Proceedings of SALT 33: 686–706, 2023

On the modeling of live possibilities*

Yichi (Raven) Zhang
Heinrich Heine University

Abstract In this paper, I evaluate two ways to model the notion of *live possibilities*: the supervaluation-based approach, and the alternative-based approach. I argue that the alternative-based approach is more promising in fulfilling several desirable constraints governing live possibilities. However, the existing alternative-based accounts fail to be fully satisfactory. To address this inadequacy, I devise a new alternative-based framework and explore its logical features.

Keywords: live possibility, update semantics, inquisitive semantics, alternative semantics

1 Introduction

Language can often draw attention to possibilities. Here is an example adapted from Dretske 1970:

(1) John: Look! It's a zebra.
Mary: It might be a zebra, but it also might be a cleverly disguised mule.
John: Oh, I knew their budget was tight but I didn't know it was this bad.

Mary's *might*-claim that the apparent zebra might be a cleverly disguised mule exemplifies what Bledin & Rawlins (2020) call a *resistance move*. Without directly asserting the opposite of what John just asserted, Mary manages to raise a new possibility which John was initially unaware of. After becoming aware of this new "live" possibility, John retracts his previous assertion.

Live possibilities also help to explain the informativeness of epistemic possibility claims like (2) which in particular poses a problem for the test conception of modals.

(2) Cheerios may reduce the risk of heart disease (Yalcin 2007: 1012).

On the test view (Veltman 1996), the meaning of a sentence is construed as an update function from input information states, usually modeled as sets of worlds, to output states, and $\Diamond\varphi$ is viewed as a test that checks whether the prejacent φ is compatible with the update input. Let $s[\Diamond\varphi]$ be the information state that results from updating the information state s with $\Diamond\varphi$; the semantics of $\Diamond\varphi$ is given as follows:

* I would like to thank Paolo Santorio, Jacopo Romoli, Patrick Elliot, Todd Snider, audience at HHU's Meaning in Language Colloquium, and audience at SALT33 for helpful discussion and comments.

©2023 Zhang

(3) $s[\Diamond\varphi] = \{w \in s \mid s[\varphi] \neq \varnothing\}$

This update only has two possible outcomes: if φ is compatible with s—that is, if updating s with φ does not return the absurd information state $\varnothing$—then the update with $[\Diamond\varphi]$ is idle and simply returns its input state; otherwise, the update returns the absurd state $\varnothing$, thereby signaling discourse inconsistency. Since an update with $[\Diamond\varphi]$ will always either be idle and thus trivial or be inconsistent, the simple test semantics cannot capture the informativeness of $\Diamond\varphi$. In response, more sophisticated test semantics (e.g., Willer 2013) have been proposed wherein a distinction between *live* and *mere plain* possibilities is made. Sentences like (2) are informative because they can convert a mere plain possibility to a live possibility.

Besides epistemic possibility claims, there are many other means to draw attention to possibilities. For instance, in the apparent zebra scenario, Mary could ask the following question to achieve a similar effect:

(4) Are you sure you are not looking at a cleverly disguised mule?

The antecedent of a conditional is also commonly associated with bringing possibilities to live, and this has been used to explain the contrast in felicity between the so-called *Sobel sequence* in (5) and the *reverse Sobel sequence* in (6) (cf. Fintel 2001; Gillies 2007; Willer 2017).

(5) a. If Alice comes, the party will be fun.
 b. But if Alice and Bob both come, the party won't be fun.

(6) a. If Alice and Bob both come, the party won't be fun.
 b. ?? But if Alice comes, the party will be fun.

In (5), since the antecedent of (5a) brings up a new possibility, it expands the set of possibilities one is considering, which in turn allows the apparent conflict between (5a) and (5b) to be reconciled. By contrast in (6), since the antecedent of (6a) has already made Alice's coming to the party a live possibility, the antecedent of (6b) does not introduce any new possibilities. Consequently, no expansion occurs and the whole sequence is perceived as infelicitous.

Despite the intuitiveness of this notion, it remains unclear how to best cash out live possibilities formally. To begin, as we have seen, if information states are simply modeled as sets of worlds, then it is impossible to distinguish live from plain possibilities. One natural maneuver is thus to add more structure by moving one level up on the set-theoretic hierarchy and represent bodies of information using sets of sets of possible worlds. I will call them *hyperstates* (notated by S).

At the level of hyperstates, p is viewed as a plain possibility iff S contains an information state s that contains a p−world. As for live possibilities, there are two distinct ways to define them: a supervaluation-based analysis (Willer 2013, 2017), and an alternative-based on analysis (Ciardelli, Groenendijk & Roelofsen 2009; Roelofsen 2013; Bledin & Rawlins 2020).

S(upervaluation)-Based: p is a live possibility in S iff $\forall s \in S\ \exists w \in s : w \in \mathcal{I}(p)$.

A(lternative)-Based: p is a live possibility in S iff $\exists s \in S\ \forall w \in s : w \in \mathcal{I}(p)$.

On the S-based account, p is a live possibility iff every information state in S is compatible with p, whereas on the A-based account, p is a live possibility iff there is an information state in S that consists exclusively of p-worlds. This paper offers a critical comparison of the two approaches. To this end, I will first delineate a few attentional constraints which I believe an adequate analysis of live possibilities should satisfy. I argue that none of the existing approaches, be it S-based or A-based, fulfills them all. I will then sketch an A-based account capable of capturing all the constraints. Lastly, I will highlight some logical features of this new framework.

2 Attentional Constraints on Hyperstates

The aforementioned attentional constraints will be stated using the notion of *support* (or *acceptance*) as employed in update semantics:

(7) **Support:** $S \vDash \varphi$ iff $S[\varphi] = S$

A hyperstate supports φ just in case the update with $[\varphi]$ is idle. We can then take $S \vDash \Diamond\varphi$ to mean that φ is already a live possibility in S. With a notion of support at hand, I postulate the following attentional constraints all of which are placed on the output states yielded by updating the minimal state S_0, a state where no factual information has been acquired and no possibilities have been brought to salience.

Restricted Idempotence (RI): $S_0[\Diamond\varphi] \vDash \Diamond\varphi$.

Closure under Negation (CN): $S_0[\Diamond\varphi] \vDash \Diamond\neg\varphi$.

Non Closure under Conjunction (NCC): $S_0[\Diamond\varphi \wedge \Diamond\psi] \nvDash \Diamond(\varphi \wedge \psi)$.

Decomposable under Conjunction (DC): $S_0[\Diamond(\varphi \wedge \psi)] \vDash \Diamond\varphi, \Diamond\psi$.

Dynamic Attentiveness (DA): Let $S = S_0[\Diamond\varphi]$ and $S' = S[\psi]$ for any ψ; then if $S' \nvDash \neg\varphi$, then $S' \vDash \Diamond\varphi$.

RI is an obvious constraint: an update with $[\Diamond\varphi]$ should make φ a live possibility. In addition, according to **CN**, the update on S_0 with $[\Diamond\varphi]$ should also make $\neg\varphi$ a live possibility. To put it another way, since updating S_0 with $[\Diamond\varphi]$ should not make φ true straight away, **CN** amounts to saying that any state where φ is a live possibility but has yet to be settled true should also deem $\neg\varphi$ as a live possibility. This seems intuitively plausible: anyone who is actively entertaining φ but falls short of accepting φ should also consider $\neg\varphi$ as a live option. For instance, (8) sounds odd as it violates **CN**.

(8) ?? I don't know whether Alice will come to the party. She might come but it's not the case that she might not come.

It is worth stressing at this stage that **CN** can be fulfilled via different means. On one hand, it could be a direct result of the update with $[\Diamond\varphi]$; on the other hand, $\Diamond\neg\varphi$ could be derived from some additional inference, for instance, as a scalar implicature of $\Diamond\varphi$. I do not intend to settle this issue here and shall consider both options viable.

Next, **NCC** and **DC** can be illustrated by the contrast between (9) and (10):

(9) a. Alice might come to the party, and Bob might come as well.

b. But they won't both come/can't both be coming.

(10) a. Alice and Bob might both come to the party.

b. ?? But Alice won't come/can't be coming.

If live possibilities were closed under conjunction, we would expect (9a) to contradict (9b), which is not the case. By contrast, (10) does sound like a contradiction, so possibilities should be decomposable under conjunction.

Lastly, **DA** states that if φ is a live possibility in S, then as long as the update with ψ does not make φ false, φ should remain as a live possibility. In other words, once a possibility has been brought to salience, it will stay salient unless it has been explicitly ruled out. Although this constraint no doubt involves some idealization in assuming that discourse participants have unlimited memory and attentional resources—that they do not lose track of what possibilities have become live—it is arguably a reasonable postulate for any formal semantic framework that does not intend to actively model discourse participants' cognitive resources.

3 Evaluating Existing Accounts

I will now evaluate some prominent existing accounts of live possibilities against the aforementioned attentional constraints. Consider the **S-based** approach from Willer 2013, 2017, under which p is a live possibility just in case it is compatible with every body of information deemed relevant in some inquiry. On this view, updating a hyperstate S with $[\varphi]$ amounts to first updating each s in S with $[\varphi]$ according to Veltman's (1996) update clauses and then collecting all the non-empty states to form the new hyperstate. **RI** is immediately fulfilled. It successfully fulfills **NCC**, because some s can contain both a p-world and a q-world but not any $(p \wedge q)$-world.

It also fulfills **DC**, because any s that contains a $(p \wedge q)$-world must contain both a p-world and a q-world.

However, it fails to capture **CN** and **DA**. For **CN**, since an update with $[\Diamond p]$ will only eliminate every information state in S that does not contain any p-world, it will leave those states that consist exclusively of p-worlds intact. Since not every state in the output hyperstate is guaranteed to contain a $\neg p$-world, $\neg p$ fails to become a live possibility.

While **CN** can be secured through either requiring an update with $[\Diamond p]$ be accompanied by an update with $[\Diamond \neg p]$ or deriving it as an implicature, there does not appear to be an easy fix for **DA**. To illustrate, consider updating S_0 with $[\Diamond p \wedge \Diamond q]$ and then with $[q]$. Given a logical space ω modeled as a set of worlds, the minimal hyperstate S_0 is defined as the power set of ω excluding the empty set: $\wp(\omega) - \{\varnothing\}$. Let pq be a $(p \wedge q)$-world, $p\bar{q}$ be a $(p \wedge \neg q)$-world, and so on. The sequential update on S_0 with $[\Diamond p \wedge \Diamond q]$ and then with $[q]$ proceeds as follows:

i. Updating S_0 with $[\Diamond p \wedge \Diamond q]$ makes both p and q salient. Assume that **CN** is satisfied; then $\neg p$ and $\neg q$ also become live. Hence, the output hyperstate S_1 contains the following information states: $\{pq, \bar{p}\bar{q}\}, \{p\bar{q}, \bar{p}q\}, \{pq, p\bar{q}, \bar{p}q\}$, $\{pq, p\bar{q}, \bar{p}\bar{q}\}, \{pq, \bar{p}q, \bar{p}\bar{q}\}$, $\{p\bar{q}, \bar{p}q, \bar{p}\bar{q}\}$, and $\{pq, p\bar{q}, \bar{p}q, \bar{p}\bar{q}\}$.

ii. The subsequent update with $[q]$ eliminates all the $\neg q$-worlds from every $s \in S_1$. The resulting hyperstate S_2 thus contains the following members: $\{pq\}$, $\{\bar{p}q\}$, and $\{pq, \bar{p}q\}$.

But since not every $s \in S_2$ contains a p-world, p is no longer a live possibility; it has changed from a live possibility to a mere plain possibility. This means **DA** fails. Here is a quick diagnosis: since updating S with $[q]$ will eliminate every $\neg q$-world from every $s \in S$, this update may inadvertently eliminate some $(p \wedge \neg q)$-worlds so that some information states in the output hyperstate will no longer contain any p-world, thereby demoting p from a live possibility to a mere plain possibility.

On an **A-based** account, p is a live possibility just in case p corresponds to a salient alternative in S, that is, an information state where p is settled true. As such, **DA** can be easily observed. Consider again the update with $[\Diamond p \wedge \Diamond q]$ and then $[q]$: since eliminating q-worlds from the output hyperstate $S[\Diamond p \wedge \Diamond q]$ will only shrink the size of each information state in it, as long as some information state still contains a p-world, p will remain as a salient alternative and a live possibility.

However, existing A-based accounts either fail to satisfy **CN** or fail to satisfy **NCC**. The most well-studied A-based system from the recent literature is inquisitive semantics (Ciardelli, Groenendijk & Roelofsen 2018). As inquisitive semantics was designed to provide a uniform analysis for declarative and interrogative sentences, adopting it to model live possibilities could be problematic. In inquisitive semantics, hyperstates are construed as downward closed sets of information states. This means that if a hyperstate contains an information state as a live possibility, it will also contain all of its subsets. Hence, if both p and q are live possibilities in a hyperstate and their conjunction has yet to be settled false, then the hyperstate must also contain all of their subsets including the set that comprises only $(p \wedge q)$-worlds, thereby making the conjunction a live possibility as well. As a result, **NCC** fails.

A similar problem also arises for accounts that use partition semantics to model live possibilities (e.g., Bledin & Rawlins 2020) according to which p is a live possibility just in case there is a cell that consists exclusively of p-worlds in the partition. Since members of a partition do not overlap, a partition that contains a cell consisting exclusively of p-worlds and a cell consisting exclusively of q-worlds must also contain a cell consisting exclusively of $(p \wedge q)$-worlds, so long as the conjunction has not been completely ruled out. As a result, **NCC** fails again.[1]

One solution to this problem is to drop the downward-closure requirement imposed by inquisitive semantics. Ciardelli et al. (2009) and Roelofsen (2013) call the resulting framework *attentive semantics*. On this account, the semantic content of an epistemic possibility claim, as with that of any other Boolean sentences, is represented as a set of information states. Let $[\![\varphi]\!]$ stand for the semantic value of φ, namely the set of information states denoted by φ; then $[\![\Diamond\varphi]\!]$ is defined as follows:

(11) $[\![\Diamond\varphi]\!] = [\![\varphi]\!] \cup \{\omega\}$, where ω is the logical space.

Although dropping downward closure allows attentive semantics to satisfy **NCC**,[2] it has difficulty capturing **CN**. In particular, the usual method of deriving $\Diamond\neg p$ from $\Diamond p$ via a scalar implicature becomes unavailable. This is so because the usual method requires negating the stronger scale mate of $\Diamond p$, namely $\Box p$. However, it is unclear how the necessity modal can be defined in existing attentive semantics. The standard way is to define $\Box$ as the dual of $\Diamond$, that is, $\Box\varphi := \neg\Diamond\neg\varphi$. But as $[\![\Diamond\varphi]\!]$ includes the whole logical space, its negation will be a contradiction in attentive semantics. While this may still leave open other ways to derive $\Diamond\neg\varphi$ (see, e.g., Roelofsen 2013: 210), the fact that negation of an epistemic possibility modal is contradictory in attentive semantics already seems problematic given that

1 Both inquisitive semantics and partition semantics do satisfy a weaker version of **NCC**:

Weak-NCC: For any S such that $S \vDash \Diamond p$ and $S \vDash \Diamond q$, it doesn't necessarily follow that $S \vDash \Diamond(p \wedge q)$.

Note that **Weak-NCC** can simply be witnessed by a hyperstate that has completely ruled out $(p \wedge q)$-worlds. By contrast, **NCC** requires that bringing p and q to salience should not automatically make their conjunction a live possibility even if the conjunction has not been ruled out. For example, in (9), the first sentence should make "Alice comes to the party" and "Bob comes to the party" live without making their conjunction live, even though the conjunction has yet to be ruled out before the uttering of the second sentence.

2 In fact, **NCC** is satisfied only by Roelofsen 2013 but not by Ciardelli et al. 2009. The former adopts a more sophisticated analysis of conjunction to achieve this result.

instances of negated epistemic possibility modals can be found in English as well as in other languages (Iatridou & Zeijlstra 2013); for example,

(12) Paul can't be home yet. $\neg \Diamond p$

To summarize, although the A-based approach has an advantage over the S-based approach, mainly due to its ability to easily satisfy **DA**, none of the existing A-accounts turns out to be fully satisfactory. In the next section, I will present a novel A-based framework which can fulfill all the attentional constraints while also preserving the duality between $\Diamond$ and $\Box$.

4 A New A-Based Account of Live Possibilities

4.1 Two-step updates

I devise an update semantics that divides an update $S[\varphi]$ into two steps: an additive update $[\varphi]_a$ which adds new alternatives to S thereby bringing certain possibilities to salience, and then a traditional eliminative update $[\varphi]_e$ which eliminates worlds incompatible with φ.

With respect to the additive update, I shall consider two versions; call them the *conservative* update and the *liberal* update. The two updates mainly differ in whether **CN** is realized as an immediate consequence of the update machinery or as an outcome of additional pragmatic inferences. On the *liberal* account, updating with $[\Diamond p]$ will bring both p and $\neg p$ to salience, whereas on the *conservative* account, this update will only make p a live possibility. Conceptually, these two versions of the additive update correspond to two different views on what attention is directed towards. On the *liberal* account, attention is directed at subject matters (Bledin & Rawlins 2020; see also Lewis 1988; Yalcin 2018). Thus, when "whether p" is a subject matter, both p and $\neg p$ become salient. On the *conservative* account, attention is directed towards possibilities and is thus non-symmetric.

To formally cash out conservative and liberal additive updates, I first define, as a common ingredient, a notion of **positive and negative attentive content** associated with a sentence:

(i) $att^+(p) := \{|p|\}$

(ii) $att^-(p) := \{|\bar{p}|\}$

(iii) $att^+(\neg\varphi) := att^-(\varphi)$

(iv) $att^-(\neg\varphi) := att^+(\varphi)$

(v) $att^+(\varphi \wedge \psi) := att^+(\varphi) \cup att^+(\psi)$

(vi) $att^-(\varphi \wedge \psi) := att^-(\varphi) \cup att^-(\psi)$

(vii) $att^+(\varphi \vee \psi) := att^+(\varphi) \cup att^+(\psi)$

(viii) $att^-(\varphi \vee \psi) := att^-(\varphi) \cup att^-(\psi)$

The attentive contents are treated bilaterally (cf. Willer 2018; Aloni 2022), where $|p|$ and $|\bar{p}|$ stand for the truth set of p (i.e., $\{w \in \omega \mid w \in \mathcal{I}(p)\}$) and the false set of p

(i.e., $\{w \in \omega \mid w \notin \mathcal{I}(p)\}$), respectively. The positive attentive content of an atomic formula p is the singleton set that contains the truth set of p; the negative attentive content is the singleton set that contains the false set of p. Negation swaps between the positive and negative attentive contents. Conjunction and disjunction simply collect all attentive contents associated with their constituent sentences.

As for the positive and negative contents associated with $\Diamond\varphi$, in order to satisfy **NCC**, we need to distinguish the attentive content of a *conjunctive possibility* $\Diamond(p \wedge q)$ from the attentive content of a *conjunction of possibilities* $\Diamond p \wedge \Diamond q$. Hence, I will define $att^{+}(\Diamond\varphi)$ and $att^{-}(\Diamond\varphi)$ as follows:

(ix) $att^{+}(\Diamond\varphi) := ||\varphi||^{+}$ (x) $att^{-}(\Diamond\varphi) := ||\varphi||^{-}$

where $||\varphi||^{+}$ and $||\varphi||^{-}$ are defined recursively as follows:

(a) $||p||^{+} := \{|p|\}$
(e) $||\varphi \wedge \psi||^{+} := \{x \cap y \mid x \in ||\varphi||^{+} \;\&\; y \in ||\psi||^{+}\}$
(b) $||p||^{-} := \{|\bar{p}|\}$
(f) $||\varphi \wedge \psi||^{-} := ||\varphi||^{-} \cup ||\psi||^{-}$
(c) $||\neg\varphi||^{+} := ||\varphi||^{-}$
(g) $||\varphi \vee \psi||^{+} := ||\varphi||^{+} \cup ||\psi||^{+}$
(d) $||\neg\varphi||^{-} := ||\varphi||^{+}$
(h) $||\varphi \vee \psi||^{-} := \{x \cap y \mid x \in ||\varphi||^{-} \;\&\; y \in ||\psi||^{-}\}$
(i) $||\Diamond\varphi||^{+} := ||\varphi||^{+}$
(j) $||\Diamond\varphi||^{-} := ||\varphi||^{-}$

$||\varphi||^{+}$ and $||\varphi||^{-}$ correspond to the conjoined positive and negative possibilities evoked by φ. More specifically, the active conjoining of possibilities only happens for $||\varphi \wedge \psi||^{+}$ and $||\varphi \vee \psi||^{-}$. Take $||p \wedge q||^{+}$ as an example. Given that $||p||^{+} = \{|p|\}$ and $||q||^{+} = \{|q|\}$, taking their *point-wise intersection* as given by (e) results in the set $\{|p \wedge q|\}$.

This enables us to differentiate between the attentive content of a conjunctive

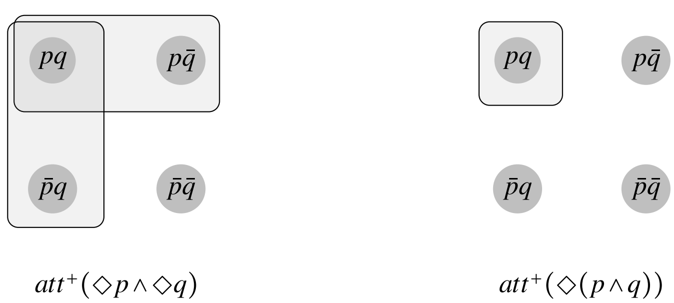


Figure 1 The attentive content of $\Diamond p \wedge \Diamond q$ and $\Diamond(p \wedge q)$ respectively. Each node in the figure stands for a possible world.

possibility $att^+(\Diamond(p \wedge q))$ and that of a conjunction of possibilities $att^+(\Diamond p \wedge \Diamond q)$, as illustrated by Figure 1.

With a notion of positive and negative attentive contents at hand, I define the conservative and liberal updates as follows:

(13) a. **Conservative additive update:** $S[\varphi]_a^c := S \cup att^+(\varphi)$

b. **Liberal additive update:** $S[\varphi]_a^l := S \cup att^+(\varphi) \cup att^-(\varphi)$

The conservative update only adds the positive attentive content of φ to its input state; by contrast, the liberal update adds both the positive and negative attentive contents to the input state, thereby directing attention to the whole subject matter.

Next, I define the eliminative update as follows:

(14) **Eliminative update:** $S[\varphi]_e := \{s[\varphi]_e^S \mid s \in S\} - \{\varnothing\}$

where $s[\varphi]_e^S$, the update with φ on an information state s with respect to a hyperstate S that contains it, is recursively defined as follows:

(i) $s[p]_e^S = s \cap |p|$

(ii) $s[\neg\varphi]_e^S := s - s[\varphi]_e^S$

(iii) $s[\varphi \wedge \psi]_e^S := s[\varphi]_e^S \cap s[\psi]_e^S$

(iv) $s[\varphi \vee \psi]_e^S := s[\varphi]_e^S \cup s[\psi]_e^S$

(v) $s[\Diamond\varphi]_e^S = \{w \in s \mid \exists s' \in S : s'[\varphi]_e^S \neq \varnothing\}$

Updating S with $[\varphi]_e$ amounts to updating every $s \in S$ with $[\varphi]_e^S$ individually and then collecting all the non-empty results. For Boolean formulas, relativizing updates to a hyperstate S has no effect. The modal $\Diamond$ receives a test semantics à la Veltman 1996 with the exception that instead of checking whether φ is compatible with the input state s, it checks whether there is any information state in the hyperstate S that is compatible with φ. Consequently, the update on a hyperstate $S[\Diamond\varphi]_e$ will always either return its input state if φ is satisfied at some world in S meaning φ is at least a *plain possibility*, or return the absurd state $\varnothing$ otherwise.

To illustrate, Figure 2 shows the sequential update on the minimal state S_0—again, defined as the singleton set containing the logical space ω—with $[\Diamond p \wedge \Diamond q]$ and then with $[q]$ using either the liberal or the conservative additive update. In both versions, the additive update with $[\Diamond p \wedge \Diamond q]_a$ adds new alternatives to the output hyperstate upon which the eliminative update with $[\Diamond p \wedge \Diamond q]_e$ is idle. Since there is no alternative that corresponds to the conjunctive possibility $p \wedge q$ in S_1 as well as in S_1', **NCC** is satisfied. In contrast to what we saw with Willer's (2013) S-based account, **DA** is preserved here as well because both S_2 and S_2' contain the information state $\{pq\}$ which keeps p as a live possibility. **CN** is immediately satisfied in S_2 on the liberal update. There is also no difficulty in deriving **CN** as an implicature on the conservative update; an update on S_2' with $[\neg\Box p]$—which is equivalent to $[\Diamond\neg p]$ by duality (see Appendix for a proof of duality)—will convert S_2' into S_2 (provided that the modifications from section 4.2 are in place).

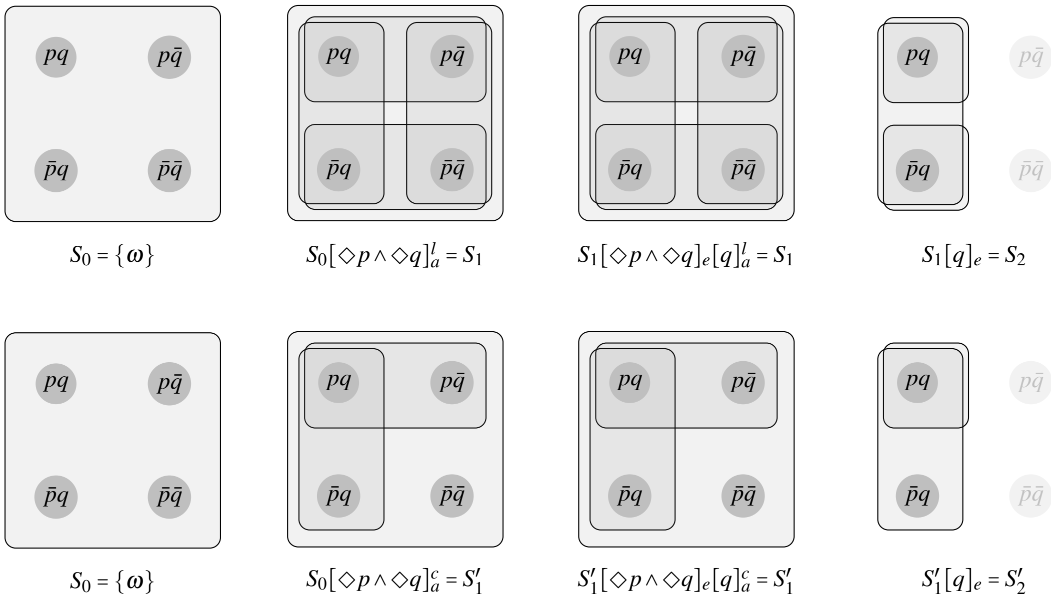


Figure 2 The top row illustrates the liberal additive update, and the bottom row the conservative update.

4.2 Fine-tuning

We still need a couple of adjustments. First, given that we defined $||\varphi \wedge \psi||^+$ and $||\varphi \vee \psi||^-$ using point-wise intersection—e.g., $||\varphi \wedge \psi||^+ := \{x \cap y \mid x \in ||\varphi||^+ \ \&\ y \in ||\psi||^+\}$—this raises a problem: the conjoined possibilities invoked by a conjunction may fail to be idempotent in the sense that $||\varphi||^+ \neq ||\varphi \wedge \varphi||^+$ (cf. Roelofsen 2013; Ciardelli, Roelofsen & Theiler 2017). For example, $||(p \vee q)||^+ \neq ||(p \vee q) \wedge (p \vee q)||^+$:

(15) a. $||(p \vee q)||^+ = \{|p|, |q|\}$

b. $||(p \vee q) \wedge (p \vee q)||^+ = \{|p|, |q|, |p \wedge q|\}$

Consequently, $att^+(\Diamond(p \vee q)) \neq att^+(\Diamond[(p \vee q) \wedge (p \vee q)])$. Hence, updating with $\Diamond(p \vee q)$ and $\Diamond((p \vee q) \wedge (p \vee q))$ will yield different results.

This failure of idempotence has negative consequences. For instance, it blocks one readily available explanation as to why $\Diamond((p \vee q) \wedge (p \vee q))$ feels redundant. When the updates with $\Diamond(p \vee q)$ and $\Diamond((p \vee q) \wedge (p \vee q))$ yield the same result, redundancy can be explained in terms of Grice's (1975) maxim of manner: since $\Diamond(p \vee q)$ and $\Diamond((p \vee q) \wedge (p \vee q))$ contribute the same semantically, the more succinct $\Diamond(p \vee q)$ should be preferred. The same explanation will not go through on the current update framework with point-wise intersection.

To restore idempotence, I adopt a version of conjunction as proposed in Roelofsen 2013. We first define a way to restrict possibilities, be they positive or negative, with respect to some information state:

(16) **Restricting Possibilities:** $||\varphi||^{+/-}_{|\psi|} = \{|\alpha| \cap |\psi| : |\alpha| \in ||\varphi||^{+/-}\} - \{\varnothing\}$

Restricting $||\varphi||^{+/-}$ to some information state $|\psi|$ amounts to intersecting every possibility invoked by φ with $|\psi|$. $||\varphi \wedge \psi||^{+}$ and $||\varphi \vee \psi||^{-}$ are redefined as follows:

- $||\varphi \wedge \psi||^{+} := (||\varphi||^{+} \cup ||\psi||^{+})_{|\varphi| \cap |\psi|}$;
- $||\varphi \vee \psi||^{-} := (||\varphi||^{-} \cup ||\psi||^{-})_{|\neg\varphi| \cap |\neg\psi|}$.

On this definition, both $||(p \vee q)||^{+}$ and $||(p \vee q) \wedge (p \vee q)||^{+}$ are equal to the set $\{|p|, |q|\}$. Idempotence is then restored.

The definitions for additive updates need to be amended as well. Take the conservative update as an example, and consider updating the hyperstate $\{\{pq, p\bar{q}\}\}$ where p has already been settled true with $[q]$. According to the definition in section 4.1, the additive update $[q]^{c}_{a}$ will add to it the possibility $\{pq, \bar{p}q\}$ thereby outputting $\{\{pq, p\bar{q}\}, \{pq, \bar{p}q\}\}$; the eliminative update with $[q]_{e}$ subsequently removes all the $\neg q$-worlds in it, yielding $\{\{pq\}, \{pq, \bar{p}q\}\}$. However, because this hyperstate now contains a $\neg p$-world, it means p is no longer settled true after the update with $[q]$.

As a first pass, we should require that any newly added live possibilities from the additive update be restricted to propositions that are already true in the input hyperstate. In our last example, since the input state $\{\{pq, p\bar{q}\}\}$ already makes p true, the additive update $[q]^{c}_{a}$ should add $\{pq\}$ instead of $\{pq, \bar{p}q\}$ to its input. Hence, I will modify the definition for the conservative update $S[\varphi]^{c}_{a}$ by restricting those possibilities invoked by φ to the *informative content* of the input S:

(17) a. **Informative content of a hyperstate**: $Info(S) := \{w \mid w \in s \text{ for some } s \in S\}$

b. **Conservative update:** $S[\varphi]^{c}_{a} := S \cup att^{+}(\varphi)_{Info(S)}$, where $att^{+}(\varphi)_{Info(S)} = \{x \cap Info(S) \mid x \in att^{+}(\varphi)\} - \{\varnothing\}$

This new definition, however, still falls short. Consider the hyperstate $\{\{pq\}, \{p\bar{q}\}\}$. Since p is already a live possibility in this hyperstate, we should expect the update with $[p]$ to return its input state as is. In spite of this, updating according to (17) produces $\{\{pq\}, \{p\bar{q}\}, \{pq, p\bar{q}\}\}$, since $att^{+}(p)_{Info(S)} = \{\{pq, p\bar{q}\}\}$. To address this problem, we do not want to introduce possibilities that are less fine-grained than those that are already live in S when we add $att^+(\varphi)_{Info(S)}$ to S. One straightforward way to achieve this is through the following bipartite definition.

(18)

$$\textbf{Conservative update: } S[\varphi]^c_a = \begin{cases} S & \text{if } \forall s' \in att^+(\varphi)_{Info(S)}, \exists s \in S : s \subseteq s' \text{ [3]} \\ S \cup att^+(\varphi)_{Info(S)} & \text{otherwise} \end{cases}$$

The additive update $[\varphi]^c_a$ will alter its input state only if the possibilities inovoked by φ are not already live in S. Analogously, the liberal update is redefined as follows:

(19)

$$\textbf{Liberal update: } S[\varphi]^l_a = \begin{cases} S & \text{if } \forall s' \in att^+(\varphi)_{Info(S)} \cup att^-(\varphi)_{Info(S)}, \exists s \in S : s \subseteq s' \\ S \cup att^+(\varphi)_{Info(S)} \cup att^-(\varphi)_{Info(S)} & \text{otherwise} \end{cases}$$

Lastly, the fact that eliminative updates with modals in the current system will always return either the input or the absurd state, though characteristic of Veltman's update semantics, is perhaps less than ideal. For instance, updating any hyperstate where p has yet to be settled either true or false with $[\Box p]_e$ will result in the absurd state, whereas intuitively we should expect the update to have a similar effect as updating with $[p]_e$. A tentative way to address this problem is to adopt the following definition for the eliminative updates instead.

(20) **Localized eliminative update:** $S[\varphi]_e := \{s[\varphi]_e \mid s \in S\} - \{\varnothing\}$, where $s[\varphi]_e$ is recursively defined exactly in accordance with Veltman's update semantics.

In contrast to the definition in (14), modals no longer impose a global check on the whole hyperstate. Thus, an update with $[\Box p]_e$ will only remove from a hyperstate information states that contain a $\neg p$-world but leave those that comprise only p-worlds intact, thereby producing a similar result as updating with $[p]_e$. On the other hand, however, this localized update only works for conservative updates. For example, applying the localized update to S_1 in Figure 2 will yield S_1' rather than S_1. As a result, liberal updates will collapse into conservative updates.

4.3 Logical features

I shall employ the following logical notions found in Veltman 1996 and Groenendijk, Stokhof & Veltman 1996:

3 Note that this condition is vacuously satisfied when $att^+(\varphi)_{Info(S)} = \varnothing$. For example, given that $att^+(\neg p)_{|p|} = \varnothing$, it follows that updating $\{|p|\}$ where p is settled true with $[\neg p]^c_a$ will return $\{|p|\}$. The subsequent eliminative update with $[\neg p]_e$ will then result in $\varnothing$ as desired.

(21) a. **Minimal-state consequence:** $\varphi_1, \ldots, \varphi_n \vDash \psi$ iff $S_0[\varphi_1]\ldots[\varphi_n] \vDash \psi$.[4]

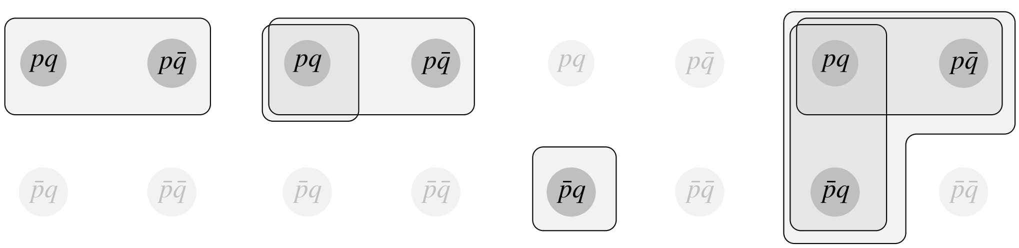
b. **Consistence:** φ is consistent iff $\exists S : S[\varphi] \neq \varnothing$.

c. **Coherence:** φ is coherent iff $\exists S : S \neq \varnothing$ & $S \vDash \varphi$.

I chose to focus on the *minimal-state consequence* in this paper for simplicity reasons. While the current system behaves like Veltman's update semantics in some respect—for example, epistemic contradictions like $\Diamond p \wedge \neg p$ and $\neg p \wedge \Diamond p$ are incoherent—there are also major differences. I will highlight two noteworthy logical features and leave a full axiomatization of the current system as well as examination of other consequence notions for future work. Although these features hold regardless of whether the additive updates involved are conservative or liberal, I will only illustrate using conservative updates in what follows.

4.3.1 Failure of disjunction introduction

First, unlike standard update semantics, the current update system does not vindicate *disjunction introduction*: $\varphi \vDash \varphi \vee \psi$. Though being classically valid, the inference from "2 is a prime number" to "2 is a prime number, or the moon is made of cheese" is undoubtedly odd. On the current account, as Figures 3(a) and 3(b) show, $S_0[p] \nvDash p \vee q$, that is, $S_0[p] \neq S_0[p][p \vee q]$. Since a disjunction draws attention to both disjuncts, updating the output from $S_0[p]$ with $[p \vee q]$ will make q a new live possibility. Hence the update will not be idle.



(a) $S_0[p]$ (b) $S_0[p][p \vee q]$ (c) $S_0[p \wedge \neg q]$ (d) $S_0[p \vee q]$

Figure 3 Failure of disjunction introduction (with conservative updates)

It is worth noting that the ability to accommodate a consequence relation that invalidates disjunction introduction is also a hallmark of truthmaker semantics (see, e.g., Fine 2012, 2017), according to which sentences are evaluated at states of affairs

4 This corresponds to Veltman's (1996) $validity_1$.

which are parts of a possible world. We call a state of affairs an *(exact) verifier* of a sentence when it incorporates just enough relevant information to make the sentence true. States of affairs are related by the mereological parthood relation: for example, the state that only verifies that Ann is at the party is part of the state that verifies that both Ann and Bob are at the party, as the latter is the fusion of the former and the state that only verifies that Bob is at the party. We say that φ entails ψ—or to use Fine's terminology, that ψ is a *conjunctive part* of φ—just in case *for every verifier x of φ, there is a verifier y of ψ such that y is part of x, and for every verifier y of ψ, there is a verifier x of φ such that y is part of x.*

To facilitate the comparison, we may replace *states of affairs* with *information states* and translate y's being part of x as saying that the corresponding information state of x is a subset of that of y; likewise, for x to be a verifier of φ is for the corresponding information state of x to be a member of the hyperstate that represents φ, which can be construed as the output hyperstate of the update $S_0[\varphi]$.

We may then translate the consequence relation of truthmaker semantics into an information state-based semantics as in (22a); a truthmaker-style support condition can also be defined as in (22b).

(22) a. $\varphi \vDash^{tm} \psi$ iff $\forall x \in S_0[\varphi], \exists y \in S_0[\psi] : x \subseteq y$ and $\forall y \in S_0[\psi], \exists x \in S_0[\varphi] : x \subseteq y$.

b. $S \vDash^{tm} \varphi$ iff $\forall x \in S, \exists y \in S_0[\varphi] : x \subseteq y$ and $\forall y \in S_0[\varphi], \exists x \in S : x \subseteq y$.

Now, although both the minimal-state consequence $\vDash$ as defined in (21a) and $\vDash^{tm}$ invalidates the inference from p to $p \vee q$, they are still distinct. For example, $p \wedge \neg q \vDash p \vee q$, whereas $p \wedge \neg q \nvDash^{tm} p \vee q$. Figure 3(c) shows the output of the update $S_0[p \wedge \neg q]$, upon which the update with $[p \vee q]$ will be idle. Unlike with the inference from p to $p \vee q$, the conclusion does not draw attention to any new atomic sentence. Hence, $p \wedge \neg q \vDash p \vee q$. By contrast, it is not the case that for every information state y in Figure 3(d), there is a state x in 3(c) such that $x \subseteq y$; the state $\{pq, p\bar{q}\}$ has none. Thus, $p \wedge \neg q \nvDash^{tm} p \vee q$.

Analogously, the support relation as defined in (7) does not coincide with (22b) but instead with (23), provided that φ is modal-free:

(23) $S \vDash^{*} \varphi$ iff $\forall x \in S, \exists y \in S_0[\varphi] : x \subseteq y$ and $\forall y \in (S_0[\varphi])_{Info(S)}, \exists x \in S : x \subseteq y$.

More specifically, we can show that each conjunct in (23) corresponds to one step in the two-step update: $S[\varphi]_a = S$ iff $\forall y \in (S_0[\varphi])_{Info(S)}, \exists x \in S : x \subseteq y$, and $S[\varphi]_e = S$ iff $\forall x \in S, \exists y \in S_0[\varphi] : x \subseteq y$ (see Appendix for a proof).

4.3.2 Free choice, ignorance inference, and double prohibition

The fact that disjunction introduction is invalid in the current framework can be exploited to give an analysis of "free choice" as well as "ignorance" inference.

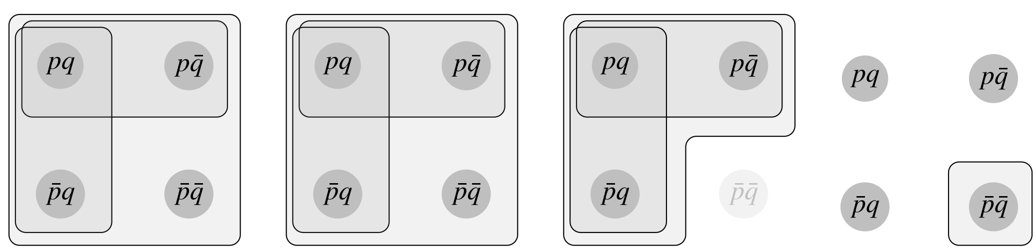
(24) a. Paul might be at the party, or Quinn might be at the party. $\Diamond p \vee \Diamond q$

b. It is possible that either Paul or Quinn is at the party. $\Diamond(p \vee q)$

c. Either Paul or Quinn is at the party. $p \vee q$

Regardless of whether the disjunction takes a wide scope as in (24a) or a narrow scope as in (24b), they both give rise to the inference that $\Diamond p \wedge \Diamond q$; similarly for the plain disjunction in (24c), it gives rise to the ignorance inference that $\Diamond p \wedge \Diamond q$.

However, free choice is in conflict with disjunction introduction (Kamp 2013; Goldstein 2019). To elaborate, if we allow the inference from $\Diamond p$ to $\Diamond p \vee \Diamond q$ by disjunction introduction, and in turn to $\Diamond p \wedge \Diamond q$ by free choice, then we will also have to allow the unwanted inference from $\Diamond p$ to $\Diamond q$. Since disjunction introduction is invalid in the current update system, this problem of explosion is successfully circumvented. Figures 4(a)–4(c) show the effects of updating on the minimal state with the sentences in (24); the output hyperstates all support $\Diamond p \wedge \Diamond q$.



(a) $S_0[\Diamond p \vee \Diamond q]$ (b) $S_0[\Diamond(p \vee q)]$ (c) $S_0[p \vee q]$ (d) $S_0[\neg\Diamond(p \vee q)]$

Figure 4 Free choice, ignorance inference, and double prohibition (with conservative updates)

Additionally, the current framework also validates the inference of "double prohibition": $\neg\Diamond(p \vee q) \vDash \neg\Diamond p \wedge \neg\Diamond q$. Figure 4(d) depicts the outcome of updating the minimal state with $[\neg\Diamond(p \vee q)]$ using the localized eliminative update as defined in (20); the output supports $\neg\Diamond p \wedge \neg\Diamond q$. Now, Goldstein (2019) has observed that free choice, double prohibition, and contraposition are jointly incompatible: by free choice, we have $\Diamond(p \vee q) \vDash \Diamond p$, which becomes $\neg\Diamond p \vDash \neg\Diamond(p \vee q)$ by contraposition, from which we can infer $\neg\Diamond q$ by double prohibition; this consequently vindicates the odd inference from $\neg\Diamond p$ to $\neg\Diamond q$. This inference is blocked in the current framework as contraposition is not in general valid, for a similar reason as to why disjunction introduction fails: $\neg\Diamond p$ fails to entail $\neg\Diamond(p \vee q)$ because while the former only directs attention towards p, the latter also directs attention towards q. Since the update with $[\neg\Diamond(p \vee q)]$ will draw attention to new possibilities, it will not be idle.

5 Open Issues

5.1 Inquisitiveness vs. attentiveness

Given that hyperstates are sets of information states as in inquisitive semantics, it brings the question of whether it is possible to model attentive and inquisitive contents simultaneously at the level of hyperstates. To begin, I follow Ciardelli (2022) and introduce a new disjunction $\mathbin{\vee\!\!\!\vee}$, called *inquisitive disjunction*, into the formal language. The attentive updates with $\mathbin{\vee\!\!\!\vee}$ will be the same as that with $\vee$; the only difference between them concerns the definition for the eliminative update.

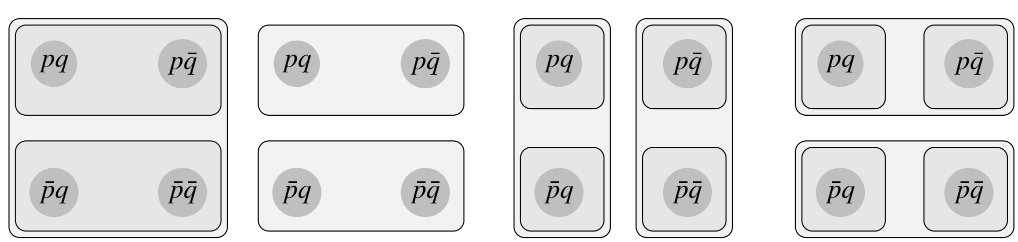
(25) **Eliminative update with $\mathbin{\vee\!\!\!\vee}$:** $S[\varphi \mathbin{\vee\!\!\!\vee} \psi]_e = S[\varphi]_e \cup S[\psi]_e$

Figures 5(a) and 5(b) show the results of updating S_0 with $[p \vee \neg p]$ and $[p \mathbin{\vee\!\!\!\vee} \neg p]$. Both updates draw attention to p and $\neg p$, but only the update with $[p \mathbin{\vee\!\!\!\vee} \neg p]$ produces an inquisitive state as only the latter contains more than one maximal element. I will follow Ciardelli et al. (2018) and define inquisitiveness as follows:

(26) **Inquisitive hyperstates:** S is inquisitive iff $Info(S) \notin S$.

Whereas the update with $[p \vee \neg p]$ merely adds p and $\neg p$ as two new live possibilities, the update with $[p \mathbin{\vee\!\!\!\vee} \neg p]$ in addition eliminates the logical space ω, thereby making the two new possibilities the maximal elements in the output hyperstate. In inquisitive semantics, $?p$, which abbreviates $p \mathbin{\vee\!\!\!\vee} \neg p$, is used to symbolize polar questions such as "Is Paul at the party?"; the two maximal elements in 5(b) thus represent the positive and negative answers to a polar question.

In addition, while both questions and epistemic possibility claims are capable of introducing live possibilities, the current framework is also able to tease them apart. For example, Figures 5(c) and 5(d) show the outcomes of updating S_0 with (27a) and (27b), respectively.



(a) $p \vee \neg p$ (b) $p \mathbin{\vee\!\!\!\vee} \neg p$ (c) $(\Diamond p \wedge \Diamond \neg p) \wedge ?q$ (d) $(\Diamond q \wedge \Diamond \neg q) \wedge ?p$

Figure 5 Updates with normal and inquisitive disjunctions.

(27) a. Paul might be at the party and he might not, but is Quinn at the party?

b. Quinn might be at the party and he might not, but is Paul at the party?

Although every possibility that is live in 5(c) is also live in 5(d), the two hyperstates embody different issues and call for different answers.

Similar to how the informative content of a hyperstate can be derived as in (17a), there is also a way to derive the inquisitive content of a hyperstate.

(28) **Inquisitive content of a hyperstate:** $Inq(S) := \{s' \mid \exists s{:}\ s \in S\ \&\ s' \subseteq s\}$

$Inq(S)$ is defined as the downward closure of S. Since every information state in $Inq(S)$ will be a subset of one of the maximal elements of S, this means every information state in $Inq(S)$ is at least as informative as a possible answer to the question at issue and is thus able to resolve it. Therefore, the current framework enables us to model both inquisitive and attentive contents at the level of hyperstates.[5]

5.2 Unawareness vs. indeterminacy

One issue that pertains to the informativeness of *might*-claims concerns sentences of the form "I don't know whether might p" (DeRose 1991; Dorr & Hawthorne 2013). For example, suppose a cancer test is run for John: a negative result completely rules out cancer; a positive result means that John *might* have cancer, but it does not guarantee that he indeed has cancer. Given this, (29) sounds natural:

(29) I don't know whether John might have cancer. We need to wait for the test result.

Similar to the problem concerning *might*-claims like (2), if the speaker's knowledge is simply modeled as a single information state, we will have trouble making sense of sentences like (29). As the speaker's knowledge state will either contain a world where John has cancer or contain no such worlds, John's having cancer will be either compatible with what the speaker knows, or incompatible. Hence, if x *knows whether might* p simply means that p is either compatible or incompatible with what x knows, which is a tautology, then sentences of the form "I don't know whether might p" can never be true.

5 Roelofsen (2013) raised a potential objection to this way of defining the inquisitive content for Ciardelli et al.'s (2009) attentive semantics. I shall briefly note that this objection does not apply to my account. On Ciardelli et al.'s account, for p to be a live possibility in S, S needs to contain the truth set $|p|$ as a member. Consequently, when the issue in S requires answers that are more fine-grained than simply settling the truth value of p, $|p|$ will fail to be part of the inquisitive content of S. By contrast, my account does not face this problem; in particular, p will be deemed as a live possibility in S so long as S contains some subset of $|p|$.

At the same time, unlike (2), we cannot explain sentences like (29) by appealing to live possibilities either. Intuitively, given the speaker's very utterance "I don't know whether John might have cancer," it is rather unlikely that John's having cancer is not yet a live possibility.

To account for ignorance about *might*-claims, an S-based account does look more promising (cf. Willer 2013; Zhang & Pacuit 2022). We will represent the speaker's knowledge as a hyperstate. To say that the speaker's knowledge does not support "John might have cancer" is to say that there are at least two information states such that one is compatible with John's having cancer while the other is not. Since not *every* information state in the speaker's knowledge state is compatible with the *might*-claim, the whole hyperstate fails to support it under an S-based approach. Again, this analysis of ignorance about *might*-claims has little to do with the attention drawing aspect of *might*-claims. Instead, what seems to be going on in such cases is that the speaker's knowledge is indeterminate between two competing epistemic spaces: one but not the other is compatible with the possibility in question, thereby resulting in ignorance.

6 Conclusion

In this paper, I evaluated two ways to model live possibilities at the level of hyperstaes (i.e., sets of information states) against a set of attentional constraints. Although the alternative-based approach fares better than the supervaluation-based approach in satisfying these constraints, none of the existing A-based accounts comes out as fully adequate. In response, I proposed a new A-based account which employs a two-step update system with two types of additive updates. The resulting framework is able to capture all the attentional constraints. The current framework has interesting logical features which I hope to further explore in the future.

A Appendix: Formal Results

Definition A.1 (Positive and negative attentive contents of $\Box\varphi$)**.** $att^{+}(\Box\varphi) := ||\varphi||^{+}$; $att^{-}(\Box\varphi) := ||\varphi||^{-}$. The additive updates $[\Box\varphi]^{c}_{a}$ and $[\Box\varphi]^{l}_{a}$ can be defined as usual.

Definition A.2 (Eliminative update on an information state s with $[\Box\varphi]^{S}_{e}$)**.** $s[\Box\varphi]^{S}_{e} = \{w \in s \mid \forall s' \in S : s'[\varphi]^{S}_{e} = s'\}$. The eliminative update $S[\Box\varphi]_{e}$ can be defined as usual.

Theorem A.3 (Duality)**.** $S[\Box\varphi] = S[\neg\Diamond\neg\varphi]$

Proof. First, we show that $S[\Box\varphi]^{c}_{a} = S[\neg\Diamond\neg\varphi]^{c}_{a}$. By definition, $att^{+}(\neg\Diamond\neg\varphi) = att^{-}(\Diamond\neg\varphi) = ||\neg\varphi||^{-} = ||\varphi||^{+} = att^{+}(\Box\varphi)$. Likewise, $S[\Box\varphi]^{l}_{a} = S[\neg\Diamond\neg\varphi]^{l}_{a}$. Next, we show that $S[\Box\varphi]_{e} = S[\neg\Diamond\neg\varphi]_{e}$. Consider the state-level update, by definition,

$s[\neg\Diamond\neg\varphi]^S_e = s - s[\Diamond\neg\varphi]^S_e$. By definition, $s[\Diamond\neg\varphi]^S_e = \varnothing$ iff $\forall s' \in S : s'[\neg\varphi]^S_e = \varnothing$ iff $\forall s' \in S : s'[\varphi]^S_e = s'$ iff $s[\Box\varphi]^S_e = s$. Hence, it follows that $s[\neg\Diamond\neg\varphi]^S_e = s$ iff $s[\Box\varphi]^S_e = s$; by parity of reasoning, $s[\neg\Diamond\neg\varphi]^S_e = \varnothing$ iff $s[\Box\varphi]^S_e = \varnothing$. Consequently, $\forall s : s[\neg\Diamond\neg\varphi]^S_e = s[\Box\varphi]^S_e$, which means that $S[\Box\varphi]_e = S[\neg\Diamond\neg\varphi]_e$. □

Theorem A.4 (Equivalence between two notions of *support*)**.** *Provided that φ is Boolean, $S \vDash \varphi$ iff $S \vDash^* \varphi$, that is, $S[\varphi] = S$ iff $\forall s \in S, \exists s' \in S_0[\varphi] : s \subseteq s'$ and $\forall s' \in (S_0[\varphi])_{Info(S)}, \exists s \in S : s \subseteq s'$.*

I will illustrate using the conservative update. This means we want to establish the following: *provided that φ is Boolean, $S \vDash \varphi$ iff $S \vDash^* \varphi$, that is, $S[\varphi]^c_a[\varphi]_e = S$ iff $\forall s \in S, \exists s' \in S_0[\varphi]^c_a[\varphi]_e : s \subseteq s'$ and $\forall s' \in (S_0[\varphi]^c_a[\varphi]_e)_{Info(S)}, \exists s \in S : s \subseteq s'$*. I will prove this via the following lemmas.

Lemma A.5. *$S \vDash \varphi$ iff $S[\varphi]^c_a = S[\varphi]_e = S$.*

Proof. Since by Definition (18), all (if any) newly added possibilities from the update with $[\varphi]^c_a$ are restricted to $Info(S)$, they will all be a subset of $Info(S)$. Given this, the eliminative update will not be able to remove them without eliminating some world in $Info(S)$. Consequently, for the output of the eliminative update to be identical to its input, no new possibilities can be added. Thus, the additive update must also be idle. □

Lemma A.6. *$S[\varphi]^c_a = S$ iff $\forall s' \in (S_0[\varphi]^c_a[\varphi]_e)_{Info(S)}, \exists s \in S : s \subseteq s'$.*

Proof. This can be proven by an induction. For the base case, we want to show that $S[p]_a = S$ iff $\forall s' \in (S_0[p]^c_a[p]_e)_{Info(S)}, \exists s \in S : s \subseteq s'$. By Definition (18), $S[p]^c_a = S$ iff $\forall s' \in att^+(p)_{Info(S)}, \exists s \in S : s \subseteq s'$. Since $att^+(p)_{Info(S)} = \{|p| \cap Info(S)\} - \{\varnothing\} = (S_0[p]^c_a[p]_e)_{Info(S)}$, the biconditional immediately follows. We can then perform an induction on the structure of φ and show that for all φ, $att^+(\varphi)_{Info(S)} = (S_0[\varphi]^c_a[\varphi]_e)_{Info(S)}$. □

Lemma A.7. *$S[\varphi]_e = S$ iff $\forall s \in S, \exists s' \in S_0[\varphi]^c_a[\varphi]_e : s \subseteq s'$.*

Proof. By definition, $S[\varphi]_e = \{s[\varphi]^S_e \mid s \in S\} - \{\varnothing\}$. Given that $s[\varphi]_e$ will always be a subset of s, it means that when $S[\varphi]_e = S$ holds, we have $\forall s \in S : s[\varphi]_e = s$ provided that φ is Boolean. This means $\forall s \in S : s \subseteq |\varphi|$. Hence, what we need to show is for any s, $s \subseteq |\varphi|$ iff $\exists s' \in S_0[\varphi]^c_a[\varphi]_e : s \subseteq s'$. Since $\omega \in S_0[\varphi]^c_a$, by the definition of the eliminative update, it follows that $\omega \cap |\varphi| \in S_0[\varphi]$, which is equivalent to $|\varphi| \in S_0[\varphi]$. Consequently, $\exists s' \in S_0[\varphi] : s \subseteq s'$ holds just in case $s \subseteq |\varphi|$ holds. □

References

Aloni, Maria. 2022. Logic and conversation: the case of free choice. *Semantics and Pragmatics* 15. 5–EA. doi:10.3765/sp.15.5.

Bledin, Justin & Kyle Rawlins. 2020. Resistance and resolution: attentional dynamics in discourse. *Journal of Semantics* 37(1). 43–82. doi:10.1093/jos/ffz015.

Ciardelli, Ivano. 2022. *Inquisitive Logic: Consequence and Inference in the Realm of Questions*. Springer Nature. doi:10.1007/978-3-031-09706-5.

Ciardelli, Ivano, Jeroen Groenendijk & Floris Roelofsen. 2009. Attention! 'might' in inquisitive semantics. In *Semantics and Linguistic Theory (SALT)*, vol. 19, 91–108. doi:10.3765/salt.v0i0.2520.

Ciardelli, Ivano, Jeroen Groenendijk & Floris Roelofsen. 2018. *Inquisitive Semantics*. Oxford University Press. doi:10.1093/oso/9780198814788.001.0001.

Ciardelli, Ivano, Floris Roelofsen & Nadine Theiler. 2017. Composing alternatives. *Linguistics and Philosophy* 40. 1–36. doi:10.1007/s10988-016-9195-2.

DeRose, Keith. 1991. Epistemic possibility. *Philosophical Review* 100. 581 – 605. doi:10.2307/2185175.

Dorr, Cian & John Hawthorne. 2013. Embedding epistemic modals. *Mind* 122. 867 – 913. doi:10.1093/mind/fzt091.

Dretske, Fred. 1970. Epistemic operators. *The Journal of Philosophy* 67(24). 1007–1023. doi:10.2307/2024710.

Fine, Kit. 2012. Counterfactuals without possible worlds. *The Journal of Philosophy* 109(3). 221–246. doi:10.5840/jphil201210938.

Fine, Kit. 2017. Truthmaker semantics. *A Companion to the Philosophy of Language* 556–577. doi:10.1002/9781118972090.ch22.

Fintel, Kai von. 2001. Counterfactuals in a dynamic context. *Current Studies in Linguistics Series* 36. 123–152. doi:10.7551/mitpress/4056.003.0006.

Gillies, Anthony S. 2007. Counterfactual scorekeeping. *Linguistics and Philosophy* 30. 329–360. doi:10.1007/s10988-007-9018-6.

Goldstein, Simon. 2019. Free choice and homogeneity. *Semantics and Pragmatics* 12. 23–EA. doi:10.3765/sp.12.23.

Grice, Herbert P. 1975. Logic and conversation. In *Speech Acts*, 41–58. Brill. doi:10.1163/9789004368811_003.

Groenendijk, Jeroen, Marin Stokhof & Frank Veltman. 1996. Coreference and modality. In S. Lappin (ed.), *The Handbook of Contemporary Semantic Theory*, 179–216. Oxford: Blackwell.

Iatridou, Sabine & Hedde Zeijlstra. 2013. Negation, polarity, and deontic modals. *Linguistic Inquiry* 44(4). 529–568. doi:10.1162/ling_a_00138.

Kamp, Hans. 2013. Free choice permission. In *Meaning and the Dynamics of Interpretation*, 169–184. Brill. doi:10.1163/9789004252882_008.

Lewis, David. 1988. Relevant implication. *Theoria* 54(3). 161–174. doi:10.1111/j.1755-2567.1988.tb00716.x.

Roelofsen, Floris. 2013. A bare bone attentive semantics for might. *The Dynamic, Inquisitive, and Visionary Life of* ϕ 190–215.

Veltman, Frank. 1996. Defaults in update semantics. *Journal of Philosophical Logic* 25. 220 – 260. doi:10.1007/bf00248150.

Willer, Malte. 2013. Dynamics of epistemic modality. *Philosophical Review* 122. 45 – 92. doi:10.1215/00318108-1728714.

Willer, Malte. 2017. Lessons from Sobel sequences. *Semantics and Pragmatics* 10(4). doi:10.3765/sp.10.4.

Willer, Malte. 2018. Simplifying with free choice. *Topoi* 37(3). 379–392. doi:10.1007/s11245-016-9437-5.

Yalcin, Seth. 2007. Epistemic modals. *Mind* 116. 983–1026. doi:10.1093/mind/fzm983.

Yalcin, Seth. 2018. Belief as question-sensitive. *Philosophy and Phenomenological Research* 97(1). 23–47. doi:10.1111/phpr.12330.

Zhang, Yichi & Eric Pacuit. 2022. Epistemic pseudo-agreement and hyper-domain semantics for epistemic modals. *Amsterdam Colloquium (AC)* 23.

Yichi (Raven) Zhang
Heinrich Heine Universität Düsseldorf
Universitätstr.1, 40225
yichi.zhang@hhu.de