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Electromagnetic Analogues in Rotating Reference Frames

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Abstract: In the absence of a physical force, the equation of motion of a particle of mass m at position $\boldsymbol{r}$ and moving with constant velocity $\dot{\boldsymbol{r}}$ in a non-inertial reference frame rotating with angular velocity $\boldsymbol{\omega}$ can be expressed in terms of analogue electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ fields and is known to assume the form $\boldsymbol{E} + \dot{\boldsymbol{r}} \times \boldsymbol{H}$ of the Lorentz force equation of electromagnetic theory. In this paper the associated analogue $\boldsymbol{E}$ and $\boldsymbol{H}$ fields are formally shown to satisfy the analogues of both Gauss's law for magnetic fields and Faraday's law of Maxwell's equations. These $\boldsymbol{E}$ and $\boldsymbol{H}$ fields are also used in Gauss's law for electric fields and Ampere's law to define rotating reference frame analogues of electromagnetic charge density and current density. Interpretations of the possible physical significance of these new analogues are provided and the relationship of Faraday's law to the work required to move a particle in a rotating frame is discussed.

Keywords: charge density; current density; analogues.

Introduction

Freeman Dyson reconstructed a proof (Dyson 1990) of a theorem shown to him by Richard Feynmann in 1948 that *if a non-relativistic quantum mechanical point particle of mass* m *satisfies canonical Cartesian position and momentum commutation relations and Newton's equation of motion, then there exist vector fields analogous to the electromagnetic electric and magnetic fields so that the associated force equation has the Lorentz form*. Interestingly, these fields also satisfy the associated Maxwell equation analogues of Gauss's law for magnetic fields and of Faraday's law.

This result was extended by Richard Hughes (Hughes 1992) to non-relativistic classical particles by replacing the quantum mechanical commutators with the corresponding Poisson brackets and observing that any equation of motion involving acceleration independent generalized forces derived from a Lagrangian can also assume the Lorentz form. Hughes used the motion of a particle in a non-inertial reference frame to illustrate this. He defined $\boldsymbol{E}$ and $\boldsymbol{H}$ analogue vector fields and stated (without proof) that these fields satisfy both Gauss's law for magnetic fields and Faraday's law.

It is noted that the relationship between electromagnetism and inertial forces predates Hughes' theory by several decades (Coisson 1973; Semon and Schmieg 1981). Coisson identified both electromagnetic fields and potentials with their analogues in rotating reference frames and stated (without proof) that these quantities satisfy Gauss's law for magnetic fields and Faraday's law, as well as Gauss's law for electric fields and Ampere's law when "suitable choices" are made for charge and current density. Semon and Schmieg used the electric and magnetic fields of a point charge in a rotating frame and employed Faraday's law to determine the work done on a point mass when moved around a closed path in the frame.

In this paper Hughes' theory is "completed" by providing formal proofs that Hughes' analogue $\boldsymbol{E}$ and $\boldsymbol{H}$ vector fields satisfy both Gauss's law for magnetic fields and Faraday's law of Maxwell's equations. Unlike Coisson's casual observations concerning "suitable choices" for charge and current densities, Hughes' analogue fields are used here in Gauss's law for electric fields and Ampere's law to define charge density and current density analogues associated with particle motion in a rotating reference frame, namely, the *rotational kinetic energy density* and the *centripetal force current*, respectively. Again, unlike Coisson, interpretations of the possible physical significance of these analogues are discussed.

For the sake of clarity and precision, the vector calculus calculations in the following sections are performed in Cartesian coordinate presentations defined by the orthonormal vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$. An Appendix containing definitions and Cartesian coordinate representations of vector quantities is provided for the reader's convenience.

Motion in a Non-inertial Reference Frame

Consider a reference frame whose origin R_0 executes a translational acceleration $\boldsymbol{a}$ and rotates with an angular velocity $\boldsymbol{\omega}$ relative to an inertial frame. If $\boldsymbol{r}$ is the position vector of a particle of mass m relative to R_0 and the particle is moving with velocity $\dot{\boldsymbol{r}}$, then its equation of motion is (Landau and Lifshitz 1976)

$$m\ddot{\boldsymbol{r}} = -m\boldsymbol{a} + m\boldsymbol{r} \times \dot{\boldsymbol{\omega}} + 2m\dot{\boldsymbol{r}} \times \boldsymbol{\omega} + m\boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega}). \quad (1)$$

Here, the second, third, and fourth terms on the right-hand side of the last equation are the Euler, Coriolis, and centrifugal inertial forces, respectively. The physical force (that force acting on m in the inertial frame) is set to zero since the interest here is strictly in the electromagnetic analogue.

Observe that these inertial forces are acceleration independent, linear in particle velocity, and that (1) is derived from the Lagrangian

$$L = \frac{1}{2}mv^2 - \left[m\boldsymbol{a} \cdot \boldsymbol{r} - m\dot{\boldsymbol{r}} \cdot \boldsymbol{\omega} \times \boldsymbol{r} - \frac{1}{2}m(\boldsymbol{\omega} \times \boldsymbol{r})^2\right], \quad (2)$$

where the first term on the right-hand side of (2) is the particle's kinetic energy and the three terms in brackets in (2) correspond to the generalized potential V. Now identify the first and third terms of V as the scalar potential

$$\phi(\boldsymbol{r}, t) = m\boldsymbol{a} \cdot \boldsymbol{r} - \frac{1}{2}m(\boldsymbol{\omega} \times \boldsymbol{r})^2$$

and the vector $\boldsymbol{\omega} \times \boldsymbol{r}$ in V as the vector potential

$$\boldsymbol{A}(\boldsymbol{r}, t) = m\boldsymbol{\omega} \times \boldsymbol{r}.$$

Using the electric and magnetic fields expressed in terms of potentials as a guide, it is found that (1) may be written in Lorentz form as (Hughes 1992)

$$m\ddot{\boldsymbol{r}} = \boldsymbol{E}(\boldsymbol{r}, t) + \dot{\boldsymbol{r}} \times \boldsymbol{H}(\boldsymbol{r}, t),$$

where

$$\boldsymbol{E}(\boldsymbol{r}, t) = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t} = -m\boldsymbol{a} + m\boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega}) - m\dot{\boldsymbol{\omega}} \times \boldsymbol{r}$$

and

$$\boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{\nabla} \times \boldsymbol{A} = 2m\boldsymbol{\omega}.$$

Verification Theorems

For the sake of completeness, this section provides formal verification of the Hughes' claim that the analogue fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$ satisfy Gauss's law for magnetic fields and Faraday's law of Maxwell's equations.

Theorem 1. $\boldsymbol{H}(\boldsymbol{r}, t)$ *satisfies Gauss's law for magnetic fields.*

Proof.

$$\begin{aligned}\boldsymbol{\nabla} \cdot \boldsymbol{H}(\boldsymbol{r}, t) &= \left(\hat{\boldsymbol{\imath}}\,\partial_x + \hat{\boldsymbol{\jmath}}\,\partial_y + \hat{\boldsymbol{k}}\,\partial_z\right) \cdot (2m\boldsymbol{\omega}) \\ &= 2m\left(\hat{\boldsymbol{\imath}}\,\partial_x + \hat{\boldsymbol{\jmath}}\,\partial_y + \hat{\boldsymbol{k}}\,\partial_z\right) \cdot \left(\hat{\boldsymbol{k}}\omega\right) \\ &= 2m\left(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{k}}\,\partial_x\omega + \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}}\,\partial_y\omega + \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}\,\partial_z\omega\right) \\ &= 2m\left(0\,\partial_x\omega + 0\,\partial_y\omega + 1\,\partial_z\omega\right) \\ &= 2m(0 + 0 + 0) \\ &= 0.\end{aligned}$$

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Theorem 2. $\boldsymbol{E}(\boldsymbol{r}, t)$ *satisfies Faraday's law.*

Proof.

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{\nabla} \times [-m\boldsymbol{a} + m\boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega}) - m\dot{\boldsymbol{\omega}} \times \boldsymbol{r}].$$

Referring to the Appendix, take the curl of each term on the right-hand side of the last equation to obtain:

$$\begin{aligned}\boldsymbol{\nabla} \times (-m\boldsymbol{a}) &= -m\big[\hat{\boldsymbol{\imath}}\left(\partial_y a_z - \partial_z a_y\right) + \hat{\boldsymbol{\jmath}}(\partial_z a_x - \partial_x a_z) \\ &\quad + \hat{\boldsymbol{k}}\left(\partial_x a_y - \partial_y a_x\right)\big] \\ &= -m\left[\hat{\boldsymbol{\imath}}(0-0) + \hat{\boldsymbol{\jmath}}(0-0) + \hat{\boldsymbol{k}}(0-0)\right] \\ &= \boldsymbol{0},\end{aligned}$$

$$\begin{aligned}\boldsymbol{\nabla} \times m[\boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega})] &= m\omega^2\boldsymbol{\nabla} \times (\hat{\boldsymbol{\imath}}x + \hat{\boldsymbol{\jmath}}y) \\ &= m\omega^2[\hat{\boldsymbol{\imath}}\left(\partial_y 0 - \partial_z y\right) + \hat{\boldsymbol{\jmath}}(\partial_z x - \partial_x 0) \\ &\quad + \hat{\boldsymbol{k}}(\partial_x y - \partial_y x) \\ &= m\omega^2\left[\hat{\boldsymbol{\imath}}(0-0) + \hat{\boldsymbol{\jmath}}(0-0) + \hat{\boldsymbol{k}}(0-0)\right] \\ &= m\omega^2\boldsymbol{0} \\ &= \boldsymbol{0},\end{aligned}$$

and

$$\boldsymbol{\nabla} \times m(\dot{\boldsymbol{\omega}} \times \boldsymbol{r}) = -m\dot{\omega}\boldsymbol{\nabla} \times (\hat{\boldsymbol{\imath}}y - \hat{\boldsymbol{\jmath}}x)$$

$$= -m\dot{\omega}\{\hat{\boldsymbol{\imath}}[\partial_y 0 + \partial_z x] + \hat{\boldsymbol{\jmath}}[\partial_z y - \partial_x 0] + \hat{\boldsymbol{k}}[-\partial_x x - \partial_y y]\}$$
$$= -m\dot{\omega}\{\hat{\boldsymbol{\imath}}[0] + \hat{\boldsymbol{\jmath}}[0] - \hat{\boldsymbol{k}}[2]\}$$
$$= 2m\dot{\omega}\hat{\boldsymbol{k}}.$$

Adding these results gives

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{0} + \boldsymbol{0} - 2m\dot{\omega}\hat{\boldsymbol{k}} = -2m\dot{\omega}\hat{\boldsymbol{k}}.$$

Also,

$$\frac{\partial \boldsymbol{H}}{\partial t} = \frac{d\boldsymbol{H}}{dt} = 2m\dot{\omega}\hat{\boldsymbol{k}}$$

because

$$(\dot{\boldsymbol{r}} \cdot \nabla)\boldsymbol{H} = (\hat{\boldsymbol{\imath}}\dot{x} + \hat{\boldsymbol{\jmath}}\dot{y} + \hat{\boldsymbol{k}}\dot{z}) \cdot (\hat{\boldsymbol{\imath}}\partial_x + \hat{\boldsymbol{\jmath}}\partial_y + \hat{\boldsymbol{k}}\partial_z)(2m\omega\hat{\boldsymbol{k}})$$
$$= (\dot{x}\partial_x + \dot{y}\partial_y + \dot{z}\partial_z)(2m\omega\hat{\boldsymbol{k}})$$
$$= \boldsymbol{0}.$$

Thus,

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial \boldsymbol{H}}{\partial t}.$$

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Two New Analogues

Recall that Gauss's law for electric fields

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \tag{3}$$

and Ampere's law

$$\nabla \times \boldsymbol{H} = \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \tag{4}$$

of Maxwell's equations involve the constant ϵ_0 - the permittivity of free space. Here, ρ is the electric charge density, $\boldsymbol{J}$ is the electric current density, and the constant μ_0 is the magnetic permeability.

To find charge density and current density analogues using these laws, substitute the rotational motion analogues $\boldsymbol{E}$ and $\boldsymbol{H}$ from above into (3) and (4) and solve the associated equations for $\frac{\rho}{\epsilon_0} \equiv \mathcal{R}$ and $\frac{\boldsymbol{J}}{\epsilon_0} \equiv \boldsymbol{\mathcal{J}}$ (as will be seen below, μ_0 divides out and is not relevant for what is to be done here). These scaled (by ϵ_0) results are the analogues of interest here and-as will be discussed below- yield several potentially interesting physical results.

Theorem 3. $\mathcal{R} = 2m\omega^2$**.**

Proof.

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}, t) = \nabla \cdot [-m\boldsymbol{a} + m\boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega}) - m\dot{\boldsymbol{\omega}} \times \boldsymbol{r}]$$

$$\nabla \cdot (-ma) = -m(\hat{\boldsymbol{\imath}}\partial_x + \hat{\boldsymbol{\jmath}}\partial_y + \hat{\boldsymbol{k}}\partial_z) \cdot (\hat{\boldsymbol{\imath}}a_x + \hat{\boldsymbol{\jmath}}a_y + \hat{\boldsymbol{k}}a_z)$$
$$= -m(\partial_x a_x + \partial_y a_y + \partial_z a_z)$$
$$= -m(0 + 0 + 0)$$
$$= 0,$$

$$\nabla \cdot \{m[\boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega})]\}$$
$$= m(\hat{\boldsymbol{\imath}}\partial_x + \hat{\boldsymbol{\jmath}}\partial_y + \hat{\boldsymbol{k}}\partial_z) \cdot [\omega^2(\hat{\boldsymbol{\imath}}x + \hat{\boldsymbol{\jmath}}y)]$$
$$= m\omega^2(\partial_x x + \partial_y y)$$
$$= m\omega^2(1 + 1)$$
$$= 2m\omega^2,$$

and

$$\nabla \cdot [m(\dot{\boldsymbol{\omega}} \times \boldsymbol{r})] = -m\dot{\omega}(\hat{\boldsymbol{\imath}}\partial_x + \hat{\boldsymbol{\jmath}}\partial_y + \hat{\boldsymbol{k}}\partial_z) \cdot (\hat{\boldsymbol{\imath}}y - \hat{\boldsymbol{\jmath}}x)$$
$$= -m\dot{\omega}(\partial_x y - \partial_y x)$$
$$= -m\dot{\omega}(0 + 0)$$
$$= 0.$$

Adding these results together yields

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}, t) = 0 + 2m\omega^2 - 0$$
$$= 2m\omega^2.$$

Thus,

$$\frac{\rho}{\epsilon_0} = 2m\omega^2$$

or

$$\mathcal{R} = 2m\omega^2.$$

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Theorem 4. $\boldsymbol{\mathcal{J}} = -2m\omega\dot{\omega}(\hat{\boldsymbol{\imath}}x + \hat{\boldsymbol{\jmath}}y)$**.**

Proof.

Since

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}, t) = 2m\{\hat{\boldsymbol{\imath}}(\partial_y \omega - \partial_z 0) + \hat{\boldsymbol{\jmath}}(\partial_z 0 - \partial_x \omega) + \hat{\boldsymbol{k}}(\partial_x 0 - \partial_y 0)\}$$
$$= \boldsymbol{0},$$

then from (4),

$$\boldsymbol{J} = -\epsilon_0 \frac{\partial \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t}.$$

But

$$\frac{\partial \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t} = \frac{d\boldsymbol{E}(\boldsymbol{r}, t)}{dt} - (\dot{\boldsymbol{r}} \cdot \nabla)\boldsymbol{E}(\boldsymbol{r}, t).$$

$$\frac{d\boldsymbol{E}(\boldsymbol{r}, t)}{dt} = \frac{d}{dt}[-m\boldsymbol{a} + m\boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega}) - m\dot{\boldsymbol{\omega}} \times \boldsymbol{r}]$$

$$\frac{d}{dt}(-m\boldsymbol{a}) = -m\frac{d\boldsymbol{a}}{dt} = \boldsymbol{0}$$

$$\frac{d}{dt}[m(\boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega}))] = \frac{d}{dt}[m\omega^2(\hat{\boldsymbol{\imath}}x + \hat{\boldsymbol{\jmath}}y)]$$
$$= 2m\omega\dot{\omega}(\hat{\boldsymbol{\imath}}x + \hat{\boldsymbol{\jmath}}y) + m\omega^2(\hat{\boldsymbol{\imath}}\dot{x} + \hat{\boldsymbol{\jmath}}\dot{y})$$
$$= m\omega[\hat{\boldsymbol{\imath}}(2\dot{\omega}x + \omega\dot{x}) + \hat{\boldsymbol{\jmath}}(2\dot{\omega}y + \omega\dot{y})]$$

$$\frac{d}{dt}[m\,(\dot{\boldsymbol{\omega}} \times \boldsymbol{r})] = \frac{d}{dt}[-m\dot{\omega}(\hat{\boldsymbol{\imath}}y - \hat{\boldsymbol{\jmath}}x)]$$
$$= -m\dot{\omega}(\hat{\boldsymbol{\imath}}\dot{y} - \hat{\boldsymbol{\jmath}}\dot{x})$$

because $\ddot{\omega} = 0$.

Summing these results gives

$$\frac{d\boldsymbol{E}(\boldsymbol{r},t)}{dt} = m[\hat{\boldsymbol{\imath}}(2\dot{\omega}\omega x + \omega^2\dot{x} + \dot{\omega}\dot{y}) + \hat{\boldsymbol{\jmath}}(2\dot{\omega}\omega y + \omega^2\dot{y} - \dot{\omega}\dot{x})].$$

But

$$(\dot{\boldsymbol{r}}\cdot\boldsymbol{\nabla})\boldsymbol{E}(\boldsymbol{r},t) = [(\hat{\boldsymbol{\imath}}\dot{x} + \hat{\boldsymbol{\jmath}}\dot{y} + \hat{\boldsymbol{k}}\dot{z})\cdot(\hat{\boldsymbol{\imath}}\partial_x + \hat{\boldsymbol{\jmath}}\partial_y + \hat{\boldsymbol{k}}\partial_z)]\boldsymbol{E}(\boldsymbol{r},t)$$
$$= [\dot{x}\partial_x + \dot{y}\partial_y + \dot{z}\partial_z][-m\boldsymbol{a} + m\boldsymbol{\omega}\times(\boldsymbol{r}\times\boldsymbol{\omega}) - m\dot{\boldsymbol{\omega}}\times\boldsymbol{r}]$$

$$[\dot{x}\partial_x + \dot{y}\partial_y + \dot{z}\partial_z](-m\boldsymbol{a}) = -m[\dot{x}\partial_x\boldsymbol{a} + \dot{y}\partial_y\boldsymbol{a} + \dot{z}\partial_z\boldsymbol{a}] = \boldsymbol{0}$$

$$[\dot{x}\partial_x + \dot{y}\partial_y + \dot{z}\partial_z]\,[m(\boldsymbol{\omega}\times(\boldsymbol{r}\times\boldsymbol{\omega}))] = m\omega^2[\dot{x}\partial_x + \dot{y}\partial_y + \dot{z}\partial_z][(\hat{\boldsymbol{\imath}}x + \hat{\boldsymbol{\jmath}}y)] = m\omega^2(\hat{\boldsymbol{\imath}}\dot{x}\partial_x x + \hat{\boldsymbol{\jmath}}\dot{y}\partial_y y) = m\omega^2(\hat{\boldsymbol{\imath}}\dot{x} + \hat{\boldsymbol{\jmath}}\dot{y})$$

$$[\dot{x}\partial_x + \dot{y}\partial_y + \dot{z}\partial_z][m(\dot{\boldsymbol{\omega}}\times\boldsymbol{r})] = -m\dot{\omega}[\dot{x}\partial_x + \dot{y}\partial_y + \dot{z}\partial_z][(\hat{\boldsymbol{\imath}}y - \hat{\boldsymbol{\jmath}}x)] = -m\dot{\omega}(\hat{\boldsymbol{\imath}}\dot{y} - \hat{\boldsymbol{\jmath}}\dot{x})$$

So that

$$(\dot{\boldsymbol{r}}\cdot\boldsymbol{\nabla})\boldsymbol{E}(\boldsymbol{r},t) = m[\hat{\boldsymbol{\imath}}(\omega^2\dot{x} + \dot{\omega}\dot{y}) + \hat{\boldsymbol{\jmath}}(\omega^2\dot{y} - \dot{\omega}\dot{x})$$

in which case

$$\boldsymbol{J} = -2m\omega\dot{\omega}(\hat{\boldsymbol{\imath}}x + \hat{\boldsymbol{\jmath}}y)\epsilon_0$$

or

$$\frac{\boldsymbol{J}}{\epsilon_0} = -2m\omega\dot{\omega}(\hat{\boldsymbol{\imath}}x + \hat{\boldsymbol{\jmath}}y)$$

or

$$\boldsymbol{\mathcal{J}} = -2m\omega\dot{\omega}(\hat{\boldsymbol{\imath}}x + \hat{\boldsymbol{\jmath}}y).$$

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Discussion

Now consider physical interpretations for $\mathcal{R}$ and $\boldsymbol{\mathcal{J}}$. Recall that the rotational kinetic energy KE_{rot} of a point mass m rotating at an angular rate ω at perpendicular distance $r_\perp$ from the axis of rotation is

$$KE_{rot} = \frac{1}{2}mr_\perp^2\omega^2.$$

It is seen from this and Theorem 3 that

$$2m\omega^2 = \frac{4KE_{rot}}{r_\perp^2} = \mathcal{R}.$$

Thus, $\mathcal{R}$ can be interpreted as a *rotational kinetic energy density*, specifically the rotational kinetic energy per unit area.

Also, observe from Theorem 4 that $\boldsymbol{\mathcal{J}}$ can be written as

$$\boldsymbol{\mathcal{J}} = -2m\frac{\dot{\omega}}{\omega}\omega^2(\hat{\boldsymbol{\imath}}x + \hat{\boldsymbol{\jmath}}y) = -2m\frac{\dot{\omega}}{\omega}[\boldsymbol{\omega}\times(\boldsymbol{r}\times\boldsymbol{\omega})].$$

Using the Appendix, recognize the quantity $m[\boldsymbol{\omega}\times(\boldsymbol{r}\times\boldsymbol{\omega})]$ as the associated centrifugal inertial force $\boldsymbol{\mathcal{F}}$, in which case

$$\boldsymbol{\mathcal{J}} = -2\frac{\dot{\omega}}{\omega}\boldsymbol{\mathcal{F}}.$$

Using the fact that $-\boldsymbol{\mathcal{F}}$ is the associated centripetal force $\boldsymbol{\mathcal{C}}$, then

$$\boldsymbol{\mathcal{J}} = 2\frac{\dot{\omega}}{\omega}\boldsymbol{\mathcal{C}}.$$

Consequently, $\boldsymbol{\mathcal{J}}$ can be interpreted as the *centripetal force current* – the centripetal force per unit time – directed towards the axis of rotation. Note that this current vanishes when $\dot{\omega} = 0$.

By substituting a mass to charge ratio into the electromagnetic result obtained from Faraday's law, Semon and Schmieg showed that Faraday's law can be used to predict the work done on a point mass moved around a closed path in the $(\hat{\boldsymbol{\imath}},\hat{\boldsymbol{\jmath}})$-plane in a rotating reference frame when ω increases in the $\hat{\boldsymbol{k}}$-direction. This result can be obtained more efficiently from Hughes' theory without having to employ a mass to charge ratio by applying Stoke's theorem to Faraday's law, using the fact that

$$\frac{\partial\boldsymbol{H}}{\partial t} = 2m\dot{\omega}\hat{\boldsymbol{k}}, \tag{5}$$

and integrating the result around a closed path in the $(\hat{\boldsymbol{\imath}},\hat{\boldsymbol{\jmath}})$-plane.

To be more precise, let $\hat{\boldsymbol{n}}$ be a unit vector normal to the $(\hat{\boldsymbol{\imath}},\hat{\boldsymbol{\jmath}})$-plane such that $\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}} = 1$. Then, using (5) in Faraday's law, taking the scalar product of the result with $\hat{\boldsymbol{n}}$, and integrating over the surface area S bounded by the closed path yields

$$\oiint_S (\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t))\cdot\hat{\boldsymbol{n}}\,d\sigma = -2m\dot{\omega}\oiint_S \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}\,d\sigma.$$

Integrating the right hand side and applying Stoke's theorem to the left hand side of this equation finally evaluates the circulation as

$$\oint_{\mathcal{C}} \boldsymbol{E}(\boldsymbol{r},t)d\lambda = -2m\dot{\omega}\mathcal{A},$$

where $\mathcal{C}$ is the closed path boundary of area $\mathcal{A}$.

Because $\boldsymbol{E}(\boldsymbol{r},t)$ is a force applied over the distance $\mathcal{C}$, then the circulation corresponds to the work W required to move m around $\mathcal{C}$:

$$W = -2m\dot{\omega}\mathcal{A}.$$

Clearly, no work is done on the mass when $\dot{\omega} = 0$.

Closing Remarks

It should be noted that - as expected - $\mathcal{R}$ and $\mathcal{J}$ agree with Coisson's choice of the "suitable values" $2\omega^2$ and $\frac{\partial \boldsymbol{E}(r,t)}{\partial t}$, respectively, for these quantities. Trivially, for fixed m and ω, the rotational kinetic energy density $\mathcal{R}$ can be considered to be a *constant of the motion* since it remains unchanged for any perpendicular distance $r_\perp$. Furthermore, since both $\mathcal{J}$ and W depend upon $\dot{\omega}$, the motion of a point mass in a rotating reference frame must experience an Euler force in order to produce a centripetal force current and have work performed upon it.

Although both Coisson and Hughes defined electromagnetic scalar and vector potential analogues for a point mass moving in a rotating reference frame, it was Hughes who noted that analogues of the well-known electromagnetic gauge transformations also apply to the scalar and vector potential analogues. In particular, for any function $f \equiv f(\boldsymbol{r},t)$, $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{H}(\boldsymbol{r},t)$ are invariant under the gauge transformations

$$\boldsymbol{A} \rightarrow \boldsymbol{A} - \boldsymbol{\nabla} f$$

and

$$\phi \rightarrow \phi + \frac{\partial f}{\partial t}$$

and that (2) is not unique and can be transformed as

$$L \rightarrow L - \frac{df}{dt}$$

without changing the associated physics (it should be noted that Hughes' theory is already in the Coulomb gauge because $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$). However, it is not clear how these transformations might be usefully applied to rotating systems.

While $\mathcal{R}$ and $\mathcal{J}$ can be viewed as real quantities because m, ω, $\dot{\omega}$, and $\boldsymbol{\mathcal{C}}$ are individually measurable, it is not clear that the current $\mathcal{J}$ itself produces an observable physical effect upon the motion of m.

Appendix

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1$$

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}} = 0$$

$$\partial_{\ell} \equiv \frac{\partial}{\partial \ell}, \ell = x, y, z$$

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\imath}}\partial_x + \hat{\boldsymbol{\jmath}}\partial_y + \hat{\boldsymbol{k}}\partial_z$$

$$\boldsymbol{a} = \hat{\boldsymbol{\imath}}a_x + \hat{\boldsymbol{\jmath}}a_y + \hat{\boldsymbol{k}}a_z;\ a_\ell\ constant, \ell = x, y, z$$

$$\boldsymbol{r} = \hat{\boldsymbol{\imath}}x + \hat{\boldsymbol{\jmath}}y + \hat{\boldsymbol{k}}z$$

$$\boldsymbol{\omega} = \hat{\boldsymbol{k}}\omega, \omega\ constant$$

$$\dot{\boldsymbol{\omega}} = \hat{\boldsymbol{k}}\dot{\omega}, \dot{\omega}\ constant$$

$$\dot{\boldsymbol{r}} \equiv \frac{d\boldsymbol{r}}{dt} = \hat{\boldsymbol{\imath}}\dot{x} + \hat{\boldsymbol{\jmath}}\dot{y} + \hat{\boldsymbol{k}}\dot{z}$$

$$\ddot{\boldsymbol{r}} \equiv \frac{d^2\boldsymbol{r}}{dt^2}$$

$$\boldsymbol{\omega} = \hat{\boldsymbol{k}}\omega, \omega\ constant$$

$$\dot{\boldsymbol{\omega}} \equiv \frac{d\boldsymbol{\omega}}{dt} = \hat{\boldsymbol{k}}\dot{\omega}, \dot{\omega}\ constant$$

$$\boldsymbol{C} \times \boldsymbol{D} = \hat{\boldsymbol{\imath}}\left(c_y d_z - c_z d_y\right) + \hat{\boldsymbol{\jmath}}(c_z d_x - c_x d_z) + \hat{\boldsymbol{k}}\left(c_x d_y - c_y d_x\right)$$

$$\begin{aligned} \boldsymbol{r} \times \boldsymbol{\omega} &= \hat{\boldsymbol{\imath}}(y\omega - z \cdot 0) + \hat{\boldsymbol{\jmath}}(z \cdot 0 - x \cdot \omega) + \hat{\boldsymbol{k}}(x \cdot 0 - y \cdot 0) \\ &= \hat{\boldsymbol{\imath}}y\omega - \hat{\boldsymbol{\jmath}}x\omega + \hat{\boldsymbol{k}}(0) \\ &= \omega(\hat{\boldsymbol{\imath}}y - \hat{\boldsymbol{\jmath}}x) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega}) &= \hat{\boldsymbol{\imath}}[0 \cdot 0 - \omega \cdot (-\omega x)] + \hat{\boldsymbol{\jmath}}[\omega \cdot (\omega y) - 0 \cdot 0] \\ &\quad + \hat{\boldsymbol{k}}[0 \cdot (-\omega x) - 0 \cdot \omega y] \\ &= \hat{\boldsymbol{\imath}}\omega^2 x + \hat{\boldsymbol{\jmath}}\omega^2 y + \hat{\boldsymbol{k}}(0) \\ &= \omega^2(\hat{\boldsymbol{\imath}}x + \hat{\boldsymbol{\jmath}}y) \end{aligned}$$

$$\begin{aligned} \dot{\boldsymbol{\omega}} \times \boldsymbol{r} &= \hat{\boldsymbol{\imath}}(0 \cdot z - \dot{\omega} \cdot y) + \hat{\boldsymbol{\jmath}}(\dot{\omega} \cdot x - 0 \cdot z) \\ &\quad + \hat{\boldsymbol{k}}(0 \cdot y - 0 \cdot x) \\ &= \hat{\boldsymbol{\imath}}(-\dot{\omega} \cdot y) + \hat{\boldsymbol{\jmath}}(\dot{\omega} \cdot x) + \hat{\boldsymbol{k}}(0) \\ &= -\dot{\omega}(\hat{\boldsymbol{\imath}}y - \hat{\boldsymbol{\jmath}}x) \end{aligned}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} = \frac{d\boldsymbol{A}}{dt} - (\dot{\boldsymbol{r}} \cdot \boldsymbol{\nabla})\mathbf{A}$$

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