Mathematics in Applied Sciences and Engineering https://ojs.lib.uwo.ca/mase
Volume **6**, Number **3**, September **2025**, pp.**280-302** https://doi.org/10.5206/mase/22681

DISTANCE, SIMILARITY AND ENTROPY MEASURES OF N-DIMENSIONAL FUZZY SETS

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Abstract. An n-dimensional fuzzy set generalizes other fuzzy structures and effectively addresses real-world problems by providing greater flexibility in assigning membership values through the selection of an arbitrarily large n. Information measures are essential tools that yield significant judgments about data expressed in fuzzy form. This paper presents the concepts of n-dimensional distance measures, similarity measures, and entropy measures, accompanied by significant examples for each, and demonstrates the interrelationships among these measures. Certain specialized measures, including σ-measure, proximity measure, and linear measure, are examined, and significant results pertaining to them are derived. A succinct approximation of n-dimensional fuzzy sets is shown through the distance measure and the notion of orderless n-dimensional fuzzy sets, which proves advantageous in addressing practical issues. Ultimately, two decision-making dilemmas are resolved utilizing the concepts presented.

1. Introduction

1.1. **Literature Review and Motivation.** Fuzzy sets significantly contribute to the growing field of general systems research by effectively modeling systems characterized by imprecise components and connections, a common trait in numerous biological and social systems. Most decision-making contexts, especially group and multi-criteria scenarios, assign a distinct membership value to each element and are complicated by the inherent ambiguities of the situations. To address this issue, one method involves substituting the singular membership value with a collection of membership values, as exemplified in interval-valued fuzzy sets [28, 42] and hesitant fuzzy sets [40, 17]. Alternatively, one could replace the individual membership function with multiple functions, as demonstrated in intuitionistic [6, 3] and picture fuzzy sets [10] cases. Numerous additional extended fuzzy structures exist, including neutrosophic sets, dual hesitant fuzzy sets, intuitionistic fuzzy numbers, η-fuzzy subgroups, and intuitionistic fuzzy complex subgroups, all of which possess a framework to encapsulate the inherent vagueness associated with each element [16, 32, 1, 36]. Shang et al. [37] produced n-dimensional fuzzy sets, analogous to the aforementioned technique. n-dimensional fuzzy sets represent a generalization of various established structures, including interval-valued fuzzy sets, and provide greater flexibility in membership values.

A fuzzy distance measure quantifies the disparity between two fuzzy sets. Numerous studies utilized it, although the most effective axioms for delineating a fuzzy distance measure were provided by Liu [30]. The distance metric provides insight into the variability of data across two distinct fuzzy sets. The similarity measure of two fuzzy sets, as established by Wang [27] and others, quantifies the similarity between fuzzy sets and is associated with the idea of distance measure. Distance and similarity metrics are clearly opposing concepts that are intricately related. The subject of similarity measures is crucial when addressing unknown data. Numerous scholars have suggested diverse similarity metrics across fuzzy sets, soft sets, fuzzy soft sets, intuitionistic fuzzy sets, and intuitionistic fuzzy soft sets in recent years [31, 21, 9, 18, 43]. Similarity measures

Received by the editors 12 March 2025; accepted 12 June 2025; published online 21 June 2025.
2010 *Mathematics Subject Classification.* Primary 05CXX, 03E72; Secondary 05C72.
Key words and phrases. n-dimensional distance measure, n-dimensional similarity measure, n-dimensional entropy, crisp approximations, orderless n-dimensional fuzzy sets, decision making problems.

have been extensively utilized in decision-making, pattern recognition, region extraction, coding theory, image processing, signal detection, security verification systems, medical diagnostics, and various other domains.

Probability has conventionally been employed to model uncertainty. Fuzziness, conceived by Zadeh, has emerged as an additional method for modeling uncertainty. Conversely, measuring the ambiguity of fuzzy sets is a crucial aspect of fuzzy systems. The entropy of fuzzy sets quantifies the degree of uncertainty among fuzzy sets. De Luca and Termini innovated the axiom formulation for fuzzy set entropy based on Shannon's probability entropy [11]. Ebanks established five criteria that the entropy of a fuzzy set should fulfill [14]. Kaufmann indicated that the entropy of a fuzzy set can be interpreted as the distance between the fuzzy set and the closest non-fuzzy set [25]. Yager characterized the fuzziness of a fuzzy set as the absence of distinction between the fuzzy set and its negation [41]. Subsequently, Xuecheng and Liu examined all these measures and established sigma measures[30].

1.2. **Problem Statement.** Shang et al. proposed and examined the notion of n-dimensional fuzzy sets utilizing a partially ordered set $I_n([0,1])$ [37]. In n-dimensional fuzzy sets, we select a dimension sufficiently large to encompass the requisite information for the analysis. This facilitates a more pragmatic approach to problem-solving. By allowing the assignment of n values ranging from 0 to 1 rather than a single value, the inaccuracy linked to inaccurate scoring can be mitigated. Real-world scenarios, such as an antenna receiving signals from n fixed transmitters and the processing of data from neuronal transmitters, exemplify contexts where data representation through n-dimensional fuzzy sets is applicable. Although we possess a tool to represent data gathered from real-life scenarios for decision-making and comparative analysis, it is essential to initially add information metrics, including distance measures, similarity measures, and entropy measures. Furthermore, as n-dimensional fuzzy sets represent a generic version of diverse fuzzy structures, the informational tools and findings presented herein can be utilized for other established fuzzy varieties. This research intends to examine diverse information metrics of $n-$dimensional fuzzy sets. The significance and implementation of such methods in practical situations are evident in publications such as [19, 20, 43]. Compared to other iterations of generalized fuzzy structures, such as type-2 fuzzy sets [33, 24], genuine sets [35, 13], and intuitionistic fuzzy sets [2], n-dimensional fuzzy sets exhibit greater efficacy in addressing practical issues due to their simplicity and the autonomy afforded to each element, independent of others. Further findings pertaining to n-dimensional fuzzy sets are available in [12, 34].

In Section 2, some preliminaries that are important for our studies are discussed. In Section 3, n-dimensional distance measures, important examples, n-dimensional similarity measures, some important theorems connecting them, proximity distance measures, and linear distance measures are all discussed. The concept of crisp approximation of an n-dimensional fuzzy set is introduced with the help of distance measures. Then the notions of the σ distance measure and the σ similarity measure of n-dimensional fuzzy sets, along with some results, are studied. In Section 4, n-dimensional entropy, σ entropy, and orderless n-dimensional fuzzy sets are studied. In Section 5, two different decision-making applications using distance measure and entropy are given, and they are compared with existing fuzzy structures in Section 6.

2. Preliminaries

Distance measure of fuzzy sets

The fuzzy distance measure evaluates the differences between two fuzzy sets and provides a number that signifies the differences in the data they provide, much like a metric in a metric space distinguishes between elements and delivers us a measure of how they differ in structure. The following axioms create the desirable characteristics of a fuzzy distance measure.

Definition 2.1. [30]. Let $\mathscr{F}(Y)$ be the collection of all fuzzy sets of Y ; $\mathscr{P}(Y)$ be the class of all crisp sets of Y; $\left[\frac{1}{2}\right]_Y$ be the fuzzy set of Y for which $\left[\frac{1}{2}\right]_Y(y)=\frac{1}{2}, \forall y \in Y$ and let $\mathscr{F}$ is a sub-class of $\mathscr{F}(Y)$ with

(1) $\mathscr{P}(Y) \subset \mathscr{F}$;
(2) $\left[\frac{1}{2}\right]_Y \in \mathscr{F}$;
(3) $K, H \in \mathscr{F} \Rightarrow K \cup H \in \mathscr{F}, K^{c} \in \mathscr{F}$.

Then a distance measure on $\mathscr{F}$ is a function $\hat{d}: \mathscr{F}^2 \rightarrow R^{+}$ that satisfies the following axioms:

($\hat{D}1$) $\hat{d}(E, F)=\hat{d}(F, E), \forall E, F \in \mathscr{F}$;
($\hat{D}2$) $\hat{d}(E, E)=0, \forall E \in \mathscr{F}$;
($\hat{D}3$) $\hat{d}\left(D, D^{c}\right)=\max_{E, F \in \mathscr{F}} \hat{d}(E, F), \forall D \in \mathscr{P}(Y)$;
($\hat{D}4$) $\forall E, F, K \in \mathscr{F}$, if $E \subset F \subset K$, then $\hat{d}(E, F) \leq d(E, K)$ and $\hat{d}(F, K) \leq \hat{d}(E, K)$.

Following are some important examples:

- the Hamming distance:

$$\hat{d}_{\mathbf{H}}(E, F)=\sum_{i=1}^{n}\left|E\left(y_i\right)-F\left(y_i\right)\right|;$$

- the normalized Hamming distance:

$$\hat{d}_{\mathrm{nH}}(E, F)=\frac{1}{n} \sum_{i=1}^{n}\left|E\left(y_i\right)-F\left(y_i\right)\right|;$$

- . the Euclidean distance ;

$$\hat{d}_{\mathrm{E}}(E, F)=\sqrt{\sum_{i=1}^{n}\left(E\left(y_i\right)-F\left(y_i\right)\right)^2};$$

- the normalized Euclidean distance measure:

$$\hat{d}_{\mathrm{nE}}(E, F)=\sqrt{\frac{1}{n} \sum_{i=1}^{n}\left(E\left(y_i\right)-F\left(y_i\right)\right)^2}.$$

Definition 2.2. [30] Let $\mathscr{P}(Y)$ be the collection of all crisp sets of Y. A σ-distance measure on $\mathscr{F}$ is a distance measure $\hat{d}$ which satisfies the following condition

$$\hat{d}(E, F)=\hat{d}(E \cap D, F \cap D)+d\left(E \cap D^{c}, F \cap D^{c}\right)$$

for any $E, F \in \mathscr{F}$ and for any $D \in \mathscr{P}(Y)$.

Similarity measure [30]

A similarity measure, is a function $\hat{s}: \mathscr{F}^2 \rightarrow R^{+}$ which satisfies the following conditions:

($\hat{S}1$) $\hat{s}(E, F)=\hat{s}(F, E), \forall E, F \in \mathscr{F}$;
($\hat{S}2$) $\hat{s}\left(D, D^{c}\right)=0, \forall D \in \mathscr{P}(Y)$;
($\hat{S}3$) $\hat{s}(E, E)=\max_{F, K \in \mathscr{F}} \hat{s}(F, K), \forall E \in \mathscr{F}$;
($\hat{S}4$) $\forall E, F, K \in \mathscr{F}$, if $E \subset F \subset K$, then $\hat{s}(E, F) \geq \hat{s}(E, K)$ and $\hat{s}(F, K) \geq \hat{s}(E, K)$.

For example, consider two continuous fuzzy sets $K, H \in \mathscr{F}$ and define

$$\hat{s}_p(K, H)=1-\left(\int_0^1|K(y)-H(y)|^p\right)^{1/p} \quad \forall K, H \in \mathscr{F}.$$

Then $\hat{s_p}$ is a similarity measure on the sub-collection of all continuous functions from $\mathscr{F}$. By normalizing distance measure more examples of similarity measure can be generated using the following theorem.

Theorem 2.1. [30] *If $\hat{d}$ is a normal distance measure then $\hat{s}=1-\hat{d}$ is a normal similarity measure and the converse is also true.*

Entropy [30]

An entropy is a function $\hat{e}: \mathscr{F} \to R^{+}$ which satisfies the following properties:

(E1) $\hat{e}(D)=0, \forall D \in \mathscr{P}(Y)$;
(E2) $\hat{e}\left(\left[\frac{1}{2}\right]_Y\right)=\max_{E \in \mathscr{F}} \hat{e}(E)$;
(E3) $\forall E, F \in \mathscr{F}$, if $F(y) \geq E(y)$ when $E(y) \geq \frac{1}{2}$ and $F(y) \leq E(y)$ when $E(y) \leq \frac{1}{2}$, then $\hat{e}(E) \geq \hat{e}(F)$;
(E4) $\hat{e}\left(E^{\mathrm{c}}\right)=\hat{e}(E), \forall E \in \mathscr{F}$.

Example 1. *Let $U=\{y_1, \ldots, y_n\}$ and define*

$$\hat{e}_1(E)=-K \sum_{i=1}^{n} S\left(E(y_i)\right) \quad \forall E \in \mathscr{F}(Y).$$

Then $\hat{e}_1$ is an entropy on $\mathscr{F}(U)$, where $S(y)=-y \ln y-(1-y) \ln (1-y), 0 \leq y \leq 1$, and we take the convention that $0 \ln 0=0$.

n dimensional fuzzy sets

The following two examples show the motivation and usefulness of n-dimensional fuzzy sets. Consider a situation where n students are being interviewed by two companies, A and B, with m and k members on their respective interview panels. Subsequently, each member may assess the students and allocate a score ranging from 0 to 1, so enabling the overall interview outcomes to be represented as an m-dimensional fuzzy set and a k-dimensional fuzzy set, respectively. Consider a scenario in which an interviewer seeks to assess the communication abilities of several students. In accordance with established fuzzy logic protocols, the evaluator will assess the student's communication skills through several methodologies and issue grades ranging from 0 to 1. Nevertheless, if the examiner is authorized to assign three values between 0 and 1 to reflect each student's score, the inaccuracy linked to designating a single value as the final grade can be mitigated.

Numerous cases such as the aforementioned can be represented using n-dimensional fuzzy sets for the entirety of the obtained data. Furthermore, $n-$dimensional fuzzy sets serve as generalizations of numerous current fuzzy extended models. Consequently, it is essential to examine distance measures, similarity measures, and entropy of n-dimensional fuzzy sets to accurately define them. The subsequent text presents the formal definition of n-dimensional fuzzy sets.

Definition 2.3. [37] Let U be a nonempty crisp set and n be a natural number, then an n-dimensional fuzzy set is given by the structure, $\mathcal{M}=\{(y, \mu_{\mathcal{M}}^{1}(y), ..\mu_{\mathcal{M}}^{n}(y)) : y \in U\}$ where $\mu_{\mathcal{M}}^{i}: U \to [0,1]$, $i=1,2 \ldots n$ are the membership functions satisfying $\mu_{\mathcal{M}}^{1}(y) \leq \cdots \leq \mu_{\mathcal{M}}^{n}(y)$.

Bedregal et al. introduced the following set [4]

$$\mathcal{I}_n([0,1])=\left\{(y_1, \ldots, y_n) \in[0,1]^n \mid y_1 \leq y_2 \leq \cdots \leq y_n\right\}.$$

Hence, we have $\left(\mu_{\mathcal{M}}^{1}(y), \ldots, \mu_{\mathcal{M}}^{n}(y)\right) \in \mathcal{I}_n([0,1])$ and there by $\mathcal{I}_n([0,1])$ contain all n-dimensional membership degrees. Hence, we can treat an n-dimensional fuzzy set as a function $\mathcal{M}$ from U to $\mathcal{I}_n([0,1])$.

Note: The element $(y, \ldots, y) \in \mathcal{I}_n([0,1])$ is denoted as $/y/n$ and it is referred to as the degenerate element for each $y \in[0,1]$.

Bedregal et al. [4] defined a partial order $\leq_n^p$ on $\mathcal{I}_n([0,1])$ called the product order, given by

$$\mathcal{Z} \leq_n^p \mathcal{V} \Leftrightarrow z_j \leq v_j \text{ for } j=1, \ldots, n.$$

We use the following standard t-norms,t-conorms, and complement functions of n-dimensional fuzzy sets for taking intersection, union, and complement of n-dimensional fuzzy sets, respectively, for our study.

$$\mathcal{Z} \cup \mathcal{V}=\left(\max\left\{z_1, v_1\right\}, \ldots, \max\left\{z_n, v_n\right\}\right),$$

$$\mathcal{Z} \cap \mathcal{V}=\left(\min\left\{z_1, v_1\right\}, \ldots, \min\left\{z_n, v_n\right\}\right),$$

and

$$\mathcal{Z}^c = (1 - z_n, ...1 - z_1).$$

For more details regarding aggregation operators in $n-$dimensional fuzzy sets, the reader may refer to [22, 23].

3. Distance and Similarity Measures

Distance and similarity measures are two essential tools in every sector of generalized sets. We can see how crucial they are for applications such as decision-making, image processing, and clustering. Distance and similarity measures help to compare two data which are represented using some fuzzy variants. For example, image processing is an important tool in AI technology, and IT, in which images captured by machines are stored as some proper data tool and then compared with existing similar data to build approximated figures. $n-$dimensional distance measures are the generalizations of various kinds of existing distance measures in several fields. For example, in the case of intuitionistic fuzzy sets, we have the following observation[38].

An intuitionistic fuzzy set I in X is given by $I = \{\langle x, \mu_I(x)\ ,\ v_I(x)\rangle \mid x \in X\}$, where

$$\mu_I(x) : X \to [0,1], v_I(x) : X \to [0,1]$$

with the condition

$$0 \leqslant \mu_I(x) + v_I(x) \leqslant 1 \quad \forall x \in X.$$

Let $\pi_I(x) = 1 - \mu_I(x) - v_I(x)$. Then the following are the two important distance measures in the collection of intuitionistic fuzzy sets.

For two intuitionistic fuzzy sets A and B over $X = \{x_1, ...x_n\}$, the Hamming distance measure for them is given by

$$d_{IFS}(A,B) = \frac{1}{2}\sum_{i=1}^{n}\Big(|\mu_A(x_i) - \mu_B(x_i)| + |v_A(x_i) - v_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|\Big).$$

The Euclidean distance measure of intuitionistic fuzzy sets A, B is given by

$$e_{IFS}(A,B) = \Big(\frac{1}{2}\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2 + (v_A(x_i) - v_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2\Big)^{\frac{1}{2}}.$$

It can be seen that our $n-$ dimensional distance measures are the generalization of these distance measures, and when $n = 3$ our Euclidean distance measure and Hamming distance measures of n-dimensional fuzzy sets coincide with the above measures, respectively. Similarly, we can see that distance measures of spherical fuzzy set [26]and Pythagorean fuzzy set [15]are also some special case of n-dimensional fuzzy sets

3.1. **Distance Measure of an n-dimensional Fuzzy Sets.** An n-dimensional distance measure is a generalized concept of fuzzy distance measure that differentiates between n-dimensional fuzzy sets depending on the data they present. Similar to how a metric in a metric space differentiates between unique elements, the distance measure on n-dimensional fuzzy sets distinguishes amongst n-dimensional fuzzy sets. A fuzzy distance measure in n dimensions can be defined using the subsequent axioms. The distance metric must exhibit symmetry, as in $D1$, as the disparity between data sets A and B is identical to that between B and A. Furthermore, there is no distinction between the data supplied by a set and that of the identical set; hence, $D2$ is equally indispensable. Given that members of $\mathbb{D}$ generalize crisp sets and that elements of D and D^c are distinctly categorized, it is our objective to assign the maximum distance measure to the pair $(D,\ D^c)$ for $D \in \mathbb{D}$. This serves as the rationale for the axiom $D3$. Since n-dimensional fuzzy sets that exhibit closer values demonstrate greater similarity in the data. It is inherent to allocate a lesser distance metric to pairs of n-dimensional fuzzy sets that are in closer proximity compared to those that are more distant. This is the principle underlying the axiom $D4$.

Definition 3.1. Let $\mathbf{U}$ be a nonempty set and let $\mathbb{F}_n$ denote the collection of all n-dimensional fuzzy sets over $\mathbf{U}$. Let $\mathbb{D}$ denote the sub collection of $\mathbb{F}_n$ which contains all n-dimensional fuzzy sets D with $D(y) = /1/n$ *or* $/0/n$ for each y in $\mathbf{U}$. Let $\mathbb{F}$ be a sub-collection of $\mathbb{F}_n$ with the following properties,

(1) Closed under finite union and complements. That is if $A, B \in \mathbb{F}_n$ then $A \cup B \in \mathbb{F}_n$ and $A^c \in \mathbb{F}_n$
(2) Let $K \in \mathbb{F}_n$ be the n-dimensional fuzzy set with $K(y) = /0.5/n$ for all y then $K \in \mathbb{F}$;
(3) $\mathbb{D} \subseteq \mathbb{F}$.

Definition 3.2. An n-dimensional distance measure on $\mathbb{F}$ is a function γ: $\mathbb{F} \times \mathbb{F} \to [0, \infty)$ which satisfies the following 4 axioms:

D1 : $\gamma(K, H) = \gamma(H, K)$, for every $K, H \in \mathbb{F}$;
D2 : $\gamma(K, K) = 0$ for all $K \in \mathbb{F}$;
D3 : $\gamma(D,D^c)$=max$\{\gamma(K, H); K, H \in \mathbb{F}\}$ for all $D \in \mathbb{D}$;
D4 : If $K, H, E \in \mathbb{F}$: $K \subseteq H \subseteq E$ then $\gamma(K, H) \leq \gamma(K, E)$ and $\gamma(H, E) \leq \gamma(K, E)$.

Result 1. *From D3 we have, If* $D_1, D_2 \in \mathbb{D}$ *then* $\gamma(D_1, D_1^c)$=$\gamma(D_2, D_2^c)$.

Following are some important examples of distance measures that can be easily verified.

Discrete distance: Define $\gamma(K, H) = 0$ if $K = H$ and 1 otherwise. Clearly, γ satisfies the axioms of distance measure on $\mathbb{F}$.

Euclidean distance: Let $|\mathbf{U}|$=m and K and H be two n-dimensional fuzzy sets over $\mathbf{U}$ with $K(y_i) = (K_1(y_i), ...K_n(y_i))$ and $H(y_i) = (H_1(y_i), ...H_n(y_i))$ then $\gamma_2(K, H)$ is defined as
$\gamma_2(K, H)$= $\sum_{j=1}^{m}(\sum_{i=1}^{n}|K_i(y_j) - H_i(y_j)|^2)^{\frac{1}{2}}$.

Proof. Clearly, γ_2 satisfies $D1, D2$. Now, we have $0 \leq K_i(y_j), H_i(y_j) \leq 1$ for every i, j. So that for any $D \in \mathbb{D}$ we have $0 \leq |K_i(y_j) - H_i(y_j)| \leq 1 = |D_i(y_j) - D_i^c(y_j)|$ and hence $\gamma(D, D^c) \geq \gamma(K, H)$. Thus $D3$ is satisfied. Now if $K \subseteq H \subseteq E$ then $|K_i(y_j) - H_i(y_j)| \leq |K_i(y_j) - E_i(y_j)|$ for every i, j and as above, we have $\gamma(K, H) \leq \gamma(K, E)$ and similarly, we can show that $\gamma(H, E) \leq \gamma(K, E)$. Thus $D4$ is also satisfied. □

Hamming distance: Considering the above conditions we define $\gamma_h(K, H)$=$\sum_{j=1}^{m}(\sum_{i=1}^{n}|K_i(y_j) - H_i(y_j)|)$.

Proof. Similar to the proof 3.1. □

pth distance: $\gamma_p(K, H) = \sum_{j=1}^{m}(\sum_{i=1}^{n}|K_i(y_j) - H_i(y_j)|^p)^{\frac{1}{p}}$ where $p \geq 1$.

Proof. Similar to the proof 3.1. □

Supremum distance: $\gamma_s(K, H) = sup\{\sum_{i=1}^{n}|K_i(y_j) - H_i(y_j)|, y_j \in \mathbf{U}\}$

3.2. **Similarity Measure Of n-dimensional Fuzzy Sets.** A similarity measure is a metric that indicates the degree of similarity between the data in two $n-$ dimensional fuzzy sets. Distance and similarity metrics are contrasting concepts that exhibit significant interrelations. The second axiom guarantees the absence of similarity between D and D^c if $D \in \mathbb{D}$. It is our inherent assumption that a n-dimensional fuzzy set exhibits the most resemblance with itself, and $S3$ guarantees this. $S4$ guarantees monotonicity, similar to distance measurements.

Definition 3.3. A similarity measure on $\mathbb{F}$ is a function ξ: $\mathbb{F} \times \mathbb{F} \to [0, \infty)$ which satisfies the following 4 axioms,

S1 : $\xi(K, H) = \xi(H, K)$;
S2 : $\xi(D, D^c) = 0$ for all $D \in \mathbb{D}$;
S3 : $\xi(K, K) = max\{\xi(H, C); H, C \in \mathbb{F}\}$;

S4 : If $K, H, E \in \mathbb{F}$: K$\subseteq H \subseteq E$ then $\xi(K,H) \geq \xi(K,E)$ and $\xi(H,E) \geq \xi(K,E)$.

From $S3$ in the above definition, we have

Result 2. *$\xi(K,K) = \xi(H,H)$ for all $K, H \in \mathbb{F}$.*

From $D3$ *and* $S3$, we can see that both γ and ξ are bounded. Hence we can normalize γ and ξ as follows,

$$\hat{\gamma}(K,H) = \frac{\gamma(K,H)}{\gamma(D,D^c)} \tag{3.1}$$

where $D \in \mathbb{D}$

$$\hat{\xi}(K,H) = \frac{\xi(K,H)}{\xi(E,E)} \tag{3.2}$$

with $E \in \mathbb{F}$.

Note: From Result 1 and Result 2 it is clear that we can choose any $D \in \mathbb{D}$ and $E \in \mathbb{F}$ for normalizing distance and similarity measure for n-dimensional fuzzy sets respectively. From now on, we will only consider normalized distance and similarity measures. The following theorem will give examples of the n-dimensional similarity measures from the n-dimensional distance measure.

Theorem 3.1. *γ is an n-dimensional normalized distance measure if and only if $\xi = 1 - \gamma$ is an n-dimensional normalized similarity measure.*

Proof. Let γ be an n-dimensional normalized distance measure and let $\xi = 1 - \gamma$. Then clearly ξ satisfies $S1$. Now from Equation (3.1) we have $\gamma(D, D^c) = 1$ for any, $D \in \mathbb{D}$ hence $\xi(D, D^c) = 0$ for all $D \in \mathbb{D}$ and thus $S2$ is satisfied. Now $\xi(K,K) = 1 - \gamma(K,K) = 1$ for every K thus $\xi(K,K) = 1 = max\{\xi(H,C); H, C \in \mathbb{F}\}$. Clearly ξ satisfies $S4$ as γ satisfies $D4$. Thus, ξ is a normalized similarity measure. Similarly, we can show that if ξ is a normalized similarity measure then $\gamma = 1 - \xi$ is a normalized distance measure. □

Definition 3.4. A distance (similarity) measure is said to be a proximity measure if $\gamma(K,H) = \gamma(K^c, H^c)$ $(\xi(K,H) = \xi(K^c, H^c))$ for all K,H $\in \mathbb{F}$.

Result 3. *If γ is a distance measure, then*

$$\hat{\gamma}(K,H) = \frac{(\gamma(K,H) + \gamma(K^c, H^c))}{2}$$

is a proximity distance measure

Proof. D_1 and D_2 follow directly. It is enough to prove D_3 and D_4.

$$\begin{aligned}\hat{\gamma}(D, D^c) &= \frac{(\gamma(D, D^c) + \gamma(D^c, D)}{2} \\ &\geq \frac{max\{\gamma(K,H);, K, H \in \mathbb{F}\} + max\{\gamma(K,H); K, H \in \mathbb{F}\}}{2} \\ &= \max\{\hat{\gamma}(K,H); K, H \in \mathbb{F}\},\end{aligned}$$

confirming D_3. To prove D_4, let $K \subseteq H \subseteq E$, implying $E^c \subseteq H^c \subseteq K^c$. Hence $\gamma(K,H) \leq \gamma(K,E)$ and $\gamma(H^c, K^c) \leq \gamma(E^c, K^c)$. So,

$$\hat{\gamma}(K,H) = \frac{(\gamma(K,H) + \gamma(K^c, H^c))}{2} \leq \frac{(\gamma(K,E) + \gamma(K^c, E^c))}{2} = \hat{\gamma}(K,E).$$

Similarly, we can prove that $\hat{\gamma}(H,E) \leq \hat{\gamma}(K,E)$ □

Result 4. *$\hat{\gamma}(K,H) = min\{\gamma(K,H), \gamma(K^c, H^c)\}$ is a proximity measure.*

Example 2. *The p^{th} distance γ_p is a proximity measure. Because, if $K(y) = (K_1(y), ...K_n(y))$ and $H(y) = (H_1(y), ...H_n(y))$ then $K^c(y) = (1-K_n(y), ...1-K_1(y))$ and $H^c(y) = (1-H_n(y), ...1-H_1(y))$. Hence*

$$\sum_{i=1}^{n} |K_i^c(y) - H_i^c(y)|^p)^{\frac{1}{p}} = \sum_{i=1}^{n} |K_i(y) - H_i(y)|^p)^{\frac{1}{p}}$$

for each y and thus $\gamma_p(K, H) = \gamma_p(K^c, H^c)$.

Note: It is clear that the supremum distance defined as

$$\$\gamma_s(K, H) = \sup\{\sum_{i=1}^{n} |K_i(y_j) - H_i(y_j)|, y_j \in \mathbf{U}\}$$

is a proximity distance measure.

Definition 3.5. Let $(\frac{1}{2})_x \in \mathbb{F}$ denote the n-dimensional fuzzy set with membership value /0.5/n for all x. Then $K, H \in \mathbb{F}$ are said to similar pair if $\xi(K, (\frac{1}{2})_x) = \xi(H, (\frac{1}{2})_x)$.

Note: Clearly K and K^c are similar pairs if ξ is a proximity measure.

Moreover, we have the following theorem.

Theorem 3.2. *If ξ is a proximity similarity measure, then K and H are similar pairs if and only if K^c and H^c are so.*

Proof. : Assume ξ is a proximity measure. If K and H are similar pairs with respect to ξ, then $\xi(K, (\frac{1}{2})_x) = \xi(H, (\frac{1}{2})_x)$, implying $s((\frac{1}{2})_x, K^c) = \xi((\frac{1}{2})_x, H^c)$(since s is a proximity measure). Hence K^c and H^c are similar pairs. Converse is similar. □

Definition 3.6. Let γ be a distance measure in $\mathbb{F}$ then $K \in \mathbb{F}$ said to linear with respect to $(\frac{1}{2})_x$, if $\gamma(K, K^c) = \gamma(K, (\frac{1}{2})_x) + \gamma(K^c, (\frac{1}{2})_x)$. If every $K \in \mathbb{F}$ is linear with respect to $(\frac{1}{2})_x$ then γ is called a linear distance measure.

Example 3. *Hamming distance γ_h is a linear distance measure.*

Note: Clearly if γ is a proximity linear distance measure then for each $K \in \mathbb{F}, \gamma(K, K^c) = 2\gamma(K, (\frac{1}{2})_x)$.

Sometimes it is important to find the crisp approximation of an n-dimensional fuzzy set. The notion of distance measure can be used well for finding the best crisp approximation of a multidimensional set. So we define the crisp approximation of $K \in \mathbb{F}$ using γ as follows.

Definition 3.7. Let $K \in \mathbb{F}$ then D $\in \mathbb{D}$ is said to crisp approximation if $\gamma(K, D) = min\{\gamma(K, D'), D' \in \mathbb{D}\}$.

Result 5. *Let γ_p be the p^{th} distance measure. Then there exits $K \in \mathbb{F}$ which does not have a unique crisp approximation.*

As an demonstrating example, let $\mathbf{U} = \{y_1, y_2, \}$ and let $n = 2$. Define a 2-dimensional fuzzy set on $\in \mathbf{U}$ by $K(y_1) = (\frac{1}{3}, \frac{2}{3})$ and $K(y_2) = (\frac{1}{4}, \frac{3}{4})$. Now define D' and $D'' \in \mathbb{D}$ by $D'(y_1) = (0, 0)$ and $D'(y_2) = (1, 1)$ $D''(y_1) = (1, 1)$ and $D''(y_2) = (0, 0)$. Then $\gamma(K, D') = \gamma(K, D'')$ and D' and D'' are clearly crisp approximation of A.

The next theorem gives sufficient conditions for the uniqueness of crisp approximation.

Theorem 3.3. *Let $K \in \mathbb{F}$ be such that $K(y) \leq_n^p /\frac{1}{2}/n$ or $\frac{1}{2}/n \leq_n^p K(y)$ for each y and strict inequality holds for every y , then K has unique crisp approximation with respect to γ_p.*

Proof. Define $D \in \mathbb{D}$ by $D(y) = \begin{cases} /0/n & if\ K(y) \leq_n^p /\frac{1}{2}/n \\ /1/n & if\ /\frac{1}{2}/n \leq_n^p K(y) \end{cases}$

Then clearly $\gamma(K, D) = min\{\gamma(K, D'), D' \in \mathbb{D}\}$ and if possible suppose that there is $D' \in \mathbb{D}$ such that $\gamma(K, D) = \gamma(K, D')$ and $D \neq D'$. Then there exist $y \in \mathbf{U}$ such that $D(y) \neq D'(y)$.

Without loss of generality assume that $K(y) \leq_n^p /\frac{1}{2}/n \Rightarrow D(y) = /0/n \Rightarrow D'(y) = /1/n$. Thus if $K(y) = (K_1, ...K_n)$ then $\sum_{i=1}^{n} |a_i - 0|^p)^{\frac{1}{p}} < \sum_{i=1}^{n} |a_i - 1|^p)^{\frac{1}{p}}$. Similarly, we get a low summation value corresponding to $D(y)$ *and* $K(y)$ when compared to $D'(y)$ *and* $K(y)$ whenever $D(y) \neq D'(y)$. Hence $\gamma(K, D) < \gamma(K, D')$ which is a contradiction.

□

The σ-distance and σ-similarity metrics are specialized measures that uniformly allocate distance or similarity between n-dimensional fuzzy sets and their complements. Subsequently, we present σ-entropy metrics for the allocation of entropy across n-dimensional fuzzy sets and crisp n-dimensional fuzzy sets. It will be demonstrated that σ-distance and σ-similarity metrics yield σ-entropy measures.

σ-distance measure:

Definition 3.8. A distance measure γ said to σ-distance measure if it satisfies $\gamma(K, H) = \gamma(K \cap D, H \cap D) + \gamma(K \cap D^c, H \cap D^c)$ for every K,H $\in \mathbb{F}$ and for every $D \in \mathbb{D}$.

Result 6. *The p^{th} distance measure γ_p is a σ measure.*

Proof. Let K, H $\in \mathbb{F}$ and $D \in \mathbb{D}$ be arbitrary. Choose $y \in \mathbf{U}$ and without loss of generality assume that $D(y) = /o/n \Rightarrow K(y) \cap D(y) = /0/n$, $H(y) \cap D(y) = /0/n$ and $K(y) \cap D^c(y) = K(y)$, $H(y) \cap D^c(y) = H(y)$. Hence the value in the sum of $\gamma(K, H)$ corresponding to y is the same as the value in the sum of $\gamma(K \cap D^c, H \cap D^c)$ corresponding to y and the value corresponding to y in $\gamma(K \cap D, H \cap D)$ is zero. Thus, we divide the sum into two without changing the actual value. □

Result 7. *The supremum distance measure γ_s is not a σ measure.*

Proof. Let $\mathbf{U} = \{y_1, y_2\}$ and $K, H \in \mathbb{F}$ be such that $K(y_1) = (0.3, 0.6), K(y_2) = (0.4, 0.7), H(y_1) = (0.4, 0.6)$ *and* $H(y_2) = (0.5, 0.8)$. Let $D \in \mathbb{D}$ be such that $D(y_1) = (0, 0), D(y_2) = (1, 1)$. Then $\gamma_s(K, H) = 0.2$ and $\gamma_s(K \cap D, H \cap D) + \gamma_s(K \cap D^c, H \cap D^c) = 0.3$. Hence γ_s is not a σ measure. □

Result 8. *The necessary and sufficient condition for a proximity distance measure γ to be a σ-distance measure is*

$$\gamma(K, H) = \gamma(K^c \cup D, H \cup D) + \gamma(K^c \cup D^c, H^c \cup D^c)$$

for every K, H $\in \mathbb{F}$ and for every $D \in \mathbb{D}$.

Proof. The proof follows directly since the standard t-norm, standard t-conorm, and standard complement satisfy De Morgan laws. Using the De Morgans law and proximity property of γ, we have $\gamma(K \cap D, H \cap D) + \gamma(K \cap D^c, H \cap D^c) = \gamma(K^c \cup D, H \cup D) + \gamma(K^c \cup D^c, H^c \cup D^c)$. The rest of the result follows directly. □

σ-similarity measure:

Definition 3.9. A similarity measure $\xi$is said to σ-similarity measure if it satisfies $\xi(K, H) = \xi(K \cap D, H \cup D^c) + \xi(K \cap D^c, H \cup D)$ for every K, H, $\in \mathbb{F}$ and for every $D \in \mathbb{D}$.

Result 9. *ξ is a σ-similarity measure if and only if it satisfies $\xi(K, H) = \xi(K \cap D, H \cup D^c) + \xi(K \cup D, H \cap D^c)$ for every K, H $\in \mathbb{F}$ and for every $D \in \mathbb{D}$.*

Proof. We prove the sufficiency part, and the necessity part follows similarly. Assume $\xi(K, H) = \xi(K \cap D, H \cup D^c) + \xi(K \cup D, H \cap D^c)$ for every K,H $\in \mathbb{F}$ and for every $D \in \mathbb{D}$. Then $\xi(K \cap D^c, H \cup D) = \xi(K \cap D^c \cap D, H \cup D \cup D^c) + \xi((K \cap D^c) \cup D, (H \cup D) \cap D^c) = \xi(D', D'^c) + \xi(K \cup D, H \cap D^c)$ where $D' \in \mathbb{D}$ defined by $D(y) = /0/n$ for every y. Hence $\xi(D', D'^c) = 0$ and thus $\xi(K \cup D, H \cap D^c) = s(K \cap D^c, H \cup D)$. □

Theorem 3.4. *Let ξ and γ be similarity measure and distance measure such that $\gamma = 1 - \xi$. Then γ is σ-distance measure if and only if ξ is a σ-similarity measure.*

Proof. Suppose γ is a σ-distance measure and let $\xi = 1 - \gamma$, we prove that ξ is a σ-similarity measure. Let ,K B $\in \mathbb{F}$ and $D \in \mathbb{D}$ then,

$$\begin{aligned}\xi(K \cap D, H \cup D^c) &= 1 - \gamma(K \cap D, H \cup D^c)\\ &= 1 - \gamma(K \cap D \cap D, (H \cup D^c) \cap D) - \gamma(K \cap D \cap D^c, (H \cup D^c) \cap D^c)\\ &= 1 - \gamma(K \cap D, H \cap D) - \gamma(D', D^c) \qquad (3.3)\end{aligned}$$

where $D'(y) = /0/n$ for all y. Similarly

$$\xi(K \cup D, H \cap D^c) = 1 - \gamma(D, D') - \gamma(K \cap D^c, H \cap D^c). \qquad (3.4)$$

From Equation (3.3) and Equation (3.4), we get

$$\begin{aligned}&\xi(K \cap D, H \cup D^c) + \xi(K \cap D^c, H \cup D)\\ &= 2 - (\gamma(D', D^c) + \gamma(D, D')) - (\gamma(K \cap D, H \cap D) + \gamma(K \cap D^c, H \cap D^c)).\end{aligned}$$

But we have,

$$\begin{aligned}1 = \gamma(D', D'^c) &= \gamma(D' \cap D, D'^c \cap D) + \gamma(D' \cap D^c, D'^c \cap D^c)\\ &= \gamma(D', D) + \gamma(D', D^c) \qquad (3.5)\end{aligned}$$

From Equation (3.5), we get

$$\begin{aligned}\xi(K \cap D, H \cup D^c) + \xi(K \cap D^c, H \cup D) &= 1 - (\gamma(K \cap D, H \cap D) + \gamma(K \cap D^c, H \cap D^c))\\ &= 1 - \gamma(K, H) = \xi(K, H).\end{aligned}$$

Thus ξ is a σ-similarity measure. Similarly, we can prove that if ξ is a σ-similarity measure, then so is $\gamma = 1 - \xi$. □

Note: Above theorem provides examples for σ- similarity measures.

4. Entropy of n-dimensional fuzzy sets

Imagine a situation in which two members of a company's interview panel assess a cohort of students and independently depict the outcomes utilizing $n-$ dimensional fuzzy sets. Consider a scenario in which, subsequent to the interview, the corporation seeks to ascertain which interviewer possesses a more robust and well-founded viewpoint concerning the students. Entropy can be efficiently employed to quantify the uncertainty of their conclusions. The n-dimensional entropy is defined as an extended concept of multiple distance measures based on four axioms, serving as a metric for fuzziness within the n-dimensional set.

Given that crisp sets exhibit no ambiguity, $E1$ guarantees the absence of fuzziness in such sets. If an element has an equal likelihood of being within or outside a set, the optimal membership value assigned to it is 0.05. Consequently, the n-dimensional fuzzy set with a membership value of 0.5 for each element exhibits the highest degree of fuzziness, as stipulated by axiom $E2$. If A is a n-dimensional fuzzy set, then $A(x)$ and $A^c(x)$ exhibit symmetry concerning $0.5/n$. Therefore, it is reasonable to presume that A and A^c possess equivalent levels of fuzziness, as stated by axiom $E3$. The motivation for $E4$ arises from the finding that n-dimensional fuzzy sets with membership values proximate to $0.5/n$ exhibit greater fuzziness than those with membership values farther distant from $0.5/n$.

Definition 4.1. An n-dimensional entropy on $\mathbb{F}$ is a function ϵ: $\mathbb{F} \to [0, \infty)$ which satisfies the following axioms:

E1 : $\epsilon(D) = 0$ for every $D \in \mathbb{D}$;
E2 : $\epsilon(K) = max\{\epsilon(H), H \in \mathbb{F}\}$ where $K(y) = /0.5/n$ for every y;
E3 : $\epsilon(K^c) = \epsilon(K)$ for every $K\ in\ \mathbb{F}$;
E4 : Let $K, H \in \mathbb{F}$ be such that $H(y) \leq_n^p K(y) \leq_n^p /0.5/n$, or $/0.5/n \leq_n^p K(y) \leq_n^p H(y)$ for every y, then $\epsilon(H) \leq \epsilon(K)$

Example 4. *Let* $\mathbb{U} = \{y_1, y_2, ...y_m\}$ *and* $K \in \mathbb{F}$. *Define* $S : \mathcal{I}_n([0,1]) \to [0, \infty)$ *by*

$$S(K_1, ...K_n) = -[\sum_{i=1}^{n} K_i ln(K_i) + \sum_{i=1}^{n}(1 - K_i)ln(1 - K_i)].$$

Then $\epsilon(K) = \sum_{j=1}^{m} S(K(y_j))$ *is an entropy on* $\mathbb{F}$.

Example 5. *Let* $K \in \mathbb{F}$. *Define* $\hat{K} \in \mathbb{F}$ *by* $\hat{K}(y) = (K_1, ...K_n)$ *where* $K_i = 1$ *if* $K_i(y) > 0.5$ *and* $K_i = 0$ *if* $K_i(y) \leq 0.5$. *Then* $\epsilon_p(K) = 1 - \gamma_p(K, \hat{K})$ *is an entropy where* γ_p *is the normalized Euclidean distance measure.*

Example 6. *Let* $K \in \mathbb{F}$ *then* $\epsilon_c(K) = 1 - \gamma_p(K, K^c)$ *is an entropy.*

From $E2$, it is obvious that the entropy is bounded and thus it can be normalized as $\hat{\epsilon}(K) = \dfrac{\epsilon(K)}{\epsilon(H)}$ where $H(y) = /0.5/n$ for all y. Thus, from now on we consider only normalized entropy. The following important theorem will give the relation between entropy, similarity measure, and distance measure.

Theorem 4.1. *If* ξ *is a similarity measure, then* $\epsilon(K) = \xi(K, K^c)$ *is an entropy. Also if* γ *is a distance measure then* $\epsilon(K) = 1 - \gamma(K, K^c)$ *is also an entropy.*

Proof. The proof follows directly from the definitions. □

Definition 4.2. Let γ be a distance measure and ϵ be an entropy then ϵ is said to symmetric with respect to γ if $\gamma(K, (\frac{1}{2})x) = \gamma(H, (\frac{1}{2})x)$ *then* $\epsilon(K) = \epsilon(H)$ for every $K, H \in \mathbb{F}$

Theorem 4.2. *Let* γ *be a linear proximity distance measure then the entropy measure generated by* γ *is symmetric with respect to* d.

Proof. Assume $\gamma(K, (\frac{1}{2})x) = \gamma(H, (\frac{1}{2})x)$. Then, $\gamma(K^c, (\frac{1}{2})x) = \gamma(H, (\frac{1}{2})x)$. Now, adding the above two equations, we get

$$\gamma(K, (\frac{1}{2})x) + \gamma(K^c, (\frac{1}{2})x) = \gamma(H, (\frac{1}{2})x) + \gamma(H, (\frac{1}{2})x) \Rightarrow \gamma(K, K^c) = \gamma(H, H^c)$$

which implies $1 - \gamma(K, K^c) = 1 - \gamma(H, H^c) \Rightarrow \epsilon(K) = \epsilon(H)$. Hence ϵ is symmetric with respect to γ. □

σ-entropy measure

Definition 4.3. An entropy ϵ is said to be a σ-entropy if $\epsilon(K) = \epsilon(K \cap D) + \epsilon(K \cap D^c)$ for each $K \in \mathbb{F}$ *and* $D \in \mathbb{D}$.

Definition 4.4. A subclass $\mathbb{E} \subseteq \mathbb{F}$ is said to comparable class if it satisfies the following conditions

(1) for each $K, H \in \mathbb{E}$, $K(y) \leq_n^p H(y)$ *or* $H(y) \leq_n^p K(y)$;
(2) $\mathbb{D} \subseteq \mathbb{E}$;
(3) $\mathbb{E}$ is closed under finite union and complement.
(4) If $K(y) = /0.5/n$ for all y, then $K \in \mathbb{E}$

Theorem 4.3. *Let* $\mathbb{E}$ *be a comparable class, then an entropy* ϵ *on* $\mathbb{E}$ *is* σ*-entropy if and only if it satisfies* $\epsilon(K) + \epsilon(H) = \epsilon(K \cup H) + \epsilon(K \cap H)$ *for every* $K, H \in \mathbb{E}$

Proof. Let $K, H \in \mathbb{E}$ and $K = \{y \in \mathbf{U} : H(y) \leq_n^p K(y)\}$. Define $D \in \mathbb{D}$ as follows,

$$D(y) = \begin{cases} /0/n & if\ y \notin K \\ /1/n & if\ y \in K \end{cases}$$

Assume that ϵ is a σ entropy then,

$$\begin{aligned} \epsilon(K \cup H) &= \epsilon((K \cup H) \cap D) + \epsilon\left((K \cup H) \cap D^c\right) \\ &= \epsilon(K \cap D) + \epsilon\left(H \cap D^c\right) \end{aligned}$$

and

$$\begin{aligned}\epsilon(K \cap H) &= \epsilon((K \cap H) \cap D) + \epsilon\left((K \cap H) \cap D^{c}\right) \\ &= \epsilon(H \cap D) + \epsilon\left(K \cap D^{c}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\epsilon(K \cup H) + \epsilon(K \cap H) &= \left(\epsilon(K \cap D) + \epsilon\left(K \cap D^{c}\right)\right) + \left(\epsilon(H \cap D) + \epsilon\left(H \cap D^{c}\right)\right) \\ &= \epsilon(K) + \epsilon(H).\end{aligned}$$

Now assume that $\epsilon(K) + \epsilon(H) = e(K \cup H) + e(K \cap H)$ for every $K, H \in \mathbb{E}$ then,

$$\begin{aligned}\epsilon(K) = \epsilon(K) + \epsilon(D) &= \epsilon(K \cap D) + \epsilon(K \cup D) \\ &= \epsilon(K \cap D) + \epsilon\left((K \cap D^{c}) \cup \mathrm{D}\right) \\ &= \epsilon(K \cap D) + \left(\epsilon\left(K \cap D^{c}\right) + \epsilon(D) - \epsilon\left((K \cap D^{c}) \cap D\right)\right) \\ &= \epsilon(K \cap D) + \epsilon\left(K \cap D^{c}\right).\end{aligned}$$

Hence, the proof is completed □

Theorem 4.4. *If ξ is a σ-similarity measure, then the entropy defined by, $\epsilon(K) = \xi(K, K^{c})$ is a σ-entropy measure.*

Proof. Consider $K \in \mathbb{F}$ and $D \in \mathbb{D}$ then we have,

$$\begin{aligned}&\epsilon(K \cap D) + \epsilon\left(K \cap D^{c}\right) \\ &= \xi\left(K \cap D, (K \cap D)^{c}\right) + \xi\left(K \cap D^{c}, (K \cap D^{c})^{c}\right) \\ &= \xi\left(K \cap D, K^{c} \cup D^{c}\right) + \xi\left(K \cap D^{c}, K^{c} \cup D\right) \\ &= \xi\left(K, K^{c}\right) \\ &= \epsilon(K).\end{aligned}$$

□

Orderless n-dimensional fuzzy sets

The notion of n-dimensional fuzzy sets can be used in two different contexts. One is that n membership value is given to each $y \in \mathbf{U}$ due to the ambiguity in providing a unique membership value. In the second case, each $y \in \mathbf{U}$ has n attributes, and we give membership value to each attribute as its measure. When we are dealing with practical problems the second case has a very important role and thus more study in this field is required.

Definition 4.5. Let $\mathbf{U}$ be a nonempty set. Then a orderless n-dimensional fuzzy set over $\mathbf{U}$ is a set of the form $\mathcal{N} = \{(y, \mu^{1}_{\mathcal{N}}(y), ..\mu^{n}_{\mathcal{N}}(y)) : y \in \mathbf{U}\}$ where $\mu^{J}_{\mathcal{N}} : U \to [0,1]$ is the membership function corresponding to the J^{th} attribute of y.

Note: Clearly, every n-dimensional fuzzy set is an orderless n-dimensional fuzzy set.

Example 7. *Let $\mathbf{U} = C_1, C_2, C_3, C_4$ be a collection of cars where each has attributes $a_1, a_2,$ and a_3. Then an orderless n-dimensional fuzzy set K that provides the information about the attributes can be expressed as follows:*

$$\begin{aligned}&K(C_1) = (0.7, 0.6, 0.5), K(C_2) = (0.6, 0.7, 0.8), \\ &K(C_3) = (0.7, 0.8, 0.5), K(C_4) = (0.4, 0.7, 0.6)\end{aligned}$$

where j^{th} entry of $K(C_i)$ gives the membership value of j^{th} attribute of C_i.

Definition 4.6. Let $\mathbf{U}$ be a nonempty set. Then the collection of all orderless n-dimensional fuzzy sets for which each of them has only zeros or ones in their membership value is defined as orderless crisp sets and denoted by $\mathbb{K}$.

Note: We can extend the notion of distance measure in n-dimensional fuzzy sets to orderless n-dimensional fuzzy sets for further studies.

Crisp approximation of orderless n-dimensional fuzzy sets

Even though $n-$dimensional fuzzy sets are good tools for data representation, we need a crisp version to make decisions. Thus, it is important to build a mechanism that can convert $n-$dimensional fuzzy sets to $n-$dimensional crisp sets. Here, a defuzzification mechanism based on the introduced distance measures is provided.

Definition 4.7. Let K be an orderless n-dimensional fuzzy set then E $\in \mathbb{E}$ is said to be a crisp approximation of K if $\gamma(K, E) = min\{\gamma(K, E'), E' \in \mathbb{E}\}$.

Definition 4.8. Let $\mathcal{N} = \{(y, \mu^1_{\mathcal{N}}(y), ..\mu^n_{\mathcal{N}}(y)) : y \in X\}$ be an orderless dimensional fuzzy sets. Then crisp neighbour of $\mathcal{N}$ is an element of $\mathbb{E}$ given by $\hat{\mathcal{N}} = \{(y, \mu^1_{\hat{\mathcal{N}}}(y), ...\mu^n_{\hat{\mathcal{N}}}(y))\}$
where

$$\mu^i_{\hat{\mathcal{N}}}(y) = \begin{cases} 0 & if\ \mu^i_{\mathcal{N}}(y) \leq 0.5 \\ 1 & if\ \mu^i_{\mathcal{N}}(y) > 0.5. \end{cases}$$

Theorem 4.5. *Let K be an orderless n-dimensional fuzzy set and let $E \in \mathbb{E}$ be the crisp neighbor of K with respect to γ_p then E is a crisp approximation of K with respect to γ_p.*

Proof. The proof is similar to the proof of theorem Theorem 3.3 □

Note: It can be seen that the crisp approximation of an $n-$dimensional fuzzy set is some kind of generalization of the defuzzification method, where the corresponding crisp set is provided by considering its distance from $n-$dimensional set. Thus, there is no loss in information as an $n-$ dimensional fuzzy set. However, there is a possibility for the existence of multiple crisp approximations for a single $n-$ dimensional fuzzy set, which may be treated as a limitation of this technique.

5. Decision-Making Methods Using n-dimensional Fuzzy Sets

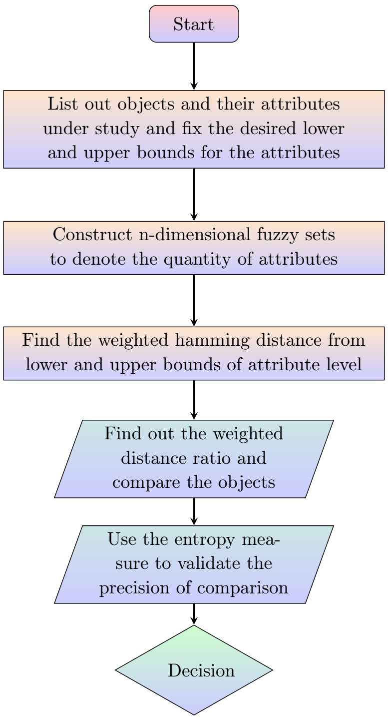
There are some situations in which we have to make decisions based on the quality or quantity of attributes of elements under study. Numerous recent relevant decision-making techniques rely on innovative fuzzy extended models, including but not limited to the Neutrosophic Multi-Criteria Model, hybrid multi-dimensional fuzzy-based approach, Pythagorean fuzzy entropy measure TOPSIS method, and others. These approaches provide valuable insights into the inner workings of decision-making methods in diverse manifestations.

Practical applications of $n-$dimensional fuzzy sets include decision-making, clustering, image processing, and so on. For example, in image processing using AI, the data is collected by instruments in the form of images, audio, etc. and they are converted into suitable $n-$dimensional fuzzy sets by evaluating their attributes such as colour combination, shapes, size ratio, etc. Then the resulting $n-$dimensional fuzzy sets are compared with existing ones using different information measures to find the best possible data match. A similar strategy is also used in scanning technology in the medical field, where the predicted image is built based on the comparison of obtained data with former ones. [5, 7]

In simple applications such as decision making under an interview, the experts can construct $n-$dimesnional fuzzy sets by evaluating their qualities such as communication skills, logical reasoning, problem solving, etc. and there by conclusions can be made by utilizing information measures properly. But when we deal with more advanced problems such as image processing through AI or signal analysising in receptors, attributes of data are evaluated by sensors within the system and $n-$dimensional fuzzy sets with proper dimension are made according to the pre-assigned values. The more the dimension the more the complexity in the computation. Here, we discuss some simple decision-making problems that illustrate the utility of $n-$dimensional fuzzy sets and their information measures.

5.1. **Decision making using orderless n-dimensional fuzzy set and weighted hamming distance measure —Method 1.** In most of the practical decision-making problems, we have to make decisions based on the quality or quantity of attributes of elements under study. Numerous recent relevant decision-making techniques rely on innovative fuzzy extended models, including but not limited to the Neutrosophic Multi-Criteria Model, hybrid multi-dimensional fuzzy-based approach, Pythagorean fuzzy entropy measure TOPSIS method, and others. These approaches provide valuable insights into the inner workings of decision-making methods in diverse manifestations. In most cases, we have a lower bound ζ and upper bound η for the membership value of attributes, such a way that, if the membership value is less than ζ then the corresponding attribute does not meet our requirements and if the membership value is greater than η then the corresponding attribute is completely apt for our need. Also, different attributes have different priorities so it is important to consider them separately.

FIGURE 1. Flowchart of the algorithm



We use the following algorithm for solving decision problems, in which the membership value is between ζ and η and the flowchart of the algorithm is given in Fig.1.

Illustration: Consider a situation of interviewing participants P_1, P_2 and P_3 by examining their quality of attributes $a_1, ...a_4$. Let Table 1 give lower bound ζ, upper bound η, of a_i, and their weights all are represented in a scale of 0-1.

Table 2 gives information about each participant's quality of attributes represented using 3-dimensional fuzzy sets. If the values of attributes are outside of the upper bound or lower bound then we can take decisions directly. Hence we only consider the situation where the attribute's value lies inside the boundaries.

Algorithm 1 Ordering Algorithm

Require:

1: Let $\mathbb{U} = \{y_1, ...y_m\}$ and let $a_1, ...a_n$ be the attributes of elements of $\mathbb{U}$ under study.

Ensure:

2: Make a table that consists of the lower bound ζ and upper bound η of the attributes along with their weights of preference or importance.

3: Make a table that consists of values of attributes for each element in $\mathbb{U}$ that is obtained after studying the elements using n-dimensional fuzzy sets for suitable n.

4: Find the weighted hamming distance of each element from the lower bounds and upper bounds of attributes say γ_h^l and γ_h^u respectively.

5: **for all do**

6: the weighted distance ratio $r = \dfrac{\gamma_h^l}{\gamma_h^u}$ and compare the elements accordingly.

7: **end for**

8: Finally find the entropy of each element as a measure of closeness to the center of the boundary to evaluate the precision we obtained.

TABLE 1. Lower and upper bound of attributes

a_i	ζ	η	weight
a_1	(0.35,0.4,0.45)	(0.5,0.60.65)	0.6
a_2	(0.3,0.4,0.45)	(0.65,0.7,0.8)	0.7
a_3	(0.45,0.5,0.65)	(0.77,0.8,09)	0.8
a_4	(0.33,0.4,0.45)	(0.55,0.6,0.65)	0.7

TABLE 2. Quality of attributes

a_i	P_1	P_2	P_3
a_1	(0.40,0.45,0.5)	(0.475,0.5,0.575)	(0.45,0.55,0.575)
a_2	(0.635,0.65,0.665)	(0.55,0.6,0.625)	(0.475,0.5,0.55)
a_3	(0.525,0.6,0.675)	(0.55,0.575,0.6)	(0.65,0.675,0.7)
a_4	(0.5,0.55,0.6)	(0.425,0.45,0.5)	(0.4,0.45,0.475)

Now we find $\gamma_h^l(P_i, l)$ and $\gamma_h^u(P_i, u)$ for $i = 1, 2$ and 3 as follows:

$$\begin{aligned}\gamma_h^l(P_1, l) &= (|0.4 - 0.35|+|0.4 - 0.45|+|0.5 - 0.45|)0.6 + (|0.3 - 0.635|+|0.4 - 0.65|+|0.45 - 0.665|)0.7 \\ &\quad + (|0.525 - 0.45|+|0.5 - 0.6|+|0.65 - 0.675|)0.8 + (|0.33 - 0.5|+|0.4 - 0.55|+|0.45 - 0.6|)0.7 \\ &= 0.39; \\ \gamma_h^u(P_1, u) &= |0.6 - 0.45|0.6 + |0.7 - 0.65|0.7 + |0.8 - 0.6|0.8 + |0.6 - 0.55|0.7 \\ &= 0.32.\end{aligned}$$

Similarly, we get

$$\gamma_h^l(P_2, l) = 0.305, \;\; \gamma_h^u(P_2, u) = 0.405$$

and

$$\gamma_h^l(P_3, l) = 0.355, \;\; \gamma_h^u(P_3, u) = 0.355.$$

Thus

$$r_1 = \frac{\gamma_h^l(P_1, l)}{\gamma_h^u(P_1, u)} = 1.21875, \;\; r_2 = \frac{\gamma_h^l(P_2, l)}{\gamma_h^u(P_2, u)} = 0.7530, \;\; r_3 = \frac{\gamma_h^l(P_3, l)}{\gamma_h^u(P_3, u)} = 1.$$

Hence the overall attribute quality of participants is in the increasing order $P_2 < P_3 < P_1$.

Now we can use the idea of entropy to find how precise our decisions are. In entropy, we use the idea that the set with membership values close to 0.5 has more fuzziness. So here we will find the people with membership values close to the center of boundary values.

TABLE 3. Centre of boundary

a_i	Centre of boundary C
a_1	(0.4,0.5,0.55)
a_2	(0.58,0.55,0.65)
a_3	(0.6,0.65,0.67)
a_4	(0.48,0.5,0.54)

Now we find the entropy measure of each individual after taking the average of membership values and using Table 3 as follows.

$\epsilon(P_1) = 1 - \gamma_h(P_1, C) = 1 - (|0.5 - 0.45| + |0.55 - 0.65| + |0.65 - 0.6| + |0.5 - 0.55|) = 0.65.$
Similarly, $\epsilon(P_2) = 0.6254$ and $\epsilon(P_3) = 0.849$.
Hence P_2 has less fuzziness in the data and P_3 has the maximum. So that our decision about P_2 is more reliable and decisions about P_3 is less reliable.

5.2. **Decision making using orderless n-dimensional fuzzy set and weighted hamming distance measure —Method 2.** The concept of distance measures of n-dimensional fuzzy sets can be used in orderless n-dimensional fuzzy sets to solve some practical problems. For instance, consider the situation in which there are some patients who are affected by disease D and show some abnormality in their level of nutrients and minerals. The distance measure can be used to find the distance between a patient's present health condition to the ideal health condition using the changes in their mineral count. Some of the abnormalities are crucial and some of them are not very crucial. Hence a weighted distance measurement of values is needed in order to study the disease. We use the following algorithm for solving such problems.

Algorithm 2 Algorithm for ordering the of health level of patients

Require:
1: Let $s_1, s_2 ... s_n$ be n values of nutrients or minerals in the body that get affected due to D. Let ζ_i and η_i be the lower bound and upper bound for the normal level of s_i, along with its weight w_i based on its importance when affected by D. Make a table that gives values of s_i,w_i, ζ_i, and η_i.

Ensure:
2: Let $P_1, P_2, ... P_m$ be m patients that are affected by D. List the current values of s_i for each patients after examining them.
3: Construct a table that gives values for the ideal condition of s_i.
4: Find the weighted hamming distance measure between each patient's data and ideal s_i data by multiplying each sum in the usual hamming distance by the corresponding weight.
5: **for all do**
6: Compare the weighted hamming distance measure of each patient and list according to the seriousness of each patient.
7: **end for**

Note: The parameters and their values in the Illustrative examples given below are purely based on general understanding and observations only and are not based on any actual clinical study or observation. Actual clinical data-derived values for parameters may have been experimented with at a possible later implementation stage.

Illustration: Let Table 4 give values of s_i about the lower bound ζ, upper bound η, and their weights with respect to D all represented in a scale of 0-1. Consider the patients P_1, P_2, and P_3 that are affected by disease D, and their current values of s_i expressed using a 3-dimensional fuzzy set as given in Table 5. We find the ideal conditions of s_i by taking the averages as in Table Table 6,

TABLE 4. lower and upper bounds

s_i	ζ	η	weight
s_1	0.3	0.6	0.5
s_2	0.4	0.8	0.6
s_3	0.4	0.8	0.7
s_4	0.3	0.6	0.8

TABLE 5. Values of s_i

s_i	P_1	P_2	P_3
s_1	(0.25,0.3,0.35)	(0.15,0.2,0.225)	(0.275,0.3,0.35)
s_2	(0.85,0.875,0.875)	(0.9,0.925,0.975)	(0.85,0.85,0.9)
s_3	(0.275,0.275,0.3)	(0.2,0.225,0.3)	(0.25,0.275,0.3)
s_4	(0.675,0.675,0.7)	(0.675,0.7,0.75)	(0.7,0.75,0.775)

TABLE 6. Ideal Values

s_i	Ideal condition I
s_1	0.45
s_2	0.6
s_3	0.6
s_4	0.45

Now we find the weighted distance of each patient from the ideal one by taking averages in each case as follows. For P_1, we have $\gamma_H(P_1, I) = |0.3 - 0.45|0.5 + |0.866 - 0.6|0.6 + |0.2833 - 0.6|0.7 + |0.6833 - 0.45|0.8 = 0.642$ Similarly for $\gamma_H(P_2, I) = 0.815$ and $\gamma_H(P_3, I) = 0.77$. Thus we can conclude that the health condition of the patients is worse in an ascending order P_1, P_3, and P_2 respectively.

Note: The same result can be obtained by using similarity measures to find the patients who have health conditions similar to the ideal condition and treat them accordingly.

6. Comparison with some existing fuzzy extensions

There are several kinds of multi-criteria decision-making methods that are widely used in different contexts, and the major problem faced by these methods is the assignment of membership values to each attribute. n-dimensional fuzzy sets allow us to give a number of membership values to each attribute and hence solve ambiguity to a certain extent. We can see this problem with important methods such as WSM, WPM, AHP, TOPSIS, etc.[39, 8]. It is obvious that any of the above decision-making methods can be represented using n-dimensional fuzzy sets, and a more realistic presentation of data can also be produced using an n-dimensional fuzzy set.

6.1. **Comaprison with Pythagorean fuzzy sets** [15]**.** A Pythagorean fuzzy set P on a universal set Y is a structure of the form

$$P = \{\langle y, \mu_P(y), \nu_P(y)\rangle \mid y \in Y\}$$

where the functions

$$\mu_P(x) : X \to [0,1] \text{ and } \nu_P(x) : X \to [0,1]$$

represents membership and non-membership degrees of the element $y \in Y$ to P, and for every $y \in Y$,

$$0 \leq (\mu_P(y))^2 + (\nu_P(y))^2 \leq 1.$$

and

$$\pi_P(y) = \sqrt{1 - \left[(\mu_P(y))^2 + (\nu_P(y))^2\right]}$$

represents the degree of indeterminacy of the element y to P.

The Hamiltonian distance measure between two Pythagorean fuzzy sets P and Q (see [15]) is given by

$$d_{PFS}(P,Q)_H = \frac{1}{2}\sum_{i=1}^{n}\{|\mu_P(y_i) - \mu_Q(y_i)| + |\nu_P(y_i) - \nu_Q(y_i)| + |\pi_P(y_i) - \pi_Q(y_i)|\} \quad (6.1)$$

Now, we convert the data given in Table 2 to different Pythagorean fuzzy sets corresponding to each participant using Table 1 as follows.

Let $Y = \{a_1, ...a_4\}$ and let $P1$ denote the Pythagorean set corresponding to the participant P_1 defined by $P1 = \{a_i, \mu_{P1}(a_i), \nu_{P1}(a_i), a_i \in Y\}$, where $\mu_{P1}(a_i)$ = sum of absolute difference between terms of $P_1(a_i)$ and the corresponding lower bound ζ. Similarly, $\nu_{P1}(a_i)$ = sum of absolute difference between terms of $P_1(a_i)$ and the corresponding upper bound η. Hence we have

$$P1 = \{(a_1, 0.15, 0.45), (a_2, 0.745, 0.15), (a_3, 0.3, 0.55), (a_4, 0.45, 0.15)\}.$$

Clearly, $\mu_{P1}(a_i)$ denotes the positive contribution of P_1 and $\nu_{P1}(a_i)$ denotes the negative contribution. Similar to the previous one we find, $P2$ and $P3$ as follows:

$$P2 = \{(a_1, 0.35, 0.25), (a_2, 0.475, 0.325), (a_3, 0.225, 0.675), (a_4, 0.175, 0.425)\},$$

$$P3 = \{(a_1, 0.375, 0.225), (a_2, 0.325, 0.575), (a_3, 0.525, 0.375), (a_4, 0.125, 0.475)\}.$$

Now, the Pythagorean fuzzy set I corresponding to the differences in the upper bound and the lower bound from Table 1 is

$$I = \{(a_1, 0.6, 0), (a_2, 0.9, 0), (a_3, 0.9, 0), (a_4, 0.6, 0)\}.$$

Next, we find the weighted Hamiltonian distance Equation (6.1) between PIs and I as follows. It can be verified that $P1, P2, P3$, and I satisfy the conditions to be a Pythagorean set and the best participant can be found out by finding the PI with the least weighted distance from I as I represents the ideal performance:

$$d_{PFS}(P1, I)_H = 0.8565, \;\; d_{PFS}(P2, I)_H = 1.6375, \;\; d_{PFS}(P1, 3)_H = 1.2216.$$

Hence participant's performance is in the increasing order $P_2 < P_3 < P_1$ which is the same that we obtained through $n-$ dimensional fuzzy sets. Thus, both structure agrees with the results in the decision-making method 1.

6.2. **Comaprison with Zadeh fuzzy sets.** Next, we compare $n-$dimensional fuzzy sets with ordinary fuzzy sets to show the advantages in data representation using $n-$dimensional fuzzy sets. Suppose that we use ordinary Zadeh fuzzy sets to record the data in illustration one; we will get the following possible Table 7 as the quality of attributes by simply taking averages of values in Table 2.

Now by performing Algorithm 1 after similarly taking the average values of bounds of attributes, we get,

$$r_1 = 1.17, r_2 = 0.95, r_3 = 1.34,$$

that gives the ordering,

$$P_2 < P1 < P3$$

which is different from the ordering we got from one which obtained using $n-$dimensional fuzzy sets.

Even if we receive the same order of participants, this does not guarantee that the difference in performance between them will be the same as when we use $n-$dimensional fuzzy sets. We can use $n-$dimensional fuzzy distance measure to evaluate the distance between their overall performance as follows. Thus, we have the following observations using weighted hamming distance measure γ_h:

$$\begin{aligned}\gamma_h(P_1, P_2) &= 0.6(|0.4 - 0.475| + |0.45 - 0.5| + |00.5 - 0.575|) + 0.7(|0.635 - 0.65| + |0.65 - 0.7| \\ &+ |0.665 - 0.725|) + 0.8(|0.525 - 0.45| + |0.6 - 0.475| + |0.675 - 0.5|) + 0.7(|00.5 - 0.425| \\ &+ |0.55 - 0.45| + |0.6 - 0.5|) \\ &= 0.6.\end{aligned}$$

Similarly we have $\gamma_h(P_1, P_3) = 0.565$ and $\gamma_h(P_2, P_3) == 0.517$.

Thus the distance between measures of performance of participants agrees with the ordering of the participants. Hence if we have a participant with desirable qualities then we can select other participants by measuring the fuzzy distance as above. Now if we use ordinary fuzzy sets and fuzzy distance measures in illustration 1, we will get data similar to the following one.(Table 7) (Here we took the averages from the $3-$dimensional data to construct the fuzzy data for the best results.)

Table 7. Quality of attributes

a_i	P_1	P_2	P_3
a_1	0.45	0.516	0.525
a_2	0.65	0.692	0.508
a_3	0.6	0.475	0.675
a_4	0.55	0.458	0.441

Now the weighted hamming distance between them is given as follows:

$$\gamma_h(P_1, P_2) = 0.2334, \;\; \gamma_h(P_1, P_3) = 0.2573, \;\; \gamma_h(P_2, P_3) = 0.3112.$$

From the preceding values, it is evident that our $1-$ dimensional fuzzy set does not provide the ordering $P_2 < P_3 < P_1$, as some data is lost during the conversion of $3-$dimensional data to ordinary fuzzy data. This shows the importance of $n-$ dimensional fuzzy sets over ordinary fuzzy sets.

Table 8 gives the comparison table when we compare the ordering and individual scores of participants from illustration one, under various fuzzy variants.

From the above comparison, we conclude that the distance and similarity of n-dimensional fuzzy sets can be used to make decisions and the concept of entropy will help us check the validity of our decisions.

TABLE 8. Comparison Table of Algorithm 1

Fuzzy Version	Change in order	Difference in Score Value
Zadeh Fuzzy Set	Yes	decreased
Pythagorean Fuzzy	No	decreased
$n-$dimesnional Fuzzy set	No	Increased

The Table 9 provides the comparison of $n-$dimensional fuzzy sets with some other existing structures.

Fuzzy Variant	Zadeh Fuzzy	Intuitionistic Fuzzy	$n-$dimensional Fuzzy
Nature of Uncertainty	Elements under study are not well defined or have some uncertainty in their attributes	Elements having membership and non-membership features	Studies different types of uncertainty by changing the dimension
Applications	Helps to represent data having uncertainty and do the analysis of the qualitative attributes	Divides the uncertainty to different cases and helps to solve decision-making problems more systematically	Generalize various fuzzy extensions and provide the more membership values according to the situation
Limitations	Single membership value, no scope of non-membership value	Computation of large data may be complex, limited membership values	Difficulty in allocation of correct dimension and complexity of computation

TABLE 9. Comparison with existing fuzzy structures

7. Sensitive Analysis

Next, we study the sensitive analysis of distance and entropy measures used in Illustration 1, by applying changes to the values in Table 2 by a unit of $+/-0.1$ and then determining the changes in the score functions. The modified attribution values are given in Table 10 and, Table 11 respectively.

TABLE 10. Modified attribution table by a unit of $+0.1$

a_i	P_1	P_2	P_3
a_1	(0.50,0.55,0.65)	(0.575,0.6,0.675)	(0.55,0.65,0.675)
a_2	(0.735,0.75,0.765)	(0.65,0.7,0.725)	(0.575,0.6,0.65)
a_3	(0.625,0.7,0.775)	(0.65,0.675,0.7)	(0.75,0.775,0.8)
a_4	(0.6,0.65,0.7)	(0.525,0.55,0.6)	(0.5,0.55,0.575)

Table 11. Modified attribution table by a unit of -0.1

a_i	P_1	P_2	P_3
a_1	(0.30,0.35,0.4)	(0.375,0.4,0.475)	(0.35,0.45,0.475)
a_2	(0.535,0.55,0.565)	(0.45,0.5,0.525)	(0.375,0.4,0.45)
a_3	(0.425,0.5,0.575)	(0.45,0.475,0.5)	(0.55,0.575,0.6)
a_4	(0.4,0.45,0.5)	(0.325,0.35,0.4)	(0.3,0.35,0.375)

Now the resulting score values of $\gamma_h^u(+)(P_i,u), \gamma_h^l(+)(P_i,l)$ and $\gamma_h^u(-)(P_i,u), \gamma_h^l(-)(P_i,l)$ are given as follows(Table 12),

Table 12. Modified Score Values

Score values	P_1	P_2	P_3
$\gamma_h^u(+)(P_i,u)$	0.28	0.393	0.312
$\gamma_h^l(+)(P_i,l)$	0.43	0.336	0.376
$\gamma_h^u(-)(P_i,u)$	0.36	0.434	0.376
$\gamma_h^l(-)(P_i,l)$	0.35	0.27	0.307

Now, by calculating $r_i(+)$ and $r_i(-)$ values, we have

$$r_1(+) = 2.34,\ \ r_2(+) = 1.93,\ \ r_3(+) = 2.15$$

and

$$r_1(-) = 0.81,\ \ r_2(-) = 0.134,\ \ r_3(-) = 0.56.$$

Thus we have the same ordering as in illustration 1, but the score values changed by an average of unit 1.17. This provides a measure of the sensitivity of the distance measure per unit change. Similarly finding the entropy by calculating the difference from centre of the boundary, we have,

$$\epsilon_+(P_1) = 0.859, \epsilon_+(P_2) = 0.473, \epsilon_+(P_3) = 0.782$$

and

$$\epsilon_-(P_1) = 0.35, \epsilon_-(P_2) = 0.94, \epsilon_-(P_3) = 0.84.$$

It can be seen that the entropy is increased for participants having low scores in the first situation, and the entropy is decreased for participants having low scores in the second situation. But, it is totally the opposite for the participants having higher scores. This is because of the deviation of attribute values from the center to the boundaries.

8. Conclusion

In this paper, the concept of n-dimensional fuzzy sets is elaborated by defining n-dimensional distance measures, n-dimensional similarity measures, n-dimensional entropy, and orderless n-dimensional fuzzy sets. n-dimensional fuzzy sets give a simple and powerful representation of the data as it enables us to give a large dimension so that the information about elements may be completely contained in the representation. But to solve practical problems using n-dimensional fuzzy sets it need several measures of the data. So in this paper, several important measures and their properties are studied. Also, the fundamental theorems involving these measures are discussed. The definitions of these measures are in such a way that their restriction to usual fuzzy sets agrees with the measures defined there. The concept of crisp approximation using distance measures and orderless n-dimensional fuzzy sets is helpful in solving practical problems. The illustrative example presented in the last section shows how it is possible to make use of the notion of n-dimensional measures and orderless n-dimensional in solving practical problems. The comparison with Pythagorean fuzzy sets and Zadeh fuzzy sets shows how $n-$ dimensional fuzzy sets give a simple presentation and powerful tools for problem-solving. But to solve more practical problems there is a need for introducing more measures and results involving them.

The primary disadvantage of decision-making methods using $n-$ dimensional fuzzy sets is that they limit the number of attributes or the cardinality of membership values to a fixed n. There are numerous situations in which various membership values and cardinalities are required for each element based on the ambiguity it contains. In an $n-$dimensional fuzzy set, each element has a membership value with n members, but there may be elements that require more than n members from $[0, 1]$ to represent its membership value, as well as elements that do not require n members to represent its membership value. Thus, these situations can only be depicted by a multidimensional fuzzy set. So we wish to extend this work to multidimensional fuzzy measures.[29, 22].

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