses under consideration and*
$A = \{x_1, x_2, x_3, x_4\}$ is the set of parameters, where for $i = 1, 2, 3, 4$, the parameters x_i represent, in order,
"beautiful", "wooden", "cheap" and "in the green surroundings". We define an interval valued fuzzy soft set $\mathcal{F}_A$ as follows:

$$\eta_A(x_1) = \{([0.7, 0.9]/u_1), ([0.6, 0.8]/u_2), ([0.5, 0.6]/u_3), ([0.6, 0.8]/u_4), ([0.8, 0.9]/u_5), ([0.8, 1.0]/u_6)\},$$
$$\eta_A(x_2) = \{([0.6, 0.7]/u_1), ([0.8, 1.0]/u_2), ([0.2, 0.4]/u_3), ([0.0, 0.1]/u_4), ([0.1, 0.3]/u_5), ([0.7, 0.8]/u_6)\},$$
$$\eta_A(x_3) = \{([0.3, 0.5]/u_1), ([0.8, 0.9]/u_2), ([0.5, 0.7]/u_3), ([0.7, 1.0]/u_4), ([0.9, 1.0]/u_5), ([0.2, 0.5]/u_6)\},$$
$$\eta_A(x_4) = \{([0.5, 0.8]/u_1), ([0.9, 1.0]/u_2), ([0.7, 0.9]/u_3), ([0.6, 0.8]/u_4), ([0.2, 0.5]/u_5), ([0.7, 1.0]/u_6)\}.$$

Two interval-valued fuzzy soft sets can be compared in the following way.

Definition 2.6. [67] Let $A, B \subseteq E$ and let $\mathcal{F}_A$ and $\mathcal{F}_B$ be two interval-valued fuzzy soft sets over the universe U. We say that $\mathcal{F}_A$ is an interval-valued fuzzy soft subset of $\mathcal{F}_B$ if the following holds:

(1) $A \subseteq B$,

(2) $\forall x \in A$, $\eta_A(x)$ is an interval-valued fuzzy subset of $\eta_B(x)$, which we denote as $\mathcal{F}_A \widetilde{\subseteq} \mathcal{F}_B$.

We say that $\mathcal{F}_A$ is an interval-valued fuzzy soft super set of $\mathcal{F}_B$ if $\mathcal{F}_B$ is an interval-valued fuzzy soft subset of $\mathcal{F}_A$, which we denote as $\mathcal{F}_A \widetilde{\supseteq} \mathcal{F}_B$.

Definition 2.7. [67] Two interval-valued fuzzy soft sets $\mathcal{F}_A$ and $\mathcal{F}_B$ are interval-valued fuzzy soft equal if $\mathcal{F}_A$ is an interval-valued fuzzy soft subset of $\mathcal{F}_B$ and $\mathcal{F}_B$ is an interval-valued fuzzy soft subset of $\mathcal{F}_A$, which we denote as $\mathcal{F}_A = \mathcal{F}_B$.

The operations complement, union, intersection, AND, and OR, as well as their properties, can be found in [63] and [67].

3. Energy of an interval-valued fuzzy soft set

Within this section, we introduce the concept of energy of an interval-valued fuzzy soft set, representing a numerical measure that encapsulates the interval-valued fuzzy soft set's attributes. These numerical values serve as parameters for scrutinizing conclusions in decision-making contexts, as further explored in Section 4.

Let's start with the tabular representation of an interval-valued fuzzy soft set.

Let $U = \{u_1, u_2, ..., u_n\}$ i $A = \{x_1, x_2, ..., x_m\} \subseteq E$. Then an interval-valued fuzzy soft set $\mathcal{F}_A$ can be represented in a tabular form provided in Table 1.

Example 3.1. *A tabular representation of the interval-valued fuzzy soft set $\mathcal{F}_A$ from Example 2.1 is given in Table 2.*

TABLE 1. The tabular representation of IVFSS $\mathcal{F}_A$ 2.1

$\mathcal{F}_A$	x_1	x_2	$\cdots$	x_m
u_1	$i_{\eta_A(x_1)}(u_1)$	$i_{\eta_A(x_2)}(u_1)$	$\cdots$	$i_{\eta_A(x_m)}(u_1)$
u_2	$i_{\eta_A(x_1)}(u_2)$	$i_{\eta_A(x_2)}(u_2)$	$\cdots$	$i_{\eta_A(x_m)}(u_2)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_n	$i_{\eta_A(x_1)}(u_n)$	$i_{\eta_A(x_2)}(u_n)$	$\cdots$	$i_{\eta_A(x_m)}(u_n)$

TABLE 2. The tabular representation of IVFSS from Example 2.1

$\mathcal{F}_A$	x_1	x_2	x_3	x_4
u_1	$[0.7, 0.9]$	$[0.6, 0.7]$	$[0.3, 0.5]$	$[0.5, 0.8]$
u_2	$[0.6, 0.8]$	$[0.8, 1.0]$	$[0.8, 0.9]$	$[0.9, 1.0]$
u_3	$[0.5, 0.6]$	$[0.2, 0.4]$	$[0.5, 0.7]$	$[0.7, 0.9]$
u_4	$[0.6, 0.8]$	$[0.0, 0.1]$	$[0.7, 1.0]$	$[0.6, 0.8]$
u_5	$[0.8, 0.9]$	$[0.1, 0.3]$	$[0.9, 1.0]$	$[0.2, 0.5]$
u_6	$[0.8, 1.0]$	$[0.7, 0.8]$	$[0.2, 0.5]$	$[0.7, 1.0]$

Using that $i_{\eta_A(x_j)}(u_i) = \left[i^-_{\eta_A(x_j)}(u_i), i^+_{\eta_A(x_j)}(u_i)\right]$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, the interval-valued fuzzy soft set $\mathcal{F}_A$ will be uniquely determined by the following two matrices.

Definition 3.1. Let $U = \{u_1, u_2, ..., u_n\}$ and $A = \{x_1, x_2, ..., x_m\} \subseteq E$. The matrix of minimum values of the interval-valued fuzzy soft set $\mathcal{F}_A$ is a matrix $\Psi^{\min}_{\mathcal{F}_A}$ of type $n \times m$ given by

$$\Psi^{\min}_{\mathcal{F}_A} = \begin{bmatrix} i^-_{\eta_A(x_1)}(u_1) & i^-_{\eta_A(x_2)}(u_1) & \cdots & i^-_{\eta_A(x_m)}(u_1) \\ i^-_{\eta_A(x_1)}(u_2) & i^-_{\eta_A(x_2)}(u_2) & \cdots & i^-_{\eta_A(x_m)}(u_2) \\ \vdots & \vdots & \ddots & \vdots \\ i^-_{\eta_A(x_1)}(u_n) & i^-_{\eta_A(x_2)}(u_n) & \cdots & i^-_{\eta_A(x_m)}(u_n) \end{bmatrix}.$$

Similarly, we define the matrix of maximum values of the interval-valued fuzzy soft set $\mathcal{F}_A$.

Definition 3.2. Let $U = \{u_1, u_2, ..., u_n\}$ and $A = \{x_1, x_2, ..., x_m\} \subseteq E$. The matrix of maximum values of the interval-valued fuzzy soft set $\mathcal{F}_A$ s a matrix $\Psi^{\max}_{\mathcal{F}_A}$ of type $n \times m$ given by

$$\Psi^{\max}_{\mathcal{F}_A} = \begin{bmatrix} i^+_{\eta_A(x_1)}(u_1) & i^+_{\eta_A(x_2)}(u_1) & \cdots & i^+_{\eta_A(x_m)}(u_1) \\ i^+_{\eta_A(x_1)}(u_2) & i^+_{\eta_A(x_2)}(u_2) & \cdots & i^+_{\eta_A(x_m)}(u_2) \\ \vdots & \vdots & \ddots & \vdots \\ i^+_{\eta_A(x_1)}(u_n) & i^+_{\eta_A(x_2)}(u_n) & \cdots & i^+_{\eta_A(x_m)}(u_n) \end{bmatrix}.$$

Let's consider Example 2.1 listed in Section 2 of this paper.

Example 3.2. *The interval-valued fuzzy soft set $\mathcal{F}_A$ from Example 2.1 can be represented using the matrix of minimum values and the matrix of maximum values as follows:*

$$\Psi^{\min}_{\mathcal{F}_A} = \begin{bmatrix} 0.7 & 0.6 & 0.3 & 0.5 \\ 0.6 & 0.8 & 0.8 & 0.9 \\ 0.5 & 0.2 & 0.5 & 0.7 \\ 0.6 & 0.0 & 0.7 & 0.6 \\ 0.8 & 0.1 & 0.9 & 0.2 \\ 0.8 & 0.7 & 0.2 & 0.7 \end{bmatrix}, \qquad \Psi^{\max}_{\mathcal{F}_A} = \begin{bmatrix} 0.9 & 0.7 & 0.5 & 0.8 \\ 0.8 & 1.0 & 0.9 & 1.0 \\ 0.6 & 0.4 & 0.7 & 0.9 \\ 0.8 & 0.1 & 1.0 & 0.8 \\ 0.9 & 0.3 & 1.0 & 0.5 \\ 1.0 & 0.8 & 0.5 & 1.0 \end{bmatrix}.$$

Now that we have defined the matrix of minimum values and the matrix of maximum values, we can also define the pessimistic and optimistic energies of the interval-valued fuzzy soft set.

Definition 3.3. The pessimistic energy of the interval-valued fuzzy soft set $\mathcal{F}_A$, denoted by $\mathbb{E}^{\min}_{\mathcal{F}_A}$, is defined as

$$\mathbb{E}^{\min}_{\mathcal{F}_A} = \sum_{i=1}^{n} \sigma_i,$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n \geqslant 0$ are the singular values of the matrix $\Psi^{\min}_{\mathcal{F}_A}$ of minimum values of the interval-valued fuzzy soft set $\mathcal{F}_A$.

Let's return to Examples 2.1 and 3.2.

Example 3.3. *The singular values of the matrix $\Psi^{\min}_{\mathcal{F}_A}$ from Example 3.2 can easily be determined by finding the matrix $\Psi^{\min}_{\mathcal{F}_A} \cdot \left(\Psi^{\min}_{\mathcal{F}_A}\right)^T$ and computing its eigenvalues. Then, the singular values of the matrix $\Psi^{\min}_{\mathcal{F}_A}$ are the square roots of the eigenvalues of $\Psi^{\min}_{\mathcal{F}_A} \cdot \left(\Psi^{\min}_{\mathcal{F}_A}\right)^T$:*

$$\sigma_1 = 2.813267, \quad \sigma_2 = 0.877801, \quad \sigma_3 = 0.475986, \quad \sigma_4 = 0.358376, \quad \sigma_5 = \sigma_6 = 0,$$

so based on the definition mentioned above,

$$\mathbb{E}^{\min}_{\mathcal{F}_A} = \sum_{i=1}^{6} \sigma_i = 2.813267 + 0.877801 + 0.475986 + 0.358376 + 0 + 0 = 4.52543.$$

Similarly, we define the optimistic energy of the interval-valued fuzzy soft set $\mathcal{F}_A$.

Definition 3.4. The optimistic energy of the interval-valued fuzzy soft set $\mathcal{F}_A$, denoted by $\mathbb{E}^{\max}_{\mathcal{F}_A}$, is defined as

$$\mathbb{E}^{\max}_{\mathcal{F}_A} = \sum_{i=1}^{n} \sigma_i,$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n \geqslant 0$ are the singular values of the matrix $\Psi^{\max}_{\mathcal{F}_A}$ of maximum values of the interval-valued fuzzy soft set $\mathcal{F}_A$.

Let's once again consider Examples 2.1 and 3.2.

Example 3.4. *The singular values of the matrix $\Psi^{\max}_{\mathcal{F}_A}$ from Example 3.2 are*

$$\sigma_1 = 3.721438, \quad \sigma_2 = 0.828772, \quad \sigma_3 = 0.364174, \quad \sigma_4 = 0.348435, \quad \sigma_5 = \sigma_6 = 0,$$

so based on the previous definition,

$$\mathbb{E}^{\max}_{\mathcal{F}_A} = \sum_{i=1}^{6} \sigma_i = 3.721438 + 0.828772 + 0.364174 + 0.348435 + 0 + 0 = 5.262819.$$

Now, to address the uniqueness of energy for two different interval-valued fuzzy soft sets, we introduce the concept of energy for an interval-valued fuzzy soft set.

Definition 3.5. The energy of the interval-valued fuzzy soft set $\mathcal{F}_A$, denoted by $\mathbb{E}^*_{\mathcal{F}_A}$, is defined as

$$\mathbb{E}^*_{\mathcal{F}_A} = \frac{\mathbb{E}^{\min}_{\mathcal{F}_A} + \mathbb{E}^{\max}_{\mathcal{F}_A}}{2}.$$

The energy of the interval-valued fuzzy soft set $\mathcal{F}_A$ from Example 2.1 is $\mathbb{E}^*_{\mathcal{F}_A} = 4.8941245$.

In the following, we present the basic properties of the introduced energies, from which we will provide upper bounds of the energies of IVFSS (with the lower bounds being zero).

Theorem 3.5. *Let $\mathcal{F}_A$ be an interval-valued fuzzy soft set, $U = \{u_1, u_2, \ldots, u_n\}$, $E = \{x_1, x_2, \ldots, x_m\}$ and $A \subseteq E$. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the singular values of the matrix $\Psi^{\min}_{\mathcal{F}_A}$ representing the matrix of minimal values of the interval-valued fuzzy soft set $\mathcal{F}_A$. Then for the pessimistic energy of $\mathcal{F}_A$ holds:*

$$\mathbb{E}^{\min}_{\mathcal{F}_A} \leqslant n\sqrt{m}.$$

Proof. Using the inequality between the arithmetic mean and the quadratic mean applied to the singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$, we obtain

$$\mathbb{E}^{\min}_{\mathcal{F}_A} = \sum_{i=1}^{n} \sigma_i \leqslant \sqrt{n \sum_{i=1}^{n} \sigma_i^2}.$$

Since the trace of a square matrix is equal to the sum of all its eigenvalues, including their algebraic multiplicities, and since the squares of the singular values of an arbitrary matrix A are equal to the eigenvalues of the matrix AA^T, we obtain:

$$\sum_{i=1}^{n} \sigma_i^2 = \operatorname{tr}\left(\Psi^{\min}_{\mathcal{F}_A} \cdot \left(\Psi^{\min}_{\mathcal{F}_A}\right)^T\right).$$

Furthermore, since the trace of the matrix AA^T is equal to the sum of the squares of all elements of an arbitrary matrix A, applying this to the matrix $\Psi^{\min}_{\mathcal{F}_A}$, as well as the fact that $i^-_{\eta_A(x_j)}(u_i) \leqslant 1$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, we get

$$\operatorname{tr}\left(\Psi^{\min}_{\mathcal{F}_A} \cdot \left(\Psi^{\min}_{\mathcal{F}_A}\right)^T\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(i^-_{\eta_A(x_j)}(u_i)\right)^2 \leqslant mn.$$

Therefore, $\mathbb{E}^{\min}_{\mathcal{F}_A} \leqslant n\sqrt{m}$. □

The same applies to the optimistic energy of $\mathcal{F}_A$.

Theorem 3.6. *Let $\mathcal{F}_A$ be an interval-valued fuzzy soft set, $U = \{u_1, u_2, \ldots, u_n\}$, $E = \{x_1, x_2, \ldots, x_m\}$ and $A \subseteq E$. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the singular values of the matrix $\Psi^{\max}_{\mathcal{F}_A}$ representing the matrix of maximum values of the interval-valued fuzzy soft set $\mathcal{F}_A$. Then for the optimistic energy of $\mathcal{F}_A$ holds:*

$$\mathbb{E}^{\max}_{\mathcal{F}_A} \leqslant n\sqrt{m}.$$

Using the previous two theorems, we obtain the upper bound of the energy of an interval-valued fuzzy soft set.

Theorem 3.7. *Let $\mathcal{F}_A$ be an interval-valued fuzzy soft set, $U = \{u_1, u_2, \ldots, u_n\}$, $E = \{x_1, x_2, \ldots, x_m\}$ and $A \subseteq E$. Then for the energy of $\mathcal{F}_A$ holds:*

$$\mathbb{E}^{*}_{\mathcal{F}_A} \leqslant n\sqrt{m}.$$

Proof. Since matrices $\Psi^{\min}_{\mathcal{F}_A}$ and $\Psi^{\max}_{\mathcal{F}_A}$ are of the same type $n \times m$, they both have the same number of singular values, so using the previous theorems and a definition of an energy of $\mathcal{F}_A$, we get

$$\mathbb{E}^{*}_{\mathcal{F}_A} = \frac{\mathbb{E}^{\min}_{\mathcal{F}_A} + \mathbb{E}^{\max}_{\mathcal{F}_A}}{2} \leqslant \frac{n\sqrt{m} + n\sqrt{m}}{2} = n\sqrt{m}.$$

□

4. Algorithm for the interval-valued fuzzy soft set decision-making and its application to a practical problem

The algorithm for the interval-valued fuzzy soft set decision-making has been developed to address specific challenges within a given context, offering efficient solutions for complex tasks. The development of the algorithm based on the energy of interval-valued fuzzy soft sets (IVFSS) is motivated by the need for a more precise analysis of the contribution of alternatives under uncertainty. Other approaches often fail to provide insight into how the removal of a single alternative affects the overall structure of the system. The proposed algorithm uses energy as a numerical characteristic that incorporates both optimistic and pessimistic estimates, enabling the assessment of the significance of each alternative within the interval-valued uncertainty model. In real-world decision-making scenarios, it is often necessary to select the best or the worst alternative. To make a proper decision, it is essential to identify the optimal alternative.

In the following, we will explain the steps of the algorithm and demonstrate how it can be applied to solve real-world problems. A practical example will illustrate the functionality of the algorithm and highlight its importance in solving similar problems across various disciplines.

Step 1 is trivial and it involves recording the initial interval-valued fuzzy soft set (IVFSS). In Step 2 of the algorithm, n new IVFSSs are generated from the initial input in Step 1. These newly created sets are similar to the original one, as they are obtained by progressively removing one alternative at a time from the initial IVFSS. In Step 3, the corresponding matrices are generated in a straightforward manner using Definition 3.1 and 3.2. Then, in Step 4, all singular values of these matrices are calculated using standard methods from linear algebra. Step 5 calculates n optimistic and n pessimistic energy values using Definition 3.3 and Definition 3.4, each indexed by their respective alternatives. These values represent the system's energy without a specific alternative. Specifically, $\mathbb{E}^{\min}_{\mathcal{F}_{A_i}}$ and $\mathbb{E}^{\max}_{\mathcal{F}_{A_i}}$ represent the pessimistic and optimistic energy of the system when element u_i is excluded from the original IVFSS. In Step 6, the energies of all n newly formed IVFSSs are computed using Definition 3.5. Finally, in Step 7, the alternative whose removal results in the greatest energy loss is identified by finding the smallest energy among n calculated values.

Algorithm 1	A Decision-Making Algorithm for IVFSS Energy
Input	An interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe $U = \{u_1, u_2, \ldots, u_n\}$.
Output	The alternative that contributes least to the system when removed.
Step 1	Input the interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe U.
Step 2	Form interval-valued fuzzy soft sets $\mathcal{F}_{A_i}$, over $U \setminus u_i$ for each $u_i \in U$, for $i = 1, 2, \ldots, n$.
Step 3	For each interval-valued fuzzy soft set from Step 2, form their corresponding matrix of minimum and matrix of maximum values.
Step 4	Determine the singular values for each obtained matrix of minimum and each obtained matrix of maximum values.
Step 5	Determine the pessimistic energies $\mathbb{E}^{\min}_{\mathcal{F}_{A_i}}$ and the optimistic energies $\mathbb{E}^{\max}_{\mathcal{F}_{A_i}}$ for each interval-valued fuzzy soft sets $\mathcal{F}_{A_i}$ based on the obtained singular values, for $i = 1, 2, \ldots, n$.
Step 6	Determine the energies $\mathbb{E}^{*}_{\mathcal{F}_{A_i}}$ for each interval-valued fuzzy soft sets $\mathcal{F}_{A_i}$, for $i = 1, 2, \ldots, n$ based on the obtained energies in Step 5.
Step 7	Determine the minimum energy among all energies of interval-valued fuzzy soft sets obtained in Step 6 and interpret the result obtained.

Now, we delve into the potential applications of the energy of interval-valued fuzzy soft sets. We aim to illustrate how this energy serve as a potent tool in devising decision-making algorithms. Understanding the interconnectedness among all factors within the system is crucial for effective decision-making. Let's consider the example where a family needs to choose an apartment to buy from the paper [53] by Qin.

Example 4.1. [53] *One family is planning to buy an apartment building for living. There are five alternative apartment candidates from five different property developers. This family hesitates about which to buy. We are able to evaluate the alternatives from four aspects:"reasonable price","excellent geographical location", "perfect facilities", "cozy environment". The model of IVFSS is chosen to describe the customer's feeling for the five candidates from four aspects. Hence, suppose that the universe U represents the set of the five different alternative apartment candidates and* $U = \{u_1, u_2, u_3, u_4, u_5\}$. *Then, A represents the set of four parameters and*

$$\begin{aligned} A &= \{x_1, x_2, x_3, x_4\} \\ &= \{\textit{reasonable price, excellent geographical location, perfect facilities, cozy environment}\}. \end{aligned}$$

Then IVFSS $\mathcal{F}_A$ on U is presented in Table 3.

Table 3. IVFSS for Example 4.1

$\mathcal{F}_A$	x_1	x_2	x_3	x_4
u_1	$[0.3, 0.5]$	$[0.6, 0.7]$	$[0.2, 0.4]$	$[0.4, 0.5]$
u_2	$[0.3, 0.4]$	$[0.4, 0.5]$	$[0.6, 0.7]$	$[0.1, 0.3]$
u_3	$[0.5, 0.6]$	$[1.0, 1.0]$	$[0.2, 0.3]$	$[0.2, 0.4]$
u_4	$[0.5, 0.7]$	$[0.0, 0.1]$	$[0.7, 0.8]$	$[0.6, 0.7]$
u_5	$[0.3, 0.6]$	$[0.3, 0.4]$	$[0.4, 0.7]$	$[0.2, 0.3]$

.

Using Example 4.1 we will illustrate how decisions are made based on the aforementioned algorithm. The interval-valued fuzzy soft set $\mathcal{F}_A$ is represented in Table 3, fulfilling the first step of the algorithm. In the second step, we need to form five interval-valued fuzzy soft sets, as described. We will illustrate how to find $\mathcal{F}_{A_1}$, and the remaining interval-valued fuzzy soft sets are determined analogously. We obtain Table 4 for $\mathcal{F}_{A_1}$ by removing the first row of the table for $\mathcal{F}_A$.

Table 4. IVFSS $\mathcal{F}_{A_1}$ for Example 4.1

$\mathcal{F}_{A_1}$	x_1	x_2	x_3	x_4
u_2	$[0.3, 0.4]$	$[0.4, 0.5]$	$[0.6, 0.7]$	$[0.1, 0.3]$
u_3	$[0.5, 0.6]$	$[1.0, 1.0]$	$[0.2, 0.3]$	$[0.2, 0.4]$
u_4	$[0.5, 0.7]$	$[0.0, 0.1]$	$[0.7, 0.8]$	$[0.6, 0.7]$
u_5	$[0.3, 0.6]$	$[0.3, 0.4]$	$[0.4, 0.7]$	$[0.2, 0.3]$

.

The corresponding matrix of minimum values and the matrix of maximum values for $\mathcal{F}_{A_1}$ are

$$\Psi^{\min}_{\mathcal{F}_{A_1}} = \begin{bmatrix} 0.3 & 0.4 & 0.6 & 0.1 \\ 0.5 & 1.0 & 0.2 & 0.2 \\ 0.5 & 0.0 & 0.7 & 0.6 \\ 0.3 & 0.3 & 0.4 & 0.2 \end{bmatrix} \quad \text{and} \quad \Psi^{\max}_{\mathcal{F}_{A_1}} = \begin{bmatrix} 0.4 & 0.5 & 0.7 & 0.3 \\ 0.6 & 1.0 & 0.3 & 0.4 \\ 0.7 & 0.1 & 0.8 & 0.7 \\ 0.6 & 0.4 & 0.7 & 0.3 \end{bmatrix},$$

thus completing the third step.

The singular values of the matrix $\Psi^{\min}_{\mathcal{F}_{A_1}}$ are

$$\sigma_1 = 1.624965,\ \sigma_2 = 0.836748,\ \sigma_3 = 0.298884,\ \sigma_4 = 0.002707.$$

Hence, the pessimistic energy of the interval-valued fuzzy soft set $\mathcal{F}_{A_1}$ is the sum of the obtained singular values, i.e. $\mathbb{E}^{\min}_{\mathcal{F}_{A_1}} = 2.763304$.

The singular values of the matrix $\Psi^{\max}_{\mathcal{F}_{A_1}}$ are

$$\sigma_1 = 2.151748,\ \sigma_2 = 0.768472,\ \sigma_3 = 0.306279,\ \sigma_4 = 0.124988.$$

Hence, the optimistic energy of the interval-valued fuzzy soft set $\mathcal{F}_{A_1}$ is the sum of the obtained singular values, i.e. $\mathbb{E}^{\max}_{\mathcal{F}_{A_1}} = 3.351487$.

Now, the energy of the interval-valued fuzzy soft set $\mathcal{F}_{A_1}$ is

$$\mathbb{E}^*_{\mathcal{F}_{A_1}} = 3.0573955.$$

Similarly, we determine the remaining energies and obtain:

$$\mathbb{E}^*_{\mathcal{F}_{A_2}} = 3.0316169,\ \mathbb{E}^*_{\mathcal{F}_{A_3}} = 2.78006545,\ \mathbb{E}^*_{\mathcal{F}_{A_4}} = 2.709927,\ \mathbb{E}^*_{\mathcal{F}_{A_5}} = 3.157358.$$

We need to draw a conclusion based on the obtained energies. Since the interval-valued fuzzy soft set $\mathcal{F}_{A_4}$ has the lowest energy, this means that the element u_4 contributes the most to the energy of the overall system, or it has the greatest influence on the systemic value, so apartment u_4 should be chosen. The obtained energy values can be linearly arranged as follows:

$$\mathbb{E}^*_{\mathcal{F}_{A_4}} \leqslant \mathbb{E}^*_{\mathcal{F}_{A_3}} \leqslant \mathbb{E}^*_{\mathcal{F}_{A_2}} \leqslant \mathbb{E}^*_{\mathcal{F}_{A_1}} \leqslant \mathbb{E}^*_{\mathcal{F}_{A_5}}.$$

Thus, by comparing all the energies, we obtain a linear order of apartments based on the priority of their selection:

$$u_4 \succ u_3 \succ u_2 \succ u_1 \succ u_5.$$

Within the proposed algorithm, the emphasis is placed on omitting a single alternative from the set U, which provides insight into its impact on the total energy of the system. This approach is simple, computationally efficient, and offers a clear interpretation of the results. However, a natural question arises: what would happen if multiple alternatives were simultaneously removed from the set U? Such an approach could reveal more complex relationships among the alternatives and their joint influence on the system, which is particularly important since the interconnection and interdependence of all alternatives within the system play a crucial role in effective decision-making. Nevertheless, due to the significant increase in the number of combinations to be analyzed - and consequently, in computational complexity - this type of analysis may be reserved for special cases and future research.

5. A comparison with some existing interval-valued fuzzy soft set decision-making methods

5.1. **Son's [63] and Qin's [53] methods and their limitations.** We will introduce the two existing algorithms, which are applied to solve decision-making problems based on IVFSS.

The algorithm proposed by Son [63] is based on means of the comparison table, while the algorithm proposed by Qin [53] is based on means of the contrast table.

If we compare these two algorithms, we can conclude that they are equivalent in terms of outcomes when making a decision. Below, we will provide an example to confirm this. Additionally, we will apply our algorithm based on the energy of IVFSS to the same example, then compare it with the two existing algorithms and highlight their weaknesses in obtaining the optimal solution.

Algorithm 2	A Decision-Making Algorithm in [63]
Input	An interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe $U = \{u_1, u_2, \ldots, u_n\}$.
Output	The alternative that has the maximum score.
Step 1	Input the interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe U.
Step 2	Determine the comparison table of $\mathcal{F}_A$, where the entry c_{ij}, for $i, j = 1, 2, \ldots, n$ is the number of objects satisfying $\frac{\inf i^-_{\eta_A(x_k)}(u_i) + \sup i^+_{\eta_A(x_k)}(u_i)}{2} \geqslant \frac{\inf i^-_{\eta_A(x_k)}(u_j) + \sup i^+_{\eta_A(x_k)}(u_j)}{2}$, for $k = 1, 2, \ldots, m$.
Step 3	Determine the row sum $r_i = \sum_{j=1}^{n} c_{ij}$ and the column sum $t_i = \sum_{j=1}^{n} c_{ji}$ for u_i, for $i = 1, 2, \ldots, n$.
Step 4	Determine score $s_i = r_i - t_i$ of u_i, for $i = 1, 2, \ldots, n$.
Step 5	Determine the maximum score for all of alternatives. The corresponding alternative is referred to as the best outcome.

Algorithm 3	A Decision-Making Algorithm in [53]
Input	An interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe $U = \{u_1, u_2, \ldots, u_n\}$.
Output	The alternative that has the maximum overall dominant score.
Step 1	Input the interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe U.
Step 2	Determine average degree of membership for every entry by the formula $\bar{i}_{\eta_A(x_j)}(u_i) = \frac{i^-_{\eta_A(x_j)}(u_i) + i^+_{\eta_A(x_j)}(u_i)}{2}$, for $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m$.
Step 3	Create the contrast table for $\mathcal{F}_A$, where rows and columns are the corresponding alternatives of $\mathcal{F}_A$ and the entries m_{ij} are the number of parameters for which the average degree of membership value of the alternative u_i goes over or equal to the average degree of membership value of the alternative u_j.
Step 4	Determine the row dominant sum $R_i = \sum_{j=1}^{n} m_{ij}$ and column dominant sum $T_i = \sum_{j=1}^{n} m_{ji}$ for every alternative u_i, for $i = 1, 2, \ldots, n$.
Step 5	Determine the overall dominant score $S_i = R_i - T_i$ for every alternative u_i, for $i = 1, 2, \ldots, n$.
Step 6	Determine the maximum of the overall dominant score for all of alternatives. The corresponding alternative is the optimal choice.

Example 5.1. *A group of friends is deciding on their next vacation and they are considering three potential destinations. They are uncertain about which destination to select. We evaluate the alternatives based on three criteria: "affordability," "scenic beauty," and "availability of activities." The model of IVFSS is used to represent their preferences for the three destinations based on these criteria. Suppose the universe U represents the set of the three vacation destinations and $U = \{u_1, u_2, u_3\}$. The set of parameters A represents the evaluation criteria and*

$$A = \{x_1, x_2, x_3\} = \{\textit{affordability, scenic beauty, availability of activities}\}.$$

The obtained IVFSS is provided in Table 5.

According to Son's algorithm, we first calculate the comparison table, then compute the row sums and column sums for each alternative, based on which we obtain the scores for all the alternatives. The corresponding results are shown in Table 6.

TABLE 5. IVFSS for Example 5.1

$\mathcal{F}_A$	x_1	x_2	x_3
u_1	$[1,1]$	$[0.7,0.8]$	$[0.4,0.6]$
u_2	$[0.5,0.8]$	$[0.7,0.9]$	$[0.5,0.5]$
u_3	$[0.6,0.7]$	$[0.7,0.8]$	$[0.5,0.8]$

TABLE 6. Results obtained by applying Son's algorithm to Example 5.1

U	r_i	t_i	s_i	Rank
u_1	7	7	0	1
u_2	7	7	0	1
u_3	7	7	0	1

According to Qin's algorithm, we first calculate the average degree of membership for each entry. Then, we construct the contrast table, from which we compute the row dominant sum and column dominant sum, and subsequently determine the overall dominant score for each alternative. The corresponding results are given in Table 7. It can be noted that none of the methods result in an optimal solution for this example.

TABLE 7. Results obtained by applying Qin's algorithm to Example 5.1

U	R_i	T_i	S_i	Rank
u_1	7	7	0	1
u_2	7	7	0	1
u_3	7	7	0	1

We will now apply our algorithm based on the energy of IVFSS to Example 5.1. We find that $\mathbb{E}^*_{\mathcal{F}_{A_1}} = 1.79291$, $\mathbb{E}^*_{\mathcal{F}_{A_2}} = 2.02576$ and $\mathbb{E}^*_{\mathcal{F}_{A_3}} = 1.980907$, which allows us to establish a linear ordering of alternatives and leads to the conclusion that alternative u_1 is the optimal solution.

With this example, we have demonstrated that methods [63] and [53] do not always provide the optimal solution or a linear ordering of alternatives. In contrast, our method based on the energy of IVFSS provides both a linear ordering of alternatives and the optimal solution, as shown in Table 8.

TABLE 8. Comparative analysis of the observed methods for Example 5.1

Algorithm	Obtained ranking	Optimal solution
Method from [63]	$u_1 = u_2 = u_3$	$\times$
Method from [53]	$u_1 = u_2 = u_3$	$\times$
Method based on $\mathbb{E}$	$u_1 \succ u_3 \succ u_2$	u_1

5.2. **Yang's [67] and Ma's [38] and their limitations.** We will now present two additional equivalent algorithms regarding decision-making outcomes. The algorithm for solving decision-making problems based on interval-valued fuzzy soft sets in [67] is the score based decision making approach, while in [38] it is the decision-making method considering the added objects. In both algorithms the question of the uniqueness of the decision-making order arises, while our method based on energy improves in terms of providing a unique solution. Let us first introduce the algorithms presented in these papers.

Algorithm 4	A Decision-Making Algorithm in [67]
Input	An interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe $U = \{u_1, u_2, \ldots, u_n\}$.
Output	The alternative that has the maximum score value.
Step 1	Input the interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe U.
Step 2	Determine the choice value c_i for each alternative u_i by the equation of $c_i = \left[c_i^-, c_i^+\right] = \left[\sum_{j=1}^{m} i^-_{\eta_A(x_j)}(u_i), \sum_{j=1}^{m} i^+_{\eta_A(x_j)}(u_i)\right]$, for $i = 1, 2, \ldots, n$.
Step 3	Determine the score value r_i of u_i by the equation of $r_i = \sum_{u_j \in U} \left((c_i^- - c_j^-) + (c_i^+ - c_j^+)\right)$, for $i, j = 1, 2, \ldots, n$.
Step 4	Determine the maximum score value, with the corresponding alternative representing the best outcome.

Algorithm 5	A Decision-Making Algorithm in [38]
Input	An interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe $U = \{u_1, u_2, \ldots, u_n\}$.
Output	The alternative that has the maximum overall choice value.
Step 1	Input the interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe U.
Step 2	Determine the choice value c_i for each alternative u_i by the equation of $c_i = \left[c_i^-, c_i^+\right] = \left[\sum_{j=1}^{m} i^-_{\eta_A(x_j)}(u_i), \sum_{j=1}^{m} i^+_{\eta_A(x_j)}(u_i)\right]$, for $i = 1, 2, \ldots, n$.
Step 3	Determine the overall choice value C_i^{overall} of u_i by the equation of $C_i^{\text{overall}} = c_i^- + c_i^+$, for $i = 1, 2, \ldots, n$.
Step 4	Determine the maximum of the overall choice value for all of alternatives. The optimal alternative has the maximum overall choice value.

The algorithm in [38] takes into account newly added objects and simplifies the computational process.

In the following example, we will highlight the problem of finding an optimal solution that arises when applying the algorithms from [67] and [38]. However, by applying our energy-based algorithm, we once again obtain a linear ordering and an optimal solution.

Example 5.2. *A small business owner is looking to buy a new laptop for their work. They have narrowed down their options to four different laptop models. These models will be evaluated based on five key criteria: "performance", "battery life", "price", "screen quality" and "portability". Hence, suppose that the universe U represents the set of four different alternative laptops and $U = \{u_1, u_2, u_3, u_4\}$. Then, A represents the set of five parameters and*

$$A = \{x_1, x_2, x_3, x_4, x_5\} = \{\textit{performance, battery life, price, screen quality, portability}\}.$$

The business owner wants to make an informed decision to choose the best laptop based on these criteria. An expert team evaluates each laptop based on these parameters, creating an interval-valued fuzzy soft set with its tabular representation provided in Table 9.

According to the algorithm in [67], we calculate the choice values and scores for all alternatives. The results are presented in Table 10, from which it can be observed that there is no optimal solution.

The algorithm in [38] allows for easier computation compared to [67], but yields the same result when it comes to selecting the optimal solution. The obtained results are presented in Table 11.

TABLE 9. IVFSS for Example 5.2

$\mathcal{F}_A$	x_1	x_2	x_3	x_4	x_5
u_1	$[0.5, 0.7]$	$[0.6, 0.7]$	$[0.8, 0.8]$	$[0.6, 0.9]$	$[0.4, 0.8]$
u_2	$[0.6, 0.8]$	$[0.2, 0.5]$	$[0.7, 0.9]$	$[0.5, 0.7]$	$[0.9, 1.0]$
u_3	$[0.8, 0.9]$	$[0.6, 0.9]$	$[0.3, 0.8]$	$[1.0, 1.0]$	$[0.2, 0.3]$
u_4	$[0.4, 0.7]$	$[0.2, 0.4]$	$[1.0, 1.0]$	$[0.3, 0.8]$	$[1.0, 1.0]$

TABLE 10. Results obtained by applying Yang's algorithm to Example 5.2

U	c_i	r_i	Rank
u_1	$[2.9, 3.9]$	0	1
u_2	$[2.9, 3.9]$	0	1
u_3	$[2.9, 3.9]$	0	1
u_4	$[2.9, 3.9]$	0	1

TABLE 11. Results obtained by applying Ma's algorithm to Example 5.2

U	c_i	C_i^{overall}	Rank
u_1	$[2.9, 3.9]$	6.8	1
u_2	$[2.9, 3.9]$	6.8	1
u_3	$[2.9, 3.9]$	6.8	1
u_4	$[2.9, 3.9]$	6.8	1

Now we will demonstrate that by applying the algorithm based on the energy of IVFSS, we obtain the optimal solution. Namely, we calculate $\mathbb{E}^*_{\mathcal{F}_{A_1}} = 3.698441$, $\mathbb{E}^*_{\mathcal{F}_{A_2}} = 3.6927665$, $\mathbb{E}^*_{\mathcal{F}_{A_3}} = 3.3661405$ and $\mathbb{E}^*_{\mathcal{F}_{A_4}} = 3.5998125$. Therefore, an optimal solution is u_3.

This example further confirms that our algorithm enhances the process by ensuring a unique solution, unlike the algorithms proposed in [67] and [38], which do not achieve an optimal result in this example, as shown in Table 12.

TABLE 12. Comparative analysis of the observed methods for Example 5.2

Algorithm	Obtained ranking	Optimal solution
Method from [67]	$u_1 = u_2 = u_3 = u_4$	$\times$
Method from [38]	$u_1 = u_2 = u_3 = u_4$	$\times$
Method based on $\mathbb{E}$	$u_3 \succ u_4 \succ u_2 \succ u_1$	u_3

5.3. **Yang's [68] methods and their limitations.** Another demonstration of the optimality of our method's solution comes from the following comparison between our energy-based algorithm and the algorithms proposed by Yang and Peng [68]. They proposed a revised Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) method and a choice value method for interval fuzzy soft sets with unknown weights.

First, we will consider TOPSIS method.

Now, we will describe an algorithm based on the choice value.

Let us apply the two algorithms from [68] to the following example.

Algorithm 6	A Decision-Making Algorithm in [68] (TOPSIS)
Input	An interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe $U = \{u_1, u_2, \ldots, u_n\}$.
Output	The alternative that has the maximum closeness coefficient.
Step 1	Input the interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe U.
Step 2	Determine the relative weight w_j of the parameter x_j by the equation of $w_j = \frac{\sum_{i=1}^{n}\sum_{k=1}^{n} d(i_{\eta_A(x_j)}(u_i), i_{\eta_A(x_j)}(u_k))}{\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k=1}^{n} d(i_{\eta_A(x_j)}(u_i), i_{\eta_A(x_j)}(u_k))}$ for $j = 1, 2, \ldots, m$, where $d(i_{\eta_A(x_j)}(u_i), i_{\eta_A(x_j)}(u_k)) = \frac{1}{2}\left\|(i^-_{\eta_A(x_j)}(u_i) + i^+_{\eta_A(x_j)}(u_i)) - (i^-_{\eta_A(x_j)}(u_k) + i^+_{\eta_A(x_j)}(u_k))\right\|$.
Step 3	Obtain the fuzzy soft set Γ_A from the IVFSS $\mathcal{F}_A$ in tabular representation with entries $b_{ij} = \frac{i^-_{\eta_A(x_j)}(u_i) + i^+_{\eta_A(x_j)}(u_i)}{2}$, for $i = 1, 2, \ldots n$, $j = 1, 2, \ldots, m$.
Step 4	Determine the fuzzy positive ideal object $b^+ = \{b^+_j \mid j = 1, 2, \ldots, m\}$ and the fuzzy negative ideal object $b^- = \{b^-_j \mid j = 1, 2, \ldots, m\}$, where $b^+_j = \max_i b_{ij}$ and $b^-_j = \min_i b_{ij}$.
Step 5	Determine the difference of each alternative from the fuzzy positive-ideal solution and fuzzy negative-ideal solution D^+_i and D^-_i, for $i = 1, 2, \ldots, n$, where $D^+_i = \sum_{j=1}^{m} w_j \lvert b_{ij} - b^+_j\rvert$ and $D^-_i = \sum_{j=1}^{m} w_j \lvert b_{ij} - b^-_j\rvert$.
Step 6	Determine the closeness coefficient of each alternative C_i, where $C_i = \frac{D^-_i}{\max_j D^-_j} - \frac{D^+_i}{\min_j D^+_j}$ for $i, j = 1, 2, \ldots, n$.
Step 7	Determine the maximum of all the closeness coefficients. The optimal alternative has the maximum closeness coefficient.

Algorithm 7	A Decision-Making Algorithm in [68] (choice value)
Input	An interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe $U = \{u_1, u_2, \ldots, u_n\}$.
Output	The alternative that has the maximum choice value.
Step 1	Input the interval-valued fuzzy soft set $\mathcal{F}_A$ over the universe U.
Step 2	Determine the relative weight w_j of the parameter x_j, for $j = 1, 2, \ldots, m$.
Step 3	Obtain the fuzzy soft set Γ_A from the IVFSS $\mathcal{F}_A$ in tabular representation with entries $b_{ij} = \frac{i^-_{\eta_A(x_j)}(u_i) + i^+_{\eta_A(x_j)}(u_i)}{2}$, for $i = 1, 2, \ldots n$, $j = 1, 2, \ldots, m$.
Step 4	Determine the choice value c_i of each alternative u_i by equation $c_i = \sum_{j=1}^{m} w_j b_{ij}$, for $i = 1, 2, \ldots, n$.
Step 5	Determine the maximum of all the choice values. The optimal alternative has the maximum choice value.

Example 5.3. *A family is deciding between four coffee makers to purchase for their home. The decision is based on four key criteria: "price", "ease of use", "energy efficiency" and "design". Suppose there are four coffee makers under evaluation and $U = \{u_1, u_2, u_3, u_4\}$. Then, the set of parameters A is*

$$A = \{x_1, x_2, x_3, x_4\} = \{\textit{price, ease of use, energy efficiency, design}\}.$$

The obtained IVFSS $\mathcal{F}_A$ is given by Table 13.

If we apply the TOPSIS algorithm and the choice value algorithm, we obtain the following results presented in Table 14.

TABLE 13. IVFSS for Example 5.3

$\mathcal{F}_A$	x_1	x_2	x_3	x_4
u_1	$[0.5, 0.8]$	$[0.7, 0.7]$	$[0.6, 0.8]$	$[0.4, 0.9]$
u_2	$[0.6, 0.8]$	$[0.3, 1.0]$	$[0.6, 0.7]$	$[0.5, 0.9]$
u_3	$[0.5, 0.8]$	$[0.4, 1.0]$	$[0.5, 0.9]$	$[0.6, 0.7]$
u_4	$[0.6, 0.8]$	$[0.4, 0.9]$	$[0.3, 1.0]$	$[0.7, 0.7]$

TABLE 14. Results obtained by applying Yang's algorithms to Example 5.3

U	C_i	c_i	Rank
u_1	0	0.675	1
u_2	0	0.675	1
u_3	0	0.675	1
u_4	0	0.675	1

We can observe that, for this case, the two methods do not provide the optimal solution. Now, let's apply the method based on the energy of IVFSS. We get $\mathbb{E}^*_{\mathcal{F}_{A_1}} = 2.7125665$, $\mathbb{E}^*_{\mathcal{F}_{A_2}} = 2.778632$, $\mathbb{E}^*_{\mathcal{F}_{A_3}} = 2.883792$ and $\mathbb{E}^*_{\mathcal{F}_{A_4}} = 2.7853795$. Therefore, we can clearly prioritize the elements and identify the optimal solution. The results obtained by applying these methods are compared in Table 15.

TABLE 15. Comparative analysis of the observed methods for Example 5.3

Algorithm	Obtained ranking	Optimal solution
Method TOPSIS from [68]	$u_1 = u_2 = u_3 = u_4$	$\times$
Choice value method from [68]	$u_1 = u_2 = u_3 = u_4$	$\times$
Method based on $\mathbb{E}$	$u_1 \succ u_2 \succ u_4 \succ u_3$	u_1

5.4. **Comparison of the results of the proposed method with other methods.** After analyzing the decision-making algorithms individually and their limitations, we now perform a comparative analysis of all methods using the following example. This analysis provides a clear evaluation of their performance and demonstrates the advantages of the proposed energy-based method.

Example 5.4. *A family wants to go out for dinner. In the town where they live, there are four restaurants, and the family is undecided about which one to choose. They evaluate these restaurants based on three key criteria: "food quality", "service" and "price". The set of possible restaurants is $U = \{u_1, u_2, u_3, u_4\}$, while A represents the set of three parameters and*

$$A = \{x_1, x_2, x_3\} = \{\textit{food quality, service, price}\}.$$

The family evaluates the restaurants based on these criteria using the IVFSS model with its tabular representation provided in Table 16.

The results, which allow us to rank the restaurants and identify the optimal alternative for all the algorithms considered, are presented in Table 17. From these values, we can easily determine the ranking and, if applicable, the optimal solution.

In Table 18 the obtained rankings of alternatives are presented, as well as the optimal solution. We can notice that the solution is the same for all methods that provide an optimal solution and that solution is u_2. Also, there is a slightly different order of alternatives for some methods.

TABLE 16. IVFSS for Example 5.4

$\mathcal{F}_A$	x_1	x_2	x_3
u_1	$[0.2, 0.3]$	$[0.7, 0.8]$	$[0.3, 0.6]$
u_2	$[0.6, 0.8]$	$[0.4, 0.7]$	$[0.7, 0.8]$
u_3	$[0.4, 0.6]$	$[0.6, 0.9]$	$[0.5, 0.7]$
u_4	$[0.5, 0.6]$	$[0.4, 0.8]$	$[0.6, 0.9]$

TABLE 17. The values based on which the mentioned algorithms make decisions

Algorithm	The decision values
Son's method [63]	$s_1 = -4,\ s_2 = 2,\ s_3 = 0,\ s_4 = 2$
Qin's method [53]	$S_1 = -4,\ S_2 = 2,\ S_3 = 0,\ S_4 = 2$
Yang's method [67]	$r_1 = -2.8,\ r_2 = 1.6,\ r_3 = 0.4,\ r_4 = 0.8$
Ma's method [38]	$C_1^{\text{overall}} = 2.9,\ C_2^{\text{overall}} = 4.0,\ C_3^{\text{overall}} = 3.7,\ C_4^{\text{overall}} = 3.8$
Yang's TOPSIS method [68]	$C_1 = -6.3016,\ C_2 = 0,\ C_3 = -2.2972,\ C_4 = -1.8465$
Yang's choice value method [68]	$c_1 = 0.431,\ c_2 = 0.682,\ c_3 = 0.5905,\ c_4 = 0.6275$
Method based on $\mathbb{E}$	$\mathbb{E}^*_{\mathcal{F}_{A_1}} = 2.223658,\ \mathbb{E}^*_{\mathcal{F}_{A_2}} = 2.1248825,\ \mathbb{E}^*_{\mathcal{F}_{A_3}} = 2.2390165,$ $\mathbb{E}^*_{\mathcal{F}_{A_4}} = 2.2076045$

TABLE 18. The table with the rankings of alternatives and the optimal solution

Algorithm	Obtained ranking	Optimal solution
Son's method [63]	$u_2 = u_4 \succ u_3 \succ u_1$	$\times$
Qin's method [53]	$u_2 = u_4 \succ u_3 \succ u_1$	$\times$
Yang's method [67]	$u_2 \succ u_4 \succ u_3 \succ u_1$	u_2
Ma's method [38]	$u_2 \succ u_4 \succ u_3 \succ u_1$	u_2
Yang's TOPSIS method [68]	$u_2 \succ u_4 \succ u_3 \succ u_1$	u_2
Yang's choice value method [68]	$u_2 \succ u_4 \succ u_3 \succ u_1$	u_2
Method based on $\mathbb{E}$	$u_2 \succ u_4 \succ u_1 \succ u_3$	u_2

When analyzing and comparing the results obtained using the methods presented in the previous subsections, we observe certain qualitative differences. All previous methods failed to provide a linear ranking of alternatives in several of the examples used. For an algorithm to be considered efficient, it must ensure a unique optimal solution. This was not the case for the algorithms presented in [63], [53], [67], [38] and [68] where multiple solutions were possible in some examples. However, our algorithm based on the energy of IVFSS consistently produced a unique solution and a linear ranking for the corresponding sets of alternatives. It is important to note that when the aforementioned algorithms also yield an optimal solution, the solution remains reliable, thereby validating the energy-based algorithm. In certain specific examples, the ranking of alternatives may completely coincide across all applied algorithms, whereas the energy-based algorithm is capable of producing a linear ranking even in situations where other algorithms are not successful. In this sense, it is not possible to draw a general conclusion about the extent to which the proposed algorithm outperforms others.

After analyzing the obtained results, we can conclude that our method based on the pessimistic energy, the optimistic energy and the energy of interval-valued fuzzy soft sets does not fully correspond to the proposed solutions in the observed papers (in terms of the overall ranking of alternatives). In the context of uncertainty, the proposed algorithm enables decision-making but does not provide an assessment of the reliability of those decisions. The level of confidence in the decision correlates with the minimum or maximum energy of the interval-valued fuzzy soft set, and the main challenge of methods relying on the energy of interval-valued fuzzy soft sets is the difficulty in determining the exact lower and upper bounds of this energy.

One of the drawbacks of our algorithm is that calculating the energy without the help of software tools presents a real challenge. However, the use of modern mathematical software systems can simplify this process, providing efficient solutions for decision-making.

Another limitation of the algorithm, although it occurs very rarely, is that the assignment of energy is not injective, meaning that two different IVFSSs can have the same energy values. This happens because all parameters are treated as equally contributing factors. This issue could be overcome by taking into account different parameter weights, as they influence the ranking results of alternatives.

The main characteristics of the methods compared in this chapter are summarized in the Table 19.

Table 19. Qualitative comparative analysis: main characteristics of the methods considered in this work in view of the computations shown in section 5

Procedure	Ranking methodology	Unique solution	Fine assessment during selection
Son's method [63]	Means of the comparison table	No	No
Qin's method [53]	Means of the contrast table	No	No
Yang's method [67]	Scores based decision-making approach	No	Yes
Ma's method [38]	Decision-making method considering the added objects	No	Yes
Yang's TOPSIS method [68]	Closeness coefficient based decision-making approach	No	Yes
Yang's choice value method [68]	Choice value method with unknown weights	No	Yes
Method based on $\mathbb{E}$	Scores based on comparison of energies	Yes	Yes

6. Summary and Conclusion

Interval-valued fuzzy soft sets are obtained by combining interval-valued fuzzy sets and soft sets. They find wide applications in numerous areas, including decision-making. The main focus is on defining a numerical characteristic of interval-valued fuzzy soft sets - the energy of IVFSS. The concept of energy is well known in graph theory and it is similar to the concept of nuclear norm. We have presented a new decision-making algorithm based on the introduced energy of IVFSS, which is a new perspective to make decision. A parallel is drawn between the decision-making algorithm based on the energy of IVFSS and several other algorithms, highlighting the advantages of our proposed algorithm, especially regarding the existence of an optimal solution.

Although our research establishes a strong foundation for practical applications, there are still opportunities for further improvement. Future research will focus on exploring the boundaries of the energy of IVFSS and optimizing the algorithm for large-scale systems with high-dimensional data. In addition, a subject of future research could be a formal proof that would confirm the obtained results, as well as examining the impact of simultaneously removing multiple alternatives from the universal set. The method introduced here can also be applied to other hybrid models, such as intuitionistic hesitant fuzzy soft sets or interval-valued bipolar complex fuzzy soft sets.

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